Collapse Dynamics, Singularity Formation and Regularization

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Motivations

- Collapse physics is fascinating
- Collapse related to singularity formation
- Mathematically challenging
- It is real physics
- Important in many disparate fields
- Numerous experimental observations - of its effects

Specifically consider the collapse dynamics governed by NLS:

Nonlinear Schrödinger Equation

Various physical contexts:

- Nonlinear optics - medium with amplitude dependent index of refraction –Self focusing
- Water waves at the free surface
- Plasma waves – Langmuir collapse (Zakharov *Sov. Phys. JETP* **35**, 908 (1972)) ……
- Bose-Einstein Condensate (BEC), often referred as the Gross-Pitaevskii (G-P) equation
- Quantum Electrodynamics EM wave collapse in Radiation background (Marklund *et al PRL* **91**, 163601 (2003))
- ………..

Some Reviews:

……………………….

E.A. Kuznetsov, A.m. Rubenchik and V.E. Zakharov, *Phys. Rep.* **142**, 103 (1986)

J. Juul Rasmussen and K. Rypdal, *Phys. Scr.* **33**, 481 (1986).

E.A. Kuznetsov, *CHAOS* **6,** 381 (1996).

L. Bergé, *Phys. Rep.* **303,** 259 (1998).

G. Fibich and G.C. Papanicolaou. *SIAM J. Appl. Math.* **60**, 183 (1999).

C. Sulem and P.-L. Sulem, *The Nonlinear Schrödinger Equation: Self-focusing and Wave Collapse* (Springer, Berlin 1999).

……………………….

What is collapse – blow-up?

$$\exists\, t_0 < \infty \quad ||\nabla\psi|| \to \infty \text{ for } t \to t_0 \qquad\qquad ||f|| \equiv \left(\int |f|^2 d\vec{r}\right)^{1/2}$$

Development of a singularity, infinite amplitude or gradient locally

Nonlinear Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} + \nabla^2\psi + |\psi|^2\psi = 0$$

Collapse when nonlinearity dominates over dispersion

Conserved quantities:

Power - Number of particles: $N = \int |\psi|^2 d\vec{r}$,

Hamiltonian: $H = \int \left(|\nabla\psi|^2 - \frac{1}{2}|\psi|^4\right) d\vec{r}$

Collapse

Collapse: mean squared width $\langle r^2 \rangle = N^{-1} \int r^2 |\psi|^2 d\vec{r} \to 0$ for $t \to t_0$

Virial theorem: $\partial_t^2 \langle r^2 \rangle = \frac{1}{N} 2 \left(4H - (D-2) \int |\psi|^4 d\vec{r}\right)$

For dimension $D \geq 2$ collapse ensured for $H < 0$

Collapse implies blow-up, e.i., sufficient condition for blow-up:

$$N \leq \langle r^2 \rangle \int |\nabla \psi|^2 d\vec{r}, \text{ vanishing of } \langle r^2 \rangle \quad \Rightarrow \text{blow-up of } ||\nabla \psi||$$

Collapse implies concentration of the energy in “one point”

Physics do not “allow” singularities some regularizations take place

NOTE: Blow-up usually appears before collapse!

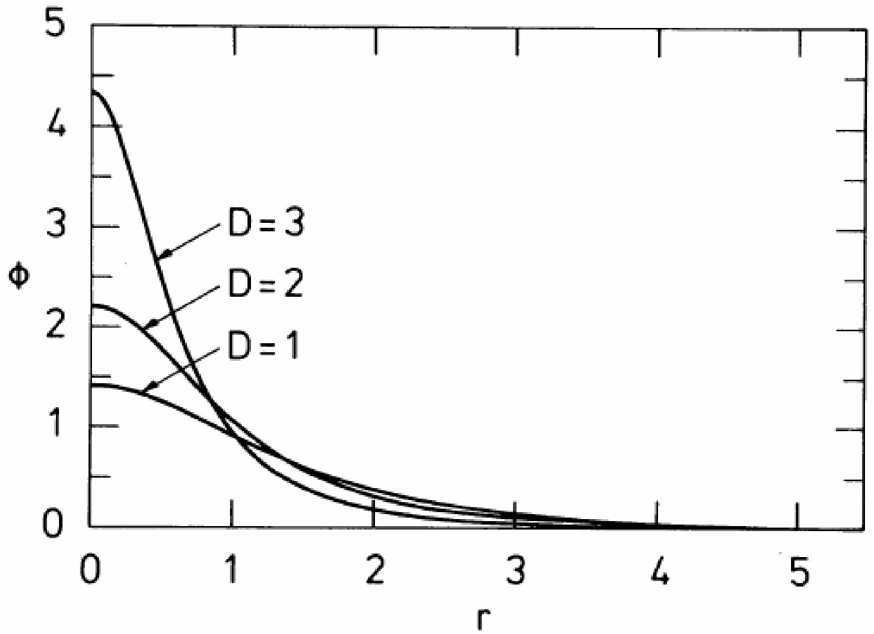
Collapse criteria

$D = 1$ NO collapse,
stable soliton solution (IST)

