The Abdus Salam
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SMR.1763- 18

**SCHOOL and CONFERENCE
on
COMPLEX SYSTEMS
and
NONEXTENSIVE STATISTICAL MECHANICS**

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Debate on Statistical Mechanics

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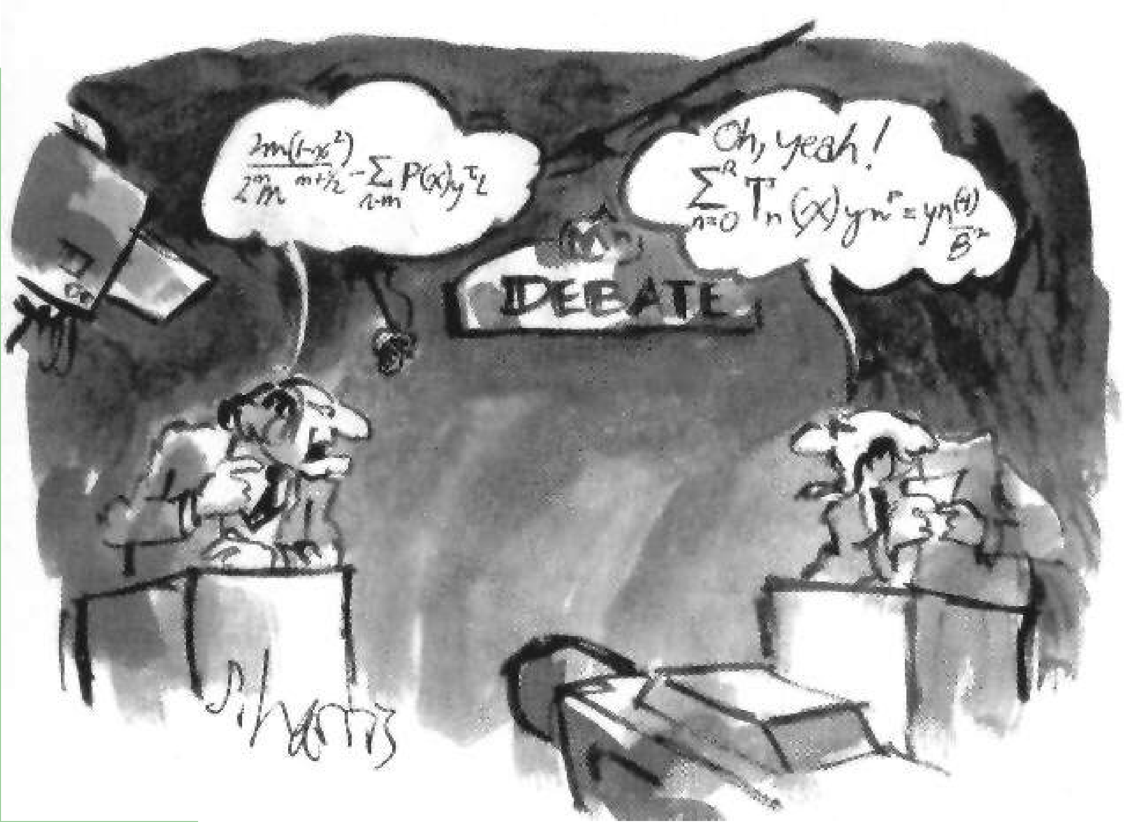
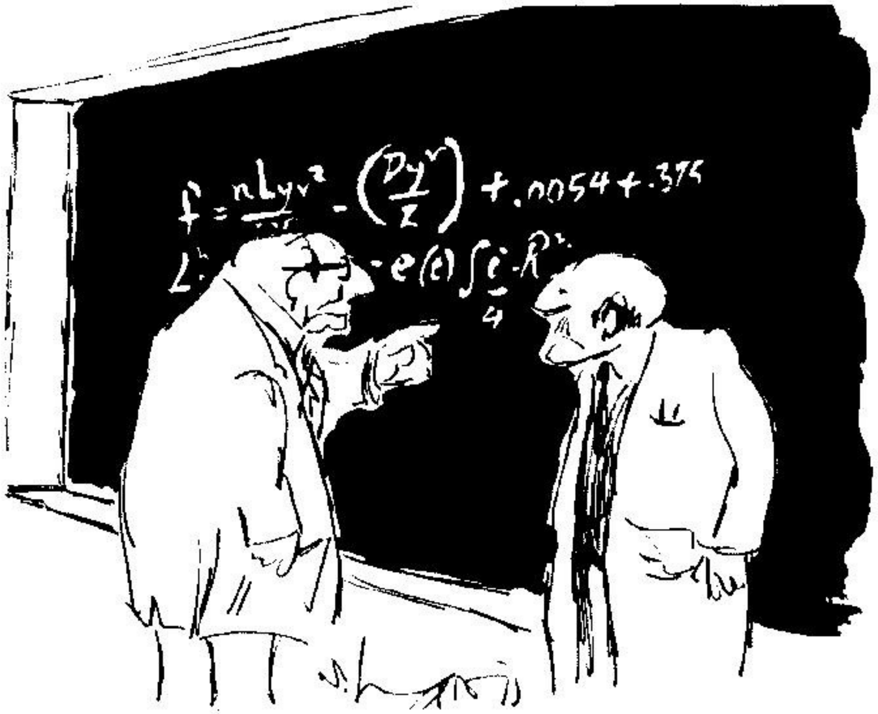
ICTP August 6 2006

Debate
on *$S_q$tatistical Mechanics*



When does a theory apply?



"Does this apply always, sometimes, or never?"

beyond Boltzmann Gibbs statistical mechanics

Why do we feel the need to go beyond Boltzmann Gibbs standard statistical mechanics, a very successful theory with so many applications?

There are several reasons

- Many systems relax very fast to equilibrium … but many others do not …and they live in long-standing quasi-stationary states which cannot be explained bt BG statistics

Moreover BG statistical mechanics has also other limitations

In fact it is based on:

- Ergodicity and equal a priori probability for cells in phase space
- Short range interactions
- Thermodynamical limit
- Small Gaussian fluctuations

Boltzmann Gibbs statistics does not work when one has

- Long range interacting systems
- Complex systems
- Systems at the edge of chaos
- Fractal phase space
- Ergodicity breaking and vanishing Lyapunov exponents
- Vanishing Lyapunov exponents
- SOC systems
- Turbulence
- Biological,geophysical and economical systems
- etc.

