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SCHOOL and CONFERENCE
on
COMPLEX SYSTEMS
and
NONEXTENSIVE STATISTICAL MECHANICS

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Globally Correlated Discrete Systems: Attractors in Probability Space and Entropic Extensivity

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J. A. Marsh

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Outline

- Discrete Systems
  - Overview of “simple” and “complex” systems
  - Discrete (binary) Systems Composition
  - Various Models
    - Independent case
    - q-correlated model
    - Stretched exponential model
    - Cutoff model
    - Power-law models
  - Summary

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"Simple" Systems

- Independent system composition yields:
  - Boltzmann-Gibbs entropy S_{BG} is extensive
    - i.e., $S_{BG}\left(N\right) \propto N \quad \text{as } N \to \infty$
  - Central Limit Theorem applies, yielding Gaussian distributions
    - e.g., Maxwellian distribution of molecular velocities, normal diffusion
- We consider a number of complex systems that illustrate various departures from this behavior

“Complex” Systems

- Global correlations among composing subsystems are known to yield complex behavior
- We consider two examples of complex (anomalous) behavior:
  - The generalized entropy S_q is extensive for $q \neq 1$, where

$$S_q \equiv \frac{1}{q-1}\left(1-\sum_{i=1}^{W} p_i^q\right) \qquad S_1 = S_{BG} = -\sum_{i=1}^{W} p_i \log p_i$$

  - In the thermodynamic limit, the attractor in probability space is non-Gaussian
    - The diffusion exponent γ (x scales like t^γ) will in general also deviate from the usual value $\gamma = 0.5$

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Entropy Growth as a Characterization of Complexity

- Consider the composition of N discrete μ-state systems
  - Total number of possible states is μ^N ($\mu \geq 1$)
  - Let the effective number of states W_{eff} be the number of states with effectively nonzero occupancy
  - "Simple" system: Quasi-independent subsystems yield

$$W_{eff} \approx \mu^N$$

  - "Complex" system: Correlations may cause phase space occupancy to be severely restricted, e.g.,

$$W_{eff} \approx N^\rho << \mu^N \quad \text{for some } \rho > 0$$

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Entropy Growth as a Characterization of Complexity

- "Simple" system:

$$W_{eff} \sim \mu^N \Rightarrow S_{BG} \sim \ln W_{eff} \sim N \ln \mu \Rightarrow q_{ent} = 1$$

- "Complex" system:

$$W_{eff} \sim N^\rho << \mu^N \quad \text{for some } \rho > 0$$

$$\Rightarrow S_q \sim \ln_q W^{eff} \equiv \frac{(W^{eff})^{1-q} - 1}{1-q} \sim N^{\rho(1-q)} \Rightarrow q_{ent} = 1 - \tfrac{1}{\rho} \neq 1$$

- In summary, the entropy S_q is extensive for

$$q_{ent} \begin{cases} = 1 & \text{for "simple" systems} \\ \neq 1 & \text{for "complex" systems} \end{cases}$$

N Identical Binary Subsystems

- Only $N+1$ probabilities are necessary to characterize the 2^N states
- $p_{N,k}$ is probability that k systems out of $N+1$ are in state "1"

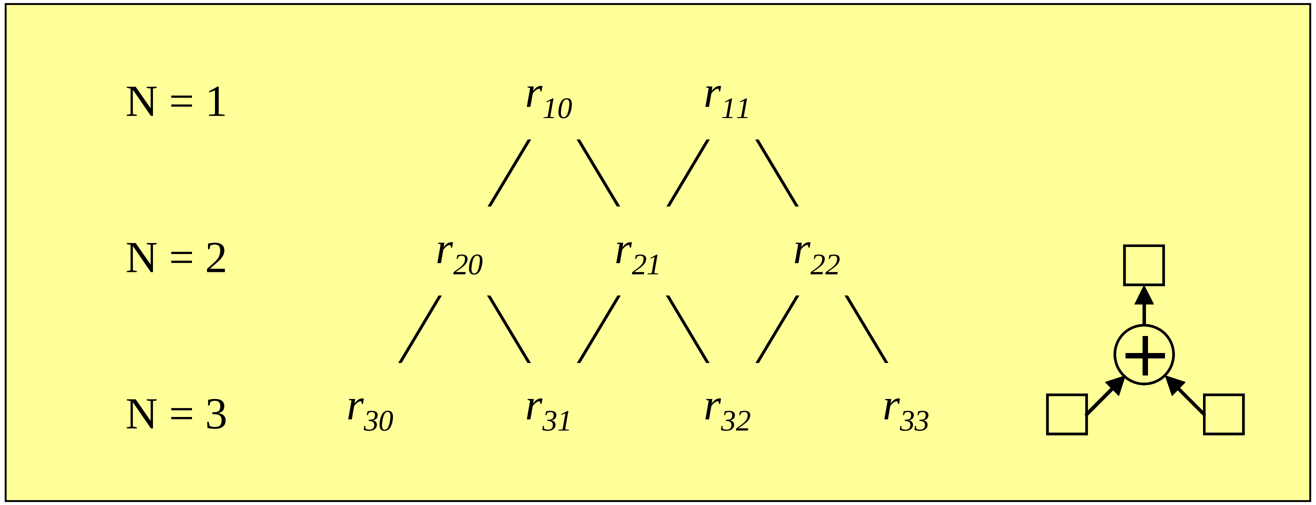
$$\sum_{k=0}^{N} p_{N,k} = 1$$

- Reduced probabilities $r_{N,k}$ defined by

$$p_{N,k} \equiv \frac{N!}{(N-k)!k!} r_{N,k}$$

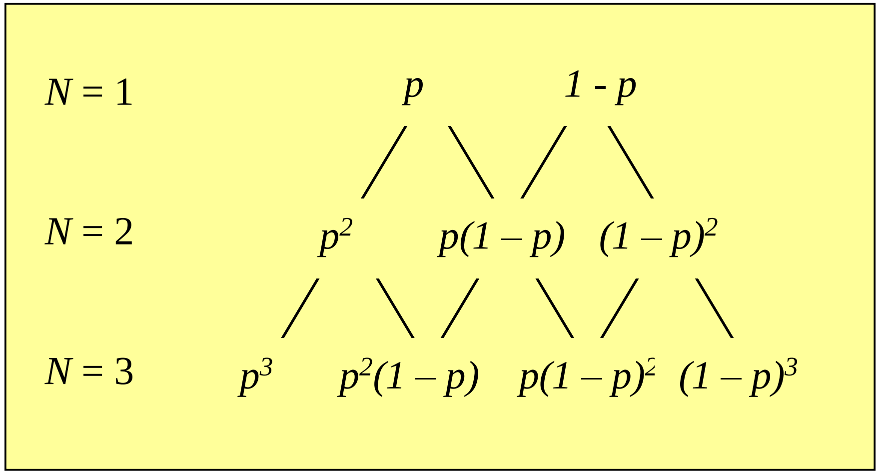
Leibnitz-Pascal Triangle

- Reduced probabilities $r_{N,k}$ can be arranged into a triangle
- Leibnitz rule: strict marginals reduction $r_{N,k} + r_{N,k+1} = r_{N-1,k}$
  - When Leibnitz rule applies, only one value in each row need be specified, typically $r_{N,0}$
  - Not all possible specifications of $r_{N,0}$ yield valid probability sets



