The Abdus Salam
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Educational, Scientific
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SCHOOL and CONFERENCE
on
COMPLEX SYSTEMS
and
NONEXTENSIVE STATISTICAL MECHANICS

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Superstatistics: Further Applications

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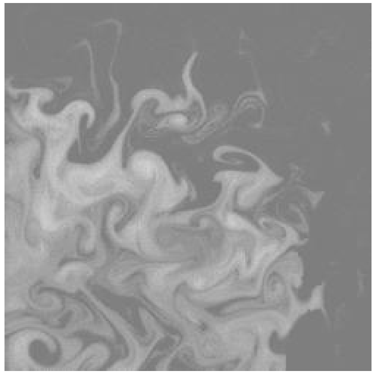
Superstatistics
Further applications



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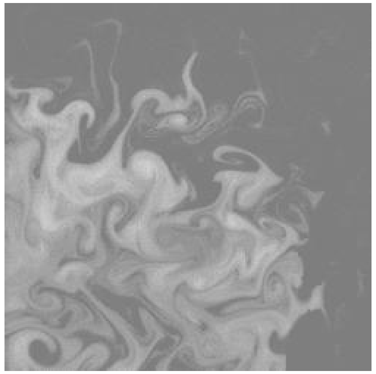

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Content

1 Short reminder: What is superstatistics?

Consider driven nonequilibrium situation with local fluctuations of the environment.
Starting point is the following formula

$$\int_0^\infty d\beta f(\beta) e^{-\beta E} = \frac{1}{(1+(q-1)\beta_0 E)^{1/(q-1)}}$$

where

$$f(\beta) = \frac{1}{\Gamma\left(\frac{1}{q-1}\right)} \left\{\frac{1}{(q-1)\beta_0}\right\}^{\frac{1}{q-1}} \beta^{\frac{1}{q-1}-1} \exp\left\{-\frac{\beta}{(q-1)\beta_0}\right\}$$

is the χ^2 distribution.

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is the χ^2 distribution.
(ordinary Boltzmann factor (with fluctuating β) $\longleftrightarrow$ generalized Boltzmann factor)

Physical interpretation: Tsallis' type of statistical mechanics relevant for nonequilibrium systems with temperature fluctuations.

Can construct dynamical realization in terms of Langevin equation

$$\dot{v} = -\gamma v + \sigma L(t)$$

with fluctuating parameters γ, σ (C.B., Phys. Rev. Lett., 2001).

Consider a Brownian particle that moves through spatial 'cells' with different local $\beta := \gamma/(2\sigma^2)$ in each cell (a nonequilibrium situation)

Assume probability distribution of β in the various cells is χ^2-distribution of degree n

$$f(\beta) \sim \beta^{n/2-1} e^{-\frac{n\beta}{2\beta_0}}$$

(e.g. $\beta = \sum_{i=1}^{n} X_i^2$)

Conditional prob. $p(v|\beta)) \sim e^{-\frac{1}{2}\beta v^2}$

Joint prob. $p(v, \beta) = f(\beta)p(v|\beta)$

Marginal prob. $p(v) = \int_0^\infty f(\beta)p(v|\beta)d\beta$

Integration yields

$$p(v) \sim \frac{1}{(1 + \frac{1}{2}\tilde{\beta}(q-1)v^2)^{1/(q-1)}}$$

(power-law Boltzmann factors with $q = 1 + \frac{2}{n+1}$, $\tilde{\beta} = 2\beta_0/(3-q)$, and $E = \frac{1}{2}v^2$)

$\beta_0 = \int f(\beta)\beta d\beta =$ average of β

Very broad interpretation—β need not be inverse temperature but can be any system parameter entering the Langevin dynamics. Similarly, v need not be a velocity. Nonlinear Langevin eq. with nontrival potential $V(v)$ also possible.

Can generalize the above example to general probability densities $f(\beta)$ and general Hamiltonians E.
Superposition of two different statistics: that of β and that of ordinary stat. mech. Short name:
Superstatistics (C.B. and E.G.D Cohen, Physica A (2003))
Consider nonequilibrium system with spatio-temporal fluctuations of an intensive parameter (e.g. inverse temperature) on a long time scale T_S.

Think, e.g., of a Brownian particle moving through different spatial cells with different temperature. Relaxation time $\gamma^{-1} << T_S$.

In the long-term run system is decribed by a mixture of different Boltzmann factors.

Define effective Boltzmann factor $B(E)$ by

$$B(E) = \int_0^\infty f(\beta) e^{-\beta E} d\beta$$

$f(\beta)$: probability distribution of β.
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- For sharply peaked $f(\beta)$ all superstatistics approach Tsallis statistics in a universal way.
- $$q = \frac{\langle \beta^2 \rangle}{\langle \beta \rangle^2}$$

  (no fluctuations: $\Rightarrow q \to 1$).
- β: can quite generally be some intensive variable or some system parameter describing environment

- Can prove superstatistical generalization of fluctuation theorems (C.B. and E.G.D. Cohen, Physica A (2004))
- Can develop variational principle for large-energy asymptotics of general superstatistics (H. Touchette and C.B., Phys. Rev. E (2005))

  (depending on $f(\beta)$, one can get not only power laws for large E but e.g. also stretched exponentials)
- Can formally define generalized entropies for general superstatistics (Tsallis and Souza, Phys. Rev. E (2003))
- Can study various theoretical extensions of the superstatistics concept (Chavanis (2005), Vignat, Plastino (2005), Grigolini et al. (2005))
- Can apply superstatistical methods to analyse statistics of 3d hydrodynamic turbulence (C.B., PRL 2001, A. Reynolds, PRL 2003, C.B., Europhysics Lett. 2003, E. Bodenschatz 2004, C.B., E.G.D. Cohen and H.L. Swinney, Phys. Rev. E 2005, ...)
- Can apply it to atmospheric turbulence (wind velocity fluctuations at Florence airport, Rizzo, Rapisarda (2004))
- Can apply superstatistical methods to finance (Bouchard 2003, Ausloos 2003), solar flares (Baiesi, Stella, Paczuski 2004), networks (Abe 2005), random matrix theory (Abul-Magd 2006), ...

2 Physically relevant superstatistical universality classes

Basically, there are 3 physically relevant universality classes:

- (a) χ^2-superstatistics (= Tsallis statistics)
- (b) inverse χ^2-superstatistics
- (c) lognormal superstatistics

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Why? Consider, e.g., case (a). Assume there are many microscopic RV ξ_j, $j = 1, \dots, J$, contributing to β in an additive way. For large J, sum $\frac{1}{\sqrt{J}} \sum_{j=1}^{J} \xi_j$ will approach a Gaussian random variable X_1 due to the Central Limit Theorem.

There can be n Gaussian random variables $X_1, \dots, X_n$ due to various relevant degrees of freedom in the system.

β positive $\Rightarrow$ $\beta = \sum_{i=1}^{n} X_i^2$ is χ^2-distributed with degree n,

$$f(\beta) = \frac{1}{\Gamma(\frac{n}{2})} \left(\frac{n}{2\beta_0} \right)^{n/2} \beta^{n/2-1} e^{-\frac{n\beta}{2\beta_0}}, \tag{1}$$

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$\Rightarrow$ Tsallis statistics is a universal limit dynamics, i.e., the details of the microscopic random variables ξ_j are irrelevant.