Tsallis generalized statistics: a possible proposal

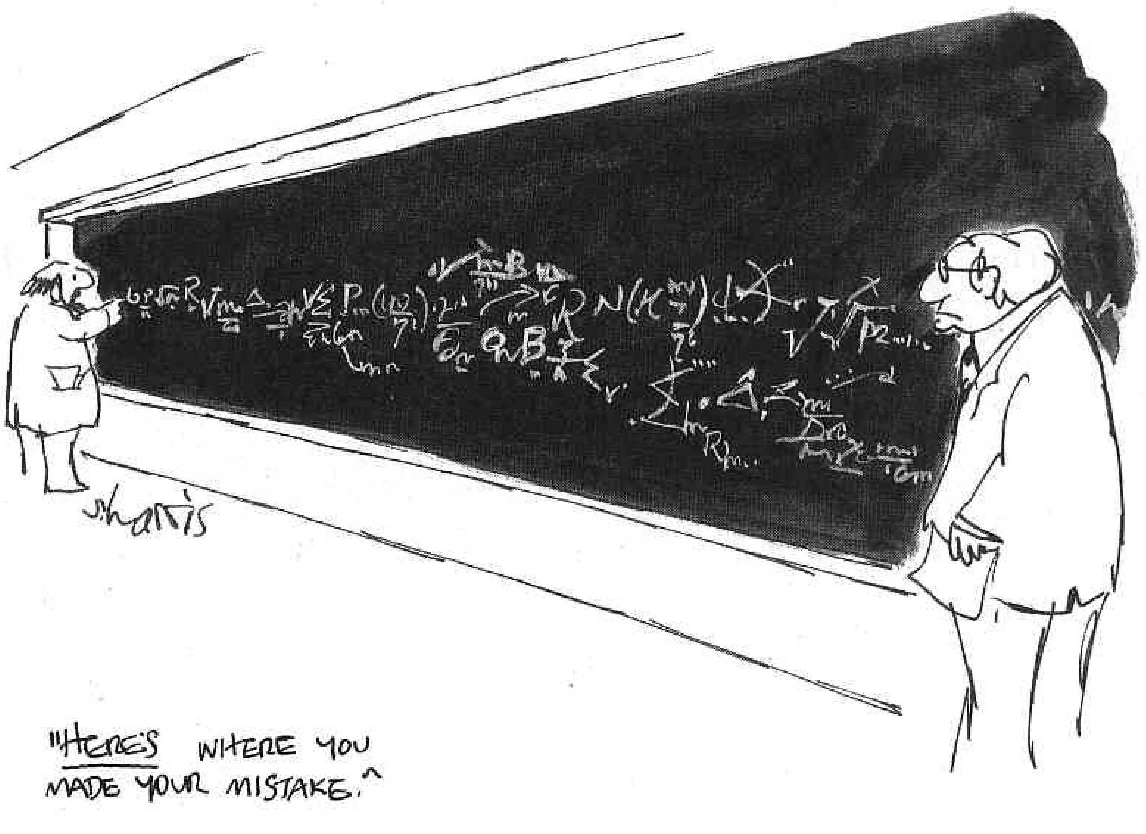
S_q entropy

$$S_q = \frac{\sum_i^N p^q{}_i - 1}{1-q} = \sum_i^N p_i \ln_q \frac{1}{p_i}$$

Maximized by the generalized weight

$$e_q(x) = \left[1 + (1-q)\,x\right]^{1/(1-q)}$$

Is this the situation….?



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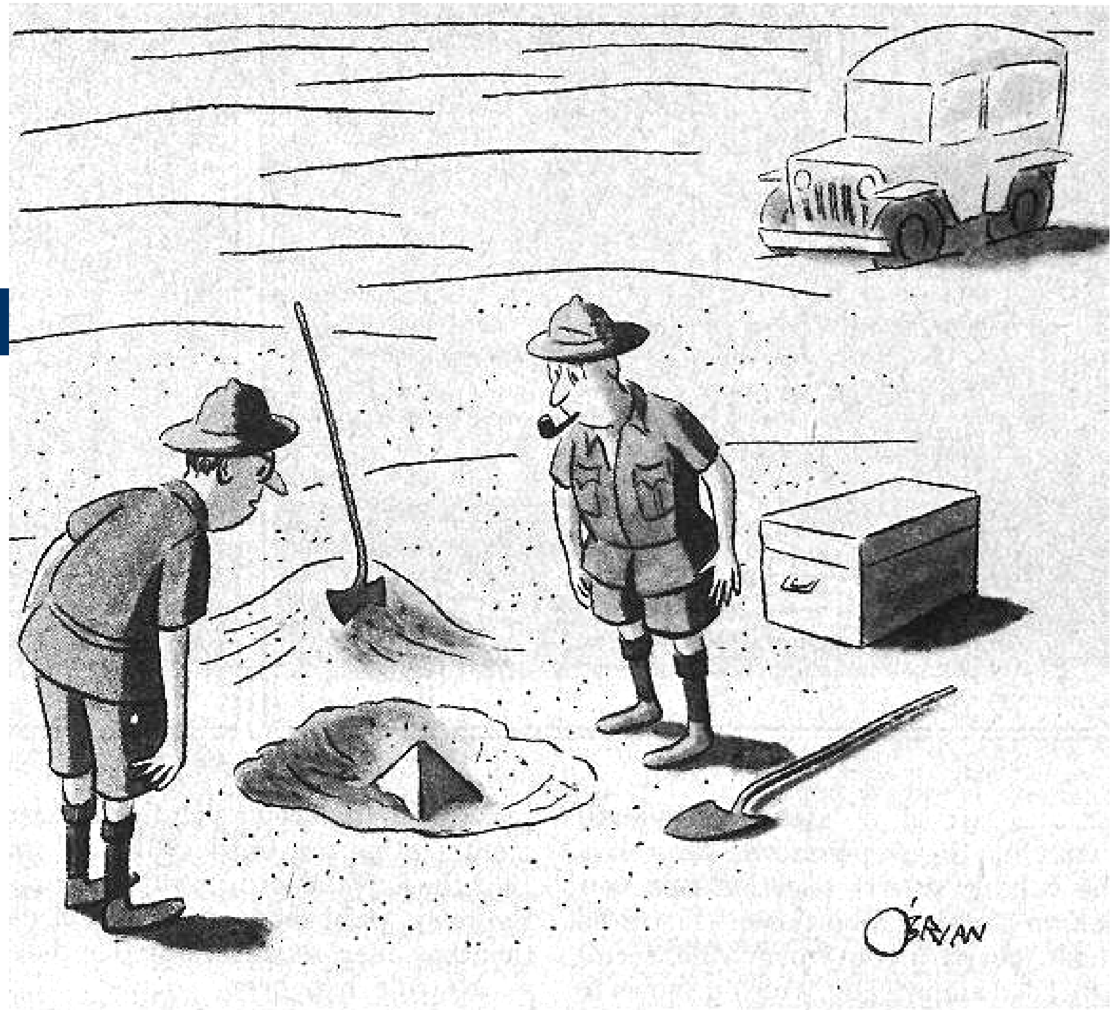
Or this…?



"This could be the discovery of the century. Depending, of course, on how far down it goes."

No a priori reasons to reject S_q statistics

- It preserves the Legendre structure of thermodynamics
- Most important theorems have been generalized for S_q
- It conserves simultaneously concavity,robustness,etc.
- It is extensive for special global correlations
- It reduces to BG formalism for $q \to 1$
- The index q characterizes the underlying dynamics and, in several cases, it has already been calculated a priori

- The point is then:

"To what extent is S_q formalism supported by experiments (real data or numerical ones) or is it useful for practical applications"?

