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**SCHOOL and CONFERENCE
on
COMPLEX SYSTEMS
and
NONEXTENSIVE STATISTICAL MECHANICS**

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Generalizing Boltzmann factors

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Outline

- Thermodynamics of complex systems ?
- What is a Boltzmann factor / experimental view ?
- Generalized Gibbs entropy
- The variational principle and consequences for the entropy
- A new entropy again ?
- Thermodynamics of this entropy
- Examples: Boltzmann-Gibbs and Tsallis are special cases
- Comments and Conclusion

Motivation

Complex Systems: distribution functions are often of power-law type. Often strong interactions among elements, c.s. violate ergodicity, they are often in states at the "edge of chaos" (weak sensitivity to initial conditions), Often not separable

In principle Gibbs entropies can take into account correlations in systems because the Hamiltonian can have potential terms

Question: can one do thermodynamics with complex systems? Can one use the maximum ignorance concept to understand the measured distributions?

Tsallis: take a modified entropy \longrightarrow understand distribution functions (\Rightarrow)

Here we want to **construct** and deduce the correct* entropy starting from a given experiment of an arbitrary statistical system, i.e.

take measured distribution functions \longrightarrow get entropy (\Leftarrow)

* Correct ? That one can do statistical physics in the way one is used to

Thermodynamics of 'complex systems' ?

Does there exist a generalized thermodynamics built upon other Boltzmann factors than the exponential ?

Start with Gibbs entropy:

$$S_G = - \int d\Gamma B(H(\Gamma)) \log B(H(\Gamma))$$

Γ are phase space variables

B is the Boltzmann factor and usually reads $B(H) \sim \exp(-\beta H)$ for the canonical distribution

What is a Boltzmann factor ?

The Boltzmann factor is a **probability to find a particular state in the bath system**. It closely relates to experiment

Remember, density of states is (classically) given by

$$\rho(E_1) = \omega_1(E_1)\omega_2(E - E_1)Z^{-1}$$

ω_1 is the **subjective** microcanonical density (**multiplicity** of states) in the ensemble of observable properties; ω_2 is the bath density

E is the energy of the total system (usually unknown)

Z is the partition function

Usually normalized $\omega_2(E - E_1)Z^{-1}$ is identified with Boltzmann factor

Let's cast the minimum requirements for a Boltzmann factor B into 'axioms'

- B is monotonous
- B can be normalized, i.e. $\int dE \omega_1(E) B(E) = 1$
- B must not explicitly depend on the total system energy. It must be possible that the E term in the argument of $\omega_2(E - E_1)$ can be factored out,

$$\omega_2(E - E_1) = F(E - E^*) B(E_1 - E^*) \quad ,$$

where the normalized version of B we call a generalized Boltzmann factor

Boltzmann factor which fulfills all requirements

$$B(H) \equiv \mathcal{E}(-\beta(H - U) - \tilde{\gamma})$$

$\tilde{\gamma}$ is a normalization constant (generalized partition function), U is the measured average energy, and β is the usual inverse temperature

Monotonicity implies the existence of an inverse function, the **generalized logarithms** $\Lambda = \mathcal{E}^{-1}$

We require the usual properties

$$\begin{aligned}\Lambda &: \mathbf{R}^+ \rightarrow \mathbf{R} \\ \Lambda(1) &= 0 \quad , \quad \Lambda'(1) = 1 \quad , \quad \Lambda' > 0 \\ \Lambda'' &< 0 \quad (\text{convexity})\end{aligned}$$

Define the **dual logarithm** as

$$\Lambda^*(x) \equiv -\Lambda(x^{-1})$$

The occurrence of dual logarithms in the context of generalized entropies has been noted recently

Philosophy: Who decides Which Boltzmann factor is right? Experiment does ! One can recover the form of the logarithm from the **measured** tail distribution of the Boltzmann factor

Generalize Gibbs entropy to arbitrary Boltzmann factors

$$S_G \equiv - \int d\Gamma B(H(\Gamma)) \Lambda(B(H(\Gamma)))$$

$$\begin{aligned} S_G &= - \int d\Gamma \mathcal{E}(-\beta(H - U) - \tilde{\gamma}) (-\beta(H - U) - \tilde{\gamma}) \\ &= - \int d\epsilon \int d\Gamma \delta(\epsilon - H) \mathcal{E}(-\beta(\epsilon - U) - \tilde{\gamma}) (-\beta(\epsilon - U) - \tilde{\gamma}) \\ &= - \int d\epsilon \omega(\epsilon) \mathcal{E}(-\beta(\epsilon - U) - \tilde{\gamma}) (-\beta(\epsilon - U) - \tilde{\gamma}) \end{aligned}$$

