The Abdus Salam
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on

High Rayleigh Number Convection

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Prediction of the center temperature for compressible non-Overbeck-Boussinesq thermal convection

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Prediction of the center temperature in compressible Non-Oberbeck-Boussinesq Rayleigh-Bénard convection

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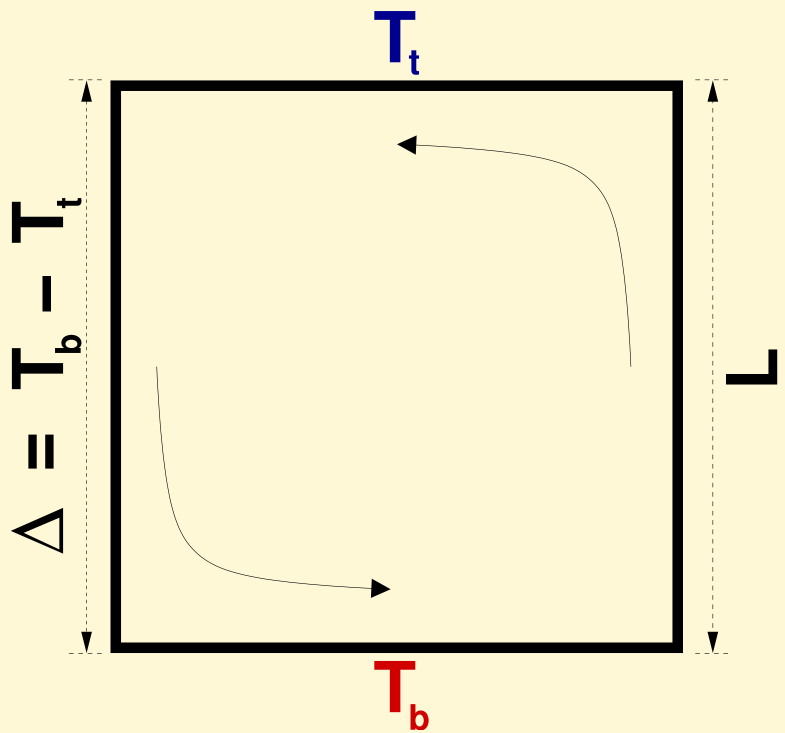
What?

Rayleigh-Bénard convection

Aspect-ratio-one container



T_m : control temperature

P_m : control pressure

Prandtl number

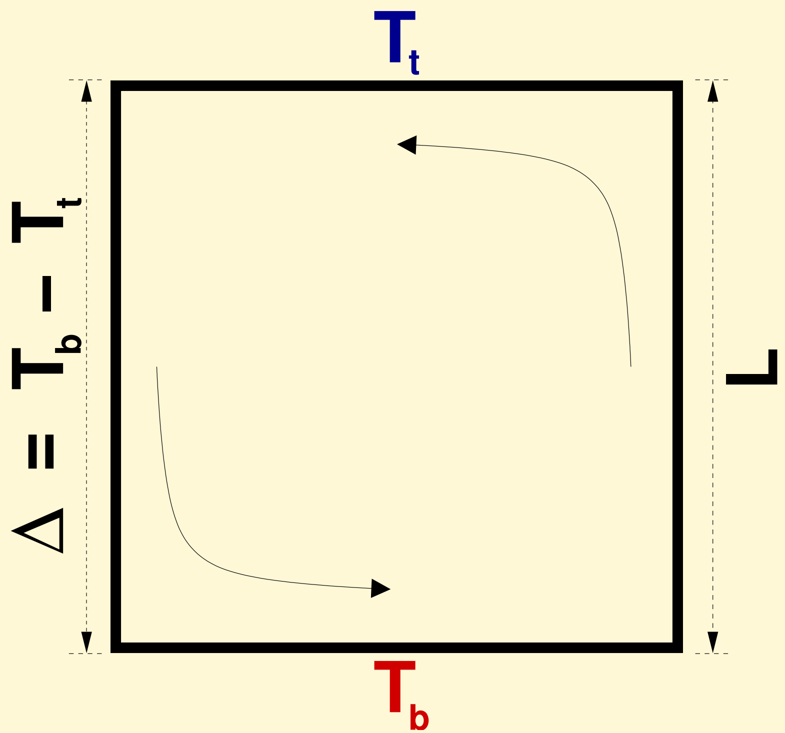
$$\mathrm{Pr} = \frac{\nu_m}{\kappa_m}$$

Rayleigh number

$$\mathrm{Ra} = \frac{\beta_m \, g \, L^3 \, \Delta}{\nu_m \, \kappa_m}$$

Oberbeck-Boussinesq (OB) approximation

Aspect-ratio-one container



OB approximation: constant properties

$$
\begin{aligned}
\eta_m &= \eta(T_m, P_m) \\
\Lambda_m &= \Lambda(T_m, P_m) \\
\beta_m &= \beta(T_m, P_m) \\
c_{P,m} &= c_P(T_m, P_m)
\end{aligned}
$$

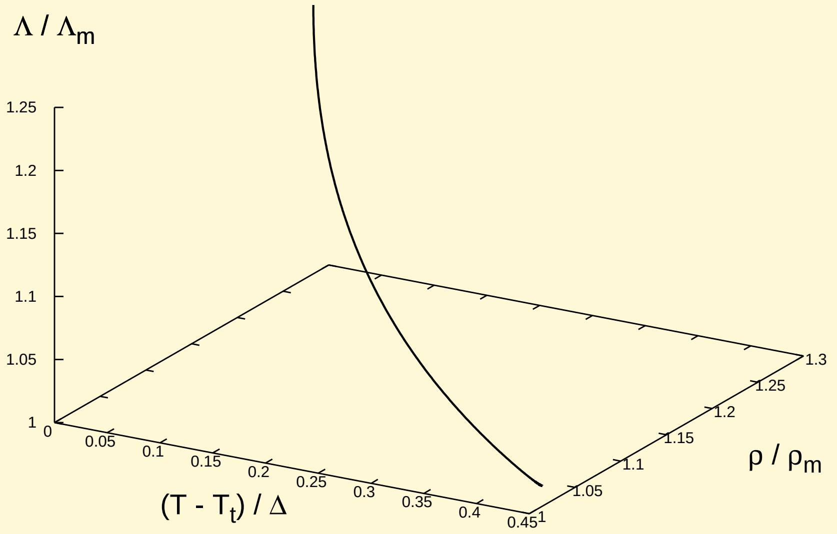
T_m : control temperature

P_m : control pressure

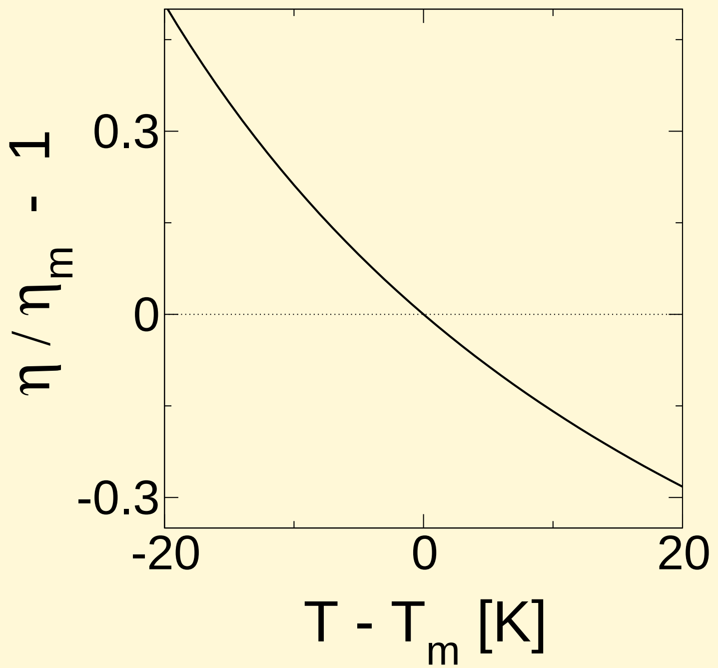
Non-Oberbeck-Boussinesq (NOB) scenario

NOB scenario:
$\eta(T,\rho)$, $\Lambda(T,\rho)$, $\beta(T,\rho)$, $c_P(T,\rho)$.

Gas
Ethane



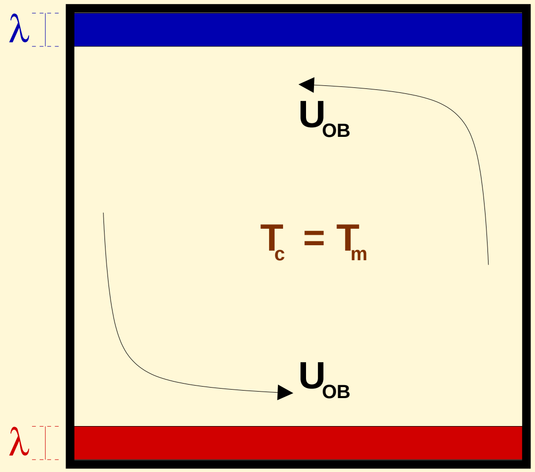
Liquid
Water



Oberbeck-Boussinesq (OB) symmetry

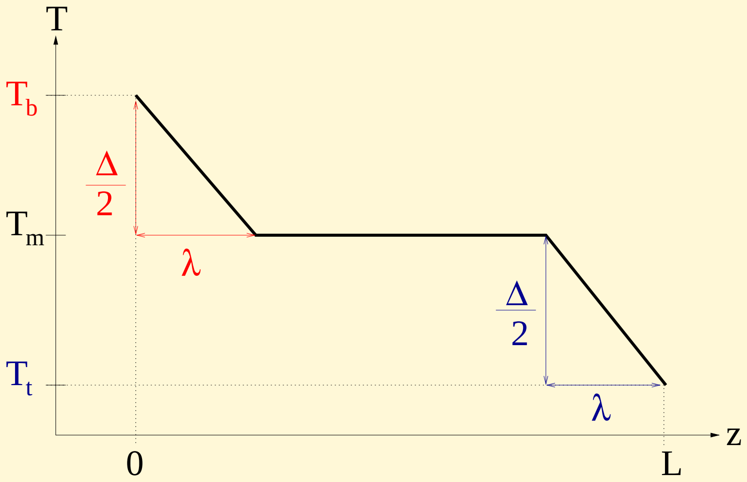
OB approximation:
η, Λ, β, c_P are constants.

Boundary-layers



$$\lambda_b = \lambda_t$$

Temperature drops

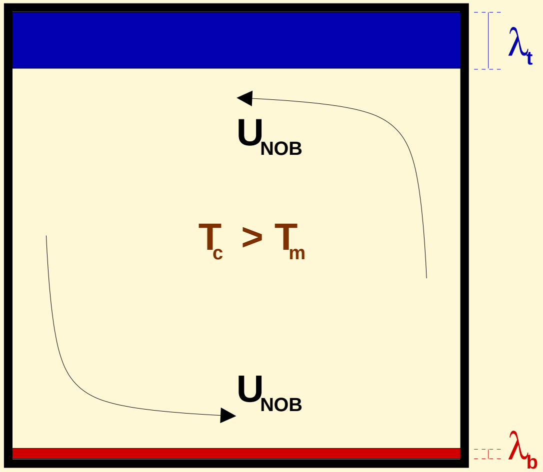


$$\Delta_b = \Delta_t$$

Non-Oberbeck-Boussinesq (NOB) asymmetry

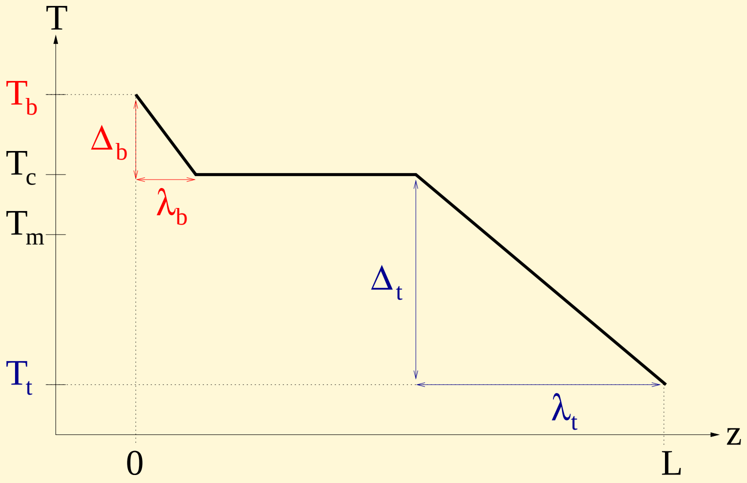
NOB scenario:
$\eta(T,\rho)$, $\Lambda(T,\rho)$, $\beta(T,\rho)$, $c_P(T,\rho)$.

