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**SCHOOL and CONFERENCE
on
COMPLEX SYSTEMS
and
NONEXTENSIVE STATISTICAL MECHANICS**

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**Temporal Extensivity of Tsallis Entropy and
Bound on Entropy Production Rate**

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I . Introduction

Boltzmann-Gibbs Statistical Mechanics

- Ergodicity

- Molecular Chaos Hypothesis

Dynamical Counterpart of Boltzmann-Gibbs-Shannon Entropy

- Kolmogorov-Sinai Entropy

- Positive Lyapunov Exponent(s)
- Constant Entropy Production Rate

Thermodynamic Limit



Long-Time Limit

A quantity $A(t)$ is “temporally extensive”, if it satisfies

$$\lim_{t \rightarrow \infty} \frac{A(t)}{t} = \text{const}$$

e.g.,

The Kolmogorov-Sinai entropy is temporally extensive for chaotic dynamical systems.

N.B.,

Thermodynamic Extensivity of
Thermodynamic Entropy



A Basic Premise for Thermodynamics
to be Constructible

Complexity at the Edge of Chaos

- The maximum Lyapunov exponent vanishes.
- The Kolmogorov-Sinai entropy fails to be temporally extensive.

II . Thermodynamics Based on Time Average

A. Carati, Physica A **348**, 110 (2005).

Connection between
“Probabilistic Process in Phase Space”
and
“Thermodynamics”

- Expected to shed new light on temporal extensivity of thermodynamic quantities

Basics of Time-Average Formalism

M : Phase Space

$$\phi: M \rightarrow M \quad \text{st.} \quad x_{n+1} = \phi(x_n) \quad (x_n \in M)$$

$\{x_n\}_{n=0,1,2,\dots}$: Time Series Generated by
the Dynamical Map

Average of a physical quantity A over
a fixed long time interval $1 \leq n \leq N$ ($N \gg 1$)

$$\bar{A}(x_0) = \frac{1}{N} \sum_{n=1}^N A(x_n)$$



still random, depending on the
initial data, x_0

"Coarse Graining"

Divide M into cells L_1, L_2, \dots, L_K ($K \gg 1$)

A_i : **representative value of A in L_i**

n_i : **number of times the system visits L_i**

$$\bar{A}(x_0) \approx \sum_{i=1}^K A_i \frac{n_i}{N}$$

where

$\{n_i\}_{i=1, 2, \dots, K}$: **random**

Sojourn Time Distribution

Cumulative Probability that L_i is visited

$n_i (\leq n)$ times by the system:

$$P(n_i \leq n) \equiv F(n_i)$$

The Average Value of A

$$\langle \bar{A} \rangle = \frac{1}{N} \sum_{i=1}^K A_i \langle n_i \rangle$$

with

$$\langle n_i \rangle = \frac{\int \prod_{j=1}^K dF(n_j) n_i \delta(N - \sum_j n_j)}{\int \prod_{j=1}^K dF(n_j) \delta(N - \sum_j n_j)}$$

Generating Function

$$Z(\lambda) = \int \prod_{j=1}^K dF(n_j) e^{-\lambda \sum_i A_i n_i} \delta(N - \sum_i n_i)$$



$$\langle \bar{A} \rangle = -\frac{1}{N} \frac{\partial}{\partial \lambda} \ln Z(\lambda) \Big|_{\lambda=0}$$

If the constraint is imposed on
the energy:

$$U = \frac{1}{N} \sum_{i=1}^K \varepsilon_i n_i$$

(ε_i : energy in L_i)



$Z(\lambda)$ is replaced by

$$\begin{aligned} Z_u(\lambda) &= \int \prod_{j=1}^K dF(n_j) e^{-\lambda \sum_i A_i n_i} \\ &\times \delta(N - \sum_i n_i) \delta(U - \sum_i \varepsilon_i n_i / N) \end{aligned}$$

and accordingly,

$$\langle \bar{A} \rangle_u = -\frac{1}{N} \frac{\partial}{\partial \lambda} \ln Z_u(\lambda) \Big|_{\lambda=0}$$

Define:

$$\int dF(n) e^{-n\zeta} = e^{\chi(\zeta)}$$

Then,

$$Z_U(\lambda) = \frac{N}{(2\pi)^2} \int_{-\infty}^{\infty} dk_1 \int_{-\infty}^{\infty} dk_2 \\ \times e^{ik_1 U N + ik_2 N + \sum_i \chi(\lambda A_i + ik_1 \varepsilon_i + ik_2)}$$