$\omega(E) \equiv \int d\Gamma \delta(E - H)$ is the multiplicity factor representing the observable system. With writing $\langle E \rangle = U$ and the expectation value given by

$$\langle f \rangle \equiv \int d\epsilon f(\epsilon) \omega(\epsilon) \mathcal{E}(-\beta(\epsilon - U) - \tilde{\gamma})$$

we get $S_G = \tilde{\gamma}$. Special case: $\beta = 0$ implies $B = Z^{-1} = (\int d\epsilon \omega(\epsilon))^{-1}$ and

$$S_G = - \int d\epsilon \omega(\epsilon) Z^{-1} \Lambda(Z^{-1}) = -\Lambda(Z^{-1}) = \Lambda^*(Z)$$

The variational principle

standard variational principle: $\delta G = 0$, with the functional

$$G = S_G[B] - \beta \int d\epsilon \omega(\epsilon) B(\epsilon) (\epsilon - U) - \gamma \left(\int d\epsilon \omega(\epsilon) B(\epsilon) - 1 \right)$$

with the generalized Gibbs entropy

$$S_G[B] = - \int d\epsilon \omega(\epsilon) B(\epsilon) \Lambda(B(\epsilon))$$

U ... measured average energy, $\omega(\epsilon)$... multiplicity, β ... usual inverse temperature, and γ ... Lagrange parameter for normalizability

Functional variation with respect to B yields the differential equation

$$\frac{d}{dB} B \Lambda(B) = -\gamma - \beta(E - U)$$

A problem

With the desired form of our Boltzmann factor $B = \mathcal{E}(-\beta(\epsilon - U) - \tilde{\gamma})$ the only solution is $\Lambda(B) = \log(B)$.

Why?

Substitute B into $\frac{d}{dB}B\Lambda(B) = -\gamma - \beta(E - U)$

$$\Lambda(B) + B\Lambda(B)' = -\gamma - \beta(E - U)$$

$$-\tilde{\gamma} - \beta(E - U) + B\Lambda(B)' = -\gamma - \beta(E - U)$$

$$B\Lambda(B)' = \text{const}$$

$$\longrightarrow \Lambda \equiv \log$$

Thus \mathcal{E} can only be the ordinary exponential Boltzmann factor

This is completely unsatisfactory !

Solve the problem

Problem arises because for any generalized Λ other than the ordinary log there exists a non-trivial extra term, $B\Lambda'(B)$. To cancel this term we suggest to **further generalize** the generalized logarithm $\Lambda(B)$ to a **functional**

$$\Lambda(B) \rightarrow \bar{\Lambda}[B] \equiv \Lambda(B) - \eta[B]$$

Substitute Λ by $\bar{\Lambda}$ into the generalized Gibbs entropy from before

$$\begin{aligned} S[B] &\equiv - \int d\epsilon \omega(\epsilon) B(\epsilon) \bar{\Lambda}([B]) \\ &= - \int d\epsilon \omega(\epsilon) B(\epsilon) (\Lambda(B(\epsilon)) - \eta[B]) \\ &= S_G[B] + \eta[B] \end{aligned}$$

used normalization condition and that η is a constant w.r.t. ϵ -integration

The Idea

After variation the additional term $\frac{\delta}{\delta B}\eta[B]$ cancels $-\omega(E)B(E)\frac{d}{dB}\Lambda(B(E))$
The corresponding condition is

$$\frac{\delta}{\delta B}\eta[B] = \omega(E)B(E)\frac{d}{dB}\Lambda(B(E))$$

which implies

$$\eta[B] = \int d\epsilon \omega(\epsilon) \int_0^{B(\epsilon)} dx \Lambda'(x)x + c$$

c is a integration constant

Yet another entropy ?