(b) Same considerations can be applied if the 'temperature' β^{-1} rather than β itself is the sum of several squared Gaussian random variables arising out of many microscopic degrees of freedom ξ_j. Resulting $f(\beta)$ is the inverse χ^2-distribution:

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It generates superstatistical distributions $p(E) \sim \int f(\beta) e^{-\beta E}$ that that decay as $e^{-\tilde{\beta}\sqrt{E}}$ for large E.

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(c) β may be generated by multiplicative random processes. Local cascade random variable $X_1 = \prod_{j=1}^J \xi_j$, where J is the number of cascade steps and the ξ_j are positive microscopic random variables. By the Central Limit Theorem, for large J the RV $\frac{1}{\sqrt{J}} \log X_1 = \frac{1}{\sqrt{J}} \sum_{j=1}^J \log \xi_j$ becomes Gaussian for large J. Hence X_1 is log-normally distributed. In general there may be n such product contributions to β, i.e., $\beta = \prod_{i=1}^n X_i$. Then $\log \beta = \sum_{i=1}^n \log X_i$ is a sum of Gaussian random variables; hence it is Gaussian as well. Thus β is log-normally distributed, i.e.,

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3 Application to Lagrangian turbulence

Navier-Stokes equation: $\dot{\boldsymbol{v}} = -(\boldsymbol{v}\boldsymbol{\nabla})\boldsymbol{v} - \frac{1}{2}\boldsymbol{\sigma}^2\boldsymbol{\Delta}\boldsymbol{v} + \boldsymbol{F}$

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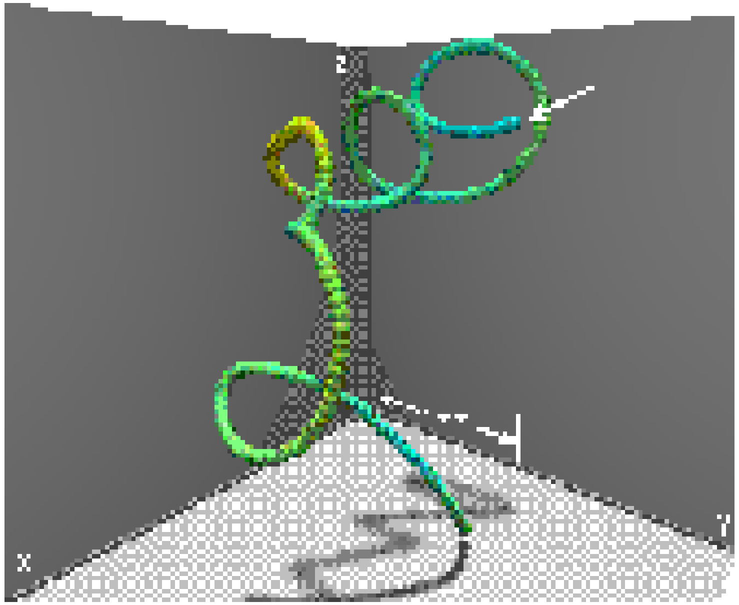
Bodenschatz et al., Nature (2001)

Measurements of acceleration $\boldsymbol{a}$ of single tracer particle in turbulent flow.



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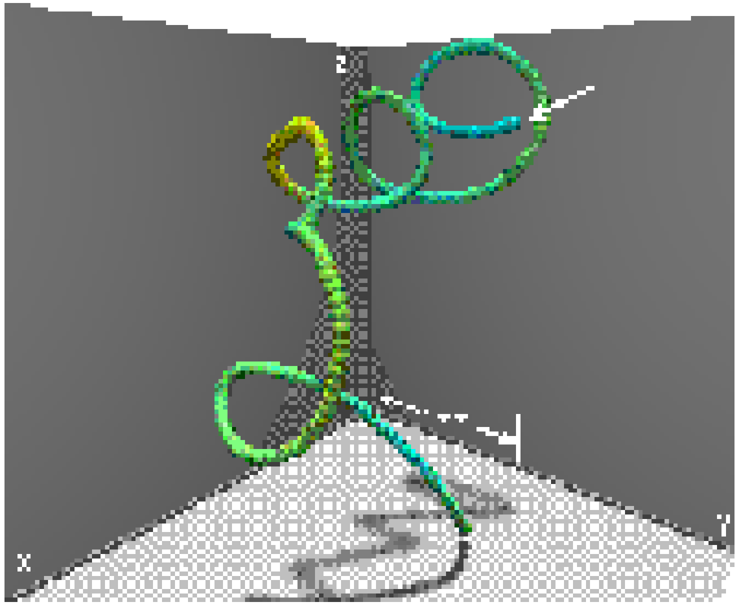
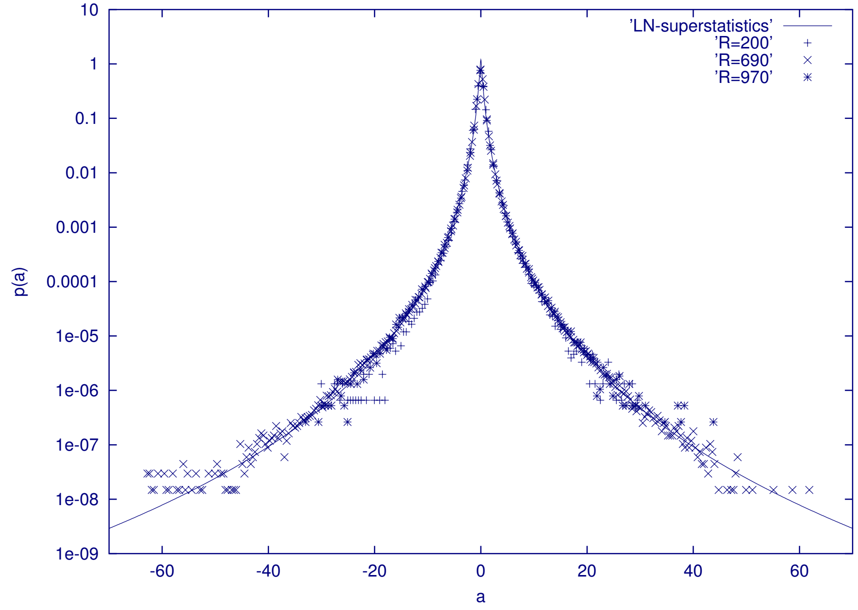


colour code:

blue.......green...yellow

$\boldsymbol{a} = 0$..................16000 $\boldsymbol{m/s^2}$

Measured probability distribution of acceleration:



strongly non-Gaussian

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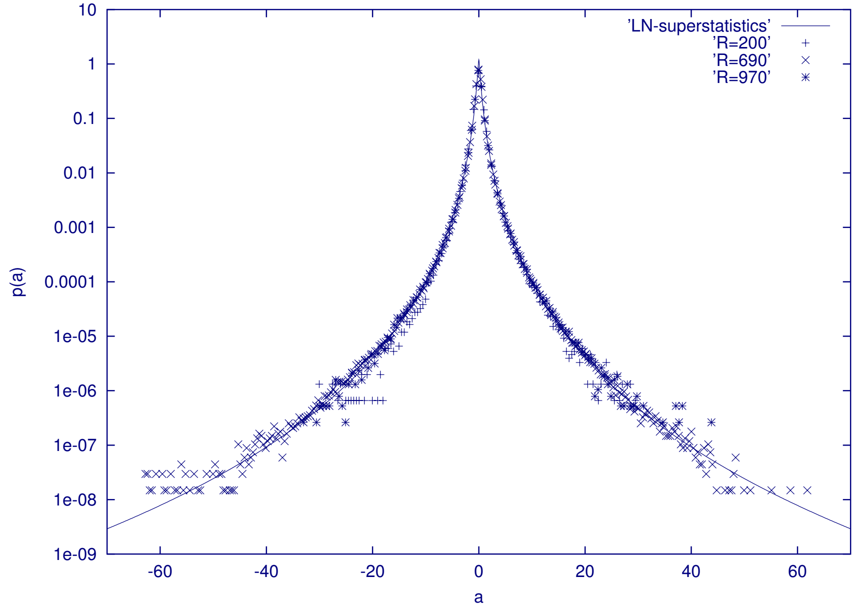

