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Quantum Ising Model: finite T correlators

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Quantum Ising Model: finite T correlators

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Ising model is as unexhaustable as atom, Nature is infinite…

Correlation functions in strongly correlated systems: 1+1-D

- Why 1+1-D ?– non-perturbative methods exist

(Inverse Quantum Scattering Problem = Bethe ansatz)

- For systems with gapless $E = v|q|$ spectrum the problem is solved:

one can use

conformal field theory and bosonization methods.

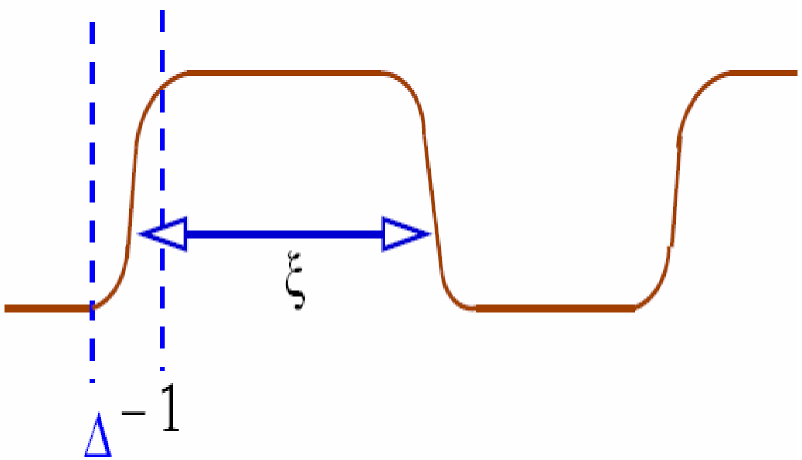
The problem exists for models with spectral gaps – models of quantum solitons.

Low T physics

- There is QFT at T=0 accompanied by a breaking of the discrete symmetry. At finite T this symmetry is restored and there is a finite density of thermally excited solitons

Scales which determine the physics

Energy	Space	Time
Δ	v/Δ	$1/\Delta$
T	$\xi \sim \frac{v}{\Delta T} e^{\Delta/T}$	$\tau_0 = \sqrt{\Delta/T}\xi$
T^*	v/T^*	$1/T^*$



Formfactor approach – standard tool to calculate correlation functions

Suggested by Karowskii et. al.in the 80-ties, developed by Smirnov in the 90-ties. It allows to calculate matrix elements of various operators

<n|A|m>

with exact eigenfunctions. These one are substituted in

the Lehmann expansion:

$$D_A^{(R)}(\omega, q) = \theta(t)\langle A(x,t)A^+(0,0)\rangle_{\omega,q}$$

$$\Im m D_A^{(R)}(\omega, q) = \frac{1 \pm \mathrm{e}^{-\beta\omega}}{Z} \times$$

$$\sum_n \mathrm{e}^{-\beta E_n} |\langle n|\hat{A}|m\rangle|^2 \delta(\omega + E_n - E_m)\delta(q + P_n - P_m)$$

(+ for fermionic and − for bosonic operators).

At $T \neq 0$ one faces difficulties related to singularities in the operator matrix elements.

Quantum Ising model – the simplest model of quantum solitons

$$H = \sum_n [-J\sigma^z_n \sigma^z_{n+1} + h\sigma^x_n] \equiv$$

$$\sum_n [-h\mu^z_{n-1/2}\mu^z_{n+1/2} + J\mu^x_n]$$

$$\equiv \sum \epsilon(p) F^+_p F_p \qquad \epsilon(p) = \sqrt{(J-h)^2 + 4Jh\sin^2(p/2)}$$

where $\mu^z_{n+1/2} = \prod_{j<n} \sigma^x_j$ is the disorder operator and F, F^+ are fermion annihilatio[n] and creation operators.

Ising model in the continuum limit

Calculations simplify in the continuum limit:

$$M = |J - h| << J$$

The spectrum is relativistic:

$$\epsilon(p) = \sqrt{M^2 + (vq)^2}, \quad v^2 = Jh$$

Convenient parametrization:

$$\epsilon = M \cosh \theta, \quad (vq) = M \sinh \theta$$

θ rapidity

Operators σ^z(alias σ) and μ^z (alias μ) have ∞-many matrix elements. The formfactors are (Karowski et al. 1986)

$$\langle\theta_1, ...\theta_n|\sigma|\theta'_1, ...\theta'_m\rangle = A\Delta^{1/8} \times$$
$$\prod_{i<j} \tanh(\theta_{ij}/2) \prod_{p<q} \tanh(\theta'_{pq}/2) \prod_{i,p} \coth[(\theta_i - \theta'_p)/2]$$

where at $h < 1$ (the ordered phase) $n + m =$ even for σ and odd for μ (for $h > 1$ it is the other way around).