Let us substitute η

$$\begin{aligned} S[B] &= \eta[B] - \int d\epsilon \omega(\epsilon) B(\epsilon) \Lambda(B(\epsilon)) \\ &= \int d\epsilon \omega(\epsilon) \left[\int_0^{B(\epsilon)} dx \Lambda'(x)x - B(\epsilon) \Lambda(B(\epsilon)) \right] + c \\ &= - \int d\epsilon \omega(\epsilon) \int_0^{B(\epsilon)} dx \Lambda(x) + \bar{c} \end{aligned}$$

Note that classical entropy is a special case; $\Lambda(x) = \log(x)$, yields the Boltzmann entropy modulo a constant, $S[B] = - \int d\epsilon \omega(\epsilon) B(\epsilon) \log B(\epsilon) + \bar{c} + 1$

Thermodynamics of this entropy

$$\begin{aligned}\frac{\partial}{\partial U} S[B] &= -\frac{\partial}{\partial U} \int d\epsilon \omega(\epsilon) \int_0^{B(\epsilon)} dx \Lambda(x) \\ &= -\int d\epsilon \omega(\epsilon) \frac{\partial B(\epsilon)}{\partial U} \Lambda(B(\epsilon)) \\ &= -\int d\epsilon \omega(\epsilon) \frac{\partial B(\epsilon)}{\partial U} (-\gamma - \beta(\epsilon - U)) \\ &= -\frac{\partial}{\partial U} \int d\epsilon \omega(\epsilon) B(\epsilon) (-\gamma - \beta(\epsilon - U)) \\ &\quad + \int d\epsilon \omega(\epsilon) B(\epsilon) \left(-\frac{\partial \gamma}{\partial U} + \beta - \frac{\partial \beta}{\partial U} (\epsilon - U) \right) \\ &= \frac{\partial}{\partial U} \int d\epsilon \omega(\epsilon) B(\epsilon) \gamma + \int d\epsilon \omega(\epsilon) B(\epsilon) \left(\beta - \frac{\partial \gamma}{\partial U} \right) \\ &= \frac{\partial \gamma}{\partial U} + \beta - \frac{\partial \gamma}{\partial U} = \beta\end{aligned}$$

Special forms of this entropy

$$S[B] = - \int d\epsilon \omega(\epsilon) \int_0^{B(\epsilon)} dx \Lambda(x)$$

Example 1: Classical Boltzmann distributions

- You measure a distribution function, $B(E) \sim \exp(-\beta E)$
- Compute inverse $\Lambda(B) \sim \log(B)$; $\int_0^{B(\epsilon)} dx \log(x) = B(\epsilon) \log B(\epsilon) - B(\epsilon)$
- Result is **Boltzmann-Gibbs** entropy: $S[B] = - \int d\epsilon \omega(\epsilon) B(\epsilon) \log B(\epsilon) + 1$

Example 2 : Classical Tsallis distributions

- You measure a q -exponential distribution, $B(E) = [1 - (1 - q)E]^{1/(1-q)}$
- The generalized logarithm is $\Lambda(B) = \log_q(B) \equiv \frac{B^{1-q} - 1}{1-q}$
- Integrate and get **Tsallis** entropy times a factor

$$S[B] = -\frac{1}{2-q} \int d\epsilon \omega(\epsilon) B(\epsilon) \log_q(B(\epsilon)) + \frac{1}{2-q} \quad , \quad (q < 2)$$

Note 1: Average of logs

Write

$$\begin{aligned} S[B] &= - \int d\epsilon \omega(\epsilon) \int_0^{B(\epsilon)} dx \Lambda(x) \\ &= - \int d\epsilon \omega(\epsilon) B(\epsilon) \frac{1}{B(\epsilon)} \int_0^{B(\epsilon)} dx \Lambda(x) \\ &\equiv - \int d\epsilon \omega(\epsilon) B(\epsilon) L(B(\epsilon)) \end{aligned}$$

with $L(a) \equiv \frac{1}{a} \int_0^a dx \Lambda(x)$ being the mean of Λ . L is not inverse of B , i.g.