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Prediction of superstatistical Lagrangian turbulence model (generalized Sawford model):

$$p(a) = \frac{1}{2\pi s} \int_0^\infty d\beta \beta^{-1/2} \exp\left\{ -\frac{(\log(\beta/\mu))^2}{2s^2} \right\} e^{-\frac{1}{2}\beta a^2}$$

C.B., Phys. Rev. Lett. (2001), Europhys. Lett. (2003)

A. Reynolds, Phys. Rev. Lett. (2003)

(based on lognormal superstatistics ($s^2 = 3.0$). β corresponds to fluctuating energy dissipation.)

Sawford model (1991): joint stochastic process $(a(t), v(t), x(t))$ of acceleration, velocity and position of a Lagrangian test particle obeys

$$\begin{aligned} \dot{a} &= -(T_L^{-1} + t_\eta^{-1})a - T_L^{-1} t_\eta^{-1} v \\ &\quad + \sqrt{2\sigma_v^2 (T_L^{-1} + t_\eta^{-1}) T_L^{-1} t_\eta^{-1}}\, L(t) \qquad (4) \\ \dot{v} &= a \qquad (5) \\ \dot{x} &= v, \qquad (6) \end{aligned}$$

$L(t)$: Gaussian white noise
T_L and t_η: two time scales, with $T_L >> t_\eta$.
$T_L = 2\sigma_v^2/(C_0\bar{\epsilon})$
$t_\eta = 2a_0\nu^{1/2}/(C_0\bar{\epsilon}^{1/2})$
$\bar{\epsilon}$: average energy dissipation
C_0, a_0: are Lagrangian structure function constants
σ_v^2 is the variance of the velocity distribution.
Taylor scale Reynolds number is $R_\lambda = \sqrt{15}\sigma_v^2/\sqrt{\nu\bar{\epsilon}}$

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σ_v^2 is the variance of the velocity distribution.
Taylor scale Reynolds number is $R_\lambda = \sqrt{15}\sigma_v^2/\sqrt{\nu\bar{\epsilon}}$
Sawford model predicts Gaussian stationary distributions for a and v, and is thus at variance with the recent measurements.

Idea: generalize Sawford model with constant parameters to a superstatistical Sawford model with fluctuating parameters

Simplification for $T_L \longrightarrow \infty$ (reasonable approximation for large Reynolds numbers). In that limit the model reduces to

$$\dot{a} = -\gamma a + \sigma L(t) \tag{7}$$

with

$$\gamma = \frac{C_0}{2a_0}\nu^{-1/2}\bar{\epsilon}^{1/2} \tag{8}$$

$$\sigma = \frac{C_0^{3/2}}{2a_0}\nu^{-1/2}\bar{\epsilon} \tag{9}$$

To construct superstatistical extension of Sawford model, replace constant energy dissipation $\bar{\epsilon}$ by a fluctuating one (e.g. log-normally distributed). One gets

$$\beta = \frac{\gamma}{\sigma^2} = \frac{2a_0}{C_0^2}\nu^{1/2}\epsilon^{-3/2}, \tag{10}$$

and for a_0 defined by

$$\langle a^2 \rangle =: a_0 \langle \epsilon \rangle^{3/2} \nu^{-1/2} \tag{11}$$

one obtains $a_0 = \frac{1}{\sqrt{2}} C_0 e^{\frac{1}{12}s^2}$ and

$$p(a) = \frac{1}{2\pi s}\int_0^\infty d\beta \beta^{-1/2} \exp\left\{-\frac{(\log(\beta/\mu))^2}{2s^2}\right\} e^{-\frac{1}{2}\beta a^2} \tag{12}$$

All this in good agreement with experimental data.

Can also calculate the Lagrangian scaling exponents ζ_j.
Let δu denote the velocity difference of the Lagrangian particle on a time scale δt

$$\langle (\delta u)^j \rangle \sim \delta t^{\zeta_j} \tag{13}$$

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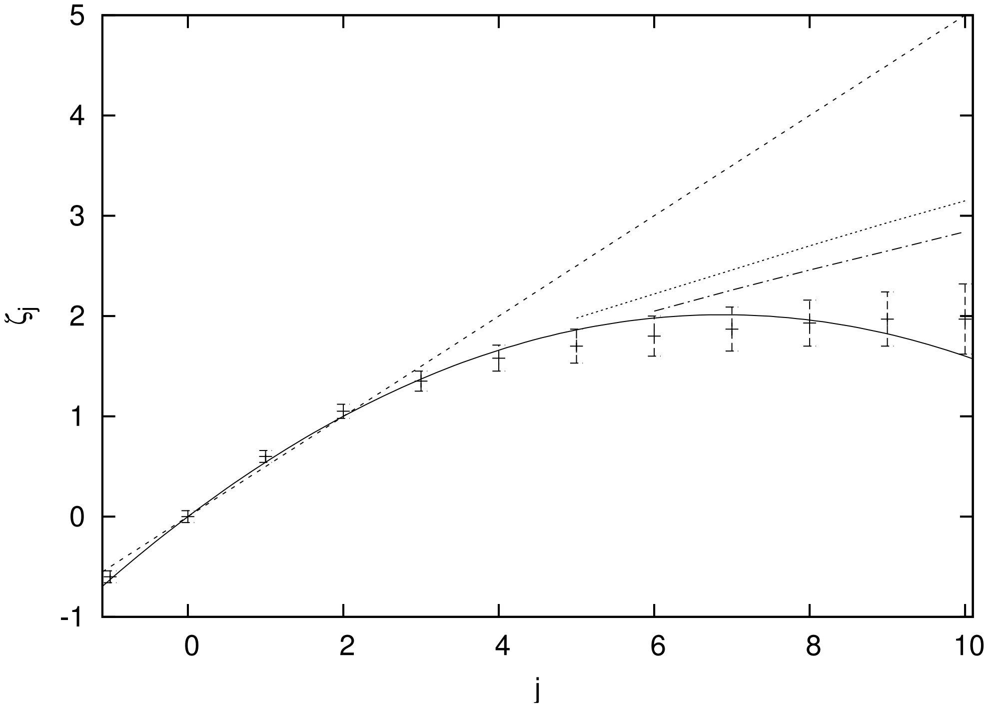

Data points: Measurements of Bodenschatz et al. (PRL 2006)
Solid line: Theoretical prediction of superstatistical model (C.B., cond-mat/0606655)
Dashed lines: Some other competing models (Biferale et al.)

Superstatistical Lagrangian model for 3-dim acceleration vector $\vec{a} = (a_x, a_y, a_z)$ (C.B., cond-mat/0606655)

$$\dot{\vec{a}} = -\gamma \vec{a} + B\vec{n} \times \vec{a} + \sigma \vec{L}(t). \tag{14}$$

Induces correlations between components: Study $R := p(a_x, a_y)/(p(a_x)p(a_y))$.