Large- N Limit

Steepest-Descent Conditions:

$$U = -\frac{1}{N} \sum_{i=1}^K \varepsilon_i \chi'(ik_1 \varepsilon_i + ik_2)$$

$$N = -\sum_{i=1}^K \chi'(ik_1 \varepsilon_i + ik_2) \\ (\chi'(\zeta) = d\chi(\zeta)/d\zeta)$$

in the limit $\lambda \rightarrow 0$

Therefore,

$$\langle \bar{A} \rangle_U = -\frac{1}{N} \sum_{i=1}^K A_i \chi'(\theta \varepsilon_i + \alpha)$$

$$= \sum_{i=1}^K A_i p_i$$

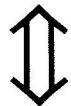
with the sojourn time probability

$$p_i = -\frac{\chi'(\theta \varepsilon_i + \alpha)}{N}$$

provided that

ik_1 and ik_2 are continued
to θ and α

Boltzmann-Gibbs Statistics



$F(n)$ is Poissonian!

$$\int dF(n) e^{-n\zeta} := \sum_{n=0}^{\infty} e^{-Np} \frac{(Np)^n}{n!} e^{-n\zeta}$$

$$= e^{Np e^{-\zeta} - Np} \quad (0 < p < 1)$$

$$\therefore \chi(\zeta) = Np e^{-\zeta} - Np$$

ζ -dependent relevant part

$$\chi_0(\zeta) = Np e^{-\zeta}$$

(anticipated as the free energy)

The Legendre Transformation

$$s(v_i) = v_i \zeta_i + \chi_0(\zeta_i)$$

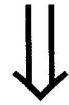
$$v_i = -\chi'_0(\zeta_i)$$

$$(\zeta_i = \theta \varepsilon_i + \alpha)$$

Then, one finds:

$$\zeta_i = -\ln(p_i / p)$$

$$v_i = N p_i$$



Boltzmann-Gibbs-Shannon Entropy

$$S = \sum_{i=1}^K s(v_i) = -N \sum_{i=1}^K p_i \ln p_i \quad (p := 1/e)$$

Boltzmann-Gibbs-Shannon Entropy

$$S \propto N$$

i.e.,

Temporally Extensive

“Thermodynamics” Entropy (Entropy Production Rate)

$$S_{\text{th}} \equiv \frac{S}{N}$$

● $\frac{\partial S_{\text{th}}}{\partial U} = \theta \equiv \frac{1}{T} := \beta$

● $p_i = \frac{e^{-\beta \varepsilon_i}}{Z(\beta)}, \quad Z(\beta) = \sum_{i=1}^K e^{-\beta \varepsilon_i}$

III. Temporal Extensivity of Tsallis Entropy

Two Points:

(i) q -Expectation Value

$$\langle \bar{A} \rangle_{U,q} = \sum_{i=1}^K A_i P_i^{(q)}$$

$P_i^{(q)} = \frac{(p_i)^q}{\sum_j (p_j)^q}$: Escort Distribution

$$P_i^{(q)} = -\frac{\chi'(\theta \varepsilon_i + \alpha)}{N}$$

(see p.13)

(ii) Deformation of Poissonian Process

Deformation of Generating Function

$$\chi_0(\zeta) = N p e^{-\zeta} \rightarrow \chi_{0,q}(\zeta) = N r_q e_q(-\zeta)$$

with the q -exponential function

$$e_q(x) = [1 + (1 - q)x]_+^{1/(1-q)} \quad (\xrightarrow{q \rightarrow 1} e^x)$$

$$([a]_+ \equiv \max\{0, a\})$$

$$v_i = -\chi'_{0,q}(\zeta) \Big|_{\zeta=\zeta_i=\theta\varepsilon_i+\alpha} = N r_q [e_q(-\zeta_i)]^q$$

$$= N P_i^{(q)},$$

$$\zeta_i = -\frac{1}{1-q} \left[\left(\frac{v_i}{N r_q} \right)^{1/q-1} - 1 \right],$$

$$\chi_{0,q}(\zeta_i) = N r_q \frac{p_i}{(c_q r_q)^{1/q}}$$

$$(c_q = \sum_{i=1}^K (p_i)^q)$$

The corresponding entropy:

$$S_q = \sum_{i=1}^K s_q(v_i) = \sum_{i=1}^K [v_i \zeta_i + \chi_{0,q}(\zeta_i)]$$

$$= \frac{N}{1-q} \left[\sum_{i=1}^K (p_i)^q - 1 \right]$$

∴

The Tsallis Entropy is also
Temporally Extensive:

$$S_q \propto N,$$

Whenever Relevant.

