Collapse Dynamics, Singularity Formation and Regularization

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Motivations

• Collapse physics is fascinating
• Collapse related to singularity formation
• Mathematically challenging
• It is real physics
• Important in many disparate fields
• Numerous experimental observations - of its effects

Specifically consider the collapse dynamics governed by NLS:

Nonlinear Schrödinger Equation

Various physical contexts:
• Nonlinear optics - medium with amplitude dependent index of refraction –Self focusing
• Water waves at the free surface
• Bose-Einstein Condensate (BEC), often referred as the Gross-Pitaevskii (G-P) equation
• Quantum Electrodynamics EM wave collapse in Radiation background (Marklund et al PRL 91, 163601 (2003))

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Some Reviews:


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What is collapse – blow-up?

\[ \exists t_0 < \infty \quad ||\nabla \psi|| \to \infty \quad \text{for} \quad t \to t_0 \quad ||f|| \equiv (\int |f|^2 dr)^{1/2} \]

Development of a singularity, infinite amplitude or gradient locally

Nonlinear Schrödinger equation:

\[ i \frac{\partial \psi}{\partial t} + \nabla^2 \psi + |\psi|^2 \psi = 0 \]

Collapse when nonlinearity dominates over dispersion

Conserved quantities:

Power - Number of particles: \( N = \int |\psi|^2 dr \),

Hamiltonian: \( H = \int (|\nabla \psi|^2 - \frac{1}{2} |\psi|^4) dr \)
Collapse

Collapse: mean squared width $\langle r^2 \rangle = N^{-1} \int r^2 |\psi|^2 d\bar{r} \to 0$ for $t \to t_0$

Virial theorem: $\partial_t^2 \langle r^2 \rangle = \frac{1}{N} 2 \left( 4H - (D - 2) \int |\psi|^4 d\bar{r} \right)$

For dimension $D \geq 2$ collapse ensured for $H < 0$

Collapse implies blow-up, e.i., sufficient condition for blow-up:

$$N \leq \langle r^2 \rangle \int |\nabla \psi|^2 d\bar{r}, \text{ vanishing of } \langle r^2 \rangle \implies \text{blow-up of } ||\nabla \psi||$$

Collapse implies concentration of the energy in “one point”

Physics do not “allow” singularities some regularizations take place

**NOTE:** Blow-up usually appears before collapse!
Collapse criteria

$D = 1$ NO collapse, 
stable soliton solution (IST)

$D = 2 : N \geq N_s$ necessary 
and sufficient condition 
$N_s = 11.68$

$D = 3 : H \leq 0$ 
sufficient condition 
sharper criterion related to $H_s$


D=2 : “Townes profile”
(Chiao et al., Phys. Rev. Lett. 13,479 (1964))
Collapse/blow-up in 2D NLS

NO global self-similar description
NO few parameter trial function governs this evolution, e.i., standard variational approach with Gaussian trial function cannot describe the full evolution
Singularity regularization

Discrete 2D NLS

Filamentation of fs laser pulse in air

Femtosecond LIDAR
Collapse types

**Weak collapse**: The power/number of particles, $N_{\text{spike}}$, tends to 0 in the singular spike

Typical for $D = 3$ (here strong collapse is unstable)

**Strong collapse**: The power/number of particles, $N_{\text{spike}}$, tends to a finite value in the singular spike

For $D = 2$ always strong collapse: $N_{\text{spike}} = N_s$

Singularity Regularization

Passive : “missing physics” in the model :
Dissipations
Extra effects, extra dimension…
Nonlocal interactions (Bang et al Phys Rev E 66, 046619 (2002))

Active : external forcing / control.
Potential landscape, optical lattice
Self-focusing of light in nonlinear materials

\[ i \partial_z \psi + \nabla_\perp^2 \psi - \beta \partial_t^2 \psi + |\psi|^2 \psi = \delta \rho \psi + \mathcal{R}(\psi) \]

\[ \partial_t \delta \rho = \sigma_K \rho_0 |\psi|^{2K} + \sigma |\psi|^2 \delta \rho - \frac{1}{\tau} \delta \rho \]

\( \beta \) group velocity dispersion:
- normal \( \beta > 0 \), anomalous \( \beta < 0 \)

\( \delta \rho \) plasma density, focused beam burns a channel
\( \mathcal{R}(\psi) \) Raman scattering etc.

Plasma defocusing, self-guiding

Normal GVD

Focusing in the transverse plane \((x,y)\), defocusing in time

Dissipation

\[ i \partial_t \psi + \nabla^2 \psi + |\psi|^2 \psi + i \eta |\psi|^4 \psi = i \gamma \psi, \]


nonlinear dissipations acts for \( |\psi|^2 \approx \eta |\psi|^4 \)

Strong collapse (D=2) power dissipated in blow-up events quantized \( \approx N_s \) independent of \( \eta \! \)

Weak collapse (D=3) power dissipated in blow-up events \( \to 0 \) for of \( \eta \to 0 \), i.e. dependent on \( \eta \)
Collapse in Bose-Einstein condensates

$^7\text{Li}$ Attractive interaction


Number of particles

Phase-contrast images, “wave-function”
85Ru: Feshbach resonance shift from repulsive to attractive interaction


Collapse in Bose Einstein Condensate: Controlled Collapse

Collapse time versus nonlinearity strength

Number of particles
BEC: Mean field theory: Gross-Pitaevskii (G.-P.) equation

\[ i\partial_t \psi = -\nabla^2 \psi + r^2 \psi + a|\psi|^2 \psi - i\eta|\psi|^4 \psi + i\gamma \psi, \]

Parabolic confinement potential

Where \( r = \tilde{r}/l_0, \ t = \tilde{t}\omega/2, \ \psi = \tilde{\psi}(8\pi l_0^2|a_0|)^{1/2} \)

\( a = \pm 1 \) for positive/negative scattering length, \( \eta \) is 3-body recombination coefficient and \( \gamma \) is source coefficient

Self-similar analysis of the spike evolution provides the number of atoms lost over a sequence of weak collapses (Bergé and Rasmussen, Phys. Lett. A 304, 136 (2002))

\[ N_n = (1 - \Delta N/N_0)^n N_0 \]
Numerical solutions of the G-P-equation


Multiple weak collapses

Collapse time versus scattering “strength”
Active Regularization

Layered nonlinear medium; nonlinearity management;

Active Regularization

Layered nonlinear medium; nonlinearity management; **experiment!**

BEC Nonlinearity management

Modulate scattering length $a$ by means of the Feshbach resonance. 2-D possible, 3-D other effects are important

No experimental observations as yet

Numerical simulations 2-D G-P equations with $a(t)$ stable solitons, 3-D?

Conclusions

- Collapse dynamics important and fascinating in many branches of physics
- Collapse can be controlled and utilized
- Covered only a limited part of investigations
- Collapse in hydrodynamics: Singularity in Navier-Stokes equations: 2-D no 3-D ??
- Wave breaking
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