For independent acceleration components this ratio would always be given by $R = 1$. However, 3-d superstatistical model yields prediction

$$R = \frac{\int_0^\infty \beta f(\beta) e^{-\frac{1}{2}\beta\tau^2(a_x^2+a_y^2)} d\beta}{\int_0^\infty \beta^{1/2} f(\beta) e^{-\frac{1}{2}\beta\tau^2 a_x^2} d\beta \int_0^\infty \beta^{1/2} f(\beta) e^{-\frac{1}{2}\beta\tau^2 a_y^2} d\beta} \tag{15}$$

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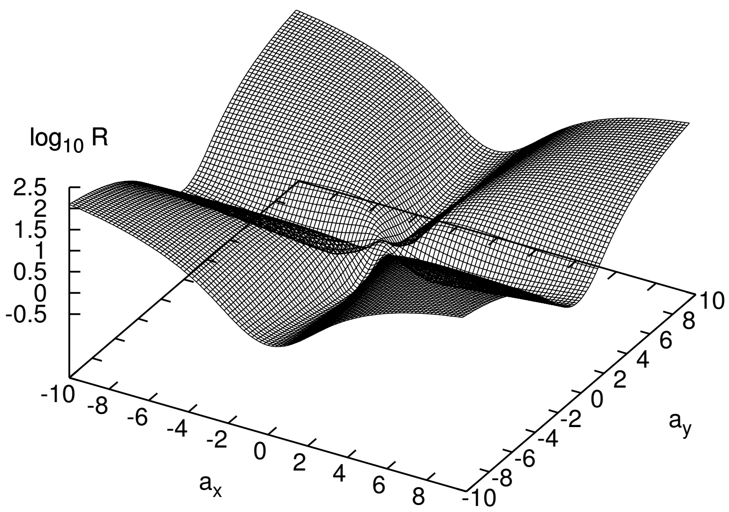
Plotted below for the example of the lognormal distribution $f(\beta)$.



Bodenschatz measures correlations of similar shape (New Journal Physics 2005)

4 Application to pattern forming systems (defect turbulence)

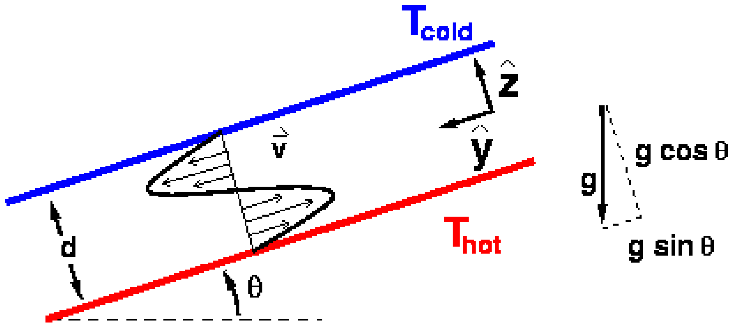
Raleigh-Benard experiment:

Heat liquid from below, cool from above.

$\Rightarrow$ Convection rolls

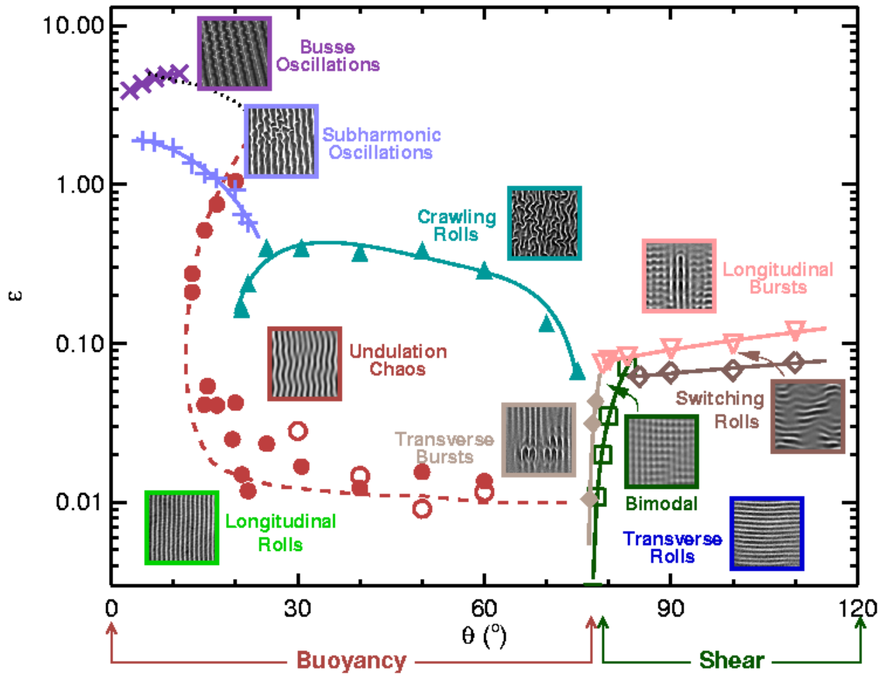
Recent experiment by Daniels and Bodenschatz (Cornell):

Tilted Raleigh-Benard (inclined layer convection)

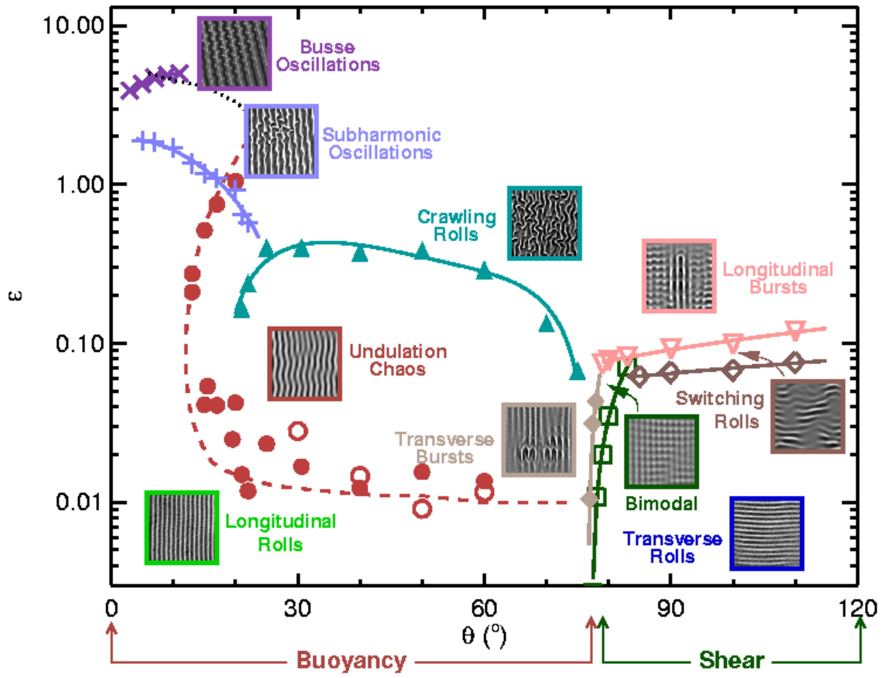


There are two parameters:

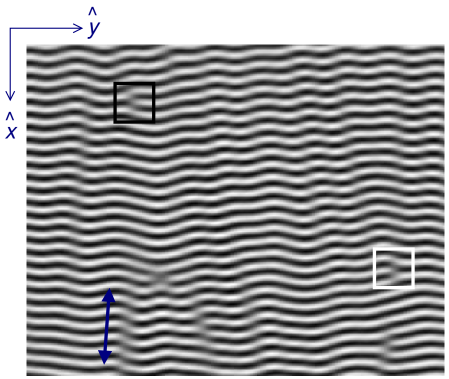
1. temperature difference ϵ
2. angle of tilt θ