Boundary-layers



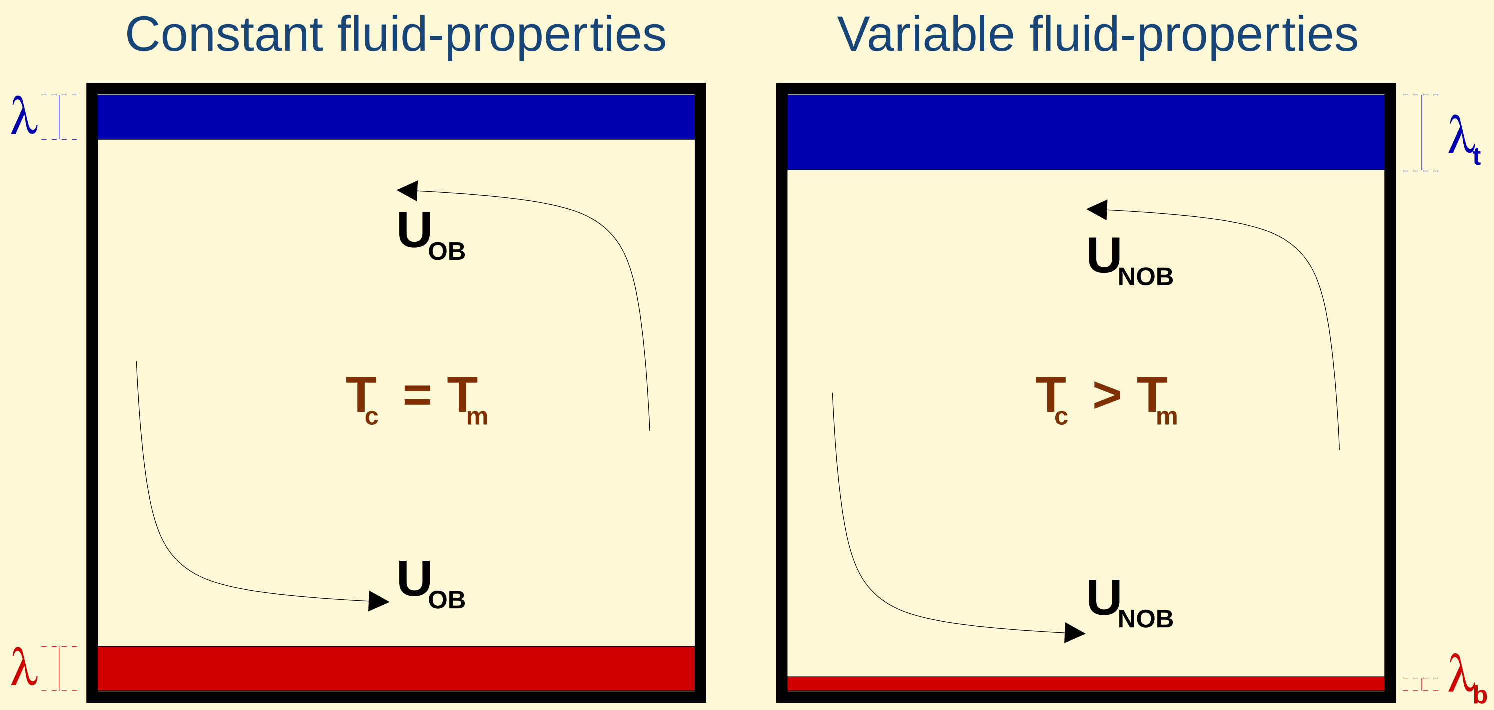
$$\lambda_b \neq \lambda_t$$

Temperature drops

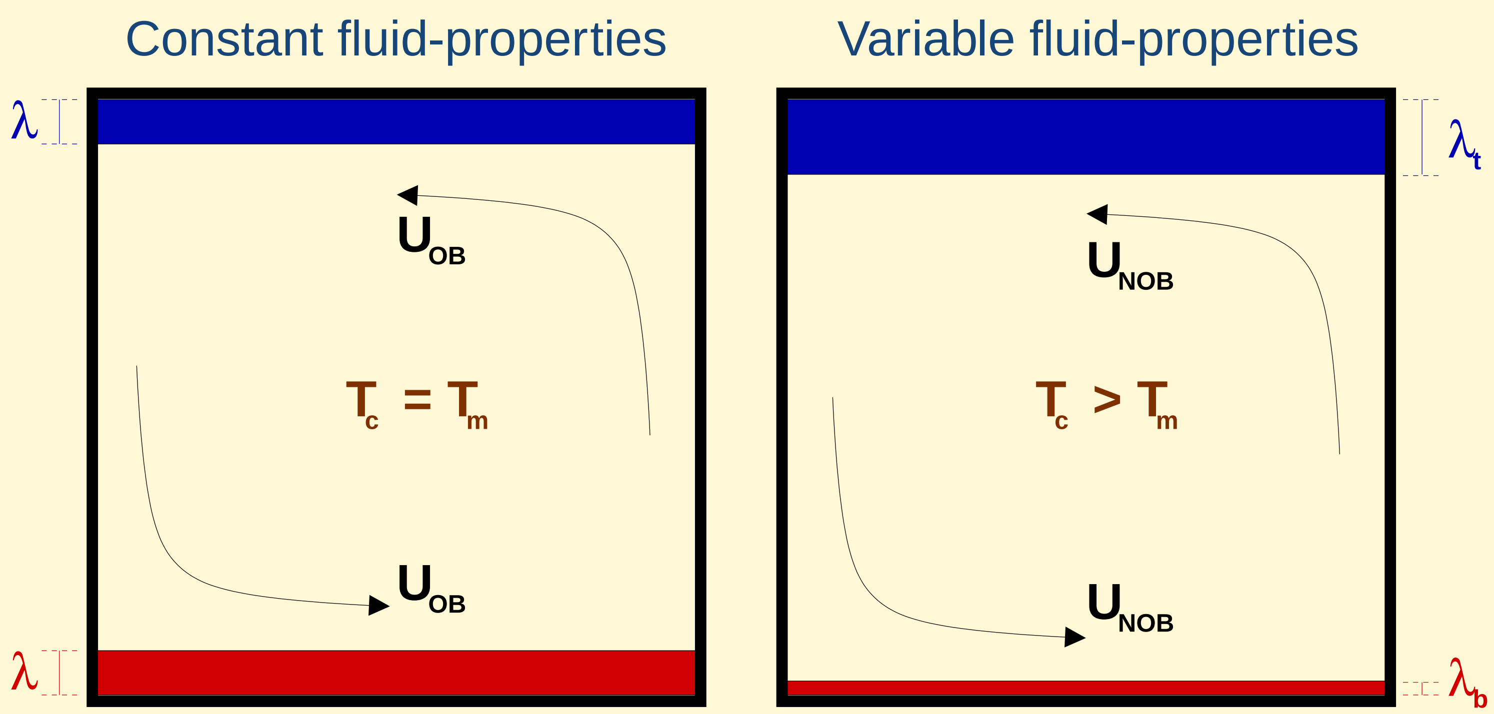


$$\Delta_b \neq \Delta_t$$

Non-Oberbeck-Boussinesq (NOB) asymmetry

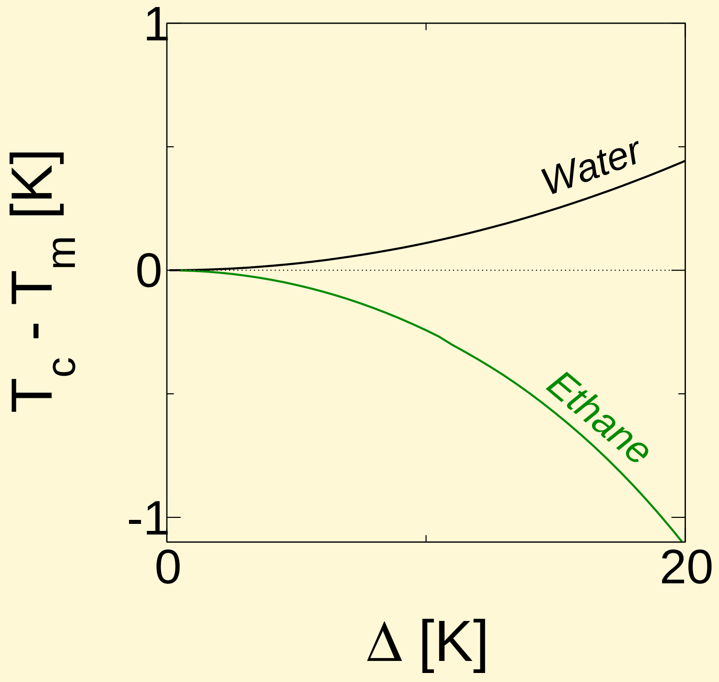


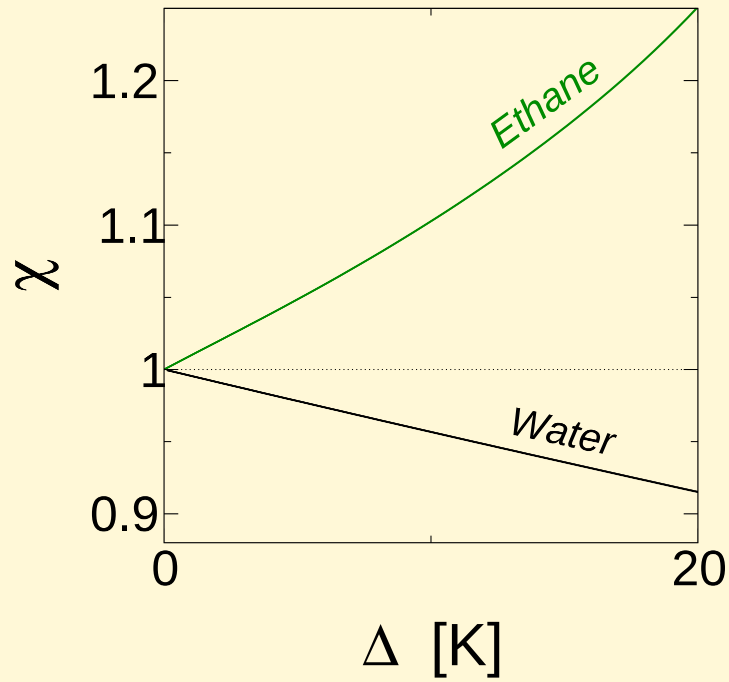
Non-Oberbeck-Boussinesq (NOB) asymmetry



Does boundary-layer theory describe NOB effects on T_c?

Non-Oberbeck-Boussinesq (NOB) asymmetry





$T_c > T_m$ in water

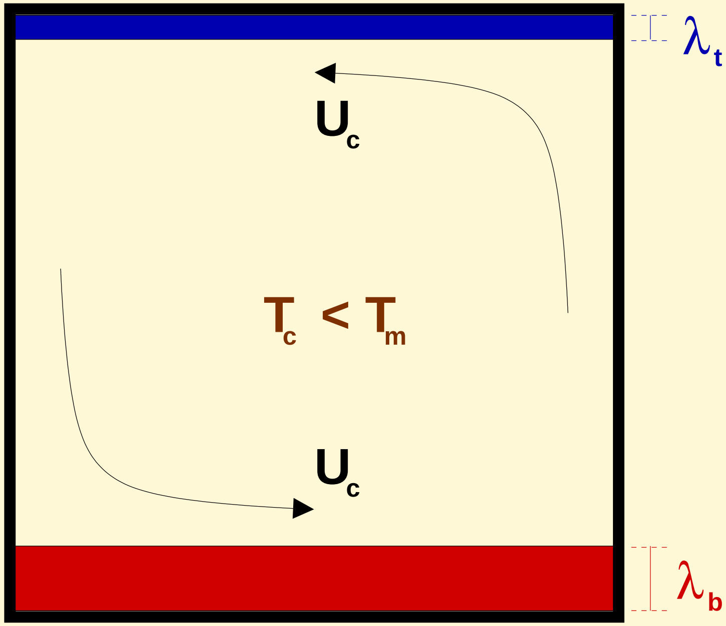
$\Delta_b < \Delta_t$ in water

$T_c < T_m$ in gaseous ethane

$\Delta_b > \Delta_t$ in gaseous ethane

BL theory

2D boundary-layers



$\lambda_{b,t}$: thermal-BL

Compressibility effects:

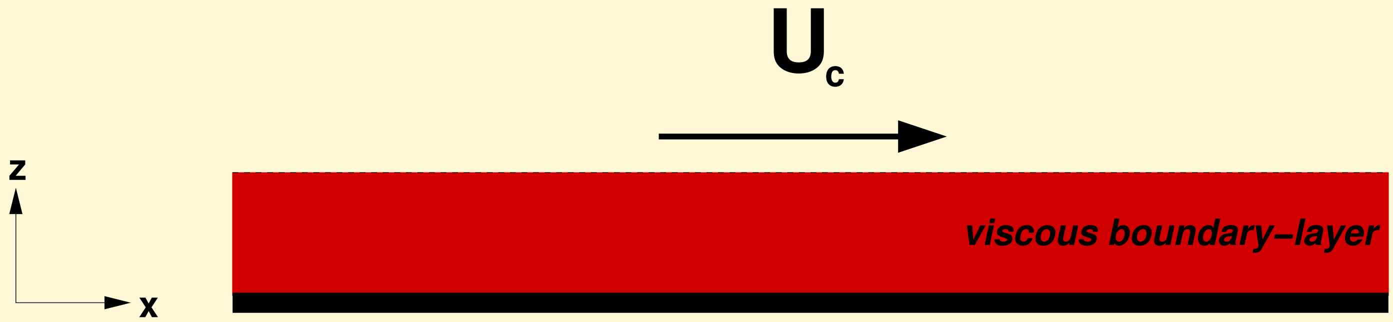
$$\rho = \rho(T, P_m)$$

Density BL!