Where S_q formalism has been successfully applied*

* just a few examples

Probability theory	**Rigorous mathematical results on the generalization of the Central limit theorem** (Umarov, Tsallis Gell-mann, Mojano)
Maps at the edge of chaos	**Rigorous mathematical results using renormalization group techniques** (Robledo et al)
Cold atoms in a dissipative optical lattice	**Very good experimental confirmation of Lutz prediction within Sq statistics** (Renzoni et al Prl 2006)
Turbulence	**Pdf of several turbulent fluids, solar wind reproduced with a very good approximation. Very good description of defect turbulence** (Beck Cohen Swinney Daniels Bodenshatz et al)
Long-range Hamiltonians	**Good reproduction of relaxation and anomalous diffusion for QSS** (Rapisarda,Anteneodo,Mojano, Pluchino, Tamarit)
Biological systems	**Very good description of anomalous diffusion and decay of correlation for Hydra viridissima.** (Upadhyaya et al)
Econophysics	**Very good description of the volatility smile** (Borland)
Computational physics	**Important applications for Optimization algorithms** (Straub, Andriociaei)
Signal and image processing	**Very performant techniques for image reconstruction** (de Albuquerque , Ben Hanza et al)

q-product

$$x \otimes_q y \equiv \left[x^{1-q} + y^{1-q} - 1 \right]^{\frac{1}{1-q}}$$

$Properties:$

$i)\quad x \otimes_1 y = x\, y$

$ii)\quad \ln_q (x \otimes_q y) = \ln_q x + \ln_q y$

$[whereas\ \ \ln_q (x\, y) = \ln_q x + \ln_q y + (1-q)(\ln_q x)(\ln_q y)]$

[L. Nivanen, A. Le Mehaute and Q.A. Wang, Rep. Math. Phys. 52, 437 (2003);
E.P. Borges, Physica A 340, 95 (2004)]

q - GENERALIZED CENTRAL LIMIT THEOREM: (mathematical proof)

S. Umarov, C.T. and S. Steinberg [cond-mat/0603593]

q-Fourier transform:

$$F_q[f](\xi) \equiv \int_{-\infty}^{\infty} e_q^{ix\xi} \otimes_q f(x)\,dx = \int_{-\infty}^{\infty} e_q^{\frac{ix\xi}{[f(x)]^{1-q}}} f(x)\,dx \qquad \text{(nonlinear!)}$$

q-correlation:

Two random variables X [*with density* $f_X(x)$] *and* Y [*with density* $f_Y(y)$]

are said q*-correlated if*

$F_q[\mathrm{X+Y}](\xi) = F_q[\mathrm{X}](\xi) \otimes_q F_q[\mathrm{Y}](\xi)$,

i.e., if

$$\int_{-\infty}^{\infty} dz\, e_q^{iz\xi} \otimes_q f_{X+Y}(z) = \left[\int_{-\infty}^{\infty} dx\, e_q^{ix\xi} \otimes_q f_X(x)\right] \otimes_q \left[\int_{-\infty}^{\infty} dy\, e_q^{iy\xi} \otimes_q f_Y(y)\right],$$

with $f_{X+Y}(z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, h(x,y)\, \delta(x+y-z) = \int_{-\infty}^{\infty} dx\, h(x, z-x) = \int_{-\infty}^{\infty} dy\, h(z-y, y)$

where $h(x,y)$ *is the joint density.*

$$\left(q\text{-correlation means} \begin{cases} \textit{independence} & \text{if } q=1\text{, i.e., } h(x,y) = f_X(x)\,f_Y(y) \\ \textit{global correlation} & \text{if } q \neq 1\text{, hence } h(x,y) \neq f_X(x)\,f_Y(y) \end{cases} \right)$$

CENTRAL LIMIT THEOREMS: $N^{1/[\alpha(2-q)]}$ - SCALED ATTRACTOR $\mathbb{F}(x)$ WHEN SUMMING $N \to \infty$ q - CORRELATED IDENTICAL RANDOM VARIABLES WITH SYMMETRIC DISTRIBUTION $f(x)$

	$q=1$ [*independent*]	$q \neq 1$ (*i.e.*, $Q \equiv 2q-1 \neq 1$) [*globally correlated*]
$\sigma_Q < \infty$ $(\alpha = 2)$	$\mathbb{F}(x) = \textit{Gaussian } G(x)$, *with same* σ_1 *of* $f(x)$ Classic CLT	$\mathbb{F}(x) = G_{\frac{3q-1}{q+1}}(x) \equiv \frac{3q-1}{q+1}$ - *Gaussian*, *with same* $\sigma_Q \left[\equiv \int dx\, x^2 [f(x)]^Q / \int dx\, [f(x)]^Q \right]$ *of* $f(x)$ $G_{\frac{3q-1}{q+1}}(x) \begin{cases} \simeq G(x) & \text{if } \lvert x \rvert << x_c(q,2) \\ \sim f(x) \sim C_q / \lvert x \rvert^{(q+1)/(q-1)} & \text{if } \lvert x \rvert >> x_c(q,2) \end{cases}$ *with* $\lim_{q \to 1} x_c(q,2) = \infty$ S. Umarov, C. T. and S. Steinberg (2006) [cond-mat/0603593]
$\sigma_Q \to \infty$ $(0 < \alpha < 2)$	$\mathbb{F}(x) = \textit{Levy distribution } \mathrm{L}_\alpha(x)$, *with same* $\lvert x \rvert \to \infty$ *asymptotic behavior* $\mathrm{L}_\alpha(x) \begin{cases} \simeq G(x) \\ \quad \text{if } \lvert x \rvert << x_c(1,\alpha) \\ \sim f(x) \sim C_\alpha / \lvert x \rvert^{1+\alpha} \\ \quad \text{if } \lvert x \rvert >> x_c(1,\alpha) \end{cases}$ *with* $\lim_{\alpha \to 2} x_c(1,\alpha) = \infty$ Levy-Gnedenko CLT	$\mathbb{F}(x) = L_{\frac{2\alpha q - \alpha + 3}{\alpha + 1}, \alpha}$ *stable distribution*, *with* $L_{\frac{2\alpha q - \alpha + 3}{\alpha + 1}, \alpha} \sim f(x) \sim C^{(L)}_{q,\alpha} / \lvert x \rvert^{(1+\alpha)/(1+\alpha q - \alpha)}$ *or* $\mathbb{F}(x) = L_{\frac{\alpha q + q - 1}{\alpha + q - 1}, \alpha}$ *stable distribution*, *with* $L_{\frac{\alpha q + q - 1}{\alpha + q - 1}, \alpha} \sim f(x) \sim C^{(*)}_{q,\alpha} / \lvert x \rvert^{2(\alpha + q - 1)/\alpha(q-1)}$ S. Umarov, C. T., M. Gell-Mann and S. Steinberg (2006) [cond-mat/0606038] and [cond-mat/0606040]

Logistic map at the edge of chaos

Entropy growth is linear only for q*=0.2425…..