10.00
1.00
0.10
0.01
ε
0
30
60
90
120
θ (°)
Busse Oscillations
Subharmonic Oscillations
Crawling Rolls
Longitudinal Bursts
Undulation Chaos
Switching Rolls
Transverse Bursts
Bimodal
Longitudinal Rolls
Transverse Rolls
Buoyancy
Shear



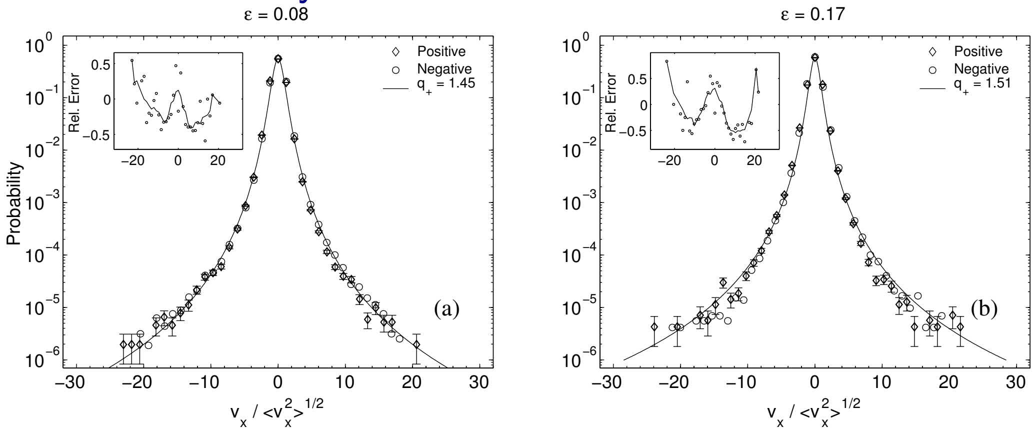
Interested in defects:



Defects are created and annihilated in pairs. They move around in a chaotic way.
Black: positive defect. White: negative defect.

Beach near St. Andrews
Similar defect patterns can be seen in sand!

Measured velocity distributions of defects:



K. Daniels, C.B. , E. Bodenschatz, Physica D (2004)

Theoretical model based on superstatistics.

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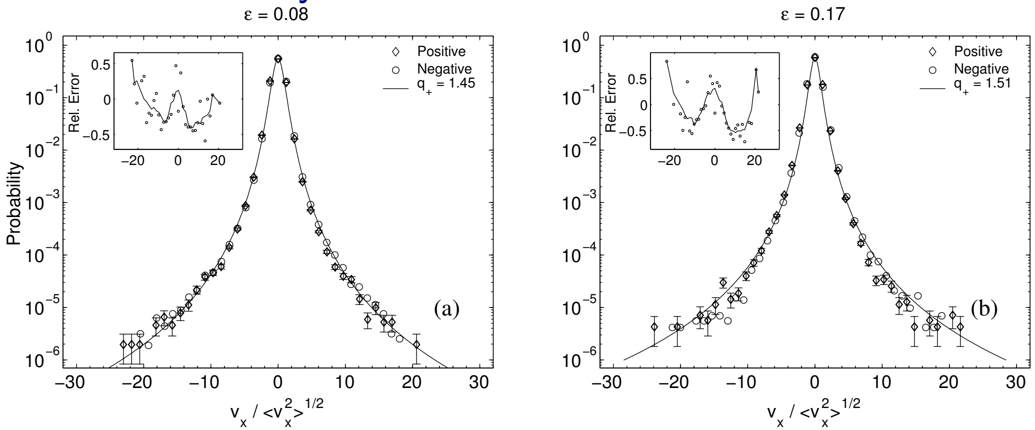

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All data for the defects (probability densities, correlation functions, anomalous diffusion exponents) well described by superstatistical Langevin model with χ^2-distributed fluctuating effective friction γ.

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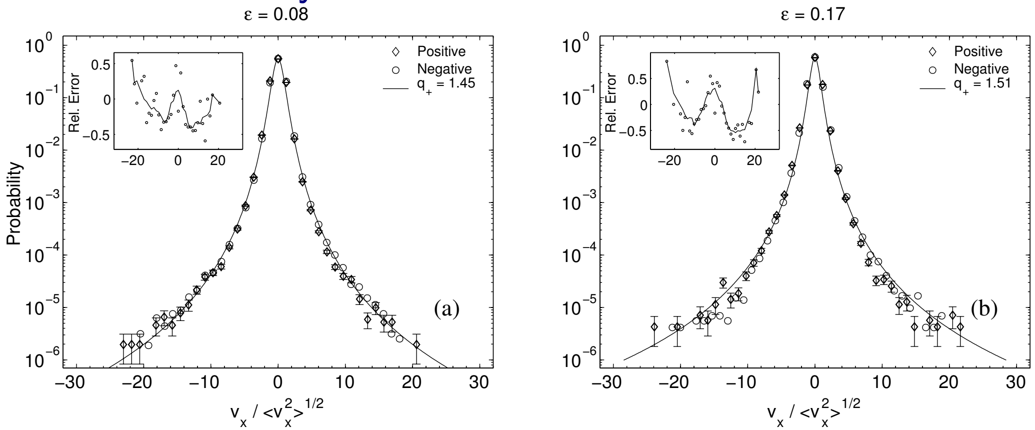
- anomalous diffusion $\langle x^2(t) \rangle \sim t^\alpha$, $\alpha \approx 1.3$

Measured velocity distributions of defects:



K. Daniels, C.B. , E. Bodenschatz, Physica D (2004)
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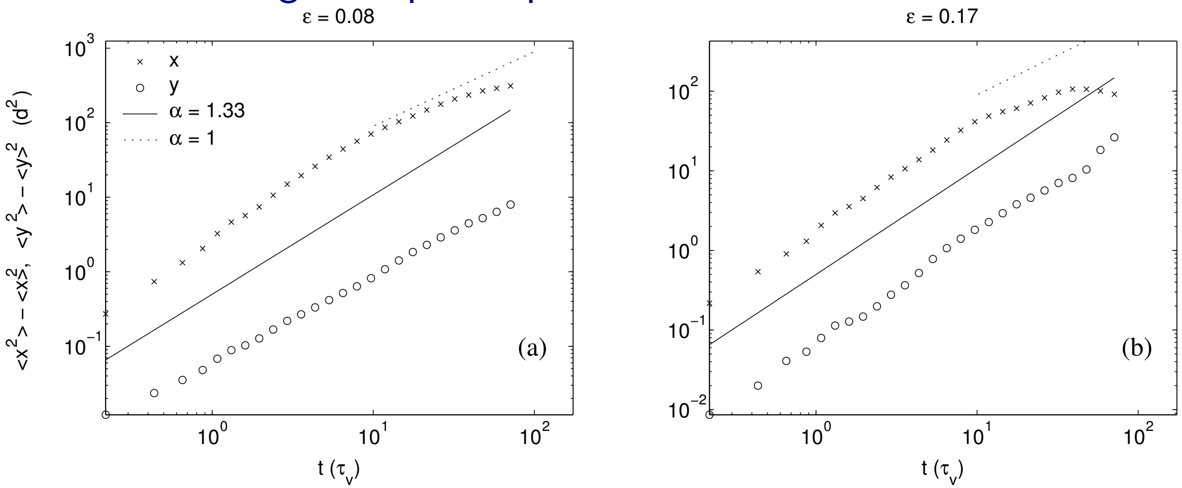
Note that for ordinary Brownian particles of size d
$\gamma = \frac{6\pi\nu\rho d}{m} = const$ (Stokes' law)
But defects are no ordinary particles!