Independent Subsystems

- N identical binary subsystems: state "1" occupied with probability p



$$r_{N,0} = p^{N}$$

$$r_{N,n} = p^{N-n}\left(1-p\right)^{n}$$

$$p_{N,n} = \frac{N!}{\left(N-n\right)!n!}\, p^{N-n}\left(1-p\right)^{n}$$

- The probabilities $p_{N,k}$ form a binomial distribution
- deMoivre-Laplace central limit theorem: binomial distribution, centered and scaled, approaches Gaussian as $N \to \infty$

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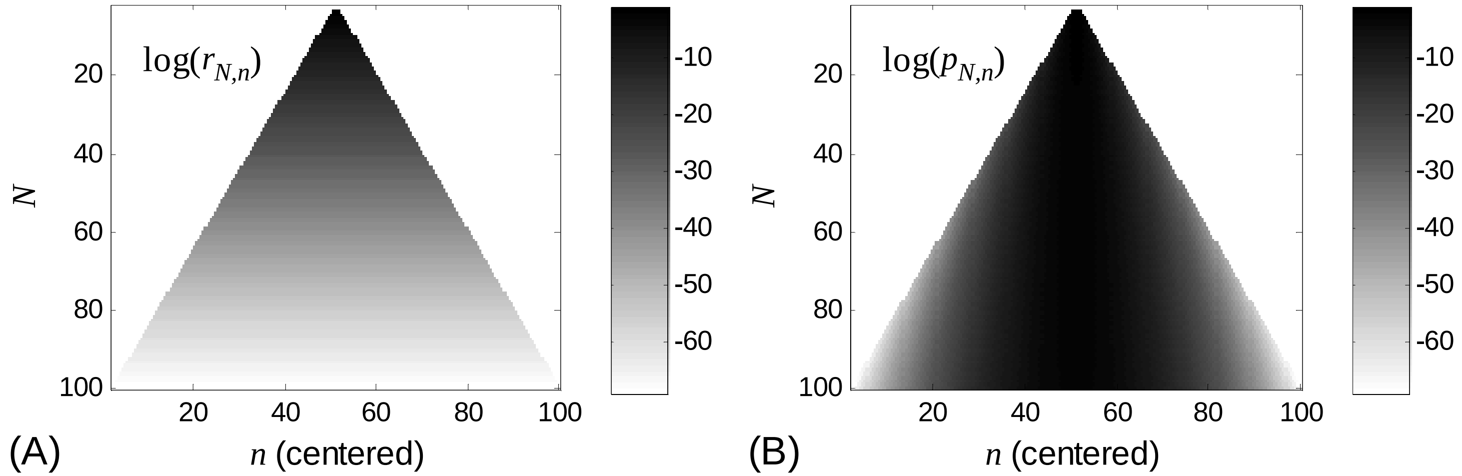
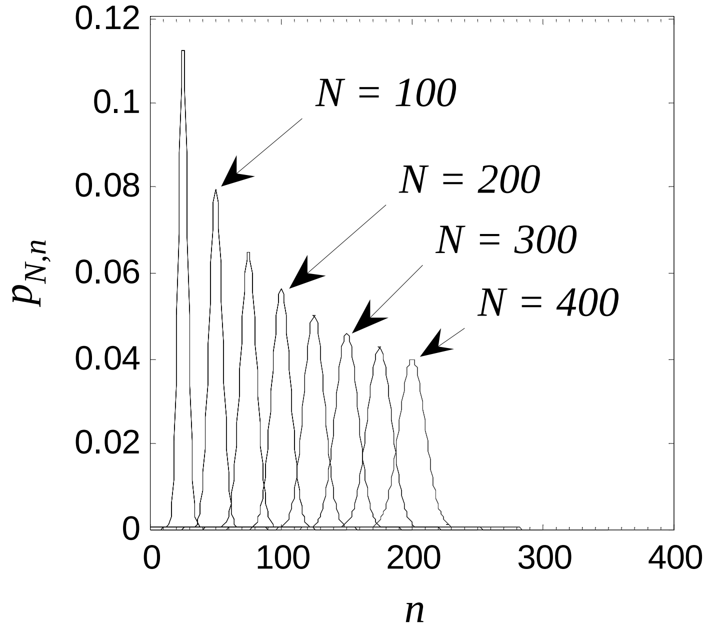
Independent case (cont).

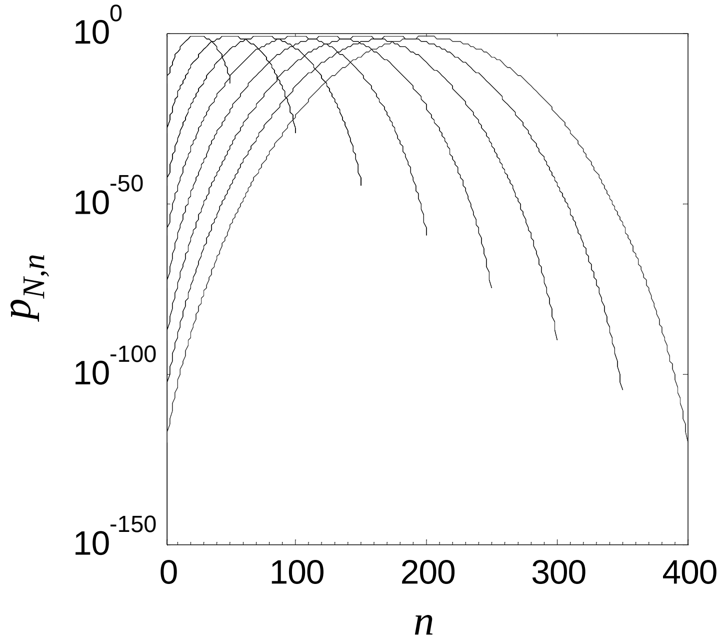
- Visualization of probabilities using grayscale images
- Case $p = 0.5$ shown



Independent case (cont).

