Variational Principle and Topology of Plasma Structures

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Preliminaries

- Hamilton's principle

$$\delta \int L dt = 0 \quad \Rightarrow \quad \partial_t u = \{H, u\} \quad or \quad \partial_t u = A \partial_u H$$

- Dirichlet's principle

$$\partial_t u = -\partial_u H$$

- Constants of motion → Constraints

a review of classical mechanics

- classical mechanics ⇔ symplectic 2-form

$$L = a_i(u)\dot{u}^i - H(u) \quad \Leftrightarrow \quad dS = a_i du^i - Hdt \text{ (canonical1form)}$$

$$\Rightarrow \quad A_{ij}(u)\dot{u}^i = \partial_{u^i} H(u) \quad (A_{ij} = \partial_{u^i} a_j - \partial_{u^j} a_i)$$

$$\omega = d(a_i du^i) = A_{ij} du^i \wedge du^j$$

- If A_{ij} has a unique inverse A^{ij}, we have

$$\dot{u}^i = A^{ij}(u)\partial_{u^j} H(u)$$

$$= \{H, u^i\} = \{u^j, u^i\}\partial_{u^j} H(u)$$

Equilibrium (stationary points)

- Characterized by $\delta H = 0$ (but, often trivial)
- Relaxation process (cf. Dirichlet's principle) drives the system towards the equilibrium.
- Constraints may yield non-trivial class of equilibria characterized by

$$\delta \tilde{H} = 0 \quad (\tilde{H} = H + \sum_j \mu_j C_j).$$

Topological constraints (*Casimir invariants*)

- A "non-canonical" Hamiltonian system admits the Poisson bracket (operator *A*) to have a kernel, i.e.,

$$\exists C \; s.t. \{G, C\} = 0 \; (\forall G).$$

- Such *C* is called a "Casimir invariant".
- Casimir invariants poses topological constraints.

Helicity (*Casimir*)

- Vortex dynamics system:

$$\partial_t \boldsymbol{u} = -\boldsymbol{\Omega} \times \boldsymbol{u} - \nabla \theta \quad \left(\boldsymbol{\Omega} = \nabla \times \boldsymbol{u} \right)$$

$$\Leftrightarrow \quad \dot{\boldsymbol{u}} = A \partial_{\boldsymbol{u}} H \quad \left(H = \tfrac{1}{2} \left\| \boldsymbol{u} \right\|^2, \; A\boldsymbol{u} = -P \boldsymbol{\Omega} \times \boldsymbol{u} \right)$$

(We assume incompressible $\boldsymbol{u}$. P is the "projection" onto the function space of incompressible fields.)

- Helicity:

$$C = \int \boldsymbol{u} \cdot \nabla \times \boldsymbol{u} \, dx \quad \Rightarrow \quad A \partial_{\boldsymbol{u}} C = 0$$

MHD case

- Canonical form of MHD

$$\frac{d}{dt}\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{B} \end{pmatrix} = A \begin{pmatrix} \partial_{\boldsymbol{v}} H \\ \partial_{\boldsymbol{B}} H \end{pmatrix}$$

$$H = \frac{1}{2}\left(\|\boldsymbol{v}\|^2 + \|\boldsymbol{B}\|^2 \right), \quad A = \begin{pmatrix} -P\boldsymbol{\Omega} \times \circ & P[(\nabla \times \circ) \times \boldsymbol{B}] \\ \nabla \times (\circ \times \boldsymbol{B}) & 0 \end{pmatrix}$$

- Casimirs (helicities)

$$C_1 = (\boldsymbol{A}, \boldsymbol{B}), \quad C_2 = (\boldsymbol{v}, \boldsymbol{B})$$

Structured equilibria

- Parameterized Hamiltonian:

$$\delta\tilde{H}(\boldsymbol{u}) = \delta\left(H(\boldsymbol{u}) + \sum \alpha_j C_j(\boldsymbol{u})\right) = 0$$

- Parameterized stationary points:

$$\left(\mathrm{curl} - \lambda_1\right)\cdots\left(\mathrm{curl} - \lambda_N\right)\boldsymbol{u} = 0$$