$D = 2 : N \geq N_s$ necessary
and sufficient condition
$N_s = 11.68$

$D = 3 : H \leq 0$
sufficient condition
sharper criterion related to H_s

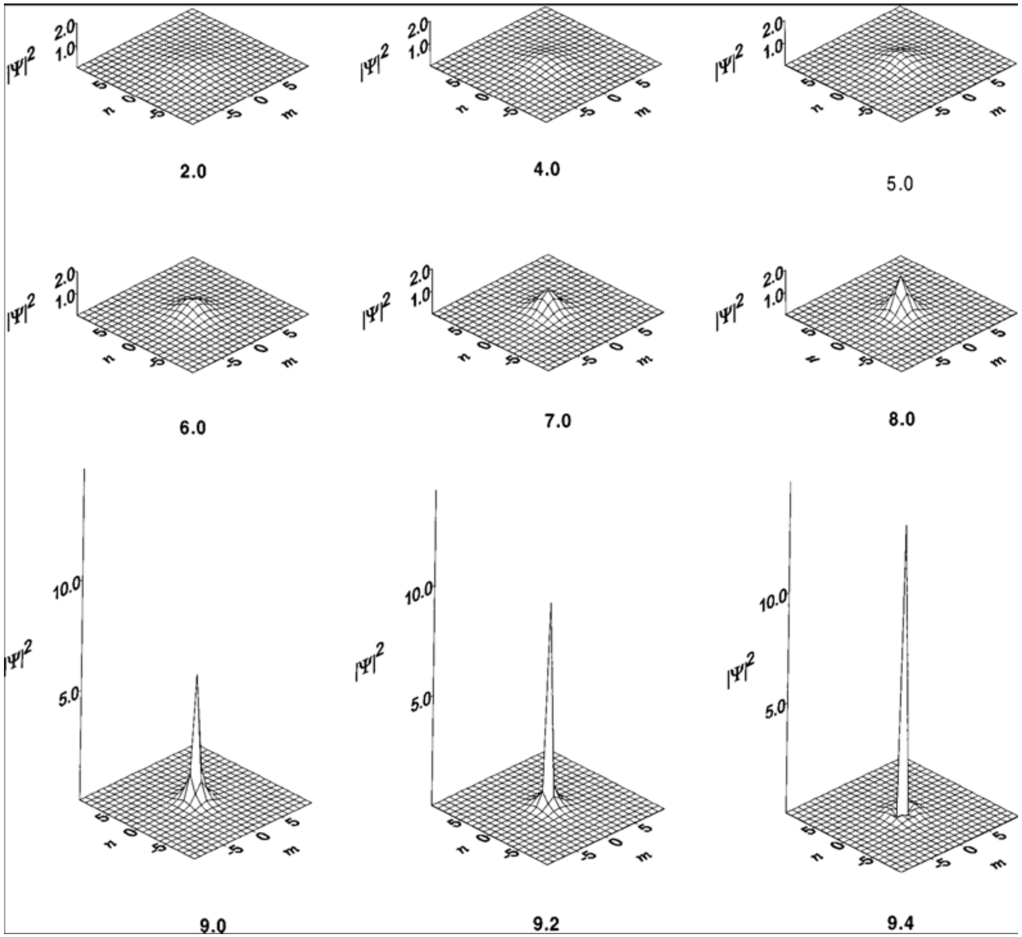
Kuznetsov *et al.* Physica D **87**, 273 (1995)