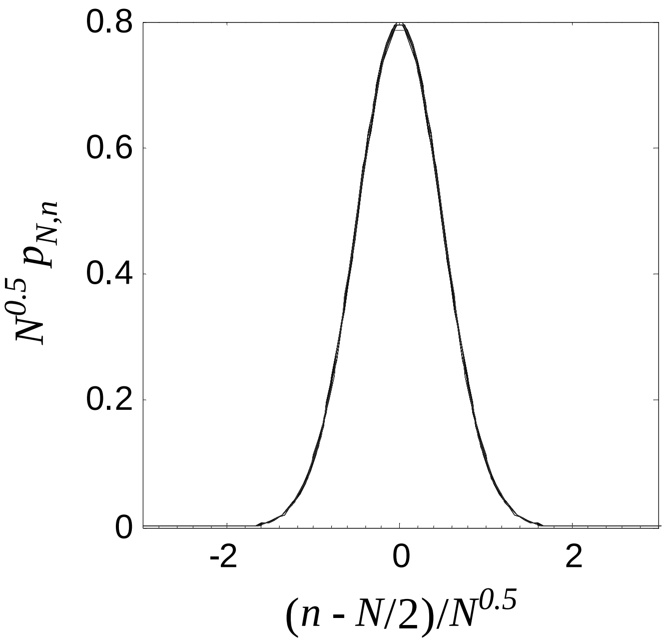
- Case $p = 0.5$ shown

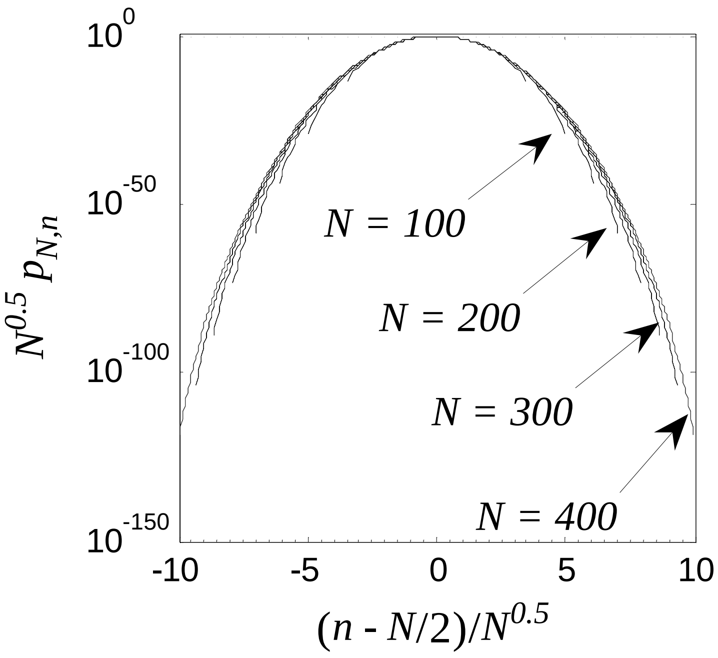




Independent Case (cont.)

- Rescaled and centered probability distributions demonstrate Gaussian attractor $p_N(n) \rightarrow p_N(n)N^{\gamma}$
- Usual diffusion exponent $\gamma = \frac{1}{2}$ $n \rightarrow \frac{(n - n_{center})}{N^{\gamma}}$





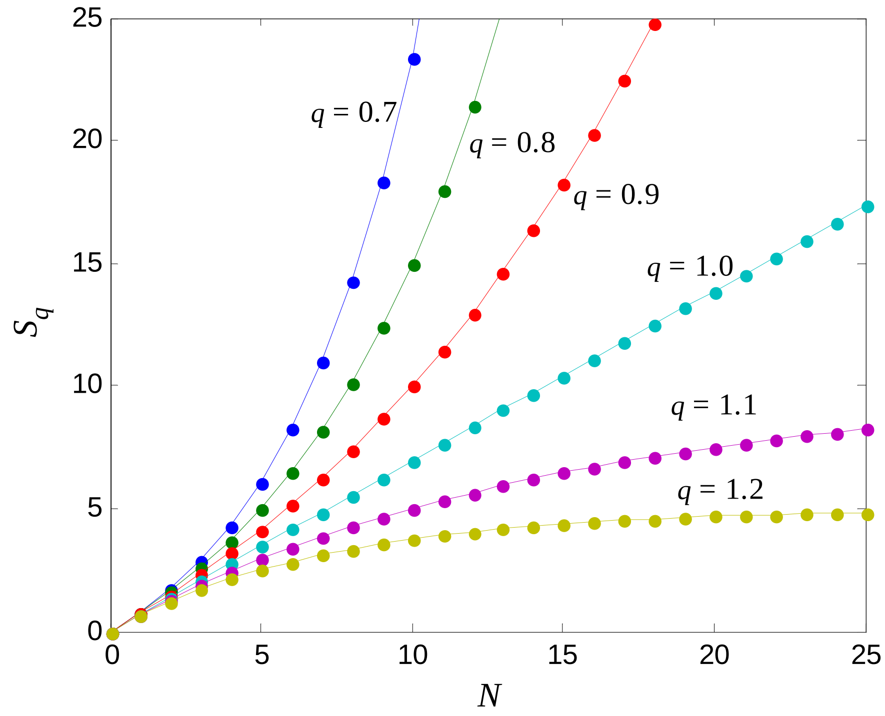
Independent Case (cont.)

- Entropy growth for generalized entropy S_q is linear only for $q = 1$

$$S_q(N) \equiv \frac{1}{q-1}\left(1-\sum_{i=1}^{2^N} p_i^q\right)$$

$$= \frac{1}{q-1}\left(1-\sum_{n=0}^{N}\frac{N!}{(N-n)!n!}\left(r_{N,n}\right)^q\right)$$

This is expected because all 2^N states are occupied with non-zero probability



q-Correlated Model*

- Generalizes the independent case

$$\frac{1}{r_{N,0}} = \left(\frac{1}{p}\right)^N$$

- By adding correlations induced by the q-product

$$x \otimes_q y \equiv \left(x^{1-q} + y^{1-q} - 1\right)^{\frac{1}{1-q}} \qquad \left(x \otimes_1 y = xy\right)$$

- Using the value q_{corr} in the q-product yields the model

$$r_{N,0} = \left[\left(\frac{1}{p}\right) \otimes_{q_{corr}} \left(\frac{1}{p}\right) \otimes_{q_{corr}} \cdots \left(\frac{1}{p}\right)\right]^{-1} = \left[Np^{q_{corr}-1} - \left(N-1\right)\right]^{\frac{1}{q_{corr}-1}}$$

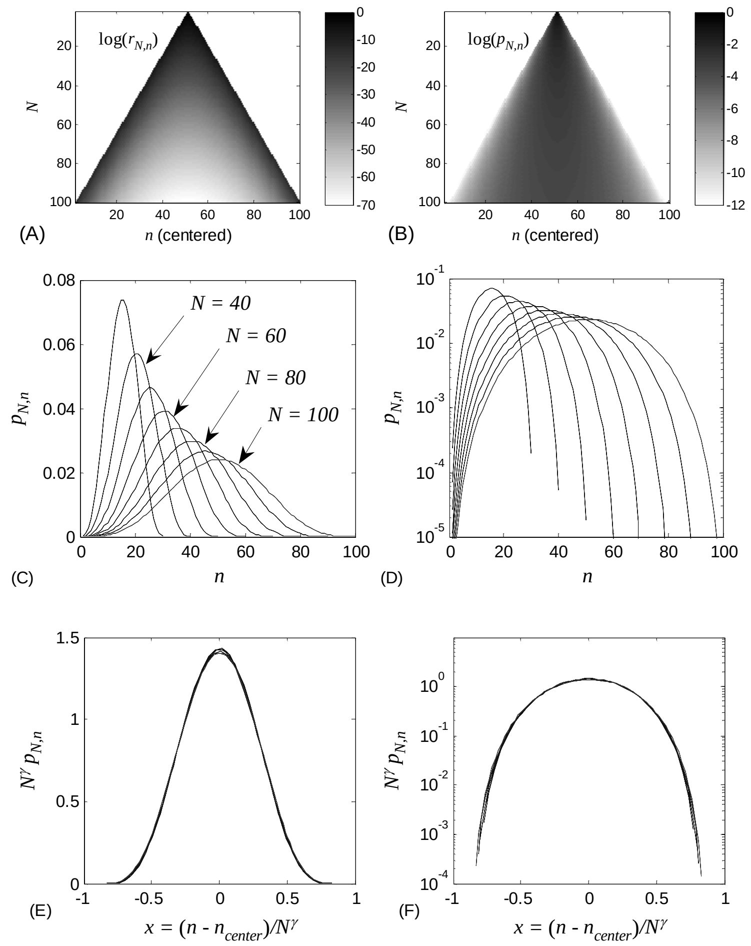
where the remaining $r_{N,n}$ are found using Leibnitz rule