N.B., r_q must be chosen to be

$$r_q = \left[\frac{q}{(c_q)^{1/q} (2 - c_q)} \right]^{q/(1-q)} \quad (\xrightarrow{q \rightarrow 1} p = 1/e)$$

Reality Condition:

$$c_q \equiv \sum_{i=1}^K (p_i)^q < 2$$

Nonextensive Statistical Mechanics

$$S_{\text{th},q} = \frac{S_q}{N}$$

● $\frac{\partial S_{\text{th},q}}{\partial U} = \theta \equiv \frac{1}{T} := \beta$

● $p_i = \frac{1}{Z_q(\beta)} e_q(-(\beta \varepsilon_i + \alpha)),$

$$Z_q(\beta) = \sum_{i=1}^K e_q(-(\beta \varepsilon_i + \alpha))$$

IV. Bound on the Tsallis-Entropy Production Rate

$$c_q \equiv \sum_{i=1}^K (p_i)^q < 2$$

(1) $q > 1$ (trivial)

$$0 < c_q < 1$$



$$\frac{S_q}{N} = \frac{1}{1-q} \left[\sum_{i=1}^K (p_i)^q - 1 \right] < \frac{1}{q-1}$$

(2) $0 < q < 1$ (nontrivial)

$$0 < c_q < 2$$



$$\frac{S_q}{N} = \frac{1}{1-q} \left[\sum_{i=1}^K (p_i)^q - 1 \right] < \frac{1}{1-q}$$

∴

Universal Bound on Tsallis Entropy Production Rate

$$\frac{S_q}{N} < \frac{1}{|1-q|}$$

$$(\forall q > 0)$$

Remark:

All known analytical results on
nonlinear dynamical systems
at the edge of chaos
are found to obey this bound.

V. Associated Probabilistic Process in Phase Space

Boltzmann-Gibbs Statistics



Poissonian

$$\int dF(n) e^{-n\zeta} := \sum_{n=0}^{\infty} e^{-Np} \frac{(Np)^n}{n!} e^{-n\zeta} = e^{Npe^{-\zeta} - Np}$$

What is the corresponding

$$f(n) = dF(n)/dn$$

for nonextensive statistics?

The z -Transformation

$$\sum_{n=0}^{\infty} f(n)z^{-n} = \tilde{f}(z)$$

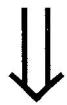
$$\tilde{f}(z) = e^{x(\zeta)}, \quad z = e^{\zeta}$$

The Inverse z -Transformation

$$f(n) = \frac{1}{2\pi i} \oint_C dz \tilde{f}(z) z^{n-1}$$

where

C is a circle centered at $z=0$
surrounding all the poles of $\tilde{f}(z)$



$$f(n) = \frac{1}{2\pi i} \int_{a-i\pi}^{a+i\pi} d\zeta e^{\chi_q(\zeta)} e^{n\zeta}$$

α : logarithm of
the radius of G

with

$$\chi_q(\zeta) = Nr_q [1 - (1-q)\zeta]^{1/(1-q)} - Nr_q$$

★★Steepest Descent Approximation★★

$$f(n) \sim \begin{cases} n^{1/(2q)-1} \exp\left[-\frac{qNr_q}{1-q}\left(\frac{n}{Nr_q}\right)^{1/q}\right] & (0 < q < 1) \\ n^{1/(2q)-1} \exp\left(-\frac{n}{q-1}\right) & (q > 1) \end{cases}$$

for large n

VI. Concluding Remarks

- **Temporal Extensivity of Tsallis Entropy**
- **Complete Derivation of Nonextensive Statistical Mechanics based on Time Average**
- **Universal Bound on Tsallis Entropy Production Rate**
- **Probabilistic Process associated with Nonextensive Statistical Mechanics**

Reference

S. A. and Y. Nakada, cond-mat/0603550

to appear in Phys. Rev. E