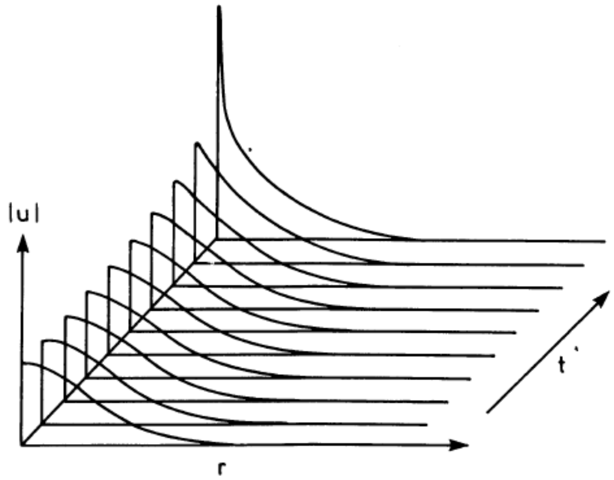


Ground-state solutions of NLS

D=2 : “Townes profile”
(Chiao *et al.*, *Phys. Rev. Lett.* **13**,479 (1964))

Collapse/blow-up in 2D NLS

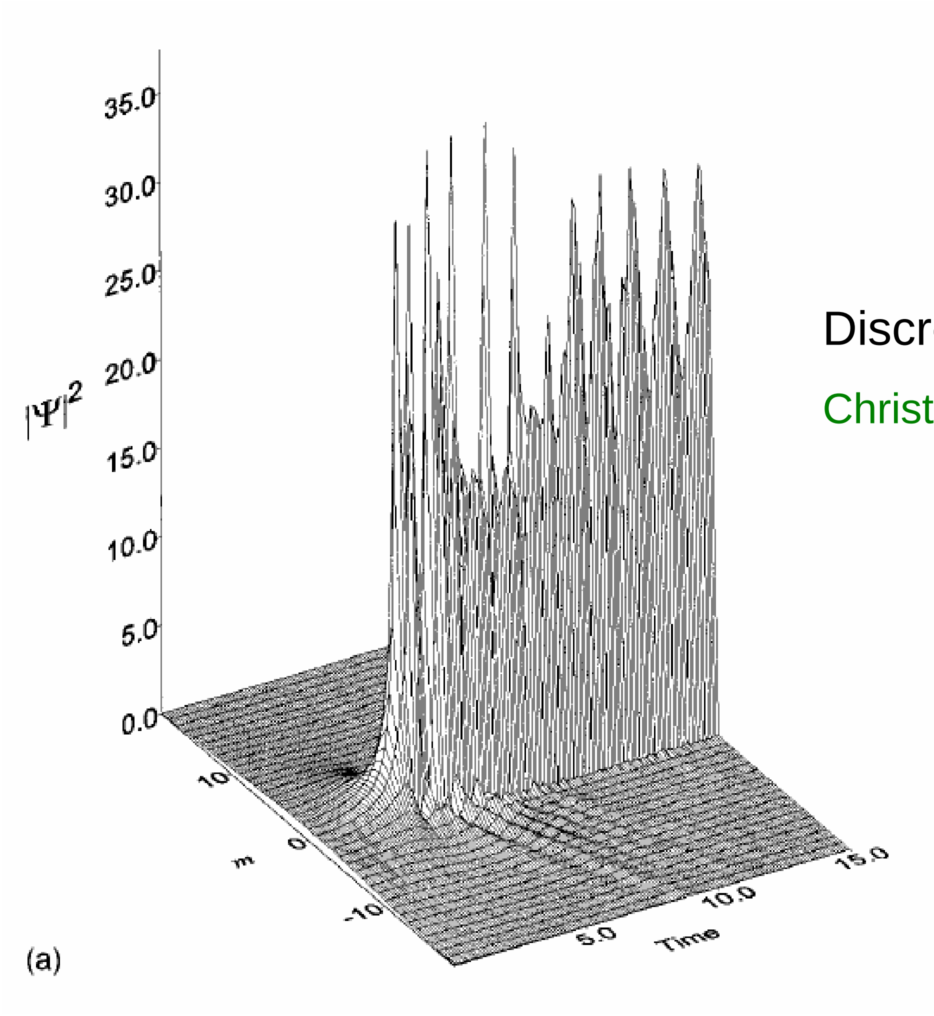




NO global self-similar description

NO few parameter trial function governs this evolution, e.i., standard variational approach with Gaussian trial function cannot describe the full evolution

Singularity regularization



Discrete 2D NLS

Christiansen *et al.* Phys. Rev. B **57**, 11303 (1998)

Filamentation of fs laser pulse in air

Femtosecond LIDAR

Collapse types

Weak collapse : The power/number of particles , N_{spike} tends to 0 in the singular spike

Typical for D = 3 (here strong collapse is unstable)

Strong collapse : The power/number of particles , N_{spike} tends to a finite value in the singular spike

For D = 2 allways strong collapse : $N_{spike} = N_s$

Zakharov and Kuznetsov, *Sov. Phys. JETP* **64**,773 (1986)

Singularity Regularization

Passive : “missing physics” in the model :

Dissipations

Extra effects, extra dimension…

Nonlocal interactions (Bang et al *Phys Rev E* **66**, 046619 (2002))

Active : external forcing / control.