A is a known numerical constant.

The singularities in the Lehmann expansion appear when some *in* and *out* rapidities coincide.

Singularities of this kind (kinematical poles) appear routinely in integrable models.

Previous results (obtained by other methods)

$$D(x,t) \equiv \langle \sigma(t,x)\sigma(0,0)\rangle =$$

$$CM^{1/4}\exp\left[-\int \frac{\mathrm{d}p}{\pi}\mathrm{e}^{-\epsilon(p)\beta}|x - t\frac{\partial\epsilon(p)}{\partial p}|\right]$$

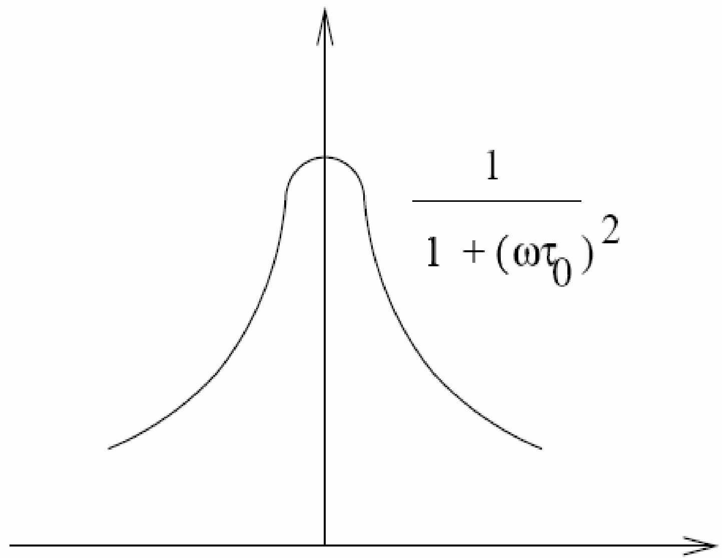
where C is a numerical constant and

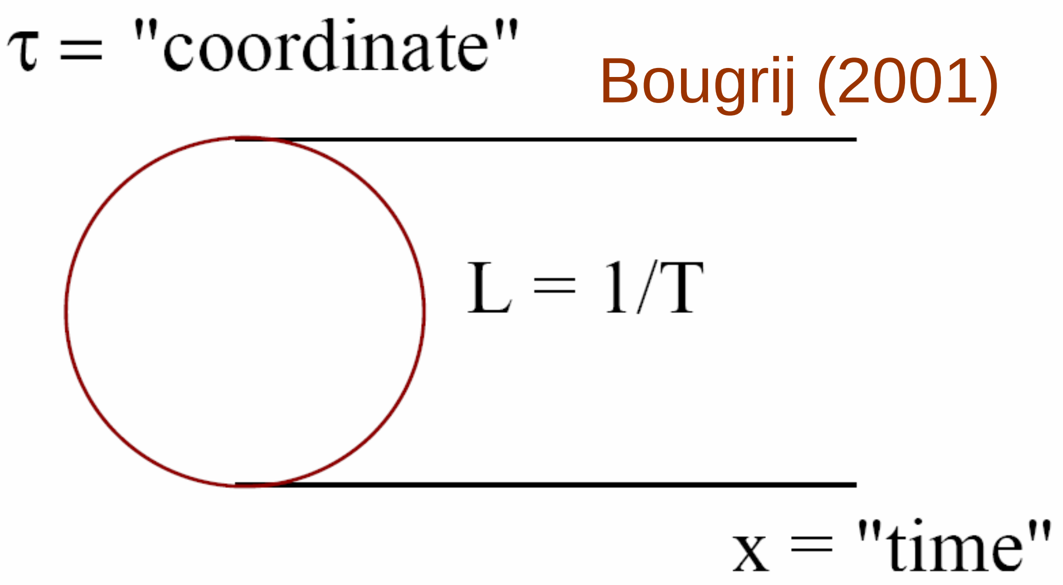
$$\epsilon = \sqrt{M^2+p^2} \approx M + p^2/2M$$

$$D(x,t) \approx C\exp[-n(T)\max(|x|, v|t|)]$$

$v = \sqrt{\pi T/2M}$ is the thermal velocity of kinks and $n(T) = \sqrt{TM/2\pi}\mathrm{e}^{-M/T}$ is the average number of solitons (fermions).