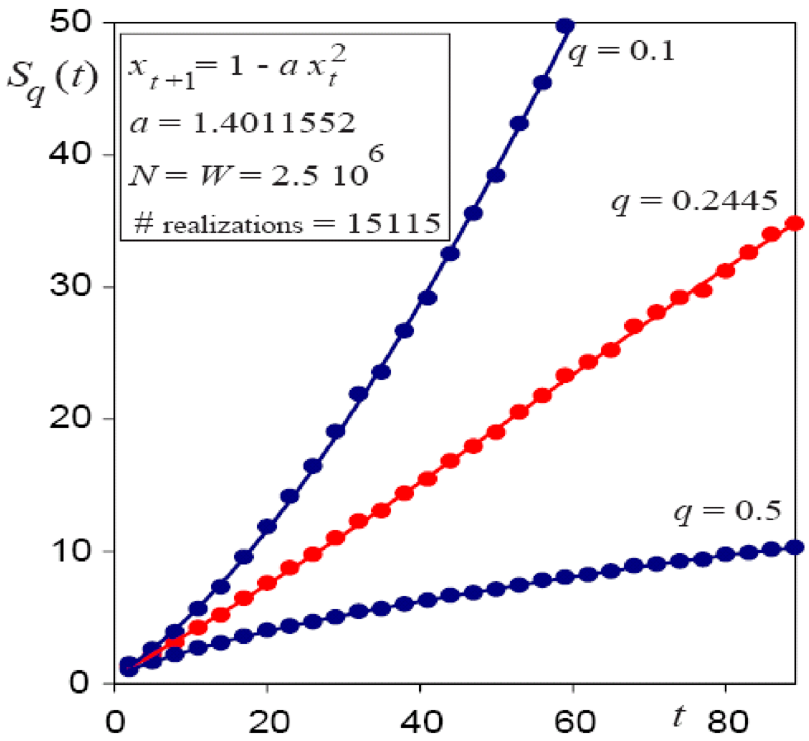
$$\frac{1}{1-q^*} = \frac{\ln \alpha_F}{\ln 2}$$

Generalized Pesin-like theorem

$$K_q = \lambda_q$$

$$\lambda_q = \frac{1}{1-q}$$

(weak chaos, i.e., **zero** Lyapunov exponent)



C. T. , A.R. Plastino and W.-M. Zheng, Chaos, Solitons & Fractals **8**, 885 (1997)
M.L. Lyra and C. T. , Phys. Rev. Lett. **80**, 53 (1998)
V. Latora, M. Baranger, A. Rapisarda and C. T. , Phys. Lett. A **273**, 97 (2000)
E.P. Borges, C. T. , G.F.J. Ananos and P.M.C. Oliveira, Phys. Rev. Lett. **89**, 254103 (2002)
F. Baldovin and A. Robledo, Phys. Rev. E **66**, R045104 (2002) and **69**, R045202 (2004)
G.F.J. Ananos and C. T. , Phys. Rev. Lett. **93**, 020601 (2004)
E. Mayoral and A. Robledo, Phys. Rev. E **72**, 026209 (2005), and references therein

Feigenbaum constant has been measured in many real experiments

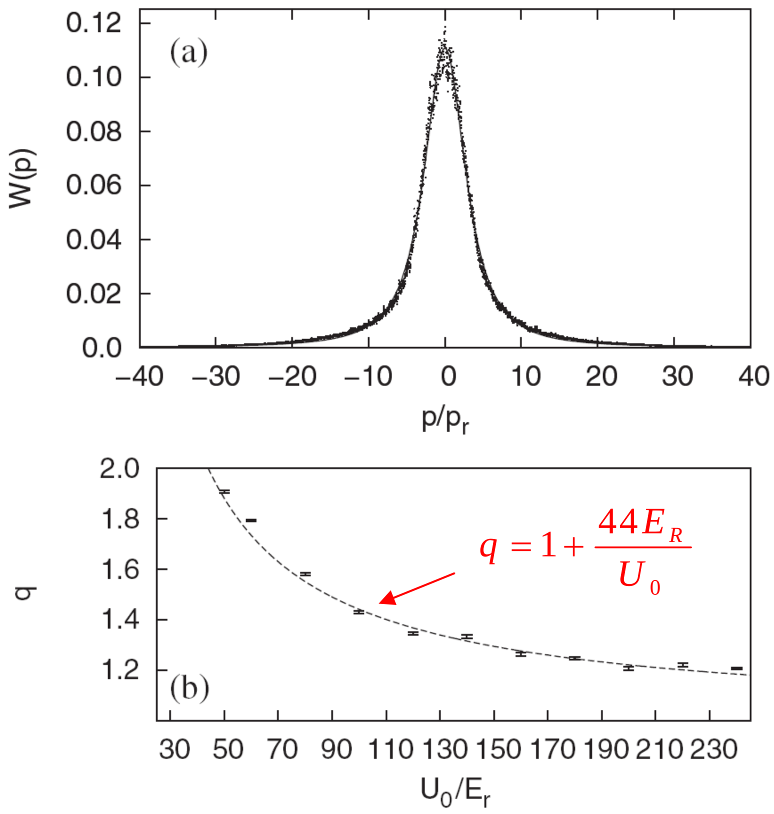
$$\alpha_F = 2.5029078....$$

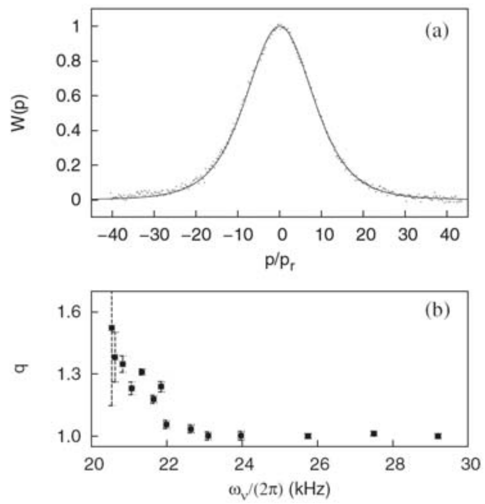
Cold atoms in a dissipative optical lattice

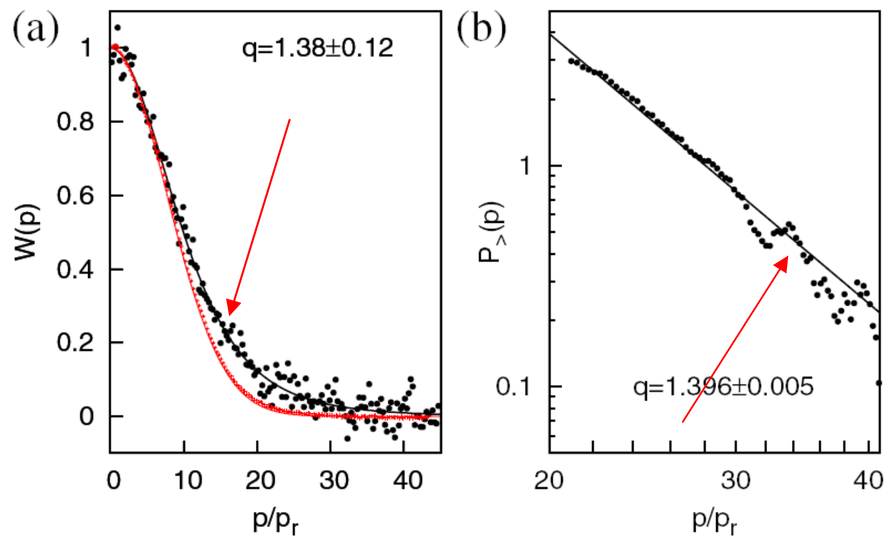
experimental verification by P. Douglas, S. Bergamini and F. Renzoni, Phys Rev Lett **96**, 110601 (2006)