Measured average of squared position of defects versus time:



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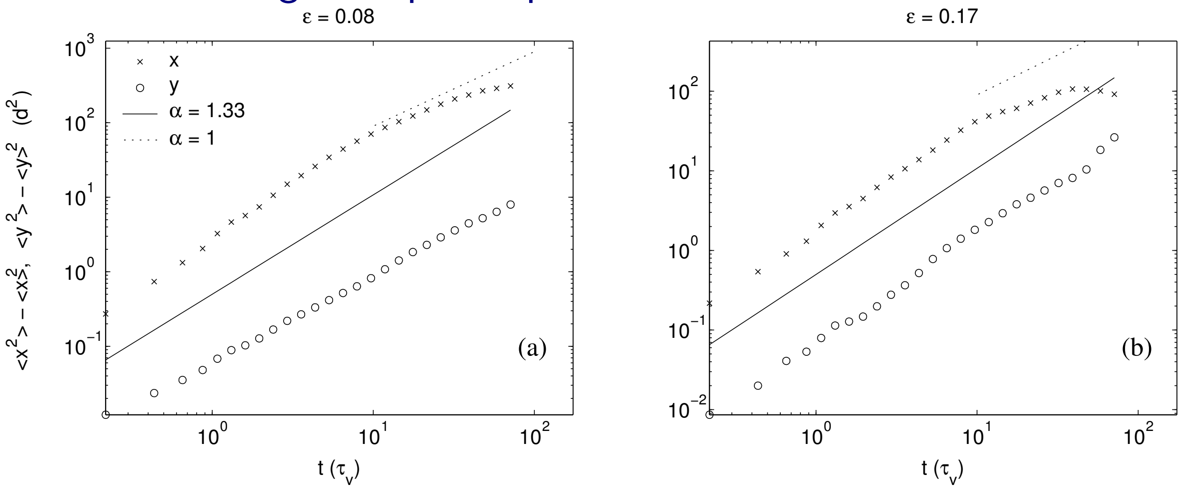

Superstatistical Langevin model predicts a relation between $\boldsymbol{\alpha}$ and $\boldsymbol{q}$:

$$\boldsymbol{\alpha = \frac{7q-9}{2q-2}} \tag{16}$$

$\boldsymbol{q = 1.45 \Rightarrow \alpha = 1.3}$

Derivation: Consider linear superstatistical Langevin equation

$$\dot{v} = -\gamma v + \sigma L(t) \tag{17}$$

where σ is constant, whereas γ fluctuates and $\beta = \frac{2\gamma}{\sigma^2}$ is χ^2-distributed with degree n.
Velocity correlation function $C(t-t') = \langle v(t)v(t')\rangle$. Can easily derive that asymptotically

$$C(t-t') \sim |t-t'|^{-\eta}, \tag{18}$$

where $\eta = \frac{n}{2} - 1$ and $\frac{1}{q-1} = \frac{n}{2} + \frac{1}{2}$
Now proceed to the position

$$x(t) = \int_0^t v(t')dt' \tag{19}$$

of the test particle. One has

$$\langle x^2(t)\rangle = \int_0^t \int_0^t \langle v(t')v(t'')\rangle dt'dt''. \tag{20}$$

Asymptotic power-law velocity correlations with an exponent $\eta < 1$ imply asymptotically anomalous diffusion of the form

$$\langle x^2(t)\rangle \sim t^\alpha \tag{21}$$

with

$$\alpha = 2 - \eta = 2 - \frac{5-3q}{2q-2} = \frac{7q-9}{2q-2} \tag{22}$$

It is interesting to compare our model with other dynamical models generating Tsallis statistics. E.g. Plastino and Plastino (Physica A, 1995) and Tsallis and Bukmann (PRE, 1996) study a generalized Fokker-Planck equation of the form

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x}(F(x)P(x,t)) + D\frac{\partial^2}{\partial x^2}P(x,t)^\nu \tag{23}$$

with a linear force $F(x) = k_1 - k_2 x$ and $\nu \neq 1$. Basically this model means that the diffusion constant becomes dependent on the probability density. The probability densities generated by eq. (23) are q-exponentials with the exponent

$$q = 2 - \nu. \tag{24}$$

The model generates anomalous diffusion with

$$\hat{\alpha} = 2/(3-q) \tag{25}$$

whereas the superstatistical Langevin model yields

$$\alpha = \frac{7q-9}{2q-2} \tag{26}$$

Interesting enough, there is a distinguished q-value where both models yield the same answer:

$$q = 1.453 \Rightarrow \alpha = \hat{\alpha} = 1.292 \tag{27}$$

These values of q and α correspond to the experimentally observed numbers in defect turbulence.

5 From time series to superstatistics (Beck, Cohen, Swinney, PRE 72, 056133 (2005))

Suppose some experimental time series $u(t)$ is given.

Goal: test the hypothesis that it is due to a superstatistics and if so, to extract the two basic time scales τ and T as well as $f(\beta)$.

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Divide time series into equal time intervals of size Δt. Define a function $\kappa(\Delta t)$ by

$$\kappa(\Delta t) = \int_0^{t_{max}-\Delta t} dt_0 \frac{\langle (u-\bar{u})^4 \rangle_{t_0,\Delta t}}{\langle (u-\bar{u})^2 \rangle_{t_0,\Delta t}^2} \tag{28}$$

Here $\langle \cdots \rangle_{t_0,\Delta t} = \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} \cdots dt$ denotes an integration over an interval of length Δt starting at t_0, and $\bar{u}$ is the average of $u(t)$

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Extract the (large) superstatistical time scale T by the condition

$$\kappa(T) = 3 \qquad (29)$$

Extract the (small) relaxation time scale τ of signal from the short-term exponential decay of correlation function $C_u(t-t') = \langle u(t)u(t') \rangle$:

$C_u(\tau) = \frac{1}{e} C_u(0)$.

Can then check whether T/τ is large (necessary consistency condition for superstatistics approach)

Once T is given, we can extract process $\beta(t)$ from

$$\beta(t) = \frac{1}{\langle u^2 \rangle_{t,T} - \langle u \rangle_{t,T}^2}. \tag{30}$$

and make histograms, calculate correlation functions of the β-process etc.

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Typical picture:

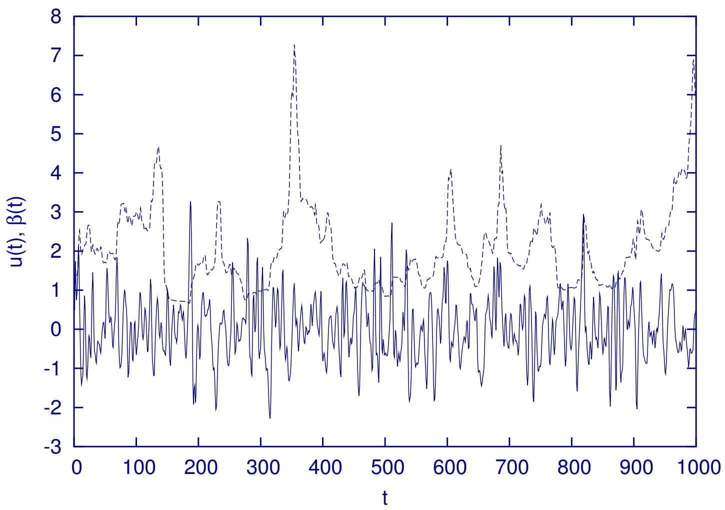

Fig.1: Turbulent velocity signal $u(t)$ evolving on time scale τ (solid line) and the corresponding inverse local variance $\beta(t)$ evolving on time scale T (dashed line).