Note 2: Relation to other solutions

Remember problematic term from variation:

$$\frac{d}{dB} B\Lambda(B) = \Lambda(B) + B\Lambda'(B) = -\gamma - \beta(E - U)$$

- Jaynes: $\Lambda \equiv \log \longrightarrow B\Lambda'(B) = 1 \longrightarrow$ absorb into constant γ
- Tsallis: $\Lambda \equiv \log_q \longrightarrow B\Lambda'(B) = 1 + (1 - q)\Lambda \longrightarrow$ absorb γ and Λ
- Kaniadakis: define logs s.t. whole expression is a scaled and shifted log: $\Lambda(B) + B\Lambda'(B) \stackrel{!}{=} a\Lambda(B/b) + c$. Result are κ -logs; no restrictions on B
- We: modify entropy such that $B\Lambda'(B)$ cancels out after variation

Conclusion

- Observe non-exponential distribution function B from a statistical system (of identical particles with maybe complicated interactions)
- Is it possible to do consistent **thermodynamics** for systems leading to such distributions, where also the **maximum entropy principle** works – with measurable constraints ?
- We construct such an entropy: $S = - \int d\epsilon \omega(\epsilon) \int_0^{B(\epsilon)} dx \Lambda(x)$. This is a **standard generalized Gibbs** entropy ($\int B\Lambda B$) with **adding a constant** which is functionally dependent on the measured distribution B . The additive term captures numbers of states of the system, which are somehow captured by the distribution
- The functional form of measured distributions, which is "knowledge" about the system, is thus naturally fed into the definition of entropy
- Demonstrate that thermodynamic relations and MEP are fine

- Demonstrate how Boltzmann-Gibbs and Tsallis entropies are special cases
- Effectively: $p \log p$ in the usual entropy is replaced by the integral $\int \Lambda(p)$
- Sidenote: generalized Boltzmann factors naturally imply dual logarithms

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Repeat: Canonical Ensemble I

Any thermodynamic system which can be measured in equilibrium must be **separable**, i.e., thermodynamic quantities of the measured system must not explicitly depend on the total system energy E

Sample: $E_1, \omega_1(E_1), H_1$ (observed system)

Reservoir: $E_2, \omega_2(E_2), H_2$

Total (isolated) system: $E = E_1 + E_2, H = H_1 + H_2$ i.e. thermal contact

Sample energy fluctuates around its **equilibrium** (extremal) value E_*

Partition function $Z(E)$ is the convolution of microcanonical densities

$$Z(E) = \int_0^E dE_1 \omega_1(E_1) \omega_2(E - E_1) \quad \text{with} \quad \omega_i(E_i) = \int d\Gamma_i \delta(H_i - E_i)$$

Repeat: Canonical Ensemble II

The **canonical** description ρ , is given by

$$\rho(E_1) = \frac{\omega_1(E_1)\omega_2(E - E_1)}{Z(E)} \quad (1)$$

Assuming the existence of a unique **extremal configuration** at $E_1 = E_*$, defined by $\delta\rho = 0$, leads to

$$\left. \frac{\omega'_1}{\omega_1} \right|_{E_1=E_*} = \left. \frac{\omega'_2}{\omega_2} \right|_{E_2=E-E_*} := \frac{1}{kT} = \beta \quad , \quad (2)$$

which defines the temperature T of the system. The usual definition of entropy $S_i = k \ln(\omega_i)$ implies that the extremal configuration is found where $S = S_1 + S_2$ is extremal with its associated temperature T

Repeat: Canonical Ensemble III

Under which circumstances does the dependence of ρ on the total energy E factorize? Look for classes of microcanonical distributions that allow for such a separation of E into a multiplicative factor. The standard way to motivate the **appearance of the Boltzmann term** in the canonical ensemble is seen as a consequence of this E -separation

$$\begin{aligned}\omega_2(E - E_1) &= \exp(\ln(\omega_2(E - E_1))) \\ &= \exp(\ln(\omega_2(E)) - \frac{\partial}{\partial E} \ln(\omega_2) E_1 + \dots) \quad (3) \\ &\approx \omega_2(E) \exp(-\beta E_1)\end{aligned}$$

Note, the approximation in Eq. (3) is exact for $\omega_2(E - E_1)$ being an exponential in E

Up to this point we have summarized textbook knowledge

Question: Is this separation the most general way?