Variable properties:

- $\eta(T, \rho)$: shear viscosity
- $\Lambda(T, \rho)$: thermal conductivity
- $\beta(T, \rho)$: thermal expansivity
- $c_P(T, \rho)$: specific heat

Viscous boundary-layer



Continuity:

$$\frac{\partial}{\partial x}\{\rho u_x\} + \frac{\partial}{\partial z}\{\rho u_z\} = 0,$$

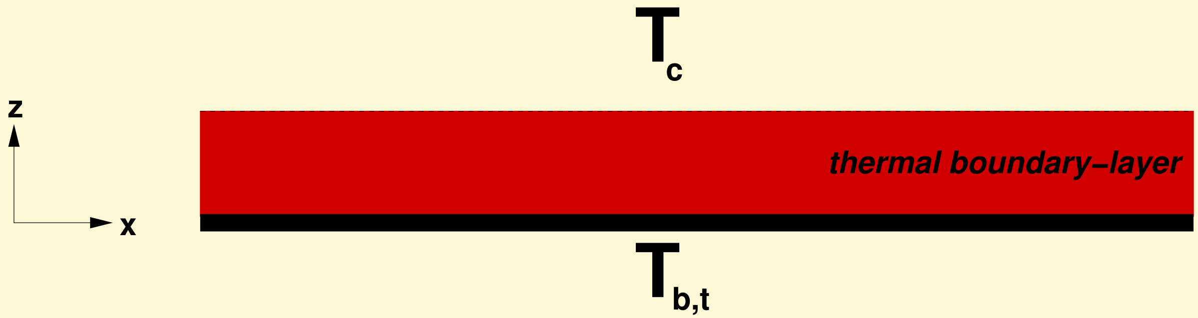
Momentum:

$$\rho\left\{u_x \frac{\partial u_x}{\partial x} + u_z \frac{\partial u_x}{\partial z}\right\} = \frac{\partial}{\partial z}\left\{\eta \frac{\partial u_x}{\partial z}\right\}$$

Boundary conditions:

$$u_x(x,0) = 0, \qquad u_z(x,0) = 0, \qquad u_x(x,\infty) = U_c.$$

Thermal boundary-layer



Temperature equation:

$$u_x \frac{\partial T}{\partial x} + u_z \frac{\partial T}{\partial z} + \frac{\gamma - 1}{\beta} \left\{ \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} \right\} = \frac{\gamma}{\rho\, c_P} \frac{\partial}{\partial z} \left\{ \Lambda \frac{\partial T}{\partial z} \right\}$$

Boundary conditions:

$$\begin{aligned} T(x,\infty) &= T_c, \\ T(x,0) &= T_{b,t}, \\ \Lambda_b \left. \frac{\partial T}{\partial z} \right|_b &= \Lambda_t \left. \frac{\partial T}{\partial z} \right|_t . \end{aligned}$$

Boundary-layer equations: summary

$$\frac{\partial}{\partial x}\{\rho u_x\} + \frac{\partial}{\partial z}\{\rho u_z\} = 0,$$

$$\rho\left\{u_x\,\frac{\partial u_x}{\partial x} + u_z\,\frac{\partial u_x}{\partial z}\right\} = \frac{\partial}{\partial z}\left\{\eta\,\frac{\partial u_x}{\partial z}\right\}$$

$$u_x\frac{\partial T}{\partial x} + u_z\frac{\partial T}{\partial z} + \frac{\gamma-1}{\beta}\left\{\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z}\right\} = \frac{\gamma}{\rho\,c_P}\frac{\partial}{\partial z}\left\{\Lambda\,\frac{\partial T}{\partial z}\right\}$$

Boundary conditions:

$$u_x(x,0) = 0, \qquad u_z(x,0) = 0, \qquad u_x(x,\infty) = U_c.$$

$$T(x,0) = T_{b,t}, \quad \Lambda_b\,\left.\frac{\partial T}{\partial z}\right|_b = \Lambda_t\,\left.\frac{\partial T}{\partial z}\right|_t, \quad T(x,\infty) = T_c.$$

Self-similarity

Stream-function

- Continuity:

$$\frac{\partial}{\partial x}\left\{\tilde{\rho} u_x\right\} + \frac{\partial}{\partial z}\left\{\tilde{\rho} u_z\right\} = 0,$$

where

$$\tilde{\rho} = \frac{\rho}{\rho_m}.$$

- Stream function:

$$\tilde{\rho}\, u_x = \frac{\partial \Psi}{\partial z},$$

$$\tilde{\rho}\, u_z = -\frac{\partial \Psi}{\partial x}.$$

- Similarity variable:

$$\tilde{z} \equiv \frac{z}{\ell_c(x)}, \qquad \text{where } \ell_c(x) \equiv \sqrt{\frac{x\,\nu_m}{U_c}}.$$

- Dimensionless stream-function:

$$\tilde{\Psi}(\tilde{z}) = \frac{\Psi(x,z)}{\ell_c\,U_c}.$$

- Velocity components:

$$u_x = U_c\,\frac{\tilde{\Psi}'}{\tilde{\rho}},$$

$$u_z = \frac{\nu_m}{2\ell_c}\left\{\tilde{z}\,\frac{\tilde{\Psi}'}{\tilde{\rho}} - \frac{\tilde{\Psi}}{\tilde{\rho}}\right\}.$$

Viscosity gradient

Viscous contributions in the momentum equation:

$$\frac{\partial}{\partial z}\left\{\eta\,\frac{\partial u_x}{\partial z}\right\} \;=\; \eta\,\frac{\partial^2 u_x}{\partial z^2} + \frac{\partial \eta}{\partial z}\,\frac{\partial u_x}{\partial z}$$

η-gradient in dimensionless form:

$$\tilde{\eta}\,' \;=\; \frac{\partial \tilde{\eta}}{\partial \tilde{\Theta}}\,\tilde{\Theta}\,' + \frac{\partial \tilde{\eta}}{\partial \tilde{\rho}}\,\tilde{\rho}\,'$$

where

$$\tilde{\Theta}(\tilde{z}) \;=\; \frac{T(x,z) - T_t}{\Delta}$$

$$\tilde{\rho}(\tilde{z}) \;=\; \frac{\rho(x,z)}{\rho_m}$$

Extended Prandtl-Blasius equation:

$$\tilde{\eta}\tilde{\Psi}''' + A_1\,\tilde{\Psi}'' + A_2\,\tilde{\Psi}' = 0$$

where

$$A_1 = \left\{ \frac{1}{2}\tilde{\Psi} + \tilde{\eta}' - 2\frac{\tilde{\rho}'}{\tilde{\rho}}\,\tilde{\eta} \right\}$$

$$A_2 = \left\{ -\frac{1}{2}\frac{\tilde{\rho}'}{\tilde{\rho}}\,\tilde{\Psi} + \left[2\left(\frac{\tilde{\rho}'}{\tilde{\rho}}\right)^2 - \frac{\tilde{\rho}''}{\tilde{\rho}} \right]\tilde{\eta} - \frac{\tilde{\rho}'}{\tilde{\rho}}\,\tilde{\eta}' \right\}$$

Boundary conditions:

$$\tilde{\Psi}(0) = 0, \qquad \tilde{\Psi}'(0) = 0, \qquad \tilde{\Psi}'(\infty) = \tilde{\rho}_c.$$

Density boundary-layer

Slope of the density profile:

$$\tilde{\rho}\,' \;=\; -\tilde{\rho}\,\tilde{\beta}\,\tilde{\Theta}\,'$$

Curvature of the density profile:

$$\tilde{\rho}\,'' \;=\; -\tilde{\rho}\,\tilde{\beta}\,\tilde{\Theta}\,'' - \tilde{\rho}\,\tilde{\beta}\,'\,\tilde{\Theta}\,' + \tilde{\rho}\,(\tilde{\beta}\,\tilde{\Theta}\,')^2.$$

Boundary conditions:

$$\begin{aligned}\tilde{\rho}(0) &= \tilde{\rho}_{b,t},\\ \tilde{\rho}\,'(0) &= -\tilde{\rho}_{b,t}\,\tilde{\beta}_{b,t}\,\tilde{\Theta}\,'_{b,t},\\ \tilde{\rho}(\infty) &= \tilde{\rho}_c.\end{aligned}$$

$$\tilde{\Lambda}\,\tilde{\Theta}'' + \left\{\frac{1}{2}\,\tilde{c}_P\,\mathsf{Pr}\,\tilde{\Psi} + \tilde{\Lambda}'\right\}\tilde{\Theta}' = 0.$$

Boundary conditions:

$$\begin{aligned}
\tilde{\Theta}(\infty) &= \tilde{\Theta}_c, && \text{(center temperature)},\\
\tilde{\Theta}(0) &= \tilde{\Theta}_{b,t}, && \text{(plate temperature)},\\
\tilde{\Lambda}_b\,\tilde{\Theta}'_b &= \tilde{\Lambda}_t\,\left|\tilde{\Theta}'_t\right|. && \text{(heat fluxes)}
\end{aligned}$$

$$\tilde{\eta}\tilde{\Psi}''' + A_1\,\tilde{\Psi}'' + A_2\,\tilde{\Psi}' = 0$$

$$\tilde{\Lambda}\,\tilde{\Theta}'' \;+\; \left\{\frac{1}{2}\,\tilde{c}_P\,\mathsf{Pr}\,\tilde{\Psi} \;+\; \tilde{\Lambda}'\right\}\tilde{\Theta}' \;=\; 0.$$

$$\tilde{\rho}' \;=\; -\tilde{\rho}\,\tilde{\beta}\,\tilde{\Theta}'$$

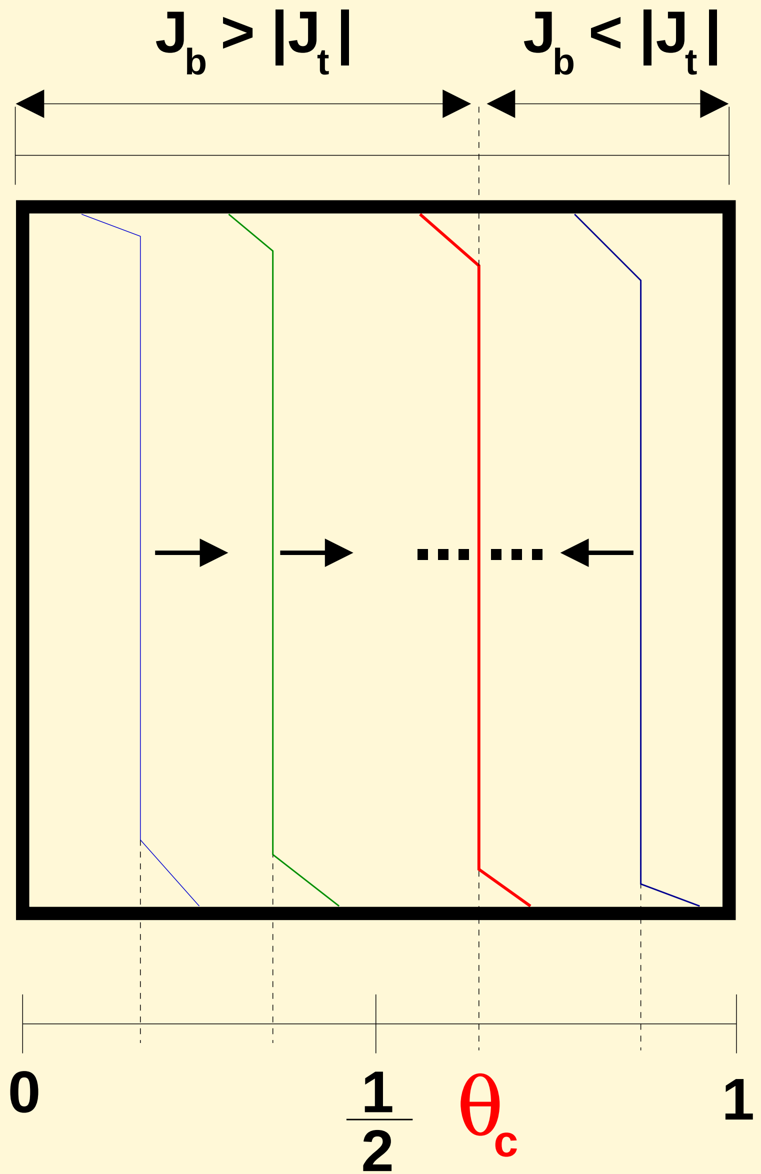
Boundary conditions:

$$\tilde{\Psi}(0) = 0, \qquad \tilde{\Psi}'(0) = 0, \qquad \tilde{\Psi}'(\infty) = \tilde{\rho}_c,$$

$$\tilde{\Theta}(0) = \tilde{\Theta}_{b,t}, \qquad \tilde{\Lambda}_b\,\tilde{\Theta}'_b = \tilde{\Lambda}_t\,\left|\tilde{\Theta}'_t\right|. \qquad \tilde{\Theta}(\infty) = \tilde{\Theta}_c,$$