Vladimir Korepin





Bougrij (2001)

One can calculate the correlation functions to get a feeling for the problem

$$\langle\sigma(\tau,x)\sigma(0,0)\rangle = CM^{1/4}\mathrm{e}^{-|x|\Delta(T)}\left\{1+\sum_{N=1}^{\infty}\frac{T^{2N}}{(2N)!}\sum_{q_1,\dots q_{2N}}\prod_{i=1}^{2N}\frac{\mathrm{e}^{-|x|\epsilon_i-\mathrm{i}\tau q_i-\eta(q_i)}}{\epsilon_i}\prod_{i>j}\left(\frac{q_i-q_j}{\epsilon_i+\epsilon_j}\right)^2\right\},$$

$$q = 2\pi T m \ (m \text{ integer}), \text{ and } \epsilon(q) = \sqrt{M^2+q^2}.$$

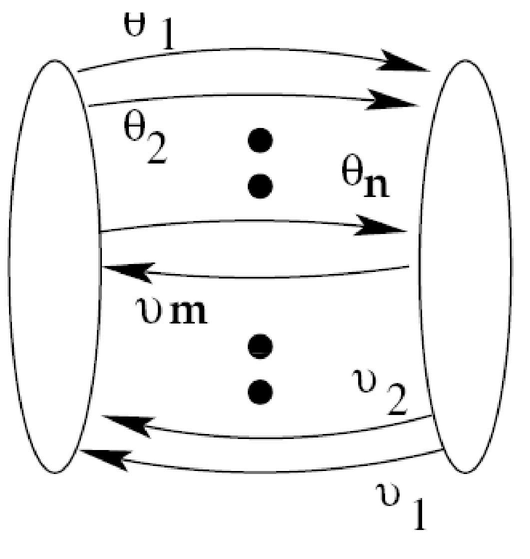
$$\Delta(T) = \int_{-\infty}^{\infty}\frac{\mathrm{d}p}{\pi}\ln\{\coth[\beta\epsilon(p)/2]\}$$

No singularities,
the Wick rotation transforms the problem into 0 temperature finite size one.

We want to analytically continue for real times. The first step: transform the sum into a contour integral.

Then each term of the sum becomes a sum of integrals – this looks like thermal formfactor expansion:

$$\sum_{n=1}^{2N}\frac{1}{n!(2N-n)!}\int\prod_{i=1}^{n}\frac{\mathrm{d}\theta_i}{2\pi}f^{(+)}(\theta_i)\mathrm{e}^{\tau\epsilon_i+\mathrm{i}xp_i}\prod_{j=1}^{2N-n}\frac{\mathrm{d}\theta'_j}{2\pi}f^{(-)}(\theta_j)\mathrm{e}^{-\tau\epsilon_j-\mathrm{i}xp_j}\frac{\prod_{i>k}\tanh^2[(\theta_i-\theta_k)/2]\prod_{j>p}\tanh^2[(\theta'_i-\theta'_p)/2]}{\prod_{i,p}\tanh^2[(\theta_i-\theta'_p+\mathrm{i}0)/2]}$$



$$f^{(+)}(\theta_i)=\frac{\mathrm{e}^{-\eta^{(+)}(\theta)}}{[\mathrm{e}^{\beta\epsilon(\theta)}-1]},\ f^{(-)}(\theta_i)=\frac{\mathrm{e}^{\eta^{(-)}(\theta)}}{[1-\mathrm{e}^{-\beta\epsilon(\theta)}]}$$

$$\eta^{(\pm)}(\theta)=\frac{\mathrm{i}M\sinh\theta}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}x\ln\{\coth[\beta\epsilon(x)/2]\}}{x^2-M^2\sinh^2(\theta\pm\mathrm{i}0)}$$

Now we can replace i tau by t

Singularities are not on the contour.

$$\frac{\prod_{i>k} \tanh^2[(\theta_i - \theta_k)/2] \prod_{j>p} \tanh^2[(\theta'_i - \theta'_p)/2]}{\prod_{i,p} \tanh^2[(\theta_i - \theta'_p + \mathrm{i}0)/2]}$$

$$= \langle \prod_i^{N-k} \mathrm{e}^{\mathrm{i}\Phi(\theta_i + \mathrm{i}0)} \prod_j^{N+k} \mathrm{e}^{-\mathrm{i}\Phi(\theta'_j - \mathrm{i}0)} \mathrm{e}^{2k\mathrm{i}\Phi(\infty)} \rangle$$

$$\langle \Phi(\theta_1)\Phi(\theta_2) \rangle \equiv G_0(\theta_{12}) = -\ln\left[\tanh^2(\theta_{12}) + a_0^2\right]$$