Theoretical prediction by

E.Lutz PRA 67 (2003) 051402(R)







(Computational verification: quantum Monte Carlo simulations)

(Experimental verification)

The solution of

$$\frac{\partial\, p(x,t)}{\partial t} = D\,\frac{\partial^2 [p(x,t)]^{2-q}}{\partial x^2} \qquad [p(x,0)=\delta(0)] \qquad (q<3)$$

is given by

$$p(x,t) \propto \left[1+(1-q)\, x^2 /\ (\Gamma t)^{2/(3-q)}\right]^{1/(1-q)} \equiv e_q^{-x^2/\ (\Gamma t)^{2/(3-q)}} \qquad (\Gamma \propto D)$$

hence

x^2 *scales like* t^{γ} (*e.g.,* $\langle x^2 \rangle \propto t^{\gamma}$)

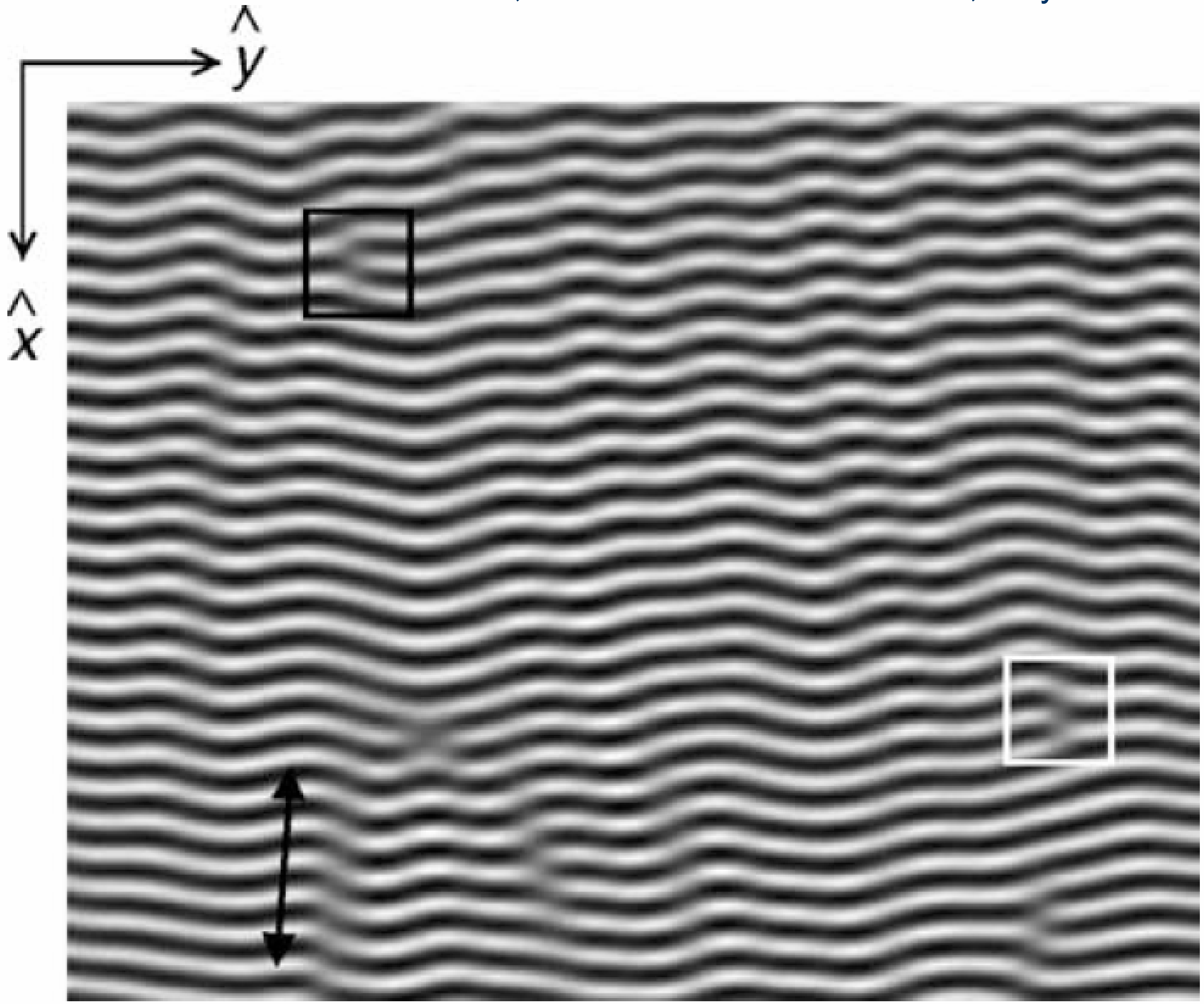
with

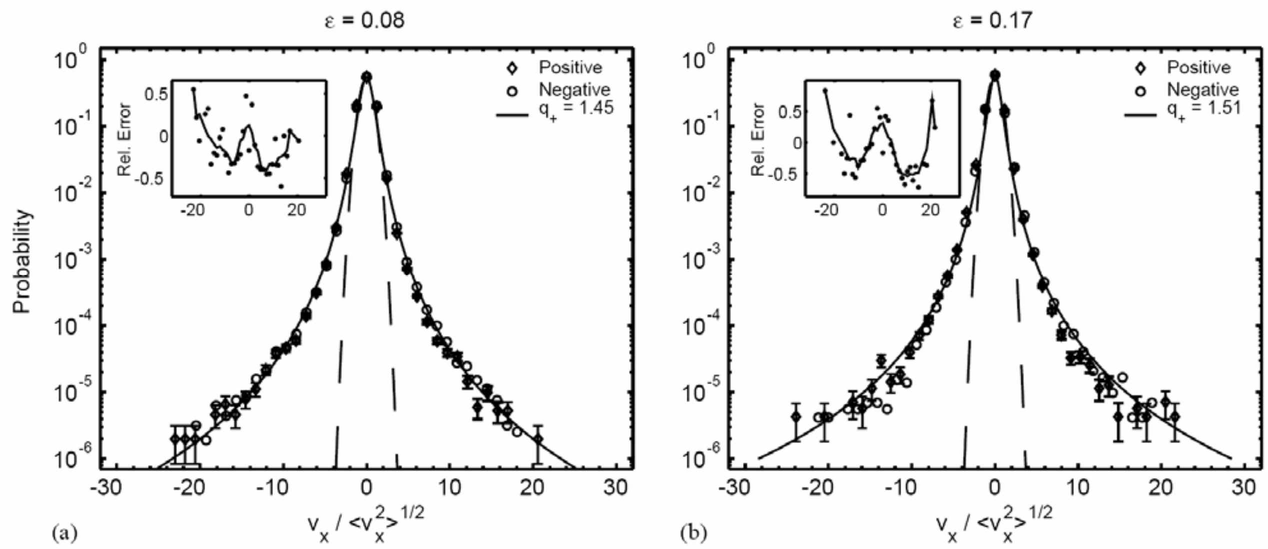
$$\gamma = \frac{2}{3-q}$$

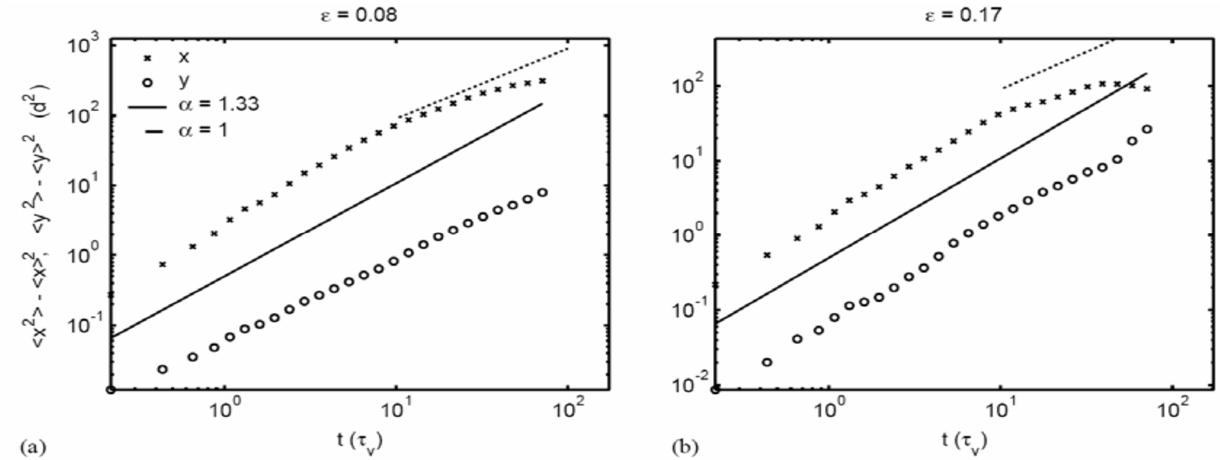
C.Tsallis and D.J. Bukman, Phys Rev E **54**, R2197 (1996)

Defect Turbulence

K.E. Daniels, C. Beck and E. Bodenschatz, Physica D 193, 208 (2004)







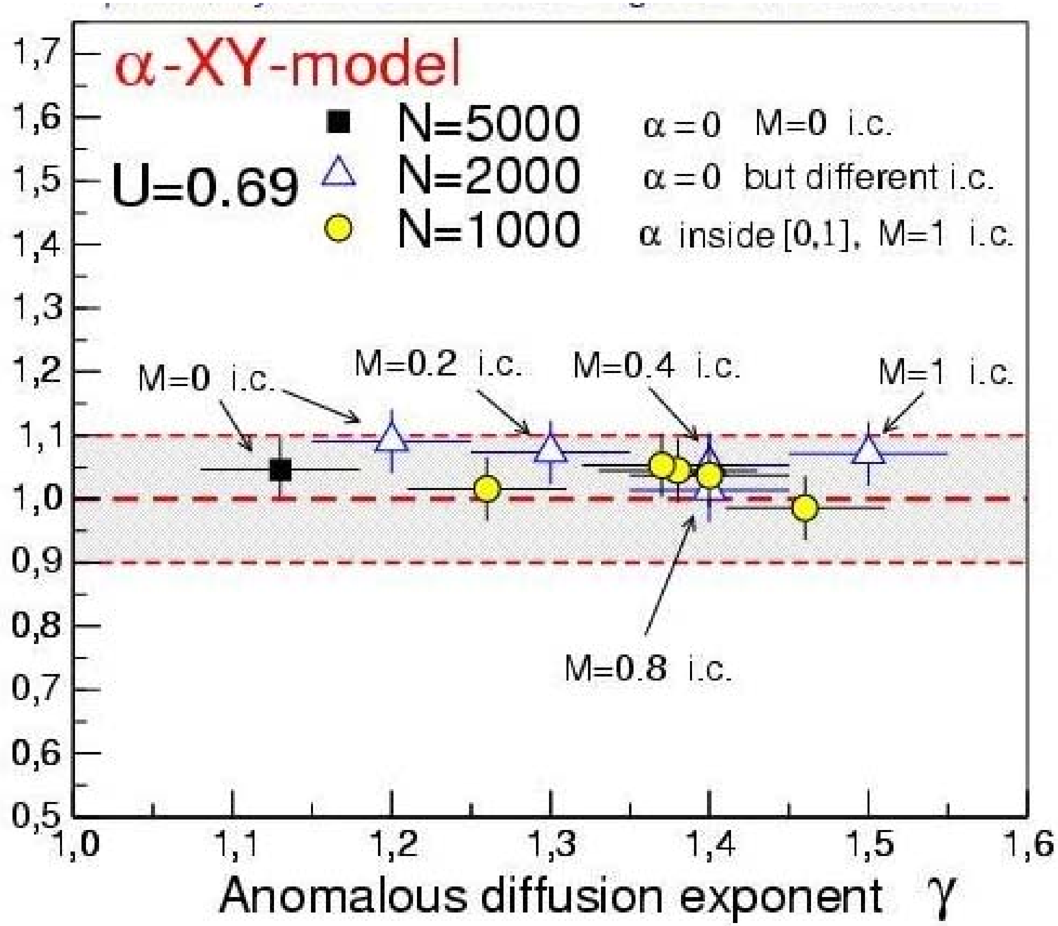
$$q \approx 1.5 \;\; and \;\; \gamma \approx 4/3 \;\; are\ consistent\ with \;\; \gamma = \frac{2}{3-q}$$

K.E. Daniels, C. Beck and E. Bodenschatz, Physica D 193, 208 (2004)