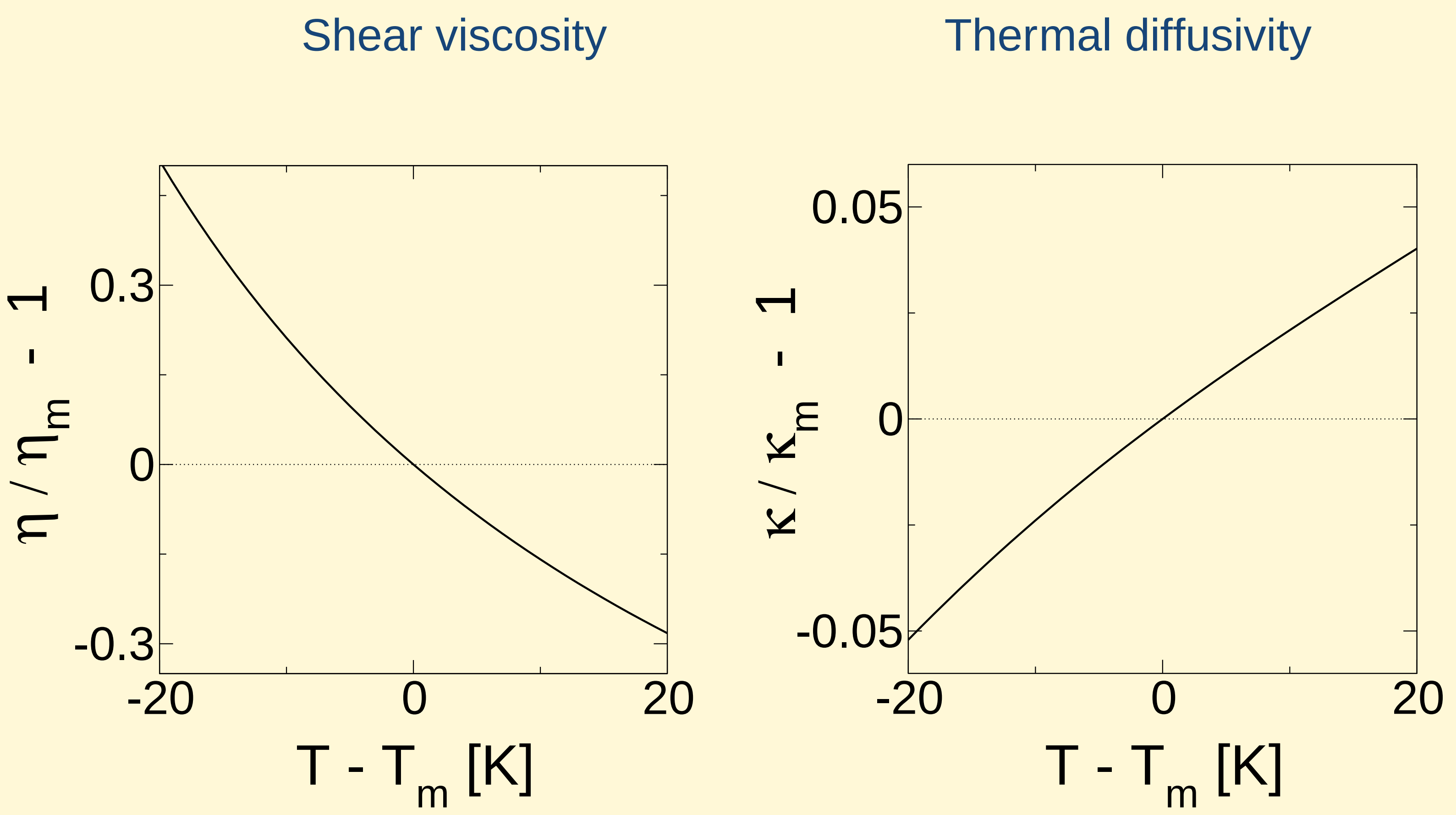
$$\tilde{\rho}(0) = \tilde{\rho}_{b,t}, \qquad \tilde{\rho}'(0) = -\tilde{\rho}_{b,t}\,\tilde{\beta}_{b,t}\,\tilde{\Theta}'_{b,t}, \qquad \tilde{\rho}(\infty) = \tilde{\rho}_c.$$