$$\langle \sigma(\tau, x)\sigma(0,0)\rangle = \quad (1$$

$$CM^{1/4}\mathrm{e}^{-|x|\Delta(T)}Z_0(x,t)\lim_{R\to\infty}\sum_{k=-\infty}^{\infty}\langle \mathrm{e}^{2\mathrm{i}k\Phi(R)}\rangle_V$$

$$Z_0 = \int D\Phi \mathrm{e}^{-S[\Phi]}/\int D\Phi \mathrm{e}^{-S_0[\Phi]} \quad = \exp[-F(x,t)]$$

where F is the free energy of the theory

Small parameter

$f^{(+)}(\theta_i)$ $\quad$ $\mathrm{e}^{-\beta M}$

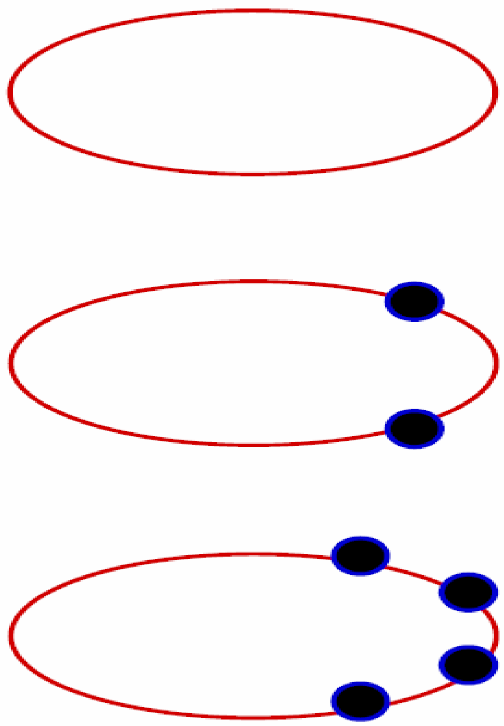
$$S = \frac{1}{2}\int \mathrm{d}\theta_1 \mathrm{d}\theta_2 \Phi(\theta_1) G_0^{-1}(\theta_{12})\Phi(\theta_2) +$$

$$\int \frac{\mathrm{d}\theta}{2\pi a_0} V[\theta, \Phi(\theta)]$$

$$V = f^{(+)}(\theta)\mathrm{e}^{\tau M \cosh\theta + \mathrm{i}xM \sinh\theta}\mathrm{e}^{\mathrm{i}\Phi(\theta+\mathrm{i}a)} +$$

$$+ f^{(-)}(\theta)\mathrm{e}^{-\tau M \cosh\theta - \mathrm{i}xM \sinh\theta}\mathrm{e}^{-\mathrm{i}\Phi(\theta-\mathrm{i}a)}$$

String theory analogy



$$\langle \sigma(\tau, x)\sigma(0,0)\rangle = \qquad (1$$

$$CM^{1/4} e^{-|x|\Delta(T)} Z_0(x,t) \lim_{R\to\infty} \sum_{k=-\infty}^{\infty} \langle e^{2ik\Phi(R)}\rangle_V$$

$$Z_0 = \int D\Phi e^{-S[\Phi]} / \int D\Phi e^{-S_0[\Phi]}$$

Expansion includes surfaces with different
 conditions at infinity.
(x,t) are parameters of the action.

Using our method we can correct Sachdev-Young result:

$$\langle\sigma(x,t)\sigma(0,0)\rangle_T = CM^{1/4}\theta(t)\exp(-\delta\Delta|x|)$$ X

$$\exp\left\{-\frac{1}{4\pi}\int \mathrm{d}p \frac{|tv(p)-x|}{\cosh^2[\beta\epsilon(p)/2]} - \frac{4\mathrm{i}}{\pi}\exp\left[-\frac{\beta M}{\sqrt{1-(x/t)^2}}\right]\right\} \quad t > |x|$$

$$CM^{1/4}\theta(t)\exp(-\Delta|x|), \quad |x| > t$$

$\delta\Delta$ $\sim \exp(-3\beta M)$

The Fermi statistics of solitons is visible.

The imaginary part attests to the quantum nature of solitons.
The linear **t** term is exact; the corrections are $\sim \exp(-2\beta M)$

Conclusions

- For the Ising model the spin-spin correlation function can be represented as a partition function of some field theory,

where (x,t) serve as parameters in the action.