Potential landscape, optical lattice

Self-focusing of light in nonlinear materials

$$i\partial_z\psi + \nabla_\perp^2\psi - \beta\partial_t^2\psi + |\psi|^2\psi = \delta\rho\psi + \mathcal{R}(\psi)$$

$$\partial_t\delta\rho = \sigma_K\rho_0|\psi|^{2K} + \sigma|\psi|^2\delta\rho - \frac{1}{\tau}\delta\rho$$

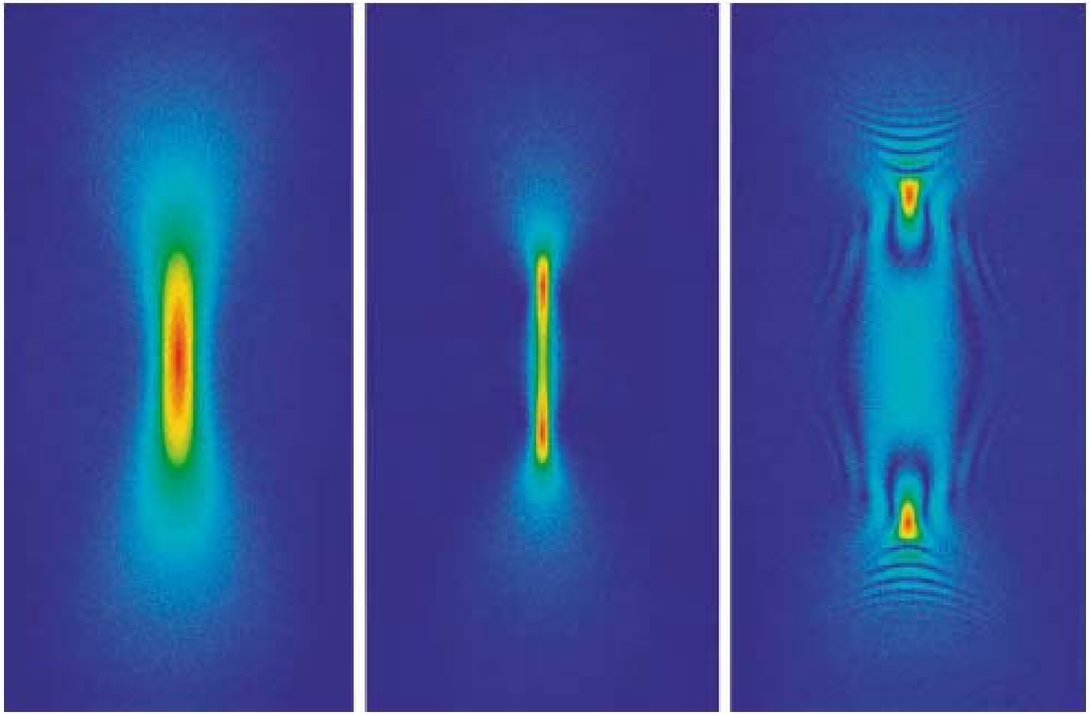
β group velocity dispersion:
normal $\beta > 0$, anomalous $\beta > 0$
$\delta\rho$ plasma density, focused beam burns a channel
$\mathcal{R}(\psi)$ Raman scattering etc.

Plasma defocusing, self-guiding

Skupin and Bergé *Physica D* **220**, 14 (2006)

Normal GVD

t



x

Focusing in the transverse plane (x,y), defocusing in time

Germaschewski *et al., Physica D* **151**, 175 (2001)

Dissipation

$$i\partial_t\psi + \nabla^2\psi + |\psi|^2\psi + i\eta|\psi|^4\psi = i\gamma\psi,$$