Center temperature

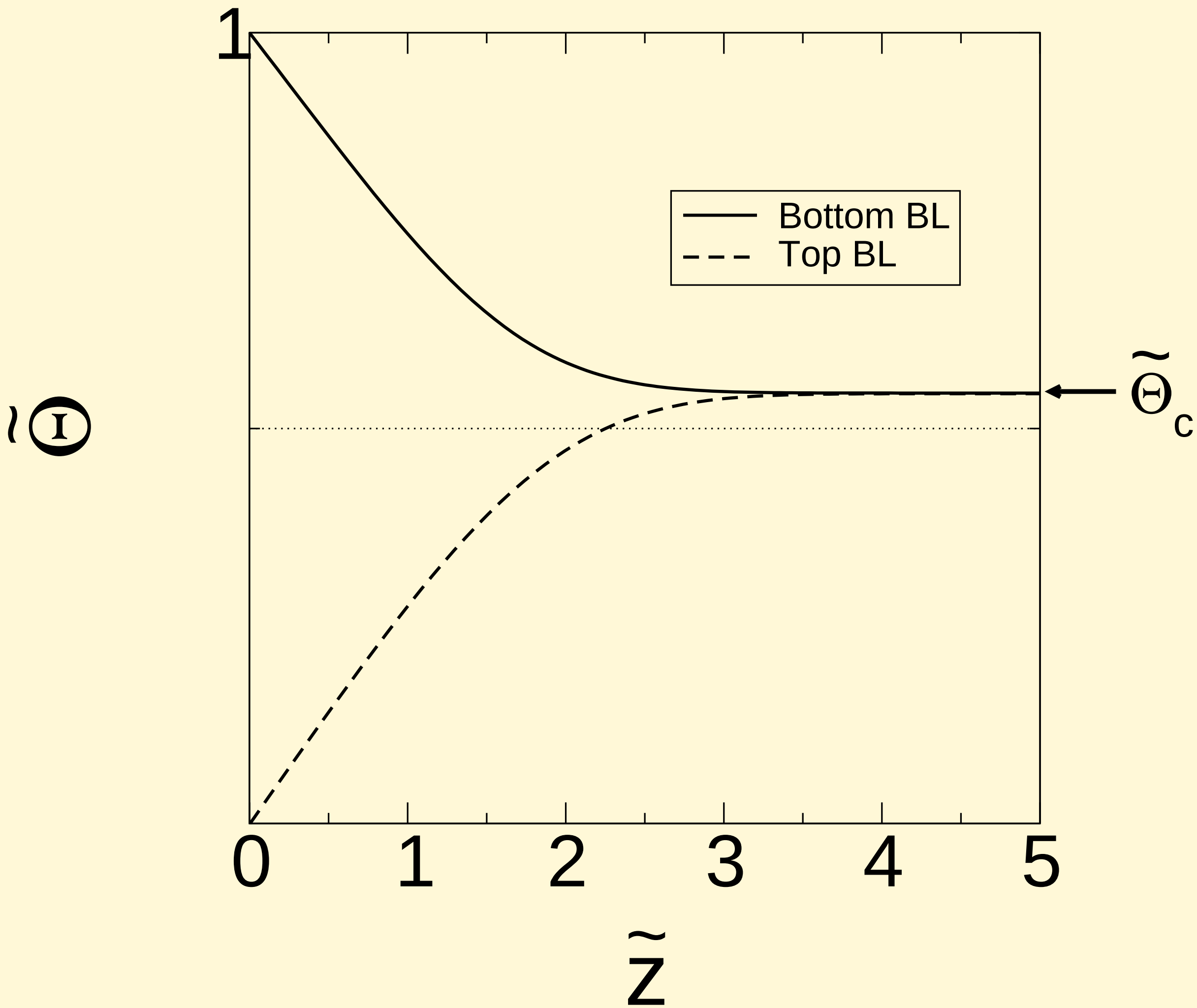


Water

Water: $\eta(T)$ and $\kappa(T)$

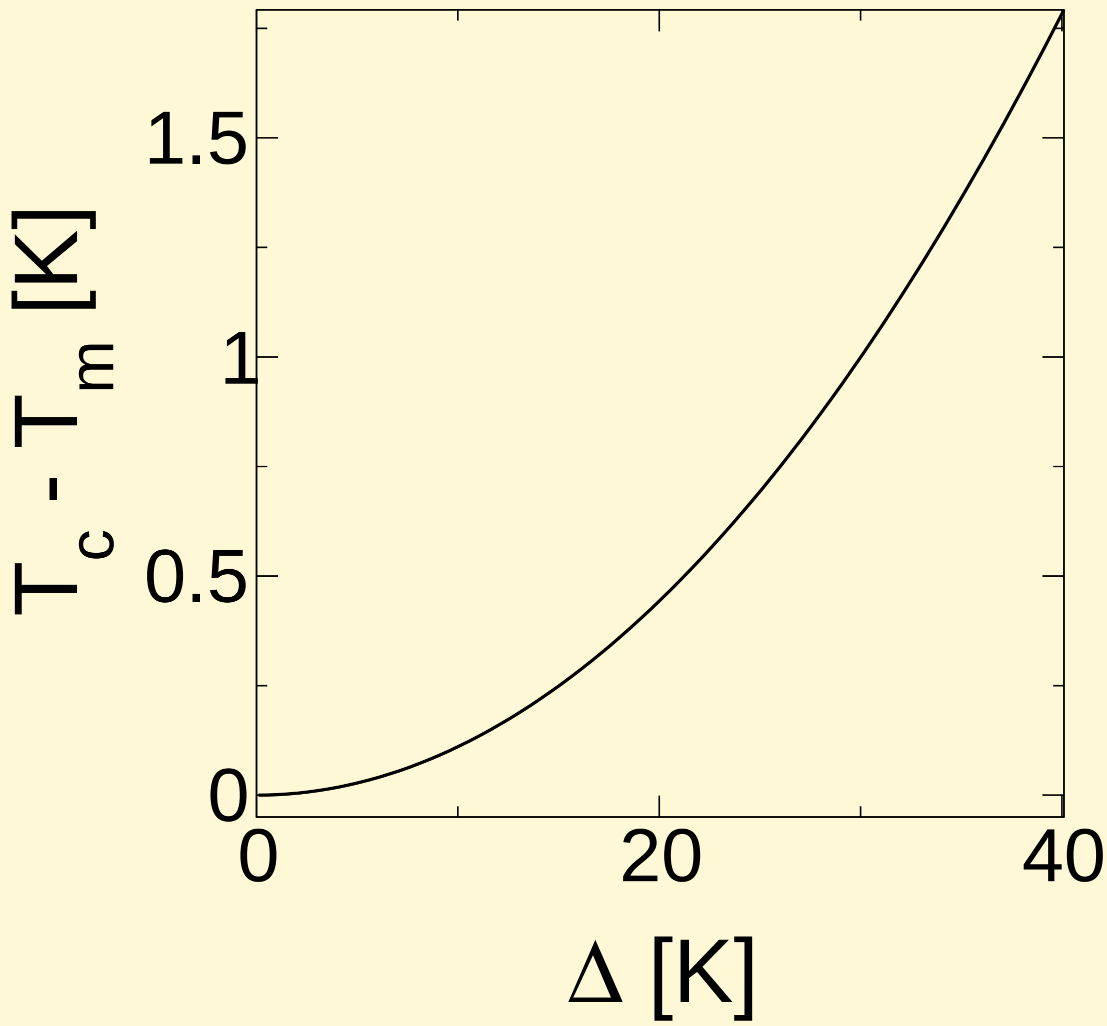


Temperature profile

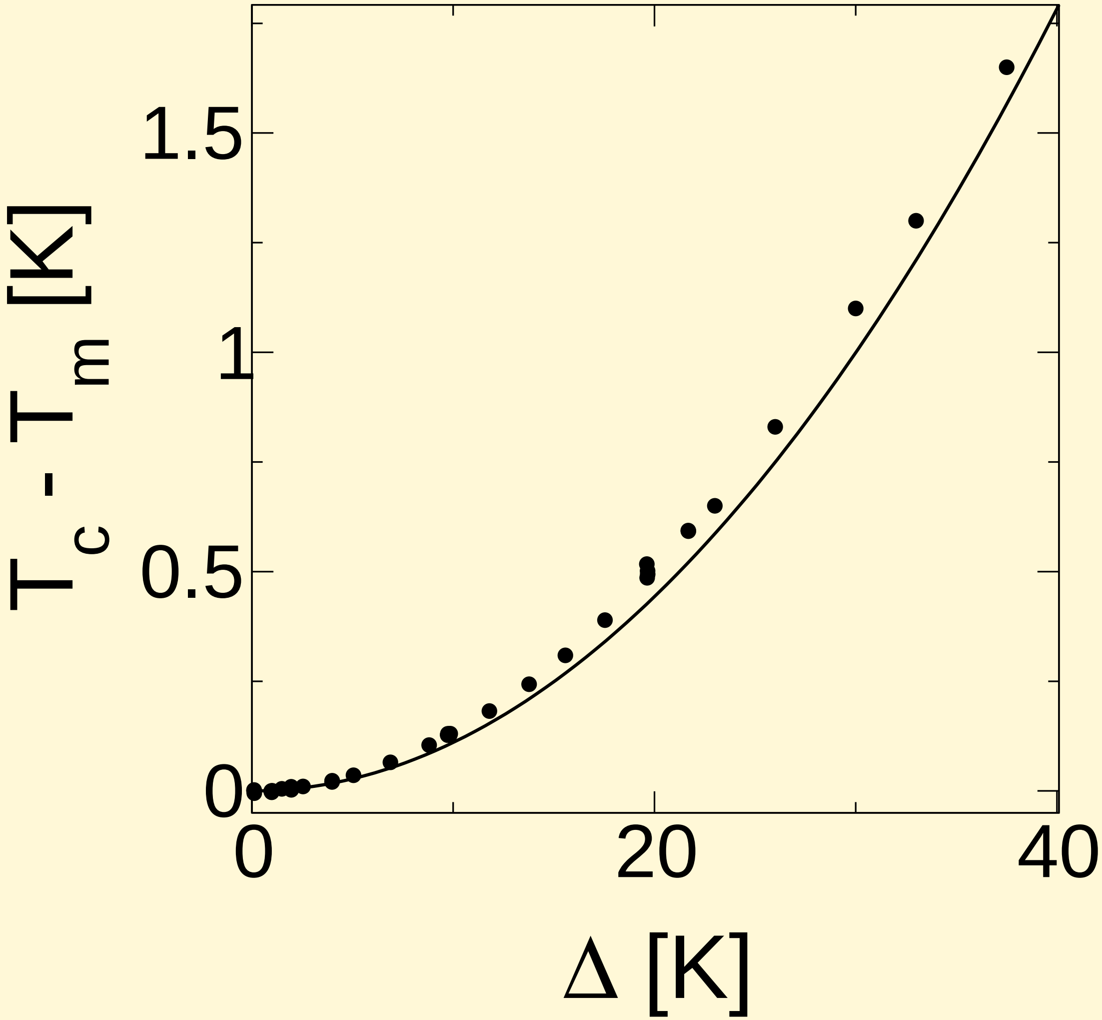


$T_m = 40\,C$ and $\Delta = 40\,C$

Center temperature



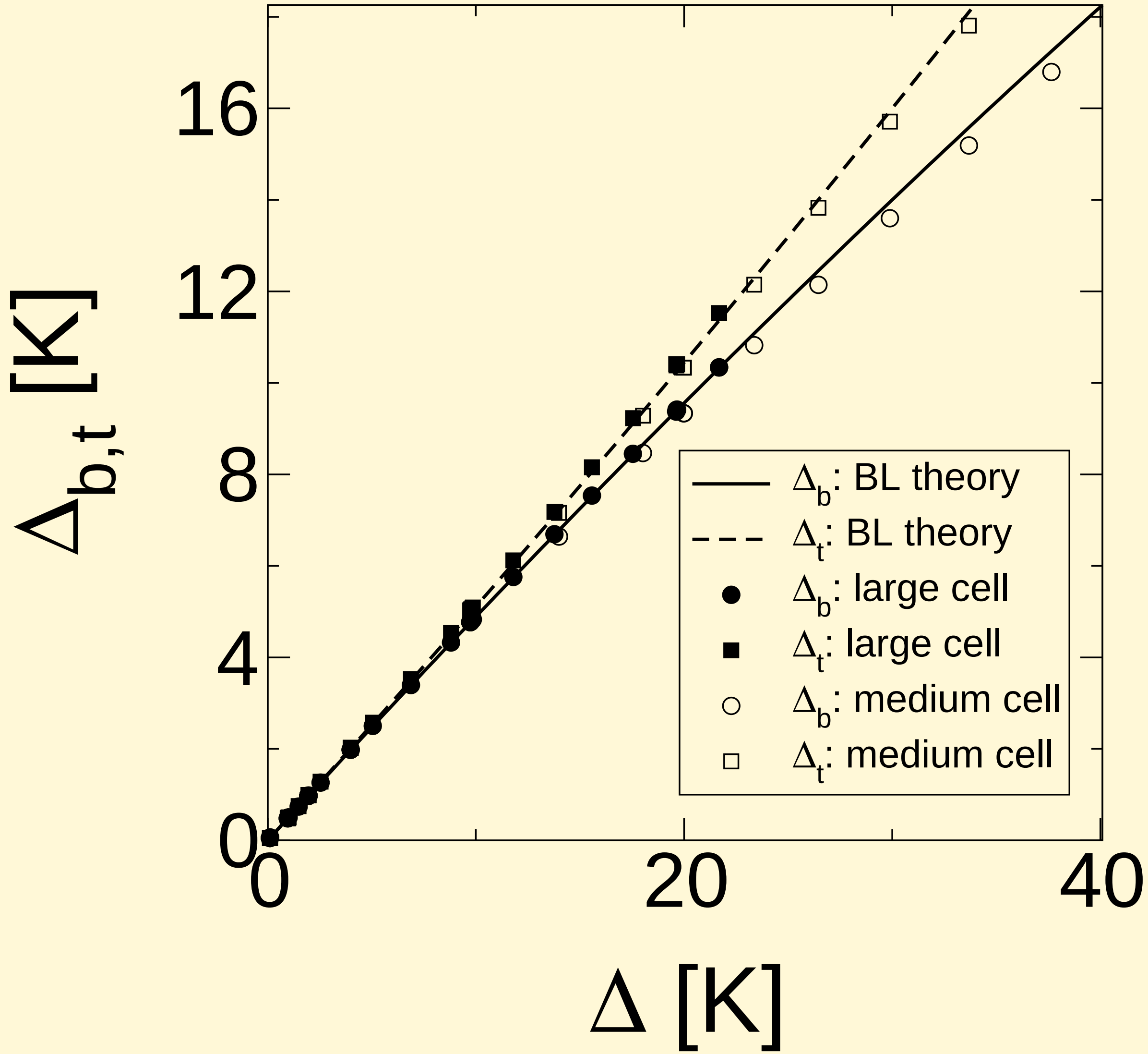
Center temperature



Experiments by Ahlers, Brown, and Funfschilling

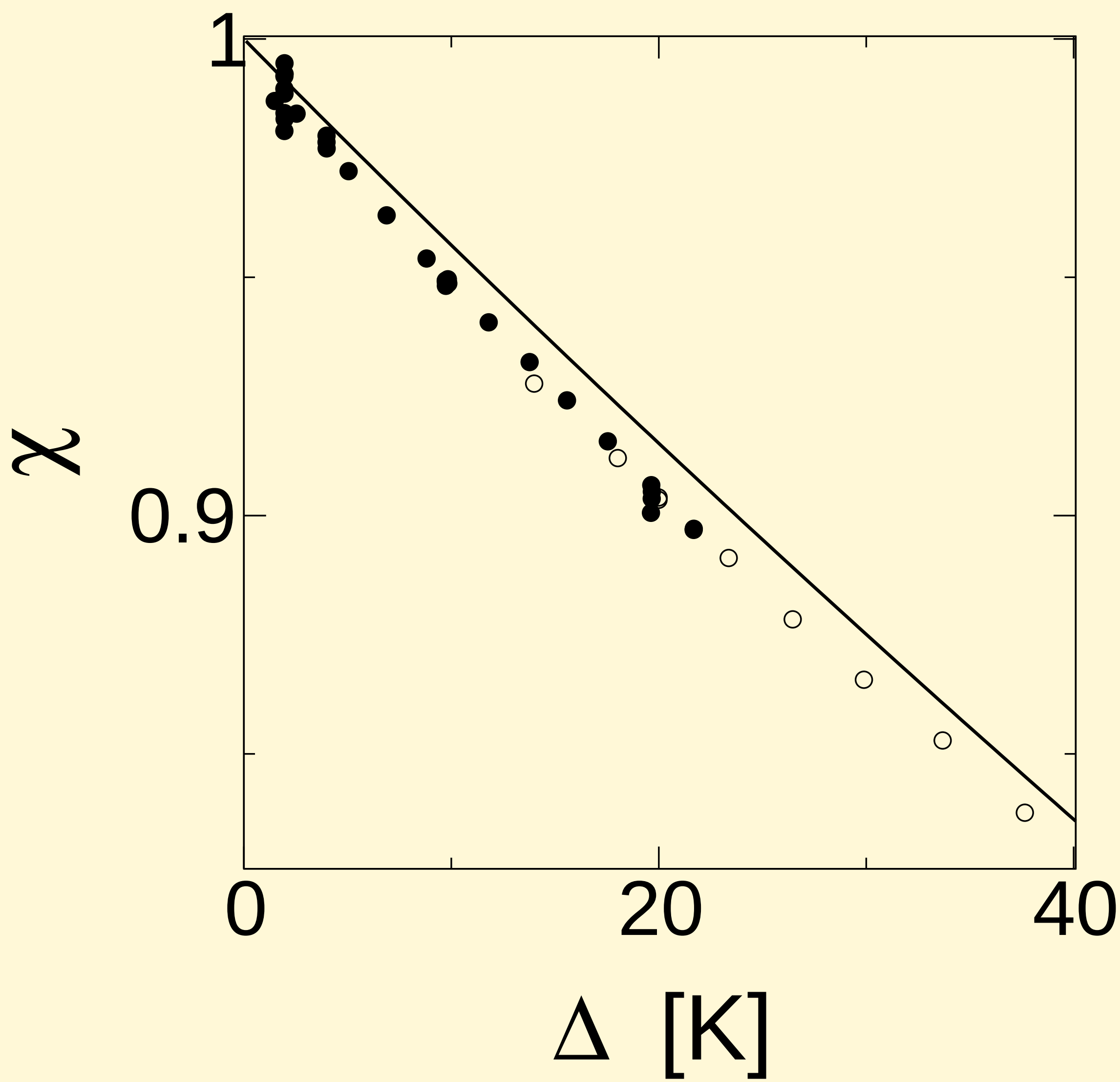
Temperature drops across the BLs

$\Delta_b = T_b - T_c$ and $\Delta_t = T_c - T_t$



Ratio of temperature drops

$$\chi = \Delta_b / \Delta_t$$



Nusselt number

- Nusselt ratio:

$$\frac{\text{Nu}_{NOB}}{\text{Nu}_{OB}} = \frac{2\lambda_{OB}^{sl}}{\lambda_t^{sl} + \lambda_b^{sl}} \frac{\kappa_t \Delta_t + \kappa_b \Delta_b}{\kappa_m \Delta}.$$

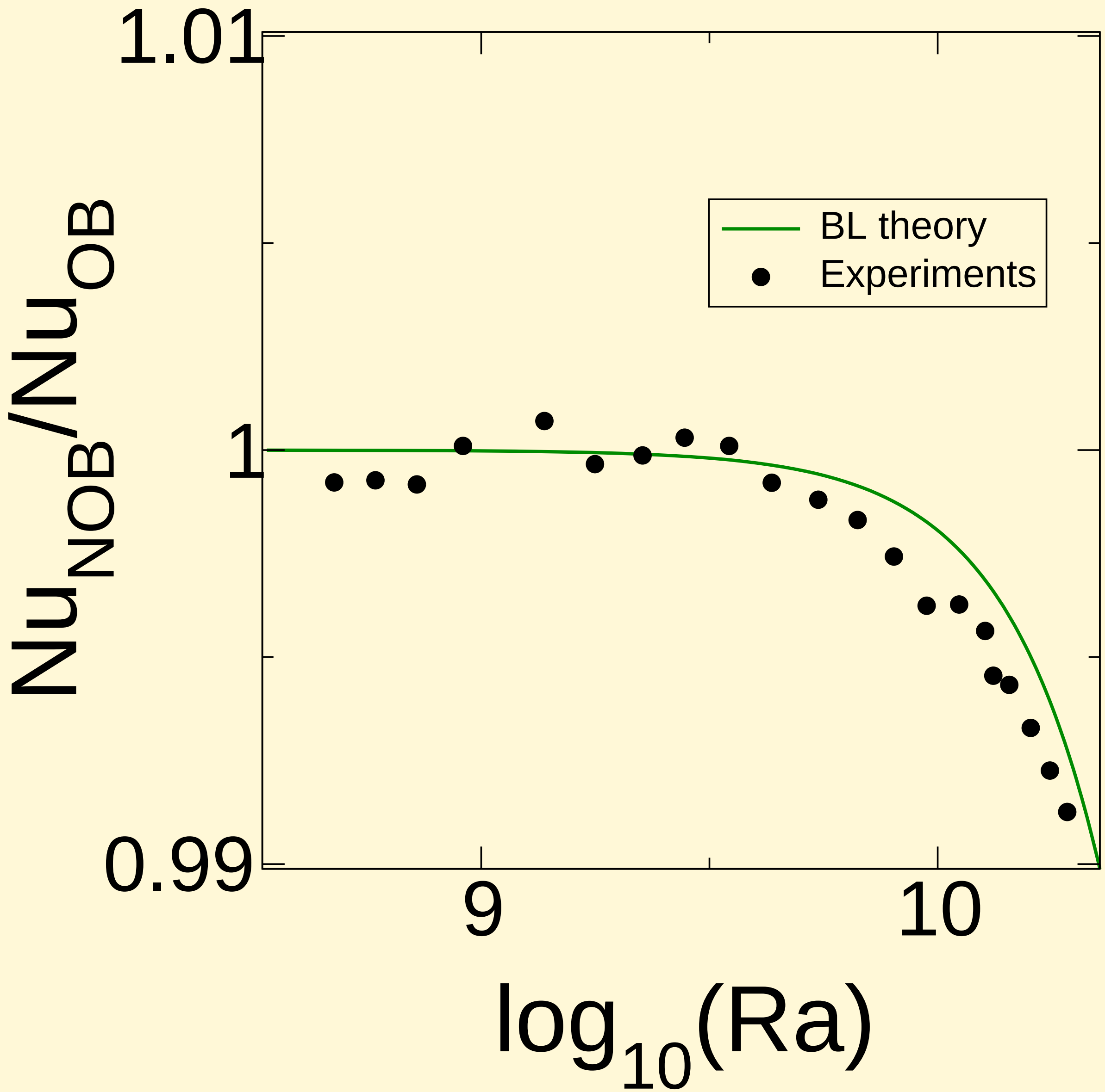
- In water experiments at $T_m = 40$ C and $\Delta \leq 40$ C:

$$\lambda_t^{sl} + \lambda_b^{sl} \approx 2\lambda_{OB}^{sl}$$

- Thus...