*L.G. Moyano, C. Tsallis and M. Gell-Mann,
Europhys. Lett. 73, 813 (2006).

q-Correlated Model (cont.)

- Case $q_{corr} = 0.8$
- Attractor is found to be a Q-Gaussian with

$$Q = 2 - \frac{1}{q_{corr}} = 2 - \frac{1}{0.8} = 0.75$$



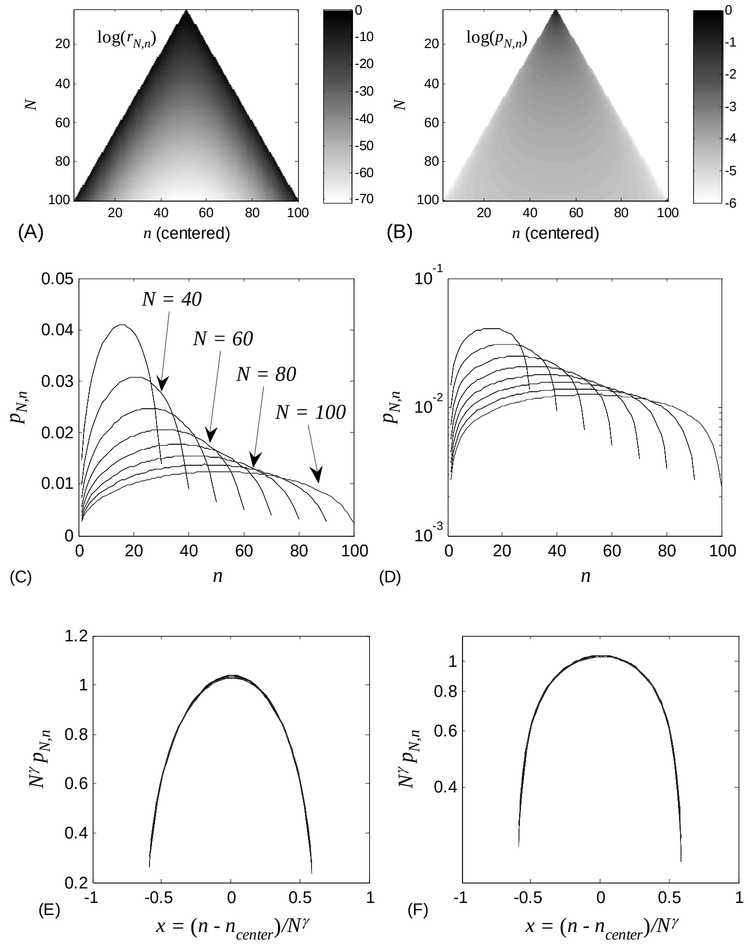
q-Correlated Model (cont.)

- Case $q_{corr} = 0.3$
- Attractor is found to be a Q-Gaussian with

$$Q = 2 - \frac{1}{q_{corr}}$$

$$= 2 - \frac{1}{0.3}$$

$$= -1.333$$



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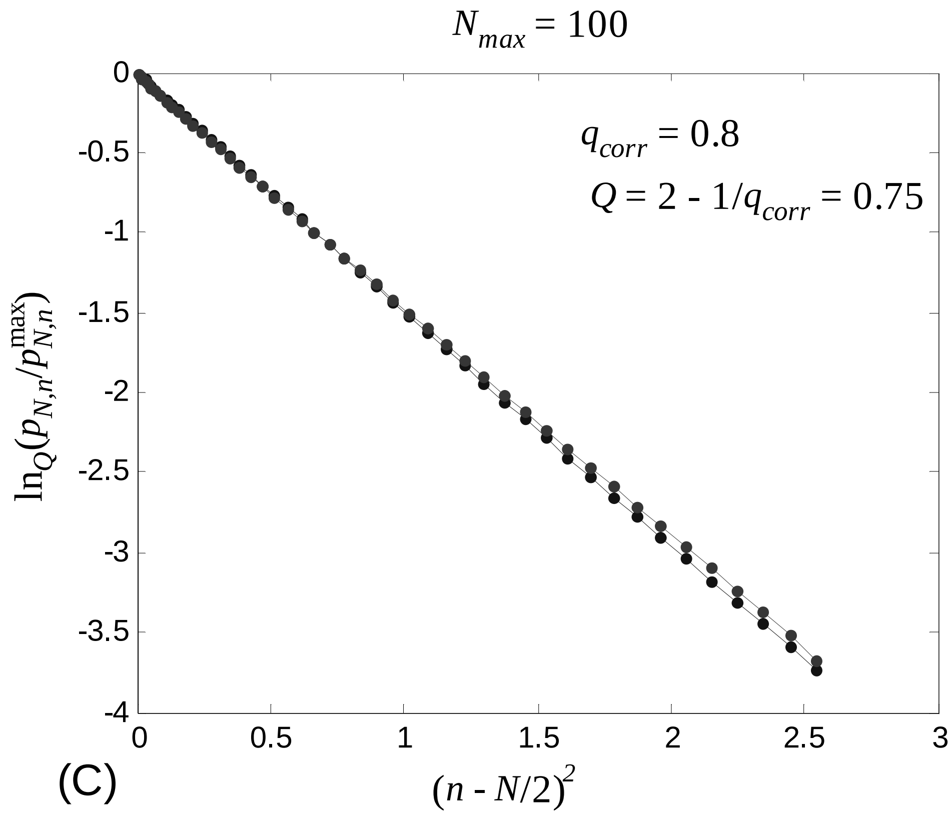
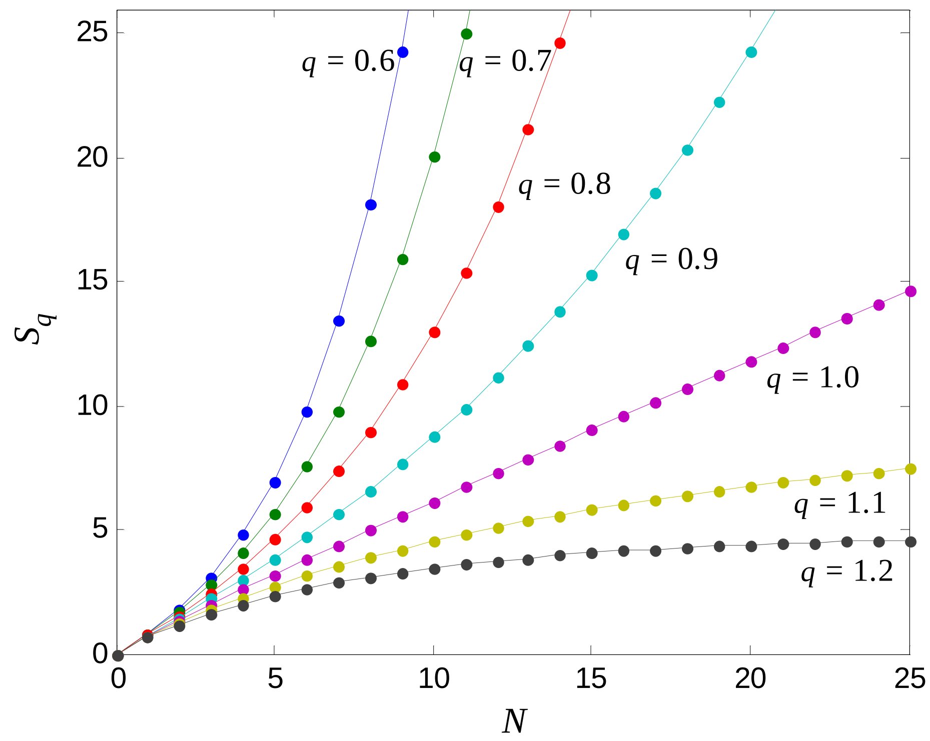
q-Correlated Model (cont.)