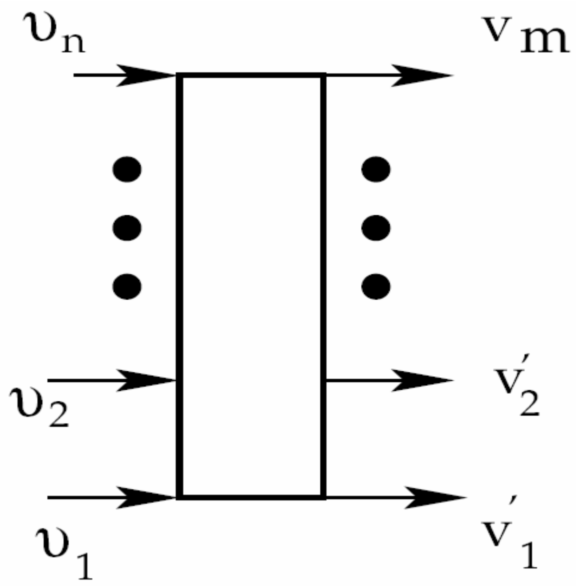
- Thus one can deal only with connected diagrams for the “free energy” which simplifies the virial expansion.

Complication: Fermi distribution function of solitons does not emerge in an in straightforward way.

General case

- Using the theory of quantum integrable systems one can isolate the leading singularities in the operator matrix elements and sum them up.

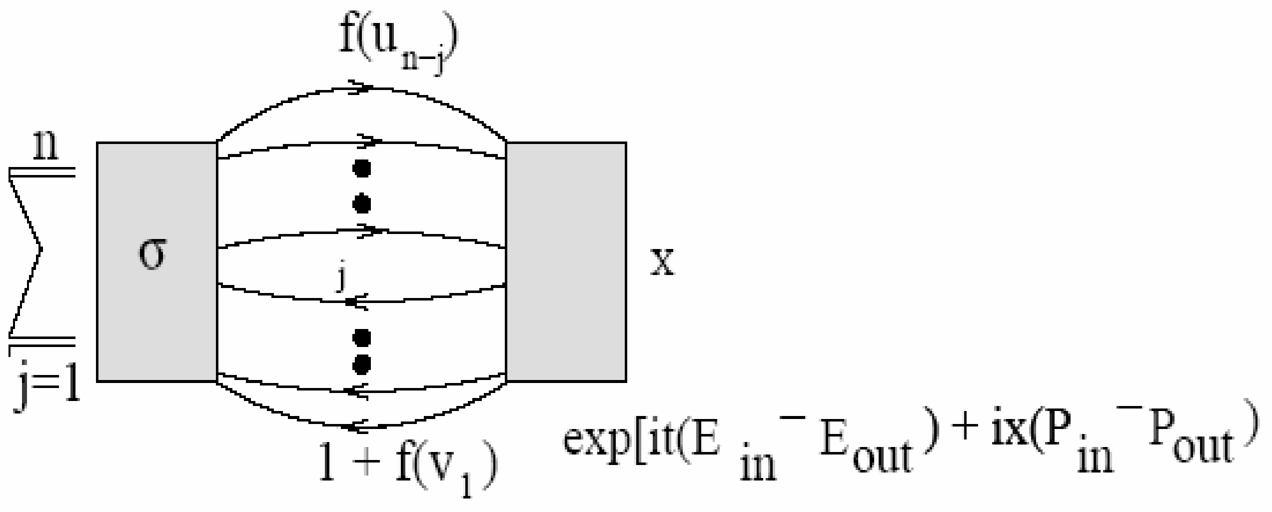
The asymptotics are determined by the behavior of the formfactors in the vicinity of *kinematic poles*



$$\langle u_1, \dots u_n | \hat{O} | v_1, \dots v_m \rangle_{a_1,..a_n; b_1,..b_m} = \sum_{P_j} \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{f(\{a\};\{b\})}{(u_i - v_{P_j} + i0)}$$

Each θ_{in} carries $e^{-\beta M} << 1$, hence one can restict summation by terms with equal number of *in* and *out* rapidities.

The leading contribution in each order comes from integration in the vicinity of each singularity.