Long-range Hamiltonian systems

Anomalous diffusion vs q-exponential decay of correlation function

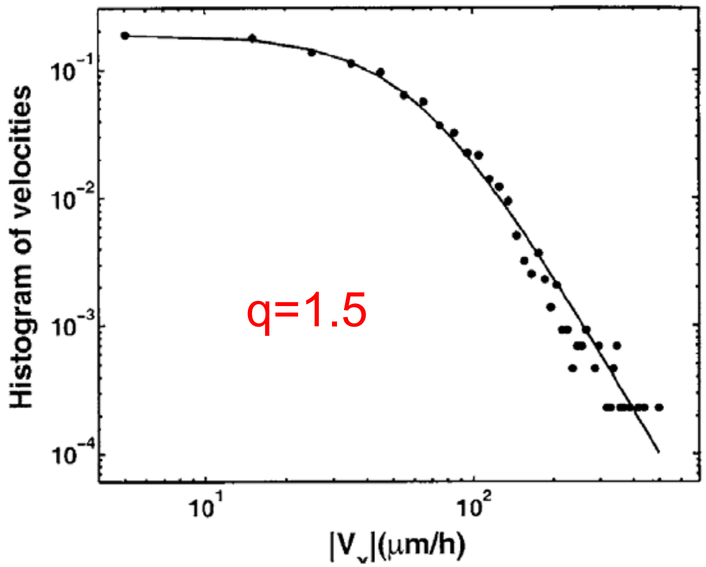
$\frac{\gamma}{2/(3-q)}$

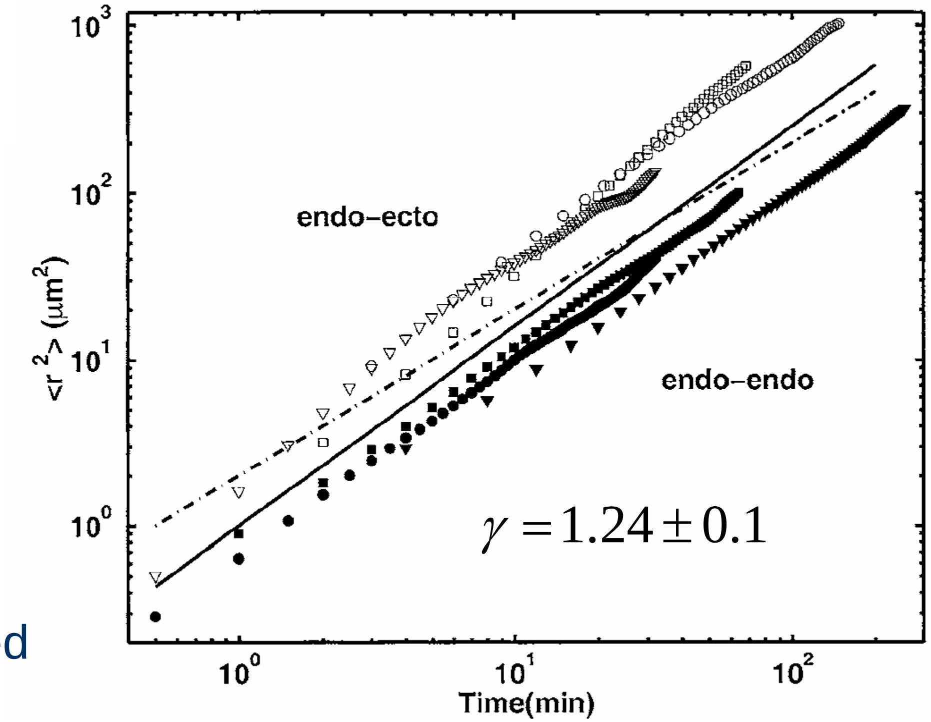


Rapisarda and Pluchino,

Europhysics News 36 (2005) 202

Anomalous diffusion for Hydra viridissima



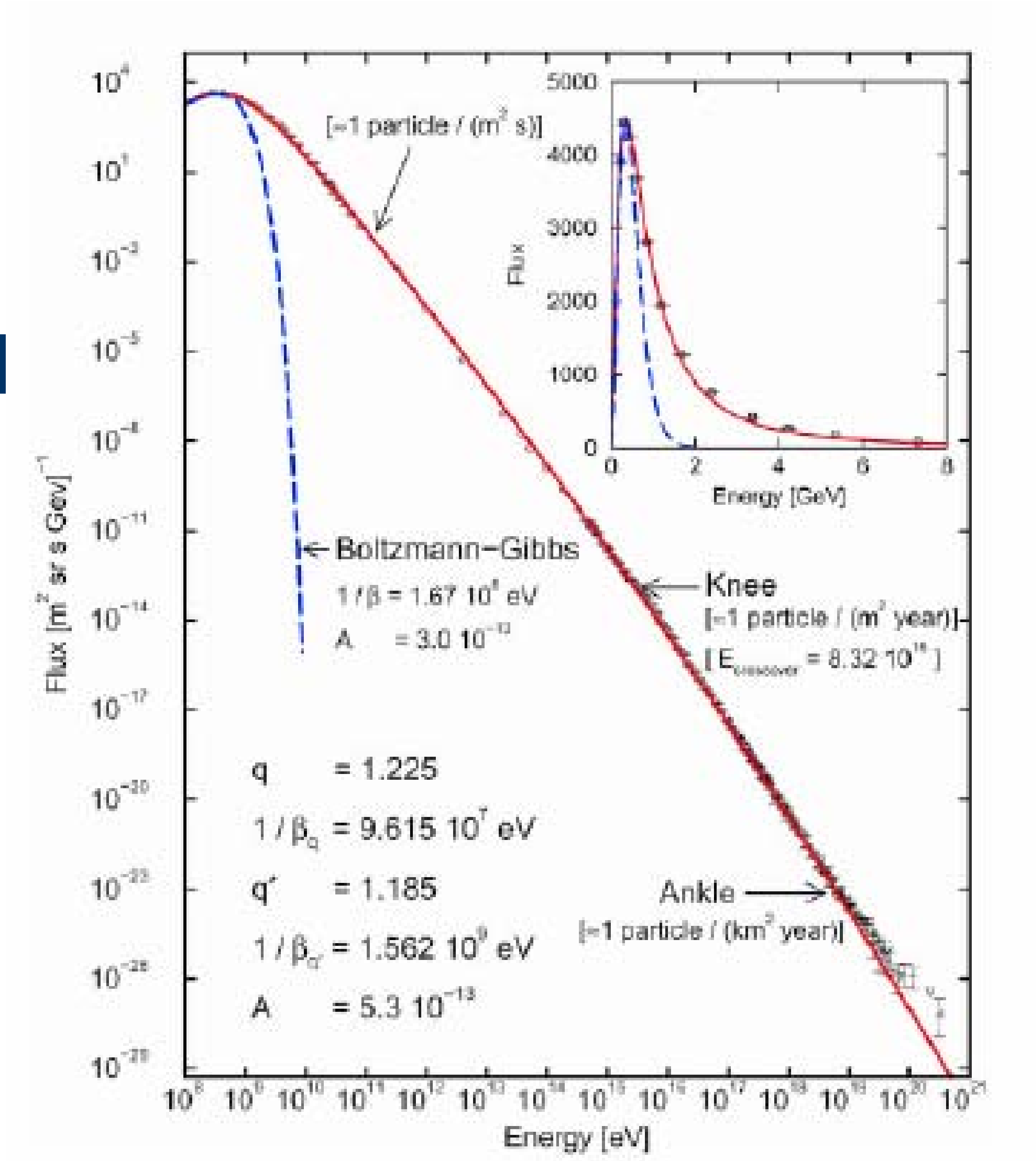


So $\gamma = \dfrac{2}{3-q}$ is confirmed

A. Upadhyaya, J.-P. Rieu, J.A. Glazier and Y. Sawada
Physica A 293, 549 (2001)

Cosmic rays

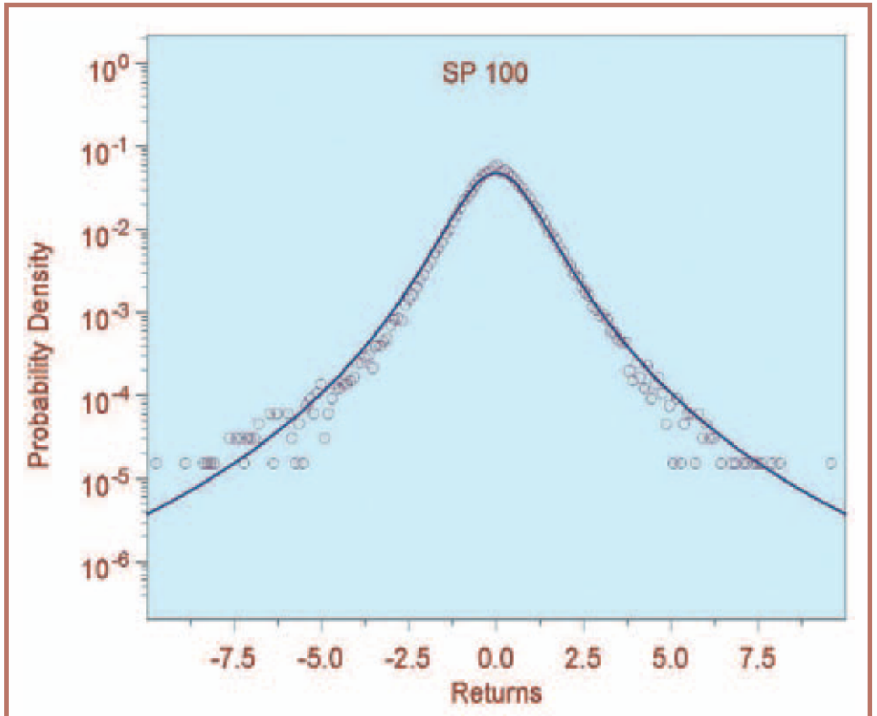
Tsallis Anjos Borges,
PLA 310,372 (2003)



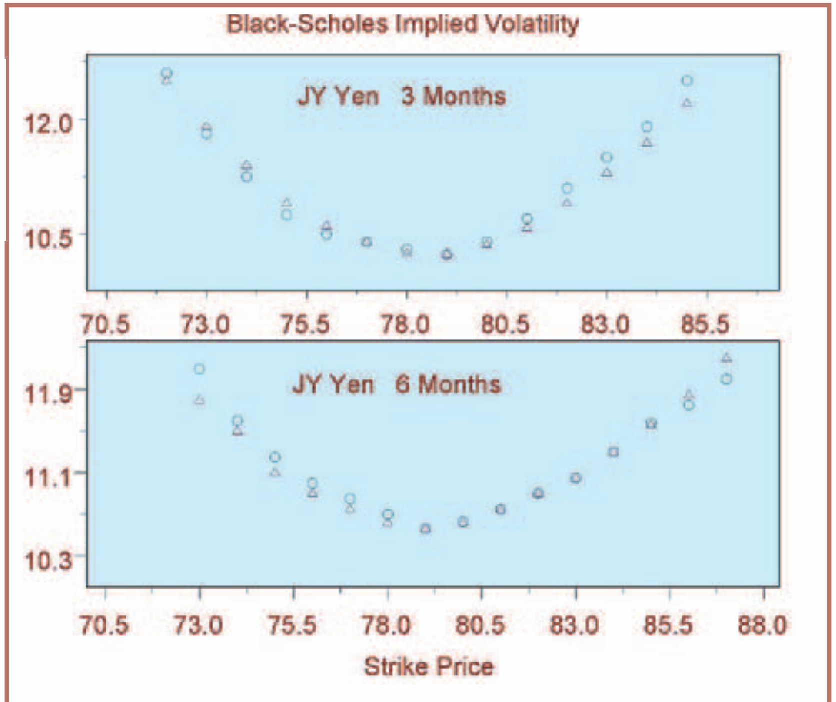
Econophysics

q-GENERALIZED BLACK-SCHOLES EQUATION:

L Borland, Phys Rev Lett **89**, 098701 (2002), and Quantitative Finance **2**, 415 (2002)
L Borland and J-P Bouchaud, cond-mat/0403022 (2004)
L Borland, Europhys News **36**, 228 (2005)
See also H Sakaguchi, J Phys Soc Jpn **70**, 3247 (2001)