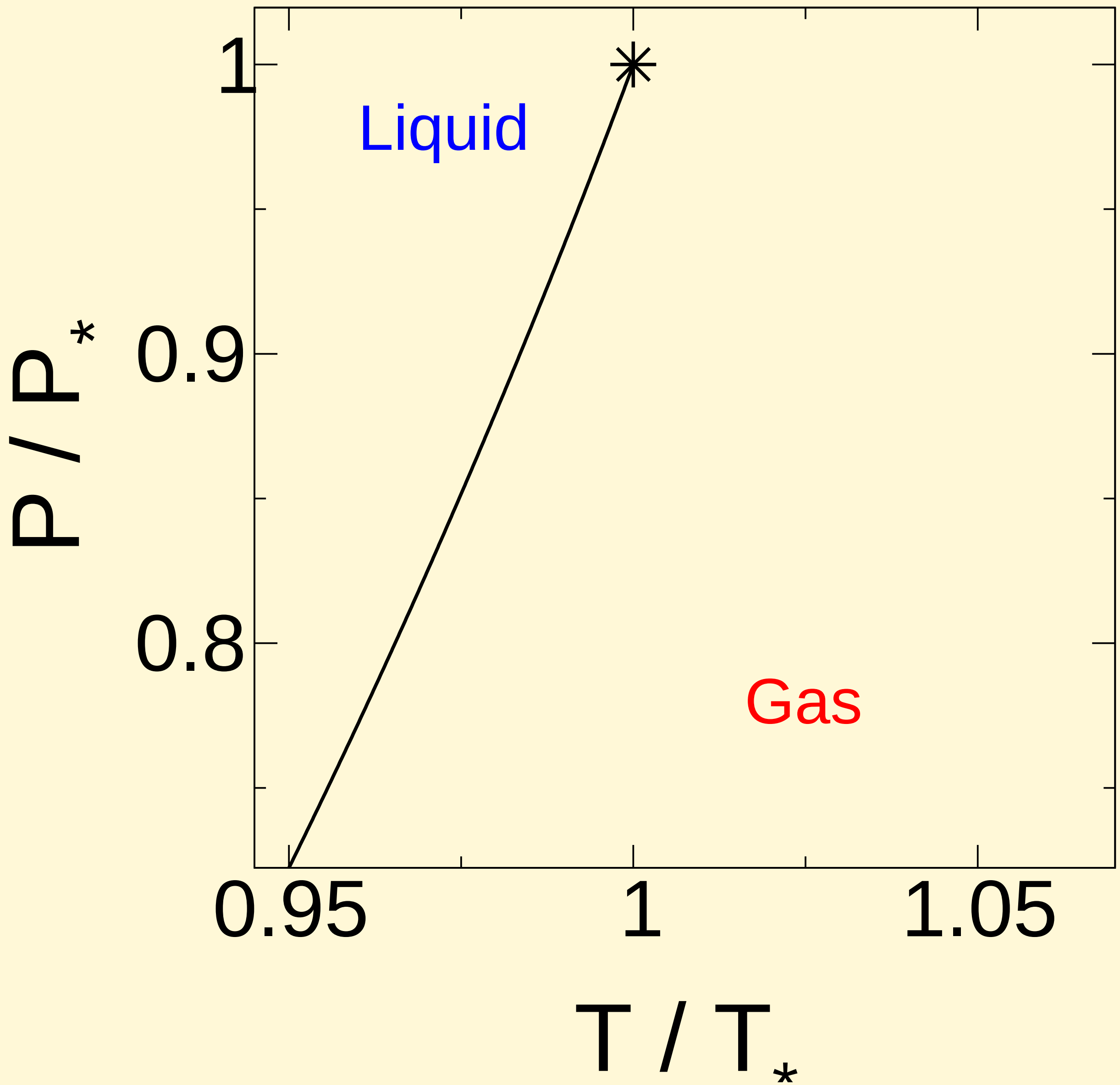
$$\frac{\text{Nu}_{NOB}}{\text{Nu}_{OB}} \approx \frac{\kappa_t \Delta_t + \kappa_b \Delta_b}{\kappa_m \Delta}$$