Each formfactor pole (the pair of lines u_j, v_j) contributes to the correlation function one power of

$$R(t,x) =$$

$$\frac{1}{2\pi^2}\int \mathrm{d}u\mathrm{d}v\mathrm{e}^{-\beta\epsilon(u)}\mathrm{e}^{\mathrm{i}t[\epsilon(u)-\epsilon(v)]/2+\mathrm{i}x[p(u)-p(v)]}$$

$$\times\left[\frac{1}{(u-v+\mathrm{i}0)^2}+\frac{1}{(u-v-\mathrm{i}0)^2}\right] =$$

$$\approx -\int\frac{\mathrm{d}p}{4\pi}\mathrm{e}^{-\beta\epsilon(p)}|x-t\partial\epsilon/\partial p|$$

Consider a generic matrix element of operator $\mathcal{O}$:

$$F^{\mathcal{O}}(\{u\},\{v\};\{a\},\{b\}) = \langle u_1, ...u_n|\hat{\mathcal{O}}|v_1, ...v_m\rangle_{a_1,..a_n;b_1,..b_m}$$

where u_i, v_j are rapidities and a_i, b_j are isotopic indices. According Balog (1994), LeClair Mussardo (1999),

$$F^{\mathcal{O}}(\{u\},\{v\};\{\bar{a}\},\{b\})_{irr} =$$
$$\langle 0|\hat{\mathcal{O}}|\{v\};\{\mathrm{i}\pi + u + \mathrm{i}0\}\rangle_{b_1,\ldots b_m;\bar{a}_1,\ldots\bar{a}_n;}$$

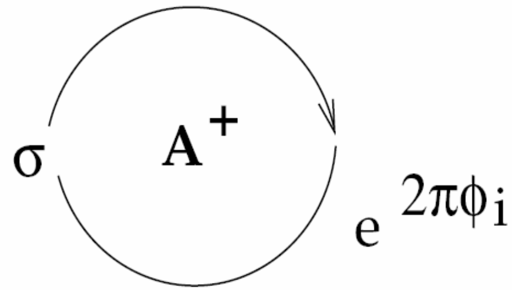
where $\bar{a}$ indices are obtained from a by charge conjugation.

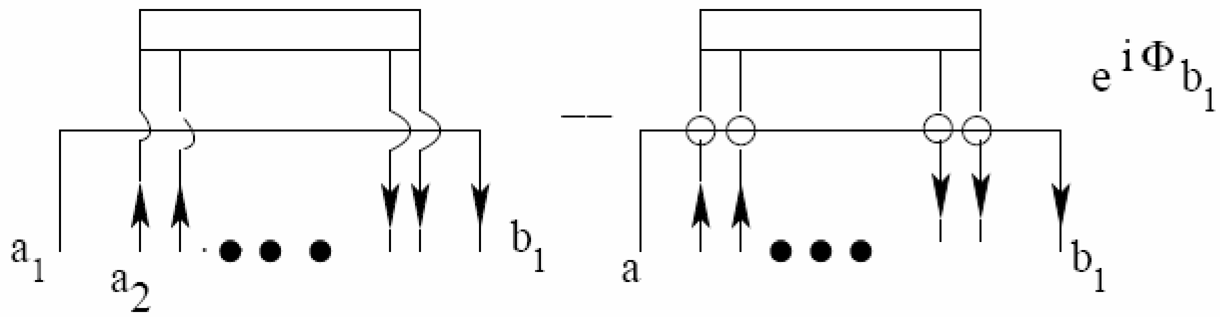
$$F^{\mathcal{O}}(0,\theta) = \langle 0|\hat{\mathcal{O}}|\theta_1, \ldots\theta_q\rangle_{c_1,\ldots c_q}$$

have poles at $\theta_i - \theta_j = \mathrm{i}\pi$ and the residues satisfy the following relation Smirnov 1992:

$$
\mathrm{iRes}_{\theta_{q-1,q}=-\mathrm{i}\pi} F^{\mathcal{O}}_{c_1,\ldots c_q}(\theta_1,\theta_2,\ldots,\theta_q) =
F^{\mathcal{O}}_{c'_1,\ldots,c'_{q-2}}(\theta_1,\ldots,\theta_{q-2})\delta_{c_q,c'_{q-1}} \times
\left[I - \mathrm{e}^{2\pi\mathrm{i}\varphi_{c_q,\mathcal{O}}} \prod_{i=1}^{q-2} S^{\tau_{i-1},c'_i}_{\tau_i,c_i}(\theta_{q-1}-\theta_i)\right],
$$

where S is the two-particle scattering matrix and $\varphi_{c,\mathcal{O}}$ is the semi-locality index between the particle creation operator A^+_c and $\mathcal{O}$.