- Q-Gaussian fit illustrated for case $q_{corr} = 0.8$



q-Correlated Model (cont.)

- The entropy index for the q-Correlated model is not anomalous: $q_{ent} = 1$
- This model shows superdiffusive behavior $\gamma > 0.5$ for $q_{corr} < 1$



Stretched Exponential Model*

- Defined by reduced probabilities $r_{N,0} = r_{1,0}^{N^{\alpha}} = p^{N^{\alpha}}$ where $0 < \alpha < 1$
- Leibnitz rule used to construct other probabilities
- Case $\alpha = 1$ reduces to the independent case
- Case $\alpha = 0$ yields an extreme case of reduced occupancy: only two of the $r_{N,k}$ are non-zero

*C. Tsallis, M. Gell-Mann and Y. Sato,
Proc. Natl. Acad. Sc. USA 102, 15377 (2005).

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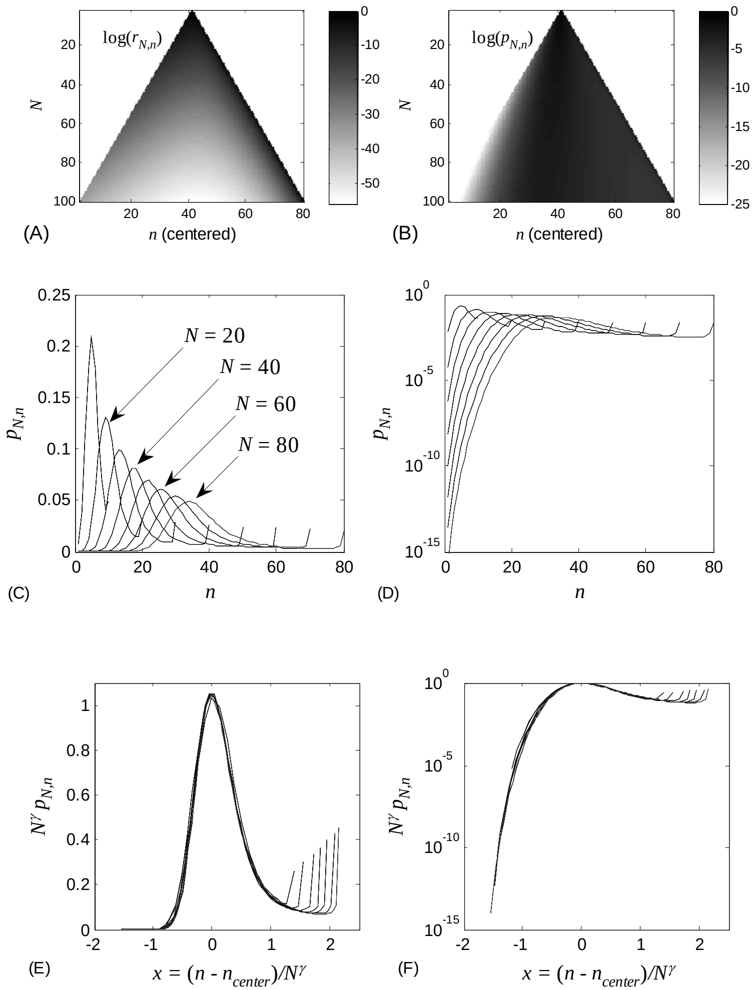
Stretched Exponential Model

- Example shown has $\alpha = 0.9$
- Model is superdiffusive for $\alpha < 1$, with diffusion exponent γ depending on α
- The entropy index for the stretched exponential model is not anomalous: $q_{ent} = 1$



Cutoff Model*

- A model with $W_{eff} \approx N^d$ can be constructed by allowing only a strip of d non-zero $r_{N,k}$ values

$$r_{N,n} = \begin{cases} \dfrac{1}{W^{eff}} = \left(\displaystyle\sum_{k=0}^{\min(d,N)} \dfrac{N!}{(N-k)!k!} \right)^{-1} & n \le d \\ 0 & \text{otherwise} \end{cases}$$

Note this model does *not* satisfy the Leibnitz rule

*C. Tsallis, M. Gell-Mann and Y. Sato,
Proc. Natl. Acad. Sc. USA 102, 15377 (2005).