Nusselt number

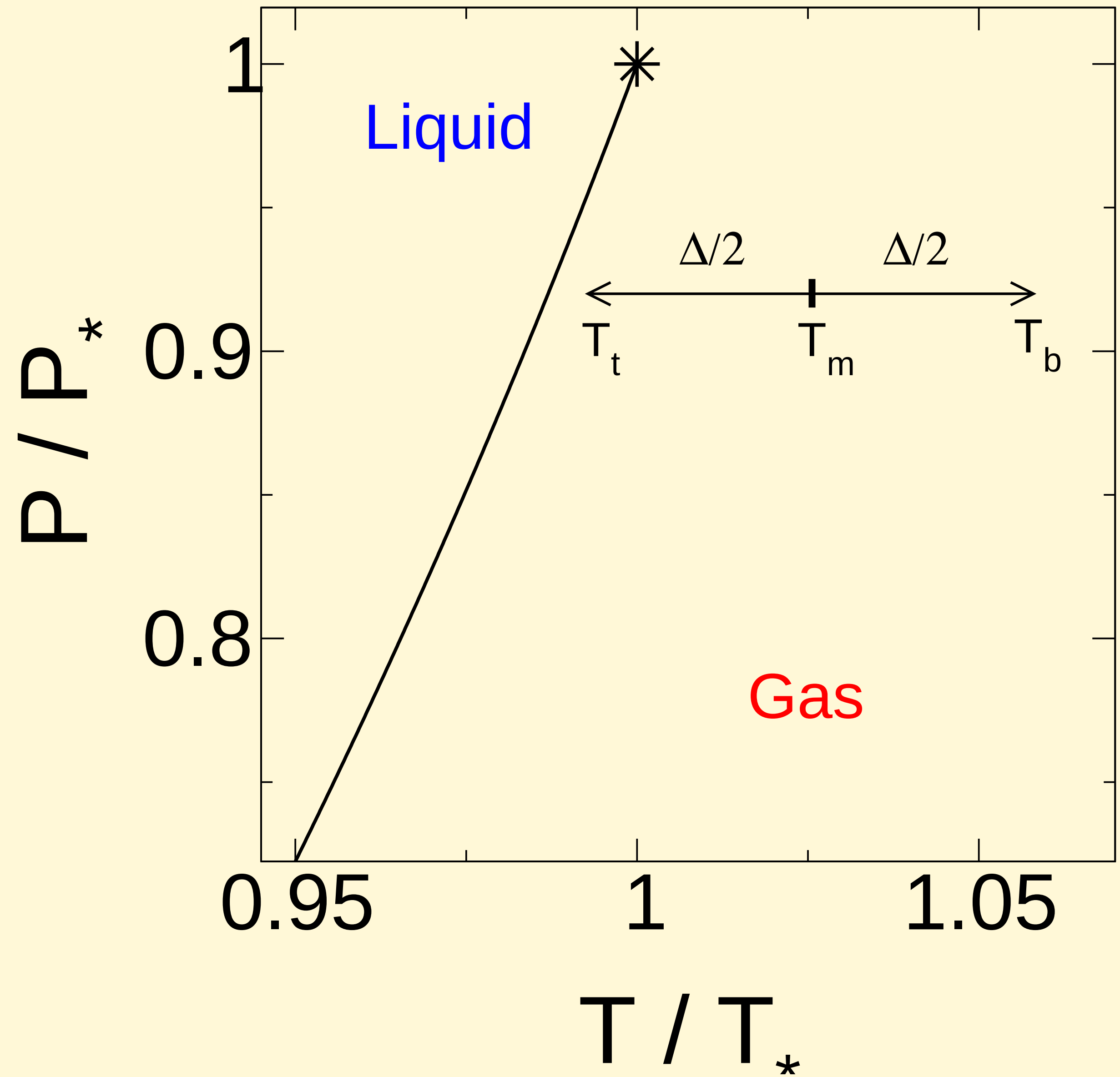


Ethane

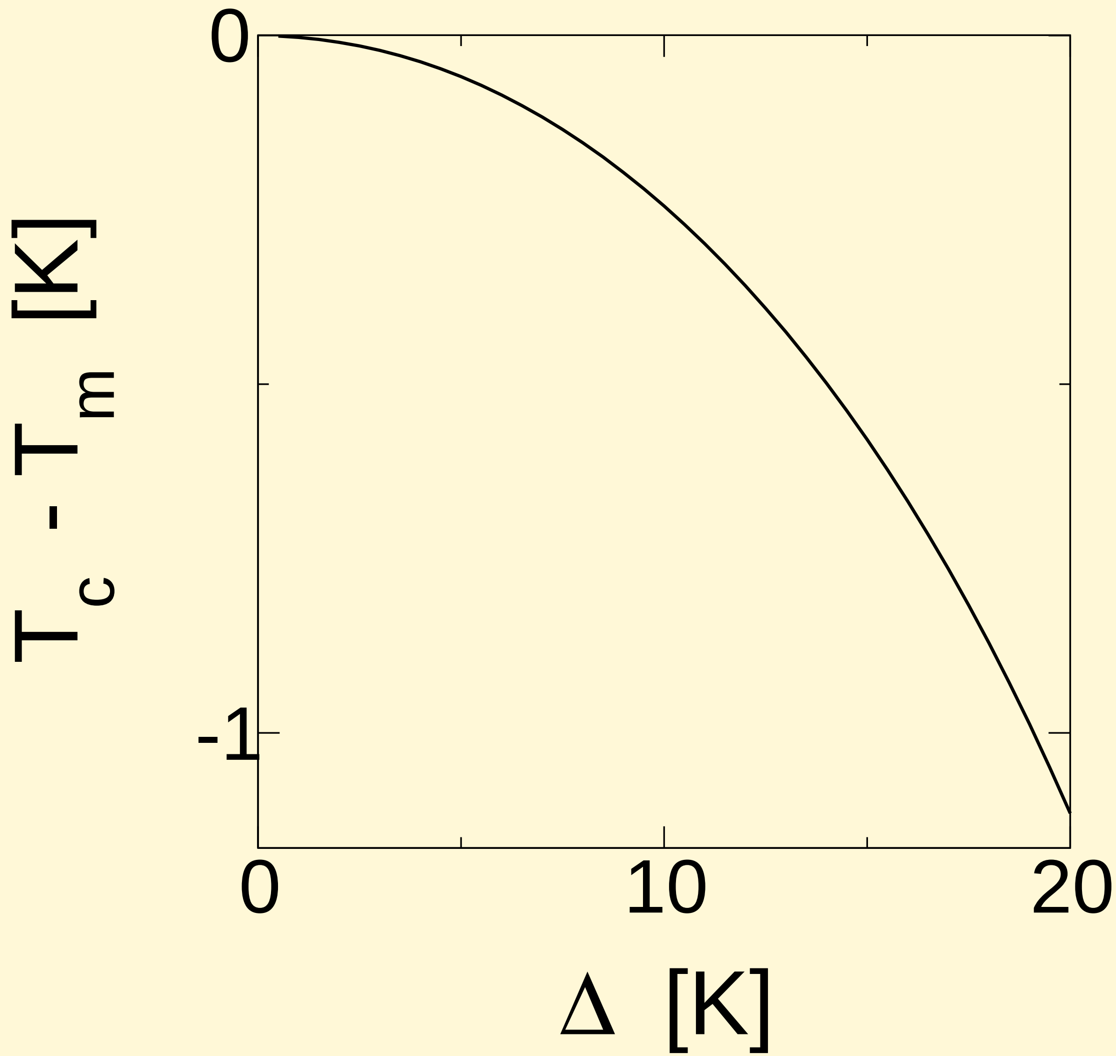
Phase diagram



Phase diagram

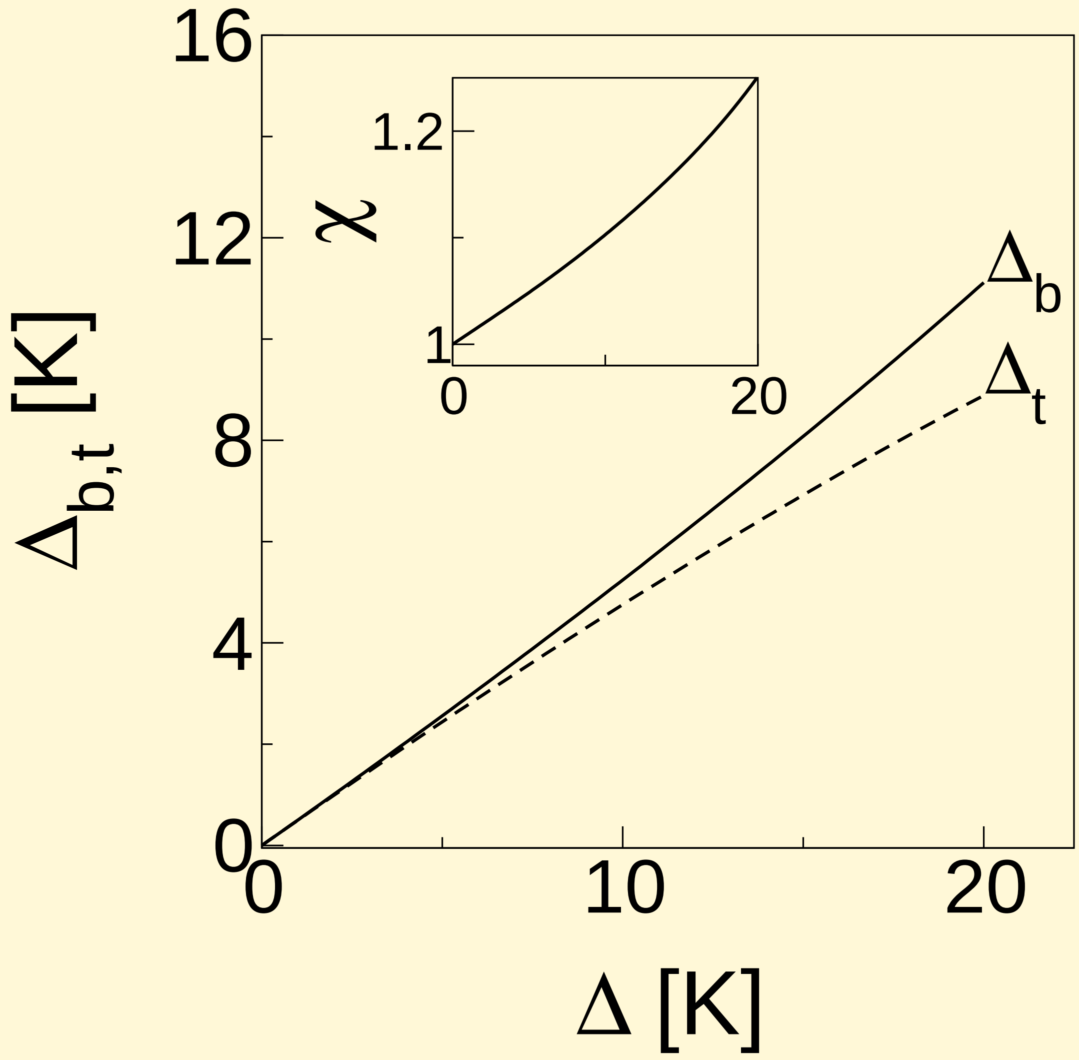


Center temperature

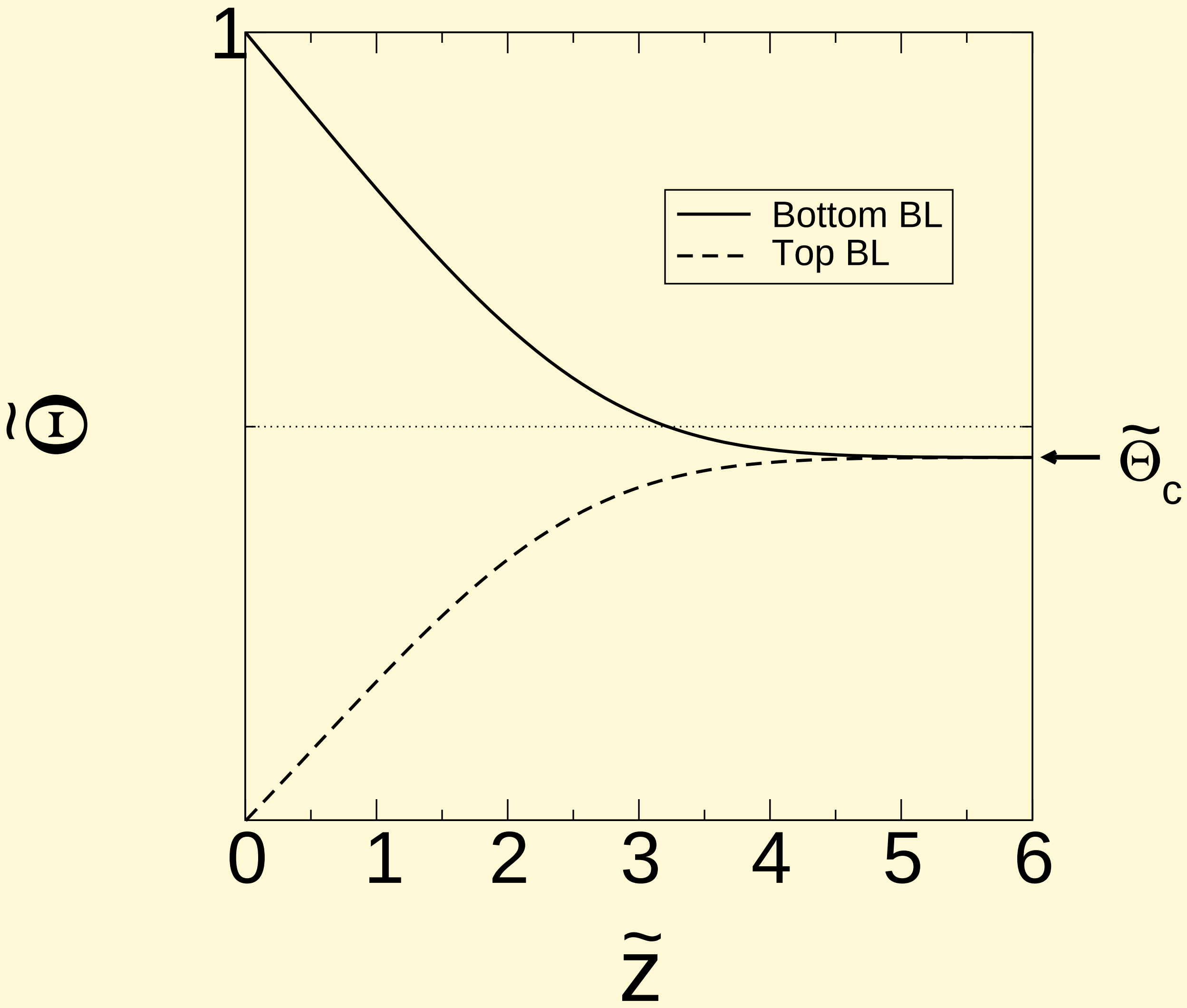


$T_m = 40^o\,C, \qquad P_m = 0.920 \cdot P_*$

Temperature drops

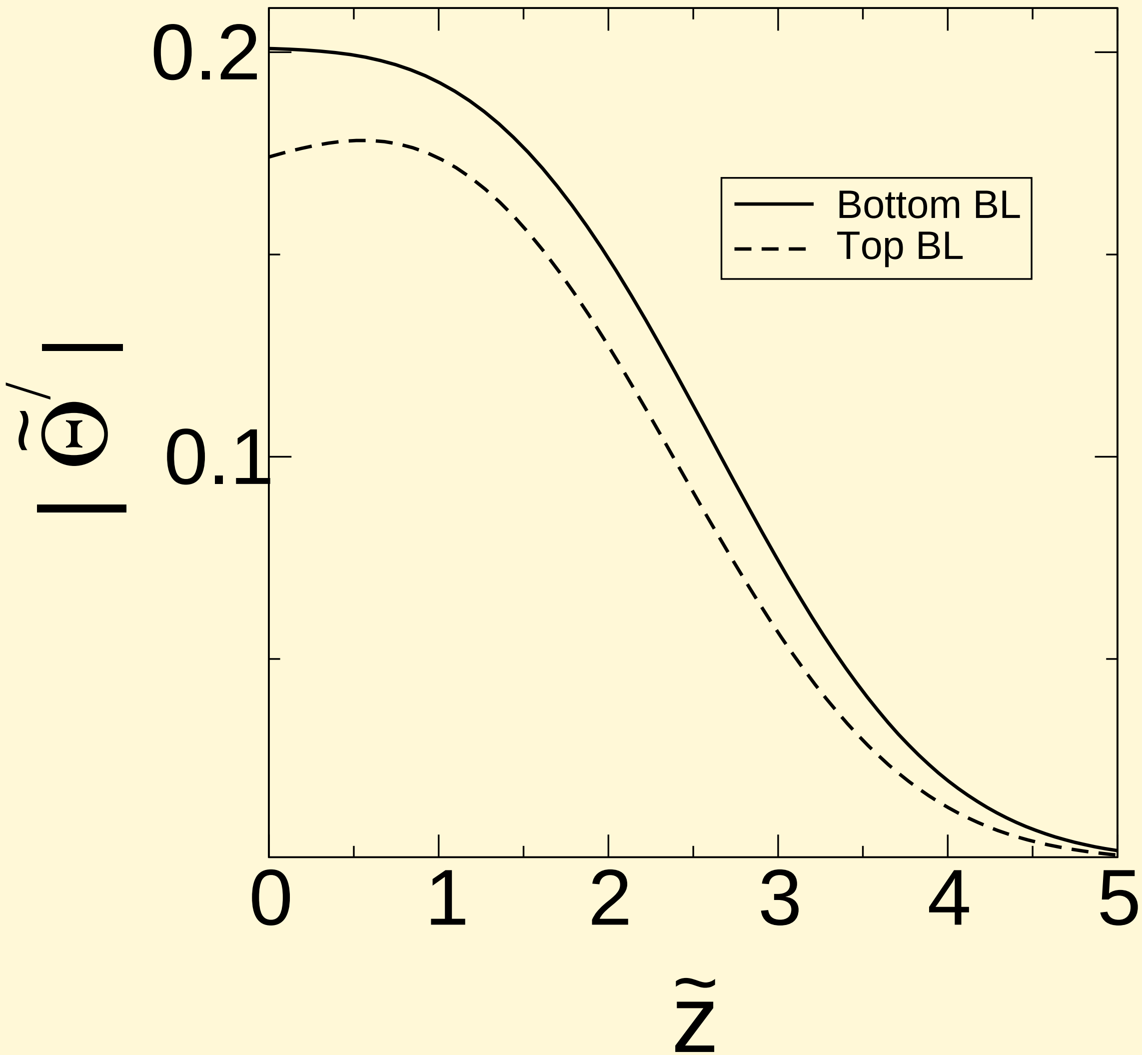


$T_m = 40^o\,C, \qquad P_m = 0.920 \cdot P_*$



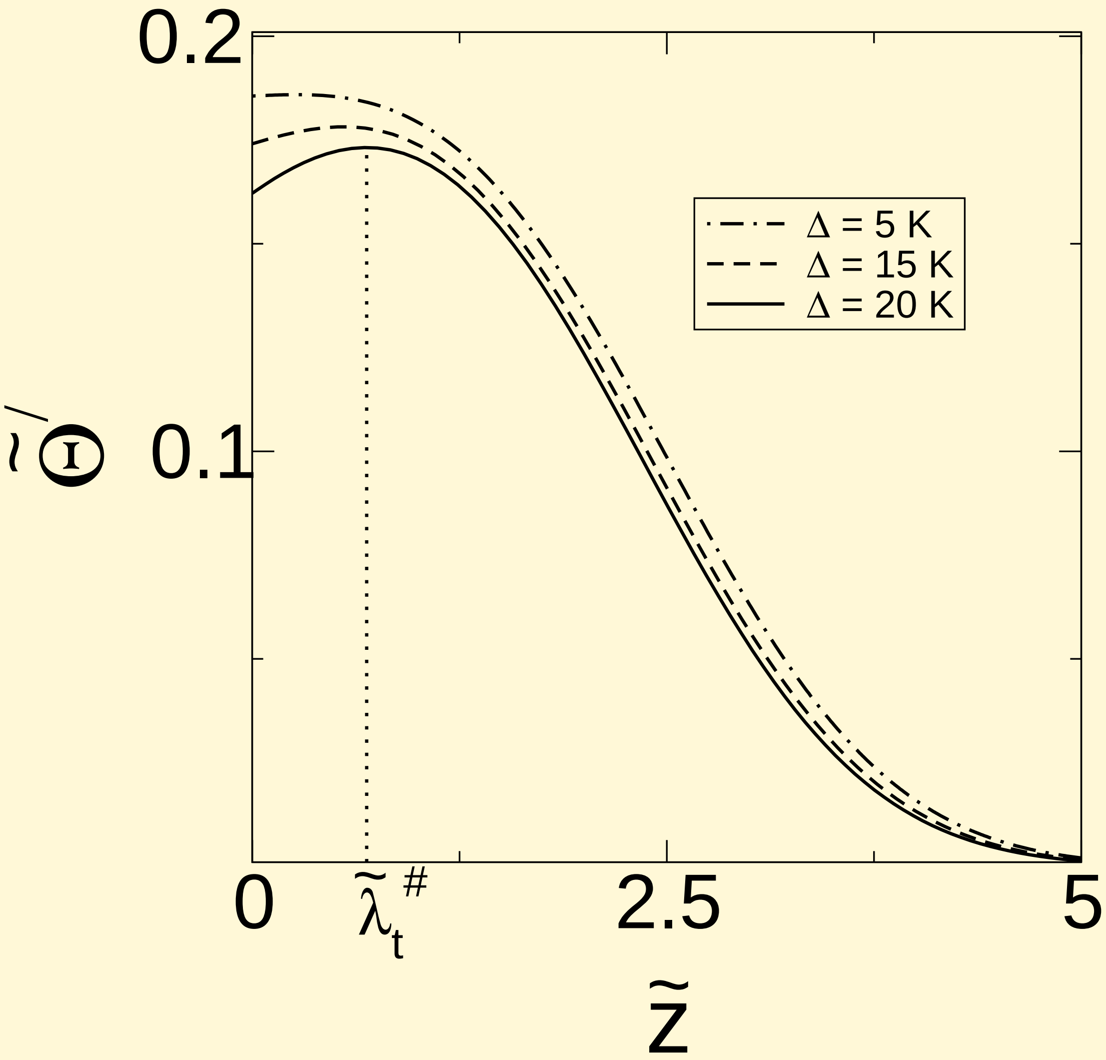
$T_m = 40^o\,C, \qquad P_m = 0.920 \cdot P_*, \qquad \Delta = 15^o\,C$

Temperature gradients across the BLs



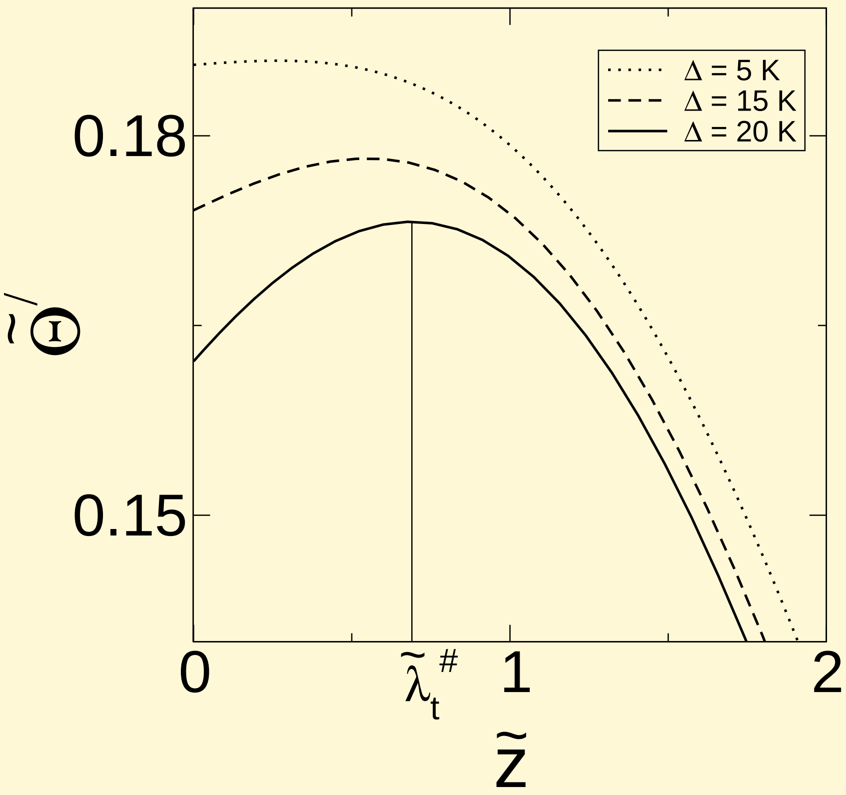
$T_m = 40^o\,C, \qquad P_m = 0.920 \cdot P_*, \qquad \Delta = 15^o\,C$