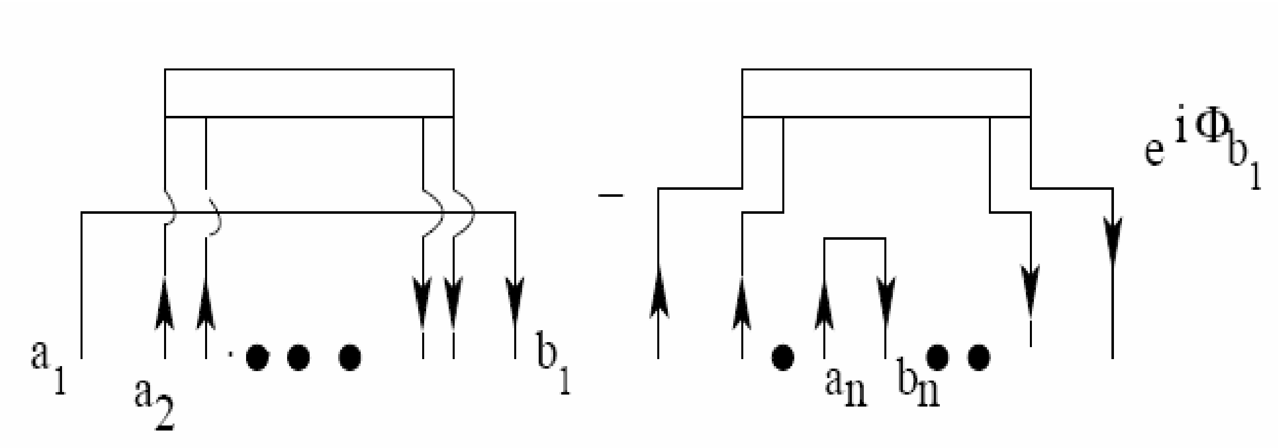


The rectangles are the formfactors. The circles are S matrices.

At low energies S-matrix is either diagonal
or equal to the permutation operator
S(0) = P. In the former case the equations for the residues become trivial

This is the Ising model universality class.

In latter case they just simplify:



And can be resolved

Sine-Gordon model as an example

The Lagrangian density

$$\mathcal{L} = \frac{1}{16\pi\gamma^2}(\partial_\mu\Phi)^2 - m^2\cos(\Phi)$$

The conserved charge

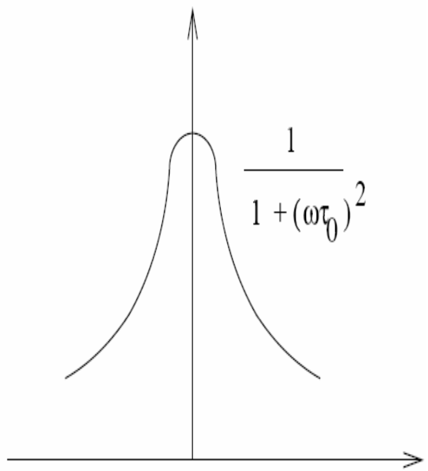
$$Q = \frac{1}{2\pi}\int \mathrm{d}x\partial_x\Phi$$

At $\gamma^2 = 1/2$ the solitons are fermions: $S = -1$. Otherwise the S matrix exhibits a crossover from diagonal at large momenta to P at small momenta.

$$\langle \mathrm{e}^{\mathrm{i}\eta\Phi(x,t)}\mathrm{e}^{-\mathrm{i}\eta\Phi(0,0)}\rangle = C^2(\gamma,\eta)\times$$

$$\exp\left[-2\sin^2(\pi\eta)\int\frac{\mathrm{d}p}{2\pi}\mathrm{e}^{-\epsilon(p)\beta}|x - t\frac{\partial\epsilon(p)}{\partial p}|\right]$$

where ϵ is the soliton energy and the factor $C(\gamma,\eta) = \langle\exp[\mathrm{i}\eta\Phi]\rangle$.



For the diagonal S matrix the transport is ballistic.

. The correlation function of $J \equiv J_x$ can be obtained from the above formula by taking $\eta \to 0$ limit and differentiating twice with respect to t:

$$\langle\langle J(x,t)J(0,0)\rangle\rangle = \int \frac{\mathrm{d}p}{\pi} v^2(p)\delta\left(x - tv(p)\right)\mathrm{e}^{-\beta\epsilon(p)}$$

where $v(p) = \partial\epsilon/\partial p$. The conductivity is ballistic with a finite Drude weight Fujimoto 1999, Affleck and Sagi 1996, Konik 2004

$$\sigma(\omega) = 2\pi D(T)\delta(\omega), \;\; D(T) = \int \frac{\mathrm{d}p}{\pi} v^2(p)\mathrm{e}^{-\beta\epsilon(p)} = 2n$$

Some operators do have ballistic dynamics, some follow the semiclassical one.