C Anteneodo and CT, J Math Phys **44**, 5194 (2003)

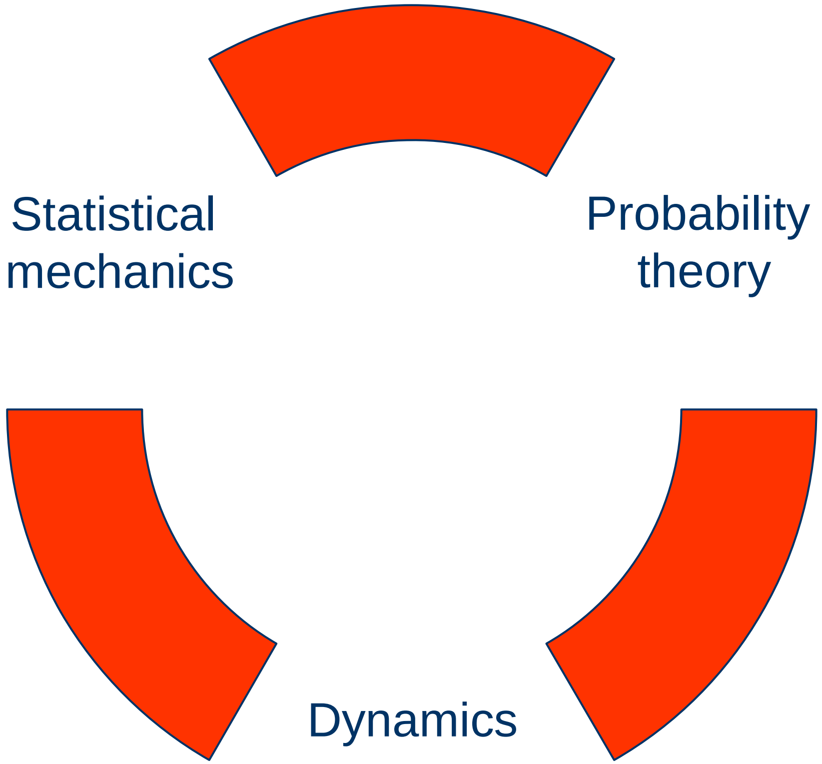


▲ Fig.2: The empirical distribution of daily returns from the stocks comprising the SP 100 (red) is fit very well by a q-Gaussian with $q = 1.4$ (blue).



▲ Fig.3: Theoretical implied Black-Scholes volatilities from the $q = 1.4$ model (triangles) match empirical ones (circles) very well, across all strikes and for different times to expiration.

[*REMARK : Student t – distributions are the particular case of q – Gaussians when* $q = \frac{n+3}{n+1}$ *with n integer*]


Statistical mechanics
Probability theory
Dynamics

A few concluding remarks

- S_q statistics is a very useful and elegant theory.
- It has also experimental and numerical support.
- There are still some open problems and further work is certainly needed.
- Future investigations will clarify its limitations and how fundamental it is.