SCHOOL and CONFERENCE on COMPLEX SYSTEMS and NONEXTENSIVE STATISTICAL MECHANICS

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The Olami-Feder-Christensen Model on a Small World topology

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Outline

- Self-organized Criticality (SOC)
- Earthquake phenomenology
  - Gutenberg-Richter Law
- SOC Model:
  - Olami-Feder-Cristensen Model (OFC)
- Power Law and Finite-Size Scaling
  - OFC on a small world topology
- Turbulence-like analysis in OFC models
Self-organized criticality (SOC) can describe emergent complex behavior in physical systems. SOC is an out-of-equilibrium mechanism that drives a system towards a critical state.

Self-organized criticality (SOC) …

- is manifested by temporal and spatial scale invariance (power laws)
- is driven by intermittent evolutions with bursts/avalanches that extend over a wide range of magnitudes

Some examples:
- Earthquakes dynamics
- Sand-pile models (Bak et al. 1987)
- Evolution (Bak and Sneppen 1993)
- Solar flares
- …etc.
It is generally believed that earthquakes result from a stick-slip dynamics involving the Earth’s crust sliding along faults.

San-Andreas Fault marks the contact between the Pacific and North American Plates.

San-Andreas fault, where the great 1906 San Francisco earthquake occurred.

There are more small earthquakes than large ones. But there is no apparent cut-off in the possible size of an earthquake; earthquakes of all sizes are possible.
A simple computer model of the earthquake fault is composed of one elastic plate and one rigid plate, for simplicity's sake.

**Burridge-Knopoff Model (1967)**

The blocks interact with the rigid plate through friction.

- The system is driven through slow uniform movement …
- Whenever the spring force exceeds a critical value, the block slides and the spring force is reduced.
- The force lost by the block is transferred to its neighbours; this may causes one or more of its neighbours to slide, and so on …. an earthquake is generated.
- The size of the earthquake in the model is defined as the number of blocks that have slid during the earthquake.

**Olami-Feder-Christensen (OFC) Model of earthquakes**

**System:** discrete system of blocks on a square lattice, each carrying a force, a real variable, $F_i$ in the range $(0, F_{th})$.

**Slow Driving:** all the forces are increased uniformly until one of them reaches the threshold value $F_{th}$ and becomes unstable ($F_i > F_{th}$).

**Earthquake:** the uniform driving is then stopped and an “earthquake” (or avalanche) starts:

$$F_i \geq F_{th} \Rightarrow \begin{cases} F'_i \to 0 \\ F_{nn} \to F_{nn} + \alpha F_i \end{cases}$$

where “nn” denotes the set of nearest-neighbor sites of $i$.

The earthquake is over when there are no more unstable sites in the system ($F_i < F_{th}$). The number of topplings during an earthquake defines its size, $s$.

**Parameters:** $\alpha$ controls the level of conservation of the dynamics and, in the case of a graph with fixed connectivity $q$, it takes values between 0 and $1/q$ ($=1/q$ corresponding to the conservative case).

**Data:** to collect the earthquake statistics, we need to skip some initial number of earthquakes (transient behaviour $\to$ critical state).

Simulations on a lattice of $L=32, 64, 128$ (NN OFC model) after $1E+09$ avalanches with OPEN boundary conditions.
The small-world behavior is characterized by the fact that the distance between any two vertices is of the order of that for a random network and, at the same time, the concept of neighbourhood is preserved, as for regular lattices. For this reason, we will expect to obtain SOC in a small world.

1998 - Watts and Strogatz (USA)
Constructing the network ...
Constructing the network ...
In order to characterize the critical behavior of the model, a finite size scaling (FSS) ansatz is used:

\[ P_N(s) \approx N^{-\beta} \cdot f\left(s / N^D\right) \]

where \( f \) is a suitable scaling function and \( \beta \) and \( D \) are critical exponents describing the scaling of the distribution function.
In a random regular graph all sites have exactly the same number of nearest neighbours $q$. In this case, it’s verified that (both for $q=4$ and $q=6$) the system organizes into a subcritical state. In order to observe scaling in the avalanche distribution, one has to introduce some inhomogeneities. For the OFC model on a (quenched) random graph, it’s found that it suffices to consider just two sites in the system with coordination $q-1$.

Is it sufficient to consider a small world graph, obtained by randomizing a fraction $p$ of the links of the regular nearest neighbour lattice, in order to obtain FSS?
Finite-size scaling for Dissipative OFC model on a Small World topology ($\alpha=0.21$)
The critical exponents are $D=2$ and $\beta=3.6$
The possibility of establishing a close analogy between 2D BTW sandpile dynamics and fully developed turbulent scaling has just been showed by Stella and De Menech in 2001.


Is it possible to find a similar connection in OFC models on a small world topology?
Turbulence-like analysis

We consider the difference between two avalanches, i.e.

\[ x = S(t + \delta) - S(t) \]

being \( S(t+\delta) \) and \( S(t) \) two successive avalanches with a time difference \( \delta \).

The time series of \( x \) is very intermittent and the pdfs are non Gaussian at criticality, i.e. when long range correlation and finite size scaling exist.

Caruso et al. cond-mat/0606118
Fat $q$-gaussian tails at criticality …

![Graph showing critical and non-critical cases]

- **Critical case**
- **Non-critical case**

Legend:
- OFC on a regular lattice (64x64)
- OFC on a Small World (64x64)
- Gaussian
- $q$-Gaussian $q=2$
Fat q-gaussian tails at criticality ...

No dependence on the time interval
Analysis of real data: World Catalog

We considered $S \sim \exp(M)$, $M$ being the earthquake magnitude.

World Catalog: 689000 events in the period 2001-2006
Analysis of real data: Northern California

Northern California: 400000 events in the period 1966-2006
A simple model

Assuming no correlation in two stochastic variables with a power law distribution and taking their difference, we get for the pdf of the difference $x = S_1(t+\Delta) - S_2(t)$ the formula

$$P(x) = \int_0^\infty dS_1 \int_x^\infty dS_2 (S_1 S_2)^{-\tau} \delta(S_1 - S_2 - x) =$$

$$= K_2 F_1 \left( \tau, 2\tau - 1; 2\tau; -\frac{|x|}{\varepsilon} \right)$$

This $P(x)$, which can be approximated by a q-Gaussian, is able to reproduce both the numerical and the experimental data once $\tau$ is given.
Thanks for the attention