The universal asymptotics corresponding S = -P

It is valid at T < T* and depends solely on the semi-locality index of the operator in question

$$\langle \mathrm{e}^{\mathrm{i}\eta\Phi(x,t)}\mathrm{e}^{-\mathrm{i}\eta\Phi(0,0)}\rangle$$

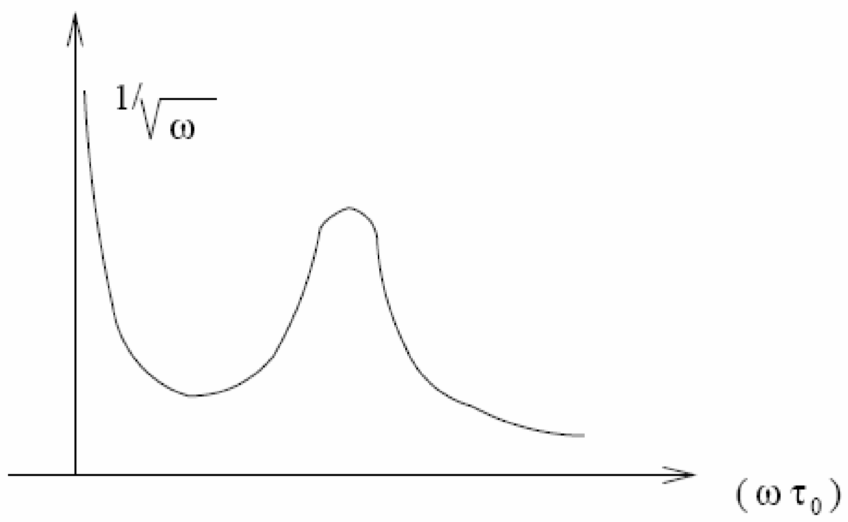
$$\frac{C^2(\eta,\gamma)}{4\pi}\int_0^{\pi/2}\frac{\sin^2(\pi\eta)\mathrm{e}^{-8R(t,x)\sin^2 y}}{\sin^2 y\cos(2\pi\eta)+\sin^4(\pi\eta)}\mathrm{d}y$$

where $C = \langle 0|\exp[\mathrm{i}\eta\Phi]|0\rangle$

$= A(\eta,\gamma)[M/\Lambda]^{\eta^2\gamma^2}$

R = max(x,vt) n(T)

At x=0, the Fourier transform is



Charge statistics in sine-Gprdon model

The conserved soliton charge is

$$Q = \frac{1}{2\pi}\int \mathrm{d}x\partial_x\Phi$$

The charge passed through point x during the time t is $Q(x,t) = (1/2\pi)[\Phi(x,t) - \Phi(x,0)]$.

The distribution function of Q:

$$P[Q(t)] \equiv \langle\delta\left(Q - \hat{Q}(x,t)\right)\rangle =$$

$$\int \mathrm{d}\eta\langle \mathrm{e}^{\mathrm{i}\eta[\Phi(t,x)-\Phi(0,x)]}\rangle \mathrm{e}^{-2\pi\mathrm{i}\eta Q}$$

In the field theory limit $M/\Lambda << 1$, the main η dependence comes from $(M/\Lambda)^{\eta^2}$ and the magnitude of the propagator has a strong maximum at small η.

At the free fermion point $\gamma^2 = 1/2$ we have

$$P[Q(t)] \sim \exp\left\{-Q^2(t)/[(t/\tau_0) + \pi^{-2}\gamma^2 \ln(\Lambda/M)]\right\}$$

where $1/\tau_0 = (2T/\pi M)\mathrm{e}^{-M/T}$

At $\gamma^2 \neq 1/2$ there is a crossover temperature $T^* \sim M|\gamma^2 - 1/2|$ below which the S-matrix is effectively equal to $-P$. Then the distribution is:

$$P[Q(t)] \sim \frac{1}{\sqrt{\bar{t}}} \int_0^\infty \frac{\mathrm{d}y}{\sqrt{A + 2\pi^2 y/\gamma}}$$

$$\exp\left[-\frac{Q^2}{4(A + 2\pi^2 y/\gamma)} - \frac{y^2\gamma^2}{\bar{t}}\right]$$

where $\bar{t} = t/\tau_0, \quad A = \pi^{-2}\gamma^2 \ln(\Lambda/\bar{M})$. At large $\bar{t} >> A$ we have

$$P[Q] \sim \exp\left[-\frac{3Q^{4/3}}{4\bar{t}^{1/3}}\right]$$

Conclusions

- At low T<< M order parameter type operators in integrable systems display universal dynamics which falls in two universality classes.

- The physics at these T is controlled by multiple particle processes whose matrix elements are singular when *in* and *out* momenta coincide.

- For the S = P universality class there are discrepancies with the semiclassical results by [Damle and Sachdev , Rapp and Zarand (2005)]. Regularization of formfactors?