Main example: Turbulent time series from a Taylor-Couette flow experiment

In turbulence: signal is longitudinal velocity difference $u(t) = v(t+\delta) - v(t)$ for a given time scale δ, β is related to local energy dissipation in the flow.

Can now evaluate function

$$\kappa(\Delta t) = \int_0^{t_{max}-\Delta t} dt_0 \frac{\langle (u-\bar{u})^4 \rangle_{t_0,\Delta t}}{\langle (u-\bar{u})^2 \rangle_{t_0,\Delta t}^2} \tag{31}$$

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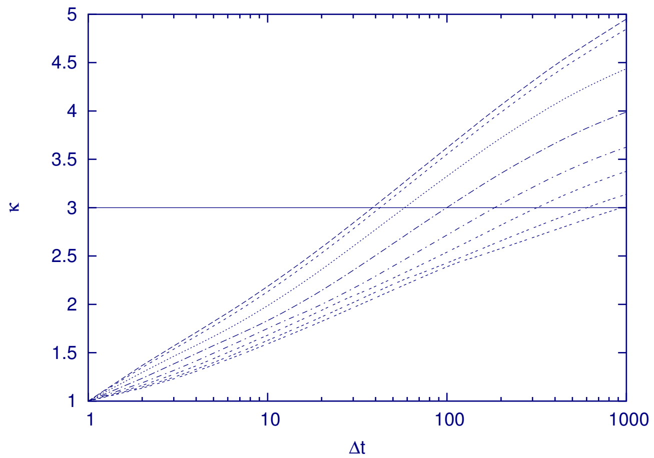

Fig.2: Determination of the long time scale T from the flatness function $\kappa(\Delta t)$, for $\delta = 2^j, j = 0, 1, 2, \ldots, 7$ (from top to bottom). The intersections with the line $\kappa = 3$ yield $T = 39, 42, 58, 100, 184, 320, 600, 948$, respectively.

Similarly, can determine the short time scales τ for each δ:

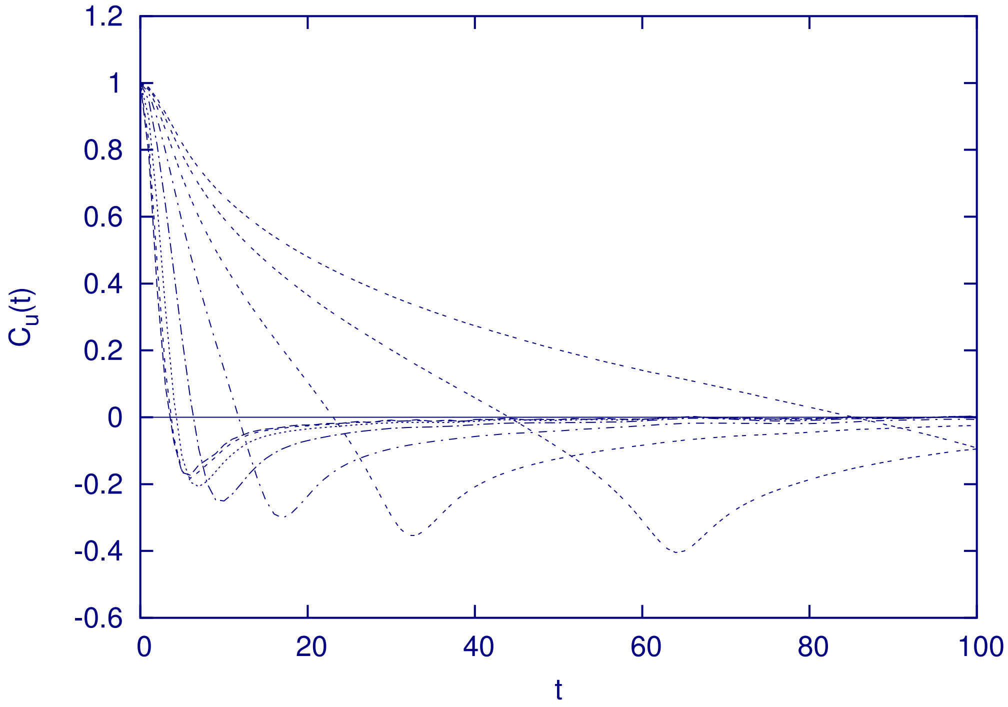

Fig.3: Determination of the short time scale τ from the decay of the correlation function $C_u(t)$ of the velocity difference. Defining τ by $C_u(\tau) = e^{-1}C_u(0)$, we obtain for $\delta = 2^j, j = 0, 1, 2, \cdots, 7$, $\tau = 2.1, 2.3, 2.8, 4.3, 7.2, 12.1, 19.9, 29.5$, respectively.

The time scale ratios T/τ for the Taylor-Couette data are large:

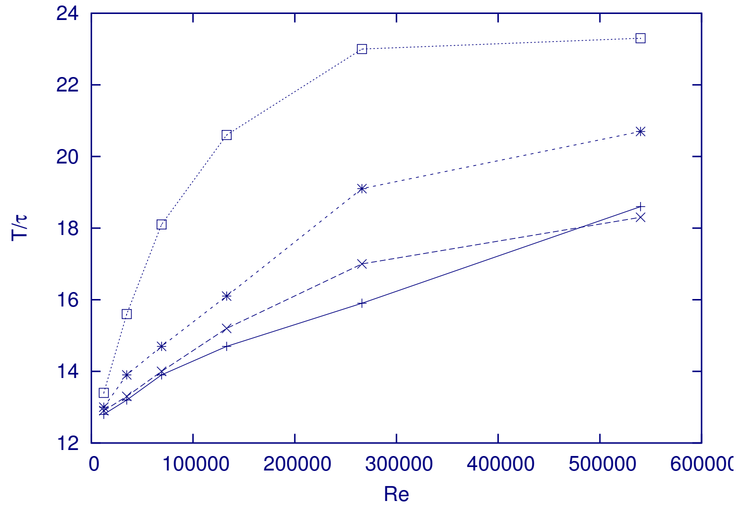

Fig.4: Time scale ratio T/τ as a function of Reynolds number Re for $\delta = 8, 4, 2, 1$ (from top to bottom).

T/τ increases with increasing Re, making the superstatistical approach more and more exact for increasing Re.