Nonlinear 3-body recombination and linear damping, both can regulate the singularity. Passot *et al.* Physica D **203**, 167 (2005); LeMesurier Physica D **138**, 334 (2000)

nonlinear dissipations acts for $|\psi|^2 \approx \eta|\psi|^4$

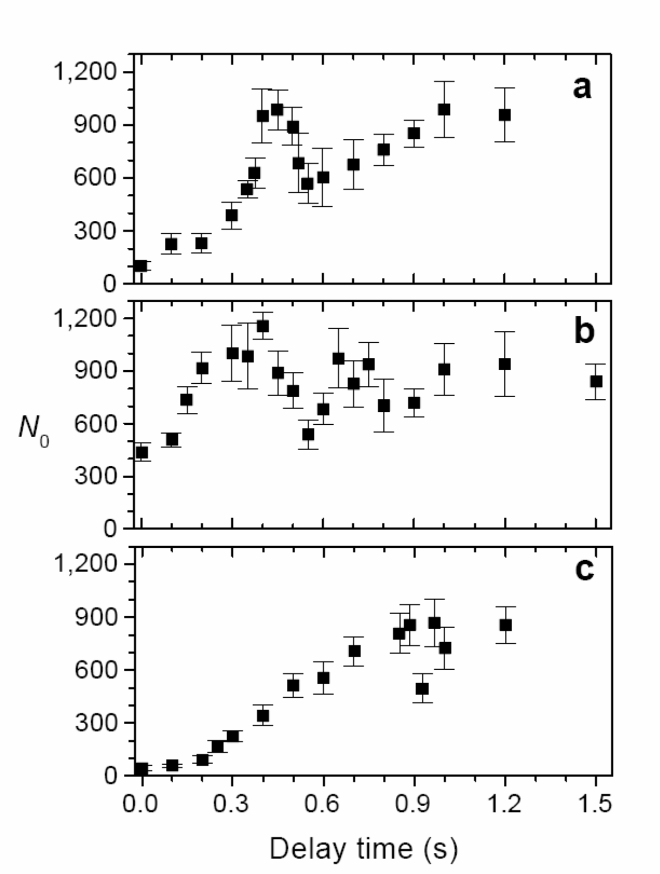
Strong collapse (D=2) power dissipated in blow-up events quantized $\approx N_s$ independent of η!

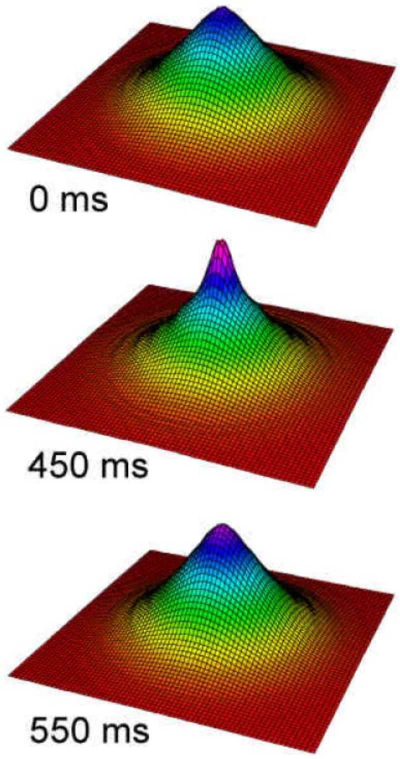
Weak collapse (D=3) power dissipated in blow-up events $\to 0$ for of $\eta \to 0$, i.e. dependent on η

Collapse in Bose-Einstein condensates

^{7}Li Attractive interaction

J.M. Gerton et al *Nature* **408,** 692 (2000)





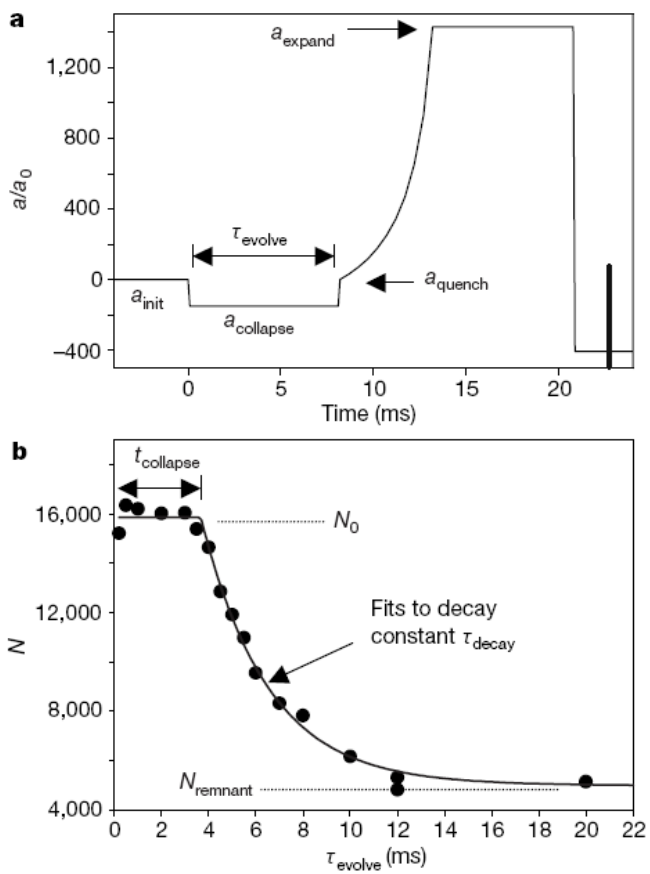
Number of particles

Phase-contrast images, “wave-function”