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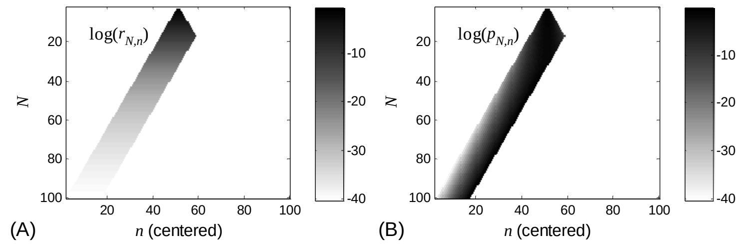
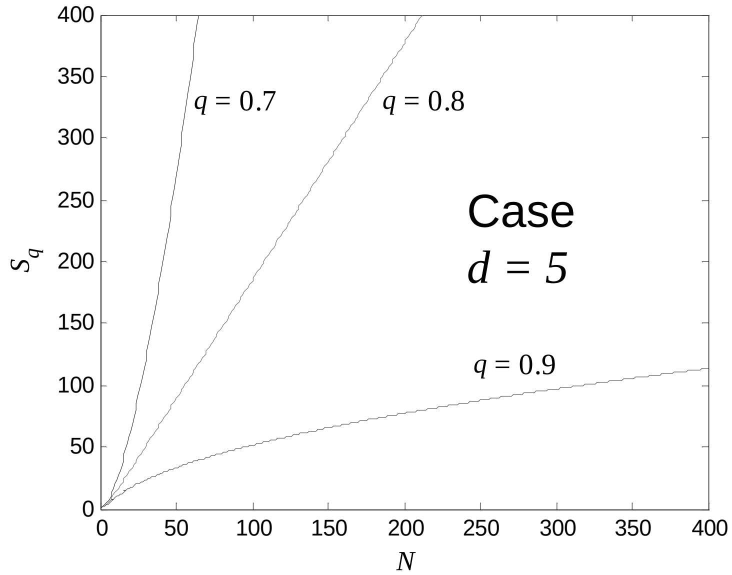
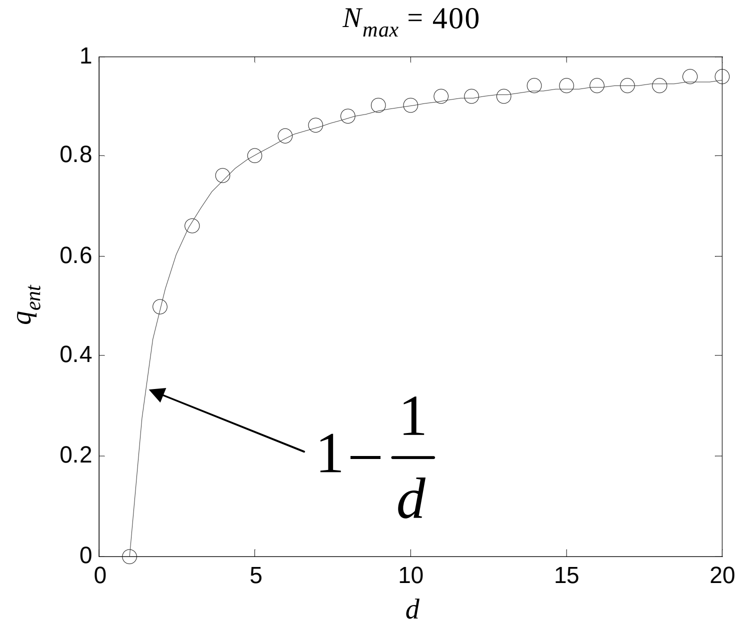
Cutoff Model (cont.)

- Attractor is a delta function (diffusion exponent $\gamma = 0$)
- We find $q_{ent} = 1 - \frac{1}{d}$ as expected

Case $d = 15$







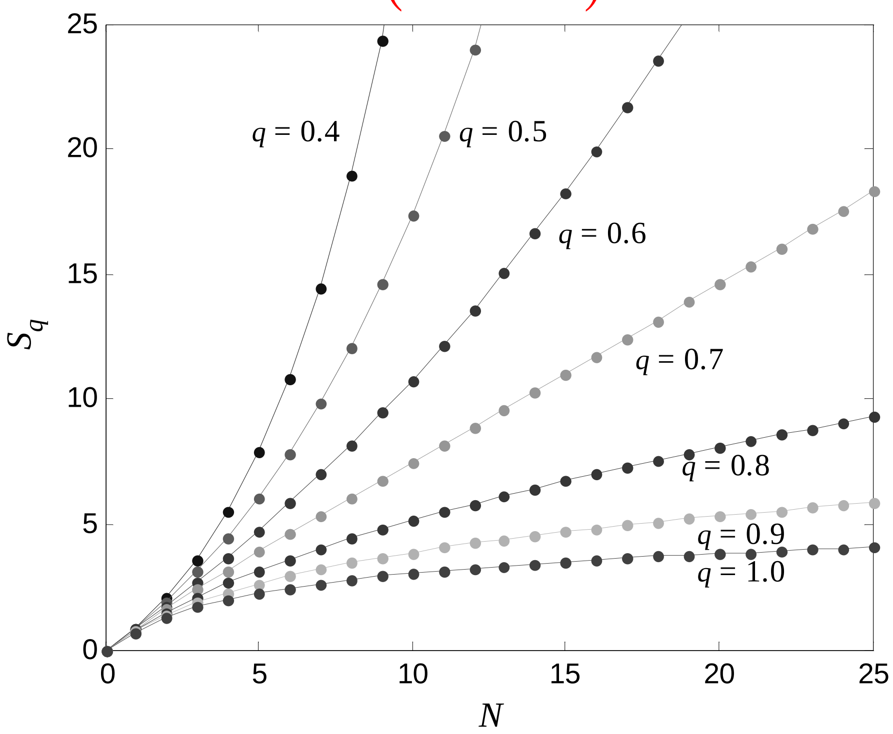
Power-Law Model*

- A new model defined by

$$r_{N,n} = \frac{N^{\alpha n}}{\sum_{k=0}^{N} \frac{N!}{(N-k)!k!} N^{\alpha k}}$$

- Due to normalization, the model is symmetric in α, only depends on $|\alpha|$
- Phase space occupation is very restricted: as N grows, fewer and fewer of the $r_{N,k}$ are effectively non-zero

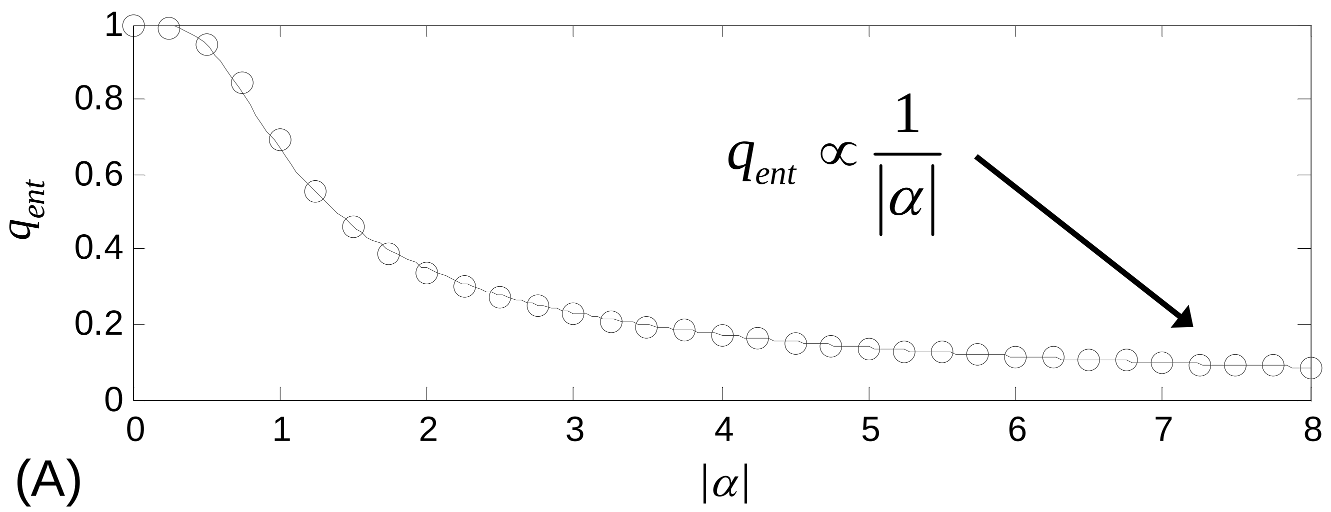
$(\alpha = -1)$



*J. Marsh, M. Fuentes, L. Moyano, and C. Tsallis, to appear in Physica A, 2006.