C.B., E.G.D. Cohen, H.L. Swinney, PRE (2005)

Our measured turbulent time series falls into the universality class of lognormal superstatistics:

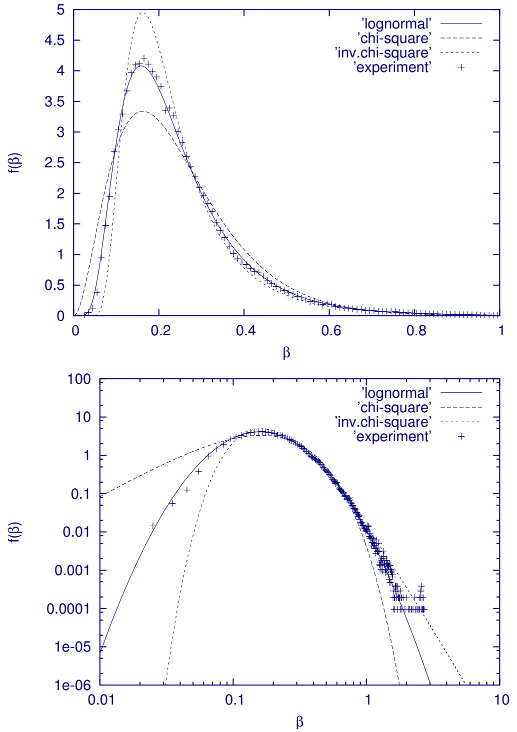

Fig.5: Probability density $f(\beta)$ extracted from the turbulent time series ($\delta = 16$), and compared with log-normal, χ^2, and inverse χ^2 distributions, on (a) linear-linear and (b) log-log plots. All distributions have the same mean and variance as the experimental data.

One can check for our data the validity of the superstatistical formula

$$p(u) = \int_0^\infty f(\beta) p(u|\beta) d\beta, \tag{32}$$

which in our case reads

$$p(u) = \frac{1}{2\pi s} \int_0^\infty d\beta \beta^{-1/2} \exp\left\{ \frac{-(\ln(\beta/\mu))^2}{2s^2} \right\} e^{-\frac{1}{2}\beta u^2}. \tag{33}$$

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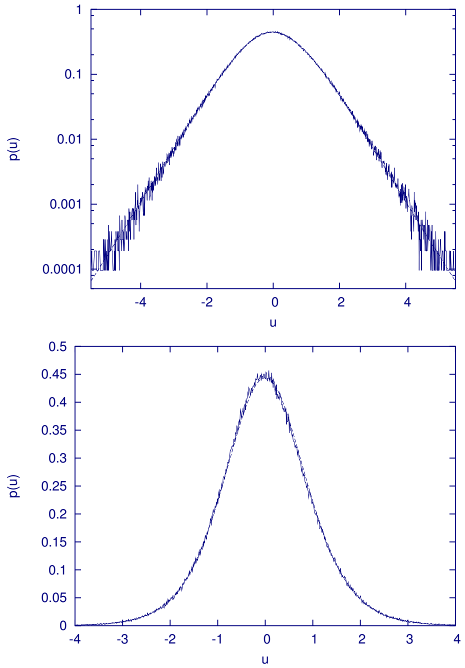

Fig.6 Comparison of the measured (fluctuating lines) and predicted (dashed lines) probability distribution $\boldsymbol{p(u)}$ for velocity differences on (a) semi-log plot, which emphasizes the tails, and (b) linear-linear plot, which emphasizes the peak.

Also, we may extract the parameter q from the data. For any superstatistics one can formally define a parameter q by

$$q := \frac{\langle \beta^2 \rangle}{\langle \beta \rangle^2}. \tag{34}$$

q measures in a quantitative way the deviation from Gaussian statistics. No fluctuations in β at all correspond to $f(\beta) = \delta(\beta - \beta_0)$ and $q = 1$, i.e., ordinary equilibrium statistical mechanics.

$q = q_{Tsallis}$ for χ^2-superstatistics.

For lognormal superstatistics one can prove that

$$q = e^{s^2} = \frac{1}{3} F, \tag{35}$$

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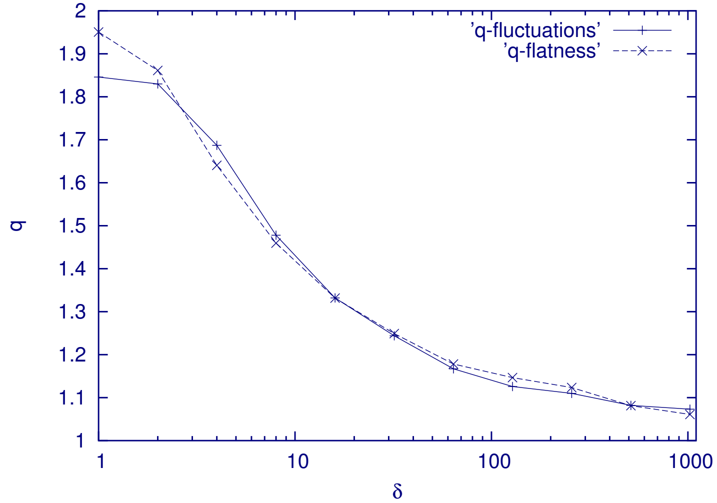

Fig.7: The parameter q as a function of δ, as evaluated from $q = \langle \beta^2 \rangle / \langle \beta \rangle^2$ (q-fluctuations) and from $q = \frac{1}{3}F$ (q-flatness).

Observed for our Taylor-Couette data: Both the correlation function of $u(t)$ as well as that of $\beta(t)$ decay with a power for very large times:

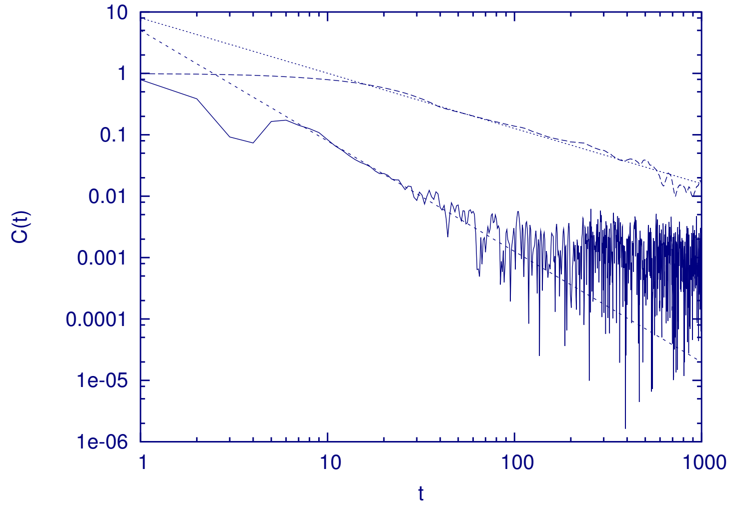

Fig.8: Correlation functions $C_\beta(t)$ (top) and $|C_u(t)|$ (bottom) for $\delta = 1$. The straight lines represent power laws with exponents -0.9 and -1.8.

Consistent with superstatistical model where γ fluctuates.

6 Application to scattering processes in high energy physics

Summary:

- Superstatistics (a 'statistics of a statistics') provides a physical reason why more general types of Boltzmann factors (e.g. q-exponentials) are relevant for nonequilibrium systems with suitable fluctuations of an intensive parameter.
- There is evidence for three major physically relevant universality classes: χ^2-superstatistics $=$ Tsallis statistics, inverse χ^2-superstatistics, and lognormal superstatistics. These arise as universal limit statistics for many different systems.
- An necessary condition for superstatistics to be a good model is a clear time scale separation: $T >> \tau$. This is indeed observed for turbulent Taylor-Couette flow.
- A general method was described how to proceed from a given experimental time series to a superstatistical description.
- Hydrodynamic turbulence well described by lognormal superstatistics (Lagrangian turbulence, Taylor-Couette flow, atmospheric turbulence).
- Defect turbulence (tilted Raleigh Benard flow) well described by χ^2-superstatistics.
- Scattering processes in particle physics seem to be correctly described by χ^2-superstatistics (power laws).