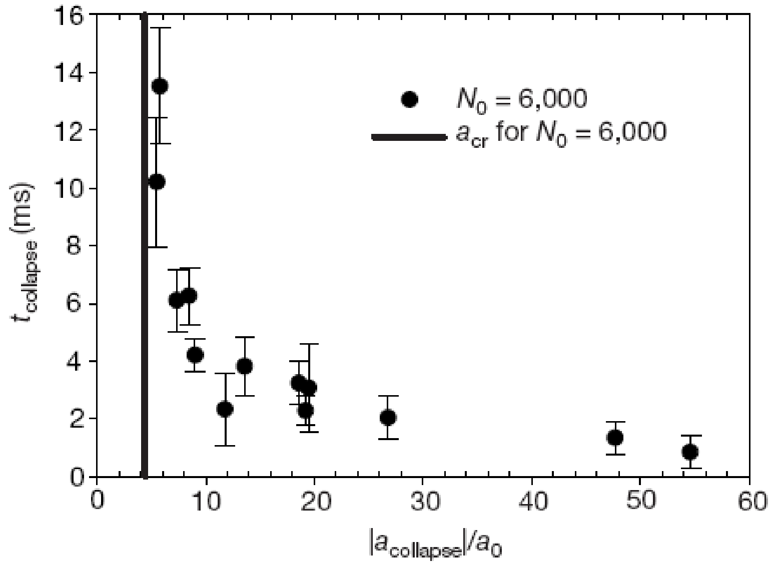
Collapse in Bose Einstein Condensate: Controlled Collapse

^{85}Ru : Feshbach resonance **shift from repulsive to attractive interaction**

E.A. Donley et al *Nature* **408,** 692 (2000)



Number of particles



Collapse time versus nonlinearity strength

BEC: Mean field theory: Gross-Pitaevskii (G.-P.) equation

$$i\partial_t \psi = -\nabla^2 \psi + r^2 \psi + a|\psi|^2 \psi - i\eta|\psi|^4 \psi + i\gamma\psi,$$

Parabolic confinement potential

Where $r = \tilde{r}/l_0$, $t = \tilde{t}\omega/2$, $\psi = \tilde{\psi}(8\pi l_0^2 |a_0|)^{1/2}$

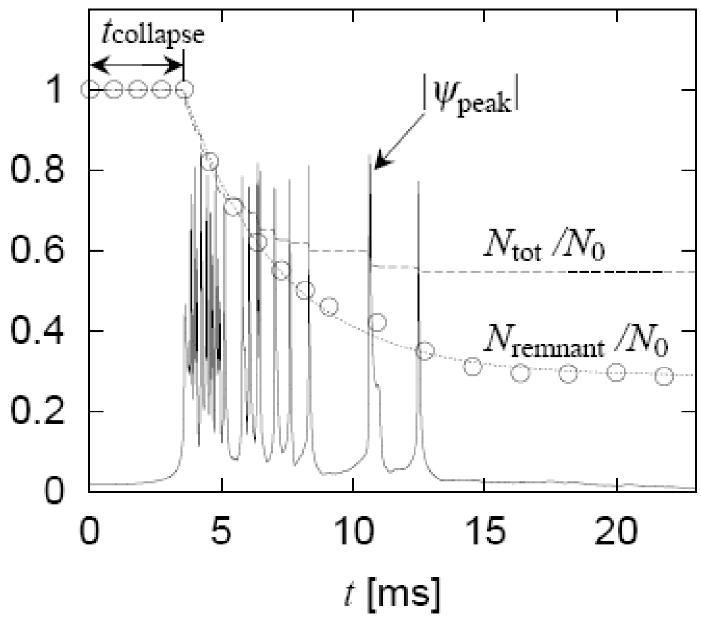
$a = \pm 1$ for positive/negative scattering length, η is 3-body recombination coefficient and γ is source coefficient