No

The General Separation

We generalize the log function before to a real function f , being strictly monotonous and twice differentiable. The idea is to write

$$\omega(E - E_1) = f^{-1} \circ f \circ \omega((E - E_*) - (E_1 - E_*)) \quad (4)$$

and to expand $f \circ \omega$ around $E - E_*$

What does energy separability mean? Suppose energy E is separable then there exist two functions g and h such that

$$\omega(E - E_1) = g(\omega(E - E_*)) h(\beta(E_1 - E_*)) \quad (5)$$

To simplify notation: $x := \beta(E_1 - E_*)$, and $\bar{\omega} := \omega(E - E_*)$. Now, use f to find the functions g and h by expanding $f \circ \omega$ to first order

$$f(\omega(E - E_1)) = f(g(\bar{\omega})h(x)) = f(\bar{\omega}) - \bar{\omega} x f'(\bar{\omega}) \quad (6)$$

which is justified for small x , i.e., the system being near equilibrium

Theorem: The most general solution to this separation Ansatz

$$f(g(\bar{\omega})h(x)) = f(\bar{\omega}) - \bar{\omega} x f'(\bar{\omega}) \quad (7)$$

is given by the family of equations $(f, g, h)_Q$, parametrized by a separation constant Q

$$\begin{aligned} f(\omega) &= C \omega^{1-Q} + C_2 \\ g(\omega) &= \omega \\ h(x) &= [1 - (1 - Q)x]^{\frac{1}{1-Q}} \end{aligned} \quad (8)$$

with C and C_2 being real constants

Proof: Set $x = 0$ and $h_0 = h(0)$, so that Eq. (7) yields $f(g(\bar{\omega})h_0) = f(\bar{\omega})$, which means $g(\bar{\omega}) = \frac{\bar{\omega}}{h_0}$. W.l.g. set $h_0 = 1$, and arrive at the equation: $f(\bar{\omega}h(x)) = f(\bar{\omega}) - \bar{\omega}x f'(\bar{\omega})$. Form partial derivatives of this expression with respect to x and $\bar{\omega}$, and eliminate the $f'(\bar{\omega}h)$ term from the two resulting equations

$$\begin{aligned} f'(\bar{\omega}h)h' &= -f'(\bar{\omega}) \\ f'(\bar{\omega}h)h &= (1 - x)f' - \bar{\omega}x f'' \end{aligned} \quad (9)$$

to arrive at the separation equation

$$1 - \frac{1}{x} \left(\frac{h}{h'} + 1 \right) = -\bar{\omega} \frac{f''(\bar{\omega})}{f'(\bar{\omega})} = Q \quad (10)$$

where Q is the separation constant. The differential equation $1 - \frac{1}{x} \left(\frac{h}{h'} + 1 \right) = Q$ is straight forwardly solved to give $h(x) = [1 - (1 - Q)x]^{\frac{1}{1-Q}}$, using $h(0) = 1$ to fix the integration constant. The equation $-\bar{\omega} \frac{f''(\bar{\omega})}{f'(\bar{\omega})} = Q$ means, $f(\bar{\omega}) = C_1 \frac{1}{1-Q} \bar{\omega}^{1-Q} + C_2$, with C_1 and C_2 integration constants. f is strictly monotonous except for $Q = 1$, where it is constant q.e.d.

The Most General Boltzmann Factor

The term of interest in the canonical distribution can now be written

$$\omega_2(E - E_1) = \omega_2(E - E_*) [1 - (1 - Q)\beta(E_1 - E_*)]^{1/(1-Q)} \quad (11)$$

- This is exactly the q exponential or Tsallis distribution
- The usual Boltzmann factor is the special case for the limit $Q \rightarrow 1$
- The separation constant Q is not specified at this level, only the choice of a particular physical system will determine Q
- For $\omega_2 \propto E^{1/1-Q}$, Eq. (11) is exact. To prove this write $\beta = \frac{\omega'_2}{\omega_2} \Big|_{E-E_*} = \frac{1}{(1-Q)(E-E_*)}$ and to compute straight forwardly

$$\begin{aligned} \omega_2(E - E_1) &= (E - E_*)^{1/(1-Q)} \left(1 - \frac{E_1 - E_*}{E - E_*}\right)^{1/(1-Q)} \\ &= \omega_2(E - E_*) [1 - (1 - Q)\beta(E_1 - E_*)]^{1/(1-Q)} \end{aligned} \quad (12)$$

The Physical Meaning of the Separation Constant Q

Why does nature fix $Q \rightarrow 1$ in so many cases, i.e. why is standard thermodynamics the most predominantly realized situation?