- Beltrami-class of equilibria:

$$\boldsymbol{u} = \sum a_j \boldsymbol{G}_j \quad \left(\nabla \times \boldsymbol{G}_j = \lambda_j \boldsymbol{G}_j\right)$$

Duality

- isoperimetric problem:
  max S for given L $\Leftrightarrow$ min L for given S

- ill-posed problem:
  min S for given L or max L for given S

problem in function spaces (∞-dimension)

- well-posed problem and its dual:

$$\min \|\nabla \times \boldsymbol{u}\|^2 \quad \text{for given } \|\boldsymbol{u}\|^2$$

$$\max \|\boldsymbol{u}\|^2 \quad \text{for given } \|\nabla \times \boldsymbol{u}\|^2$$

- ill-posed problem:

$$\min \|\boldsymbol{u}\|^2 \quad \text{for given } \|\nabla \times \boldsymbol{u}\|^2$$

$$\max \|\nabla \times \boldsymbol{u}\|^2 \quad \text{for given } \|\boldsymbol{u}\|^2$$

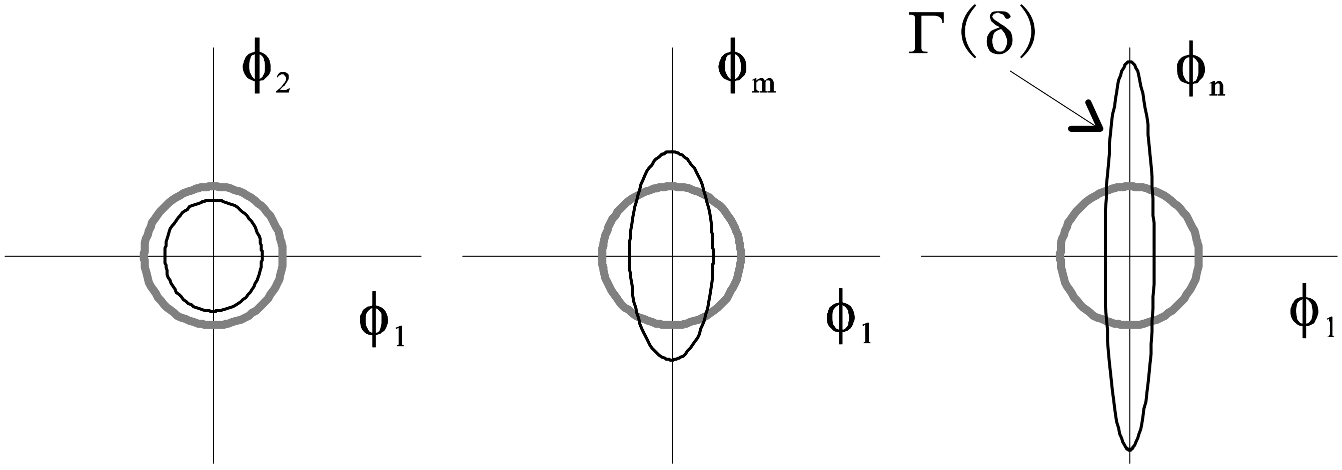
∞-dimension dynamical system

- If $\min \tilde{H}$ occurs at an isolated point x_0, this x_0 is an equilibrium.
- x_0 is surrounded by contours of $\tilde{H}$.
- The $\tilde{H}$ is a Lyapunov function.
- In an infinite dimension space (Hilbert space), the $\tilde{H}$ must be coercive.

Z. Yoshida, S. Ohsaki, A. Ito and S.M. Mahajan, J. Math. Phys. **44** (2003) 2168.

Coercivity and well-posedness

- Some Casimirs may impose ill-posed constraints, and destroy the coercivity.



Z. Yoshida and S.M. Mahajan, Phys. Rev. Lett. **88** (2002) 095001.

Summary

- Variational principles → self-organization
- Topological constraints on dynamics: helicity, Casimir invariants, singularity
- Lyapunov functions ← coercivity