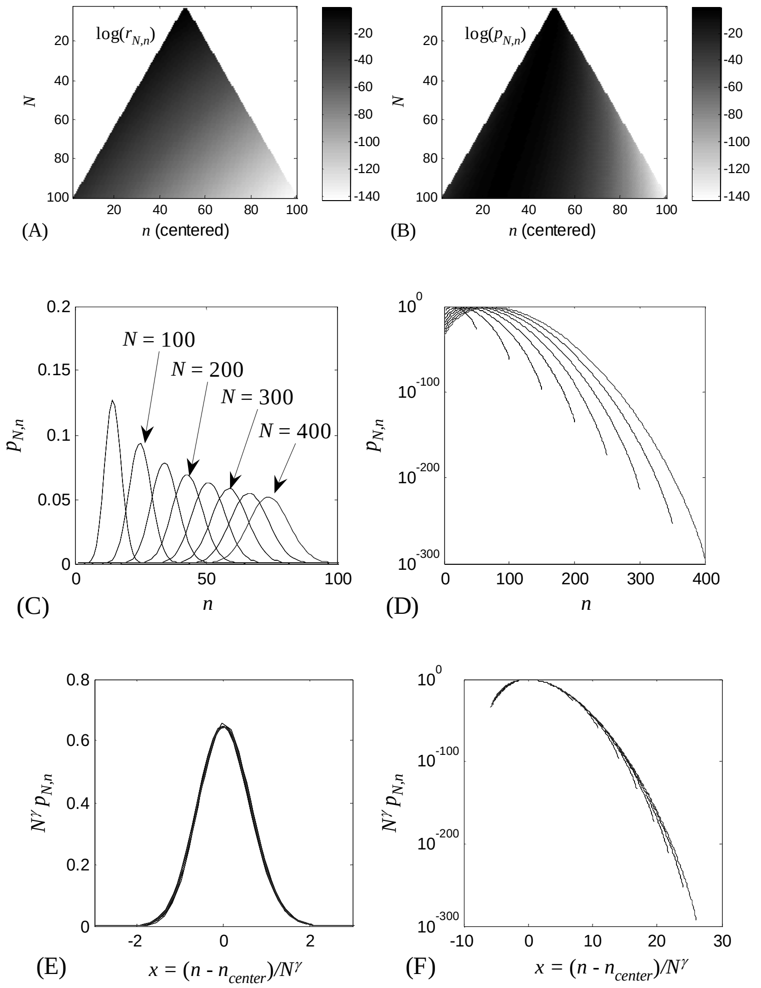
Power-Law Model

- The entropy index for the Power-Law model is anomalous: $q_{ent} < 1$



Power-Law Model

- The example shows the case α = -0.25
- In the example shown $\gamma = 0.42$



Subdiffusion observed for all $\alpha \neq 0$

Symmetric Power-Law Model*

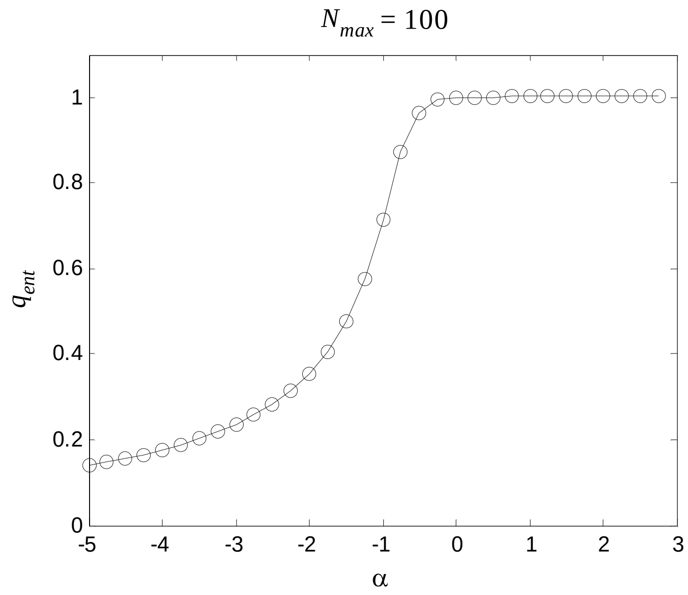
- A symmetrized version of the Power-Law model is defined as follows

$$r_{N,n} = \frac{N^{\alpha n'}}{Z}$$

where

$$n' = \begin{cases} n & n \le \lfloor N/2 \rfloor \\ N-n & otherwise \end{cases} \qquad n = 0, \ldots, N$$

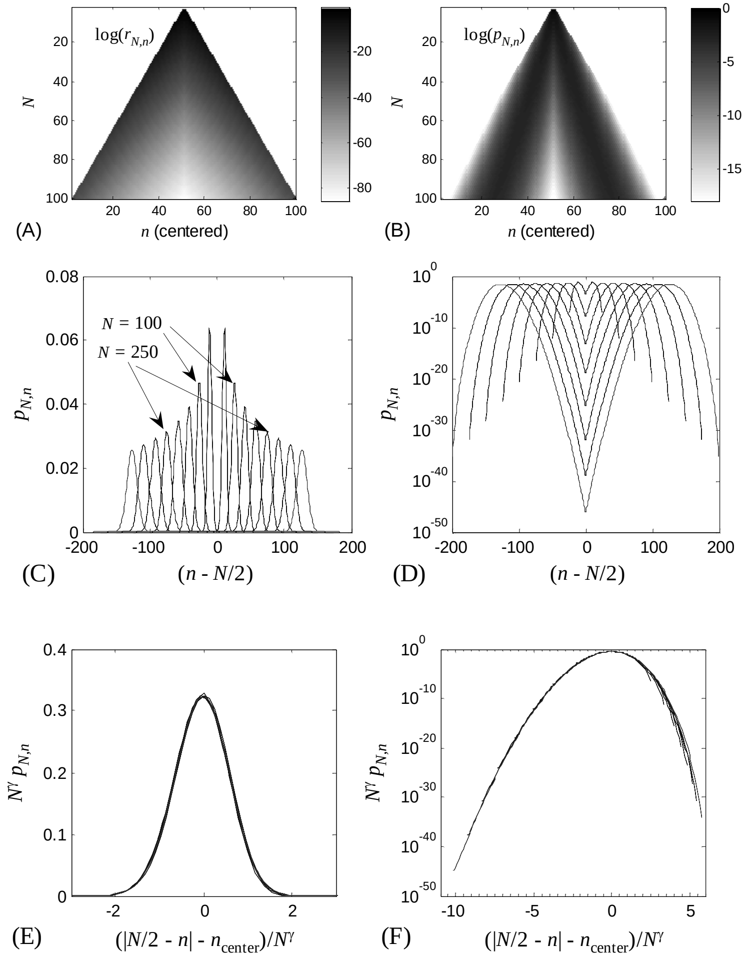
This definition adds symmetry to the probabilistic triangle, but breaks the symmetry in α observed in the Power-Law model



*J. Marsh, M. Fuentes, L. Moyano, and C. Tsallis, to appear in Physica A, 2006.