We demonstrate how the separation constant Q is related to system size and interaction parameters of real physical systems

This explains the ubiquity of $Q = 1$ in nature

An Example

Examples of this kind have been given in a different context before (Tsallis 2001 and Almeida 2002)

Let us specify the N particle **Hamiltonian for pair-potentials** governing the sample in D space dimensions. Use $n = DN$

$$H(x, p) = \sum_i^N \frac{p_i^2}{2m} + \sum_{i < j}^N |x_i - x_j|^\alpha \quad (13)$$

The energy density is given by the phase space integral

$$\begin{aligned} \omega(E) &= \int d^n p d^n x \delta \left(\sum_i \frac{p_i^2}{2m} + \sum_{i < j} |x_i - x_j|^\alpha - E \right) \\ &= \int_0^E dE_1 \int d^n p d^n x \delta \left(\sum_i \frac{p_i^2}{2m} - E_1 \right) \\ &\times \delta \left(\sum_{i < j} |x_i - x_j|^\alpha - (E - E_1) \right) \end{aligned} \quad (14)$$

We compute the **kinetic term**

$$\begin{aligned}
 \int d^n p \delta \left(\sum_{i=1}^N \frac{|\vec{p}_i|^2}{2m} - E \right) &= \int_{\frac{\vec{p}^2}{2m}=E} d\mathcal{O}_n \left| \vec{\nabla} \frac{\vec{p}^2}{2m} \right|^{-1} \\
 &= \int_{|\vec{p}|=\sqrt{2mE}} d\mathcal{O}_n \frac{m}{|\vec{p}|} = \sqrt{\frac{m}{2E}} \int_{|\vec{p}|=\sqrt{2mE}} d\mathcal{O}_n \\
 &= \mathcal{O}_n \sqrt{\frac{m}{2E}} (2mE)^{\frac{n-1}{2}} \propto E^{\frac{n}{2}-1}
 \end{aligned} \tag{15}$$

and the **potential** contribution

$$\begin{aligned}
 \int d^n x \delta \left(\sum_{i<j} |x_i - x_j|^\alpha - E \right) \\
 &= \int d^n x \delta \left(\sum_{j=2}^N \sum_{i=1}^{j-1} \left| \sum_{k=1}^D (x_i^k - x_j^k)^2 \right|^{\frac{\alpha}{2}} - E \right) \\
 &= \int d^n x \delta \left(E \left[\sum_{j=2}^N \sum_{i=1}^{j-1} \left| \sum_{k=1}^D \left(\frac{x_i^k - x_j^k}{E^{\frac{1}{\alpha}}} \right)^2 \right|^{\frac{\alpha}{2}} - 1 \right] \right) \\
 &= E^{\frac{n}{\alpha}} \int d^n y \delta \left(E \left[\sum_{j=2}^N \sum_{i=1}^{j-1} \left| \sum_{k=1}^D (y_i^k - y_j^k)^2 \right|^{\frac{\alpha}{2}} - 1 \right] \right) = E^{\frac{n}{\alpha}-1} \cdot \text{const.}
 \end{aligned} \tag{16}$$

used substitution $y_i = x_i/E^{\frac{1}{\alpha}}$ and $\int dx \delta(\lambda x) = \int dx \lambda^{-1} \delta(x)$

We finally get

$$\omega(E) \propto \int_0^E dE_1 E_1^{\frac{n}{2}-1} (E - E_1)^{\frac{n}{\alpha}-1} \propto E^{\frac{(\alpha+2)n}{2\alpha}-1} \quad (17)$$

This allows us to compare exponents (and coefficients) in Eq. (17) with the general Boltzmann factor $\omega_2(E - E_*) [1 - (1 - Q)\beta(E_1 - E_*)]^{\frac{1}{1-Q}}$ to arrive at the relation

$$\frac{1}{1-Q} = \frac{(\alpha+2)n}{2\alpha} - 1 \quad (18)$$

which fixes the separation constant in terms of physical variables

Interpretation of Example

- This equation establishes the connection between the interaction term in the Hamiltonian α , the dimensionality of the phase space $n = DN$, and the separation constant Q
- From this equation it is immediately clear that for large systems the separation constant is always $Q \rightarrow 1$, i.e. the classical Boltzmann term is recovered
- For very small systems with a fixed number of particles Q depends on the interaction between the particles. For an ideal gas $\alpha \rightarrow -\infty$ the separation constant is $Q = \frac{n-4}{n-2}$
- Nontrivial $Q \neq 1$ should be expected for strongly interacting and/or small systems, i.e., $|\alpha|/|\alpha + 2| \sim n$