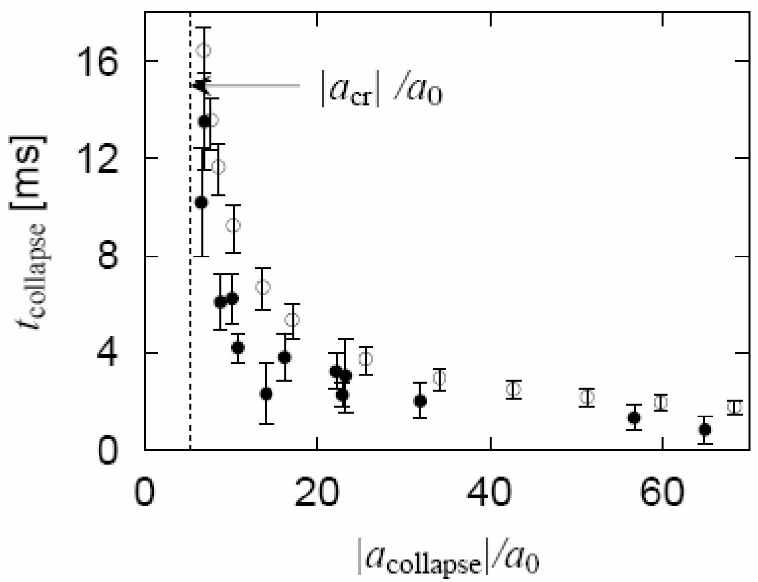
Self-similar analysis of the spike evolution provides the number of atoms lost over a sequence of weak collapses (Bergé and Rasmussen, Phys. Lett. A **304**, 136 (2002))

$$N_n = (1 - \Delta N/N_0)^n N_0$$

Numerical solutions of the G-P-equation

M. Ueda, H. Saito, J. Phys. Soc. Jpn. Vol. 72 (2003) Suppl. C pp.127-133



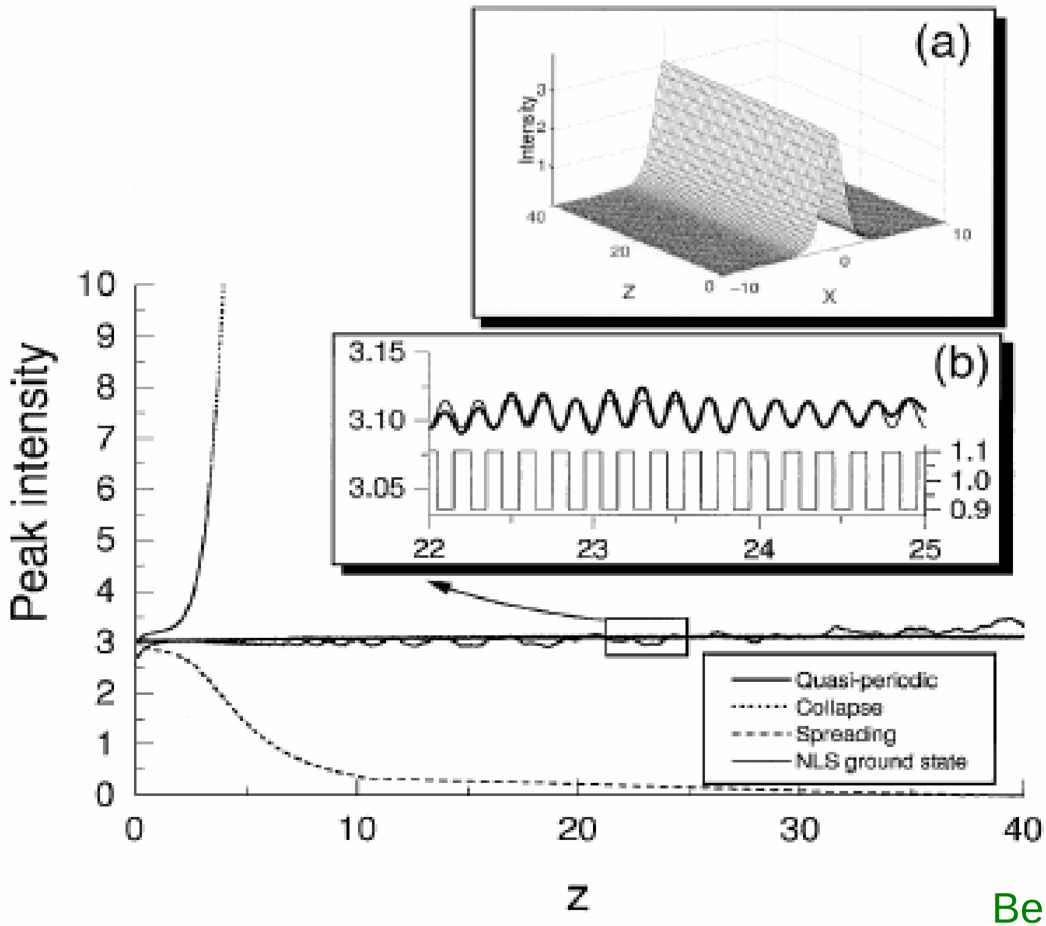


Multiple weak collapses

Collapse time versus scattering "strength"