Symmetric Power-Law Model

- Example shown is at α = -0.25
- For $\alpha < 0$ we find the essentially same diffusion exponents as in the Power-Law model



Cutoff Power Law Model*

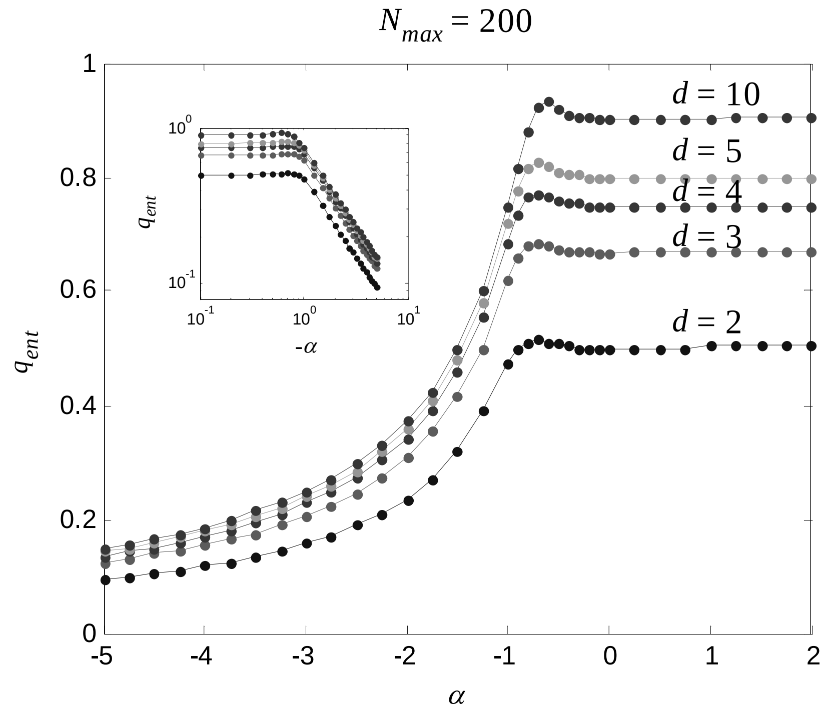
- Now we define a cutoff power law model as follows

$$r_{N,n} = \begin{cases} \dfrac{N^{\alpha n}}{Z} & n \le d \\ 0 & \text{otherwise} \end{cases}$$

  where the normalization constant Z is dependent on N
- Fitted q_{ent} values show a rich behavior, reducing to

$$q_{ent} = 1 - \frac{1}{d}$$

  when $\alpha = 0$, and showing power law decay as $\alpha \to \infty$



*J. Marsh, M. Fuentes, L. Moyano, and C. Tsallis, to appear in Physica A, 2006.

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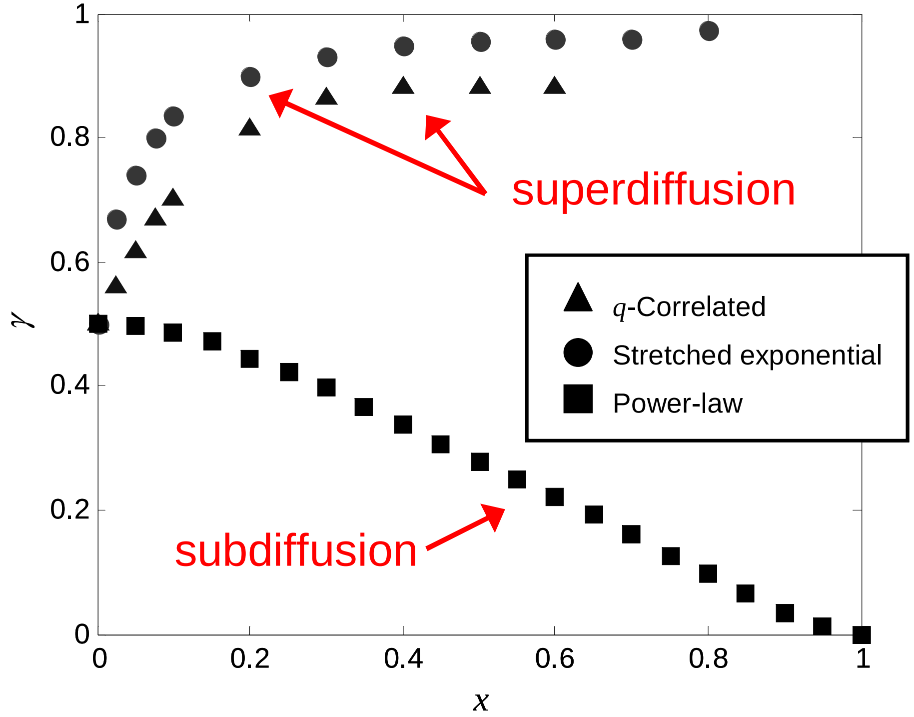
Summary of Diffusion Coefficients

- For all models, $x = 0$ corresponds to independent subsystem composition



$x = 1 - q_{corr}$ *q*-Correlated

$x = 1 - \alpha$ Stretched exponential

$x = |\alpha|$ Power-law

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Summary (cont.)

	q_{ent}	ATTRACTOR	DIFFUSION γ
INDEP	1	Gaussian	0.5
q-Correlated	1	Q-Gaussian ($Q\leq1$)	$0.5 \leq \gamma < 1$
Stretched Exponential	1	non Q-Gaussian	$0.5 \leq \gamma < 1$
Cutoff	$q_{ent} = 1 - 1/d < 1$	$\delta -$ function	0
ASF Cutoff	$q_{ent} = 1 - 1/d < 1$	$\delta -$ function	0
Power Law	$0 < q_{ent} \leq 1$	non Q-Gaussian	$0 < \gamma \leq 0.5$
Symmetric Power Law	$0 < q_{ent} \leq 1$	non Q-Gaussian	$0 < \gamma \leq 0.5$
Cutoff Power Law	$0 < q_{ent} \leq 1$	$\delta -$ function	0

superdiffusive (q-Correlated, Stretched Exponential)

subdiffusive (Cutoff through Cutoff Power Law)

Summary (cont.)