Active Regularization

Layered nonlinear medium; nonlinearity management;



2-D soliton

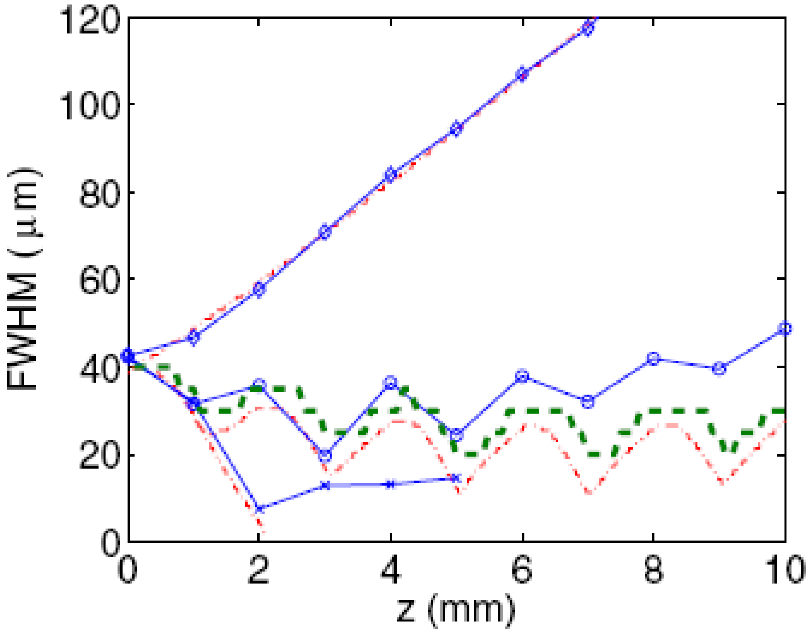
3-D nonlinearity managements possible??

Bergé *et al. Optics Letters* **25,** 1037 (2000)

Active Regularization

Layered nonlinear medium; nonlinearity management; **experiment!**

Centurion *et al. Phys. Rev. Letters* **97,** 033903 (2006)



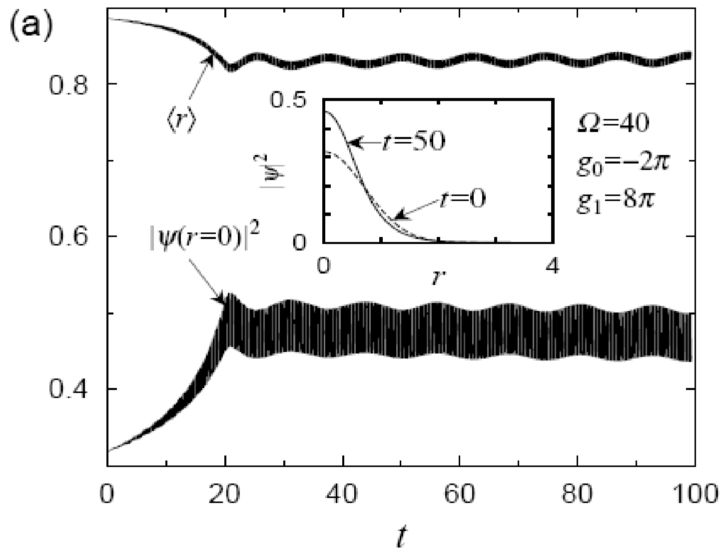
BEC Nonlinearity management

Modulate scattering length a by means of the Feshbach resonance. 2-D possible, 3-D other effects are important

No experimental observations as yet

Numerical simulations 2-D G-P equations with a(t) stable solitons, 3-D?

Saito and Ueda *Phys. Rev. Lett.* **90**, 040403 (2003)



Conclusions

- Collapse dynamics important and fascinating in many branches of physics
- Collapse can be controlled and utilized
- Covered only a limited part of investigations
- Collapse in hydrodynamics : Singularity in Navier-Stokes equations: 2-D no 3-D ??
- Wave breaking
- ....