Temperature gradient across the top BL



$$T_m = 40^o\,C, \qquad P_m = 0.920 \cdot P_*$$

Why does $\tilde{\Theta}_t{}'$ decrease?



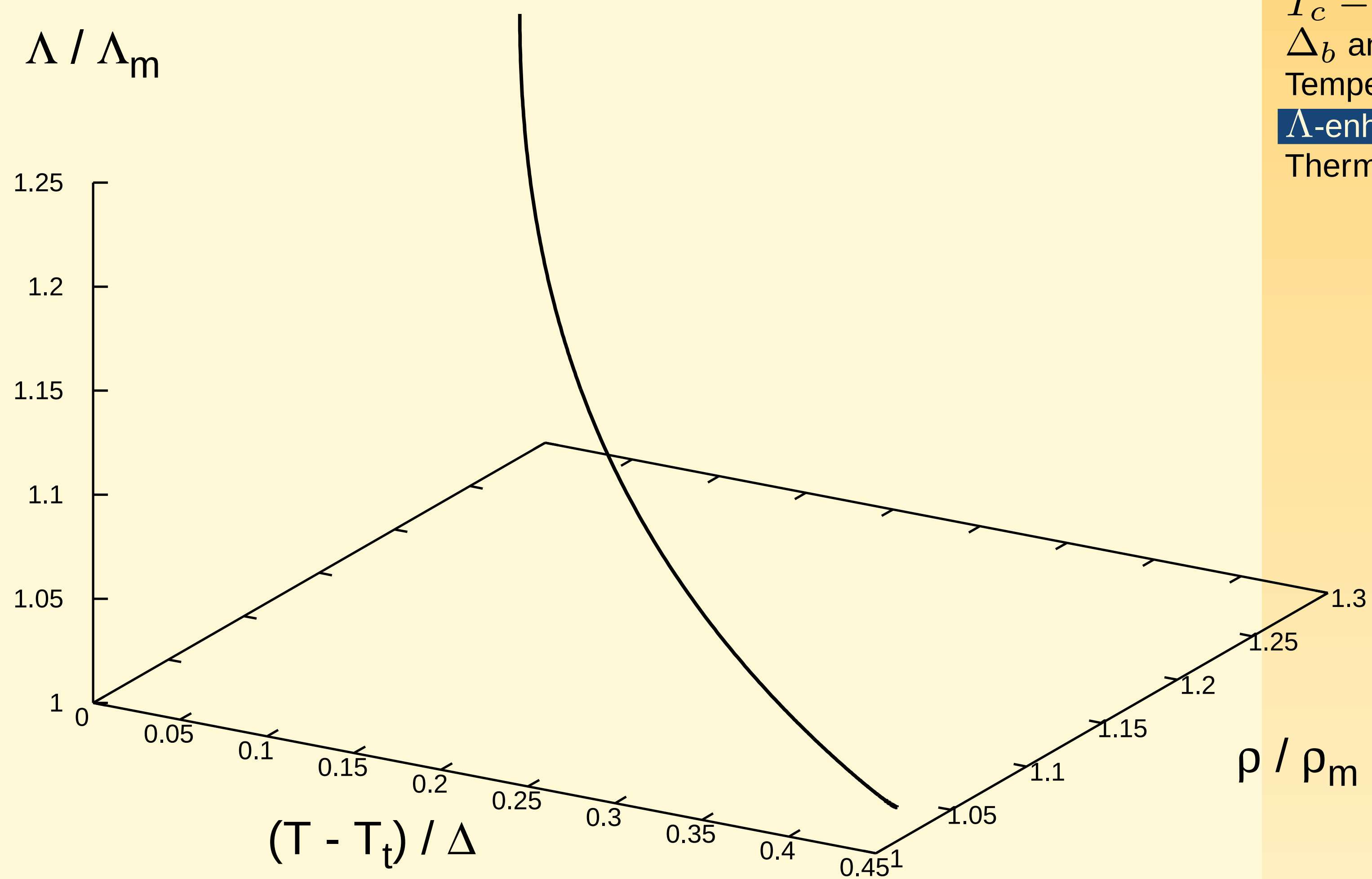
Heat fluxes:

$$\tilde{\Lambda}_t \, \tilde{\Theta}_t' = \tilde{\Lambda}_b \, \tilde{\Theta}_b'$$

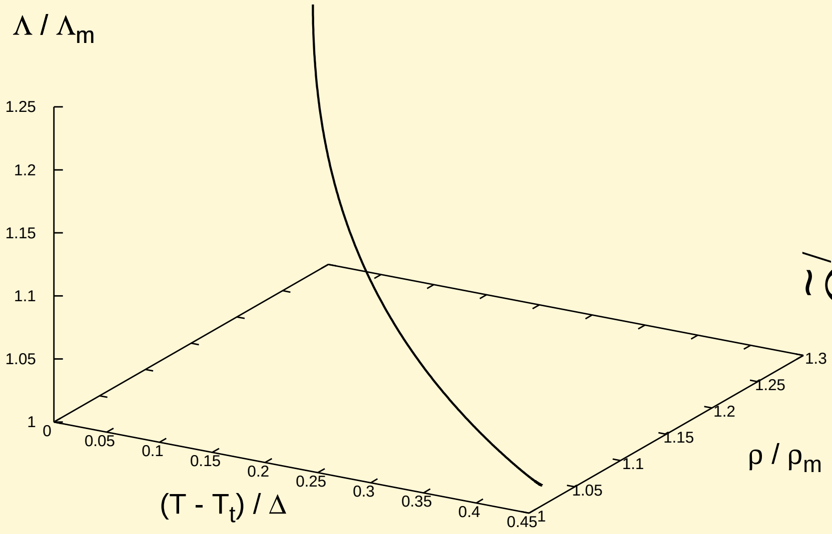
$$\tilde{\Theta}_t' = \frac{\tilde{\Lambda}_b}{\tilde{\Lambda}_t} \, \tilde{\Theta}_b'$$

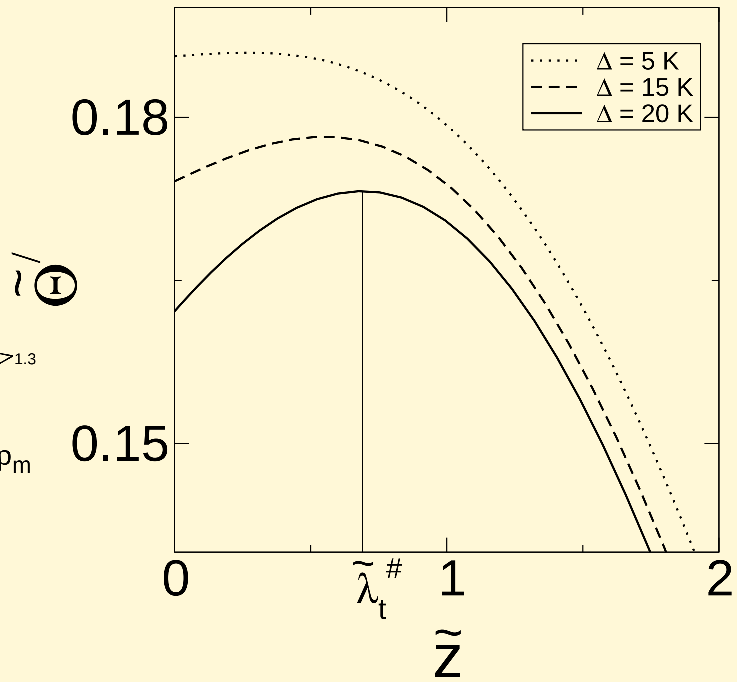
Thermal-conductivity enhancement

Near the critical point, Λ is enhanced.



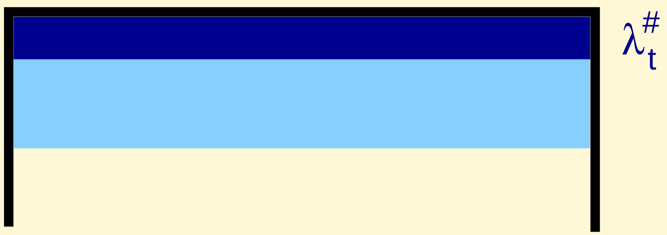
Thermal sublayer





$$\Lambda = \Lambda_{reg} + \Lambda_{sing}$$

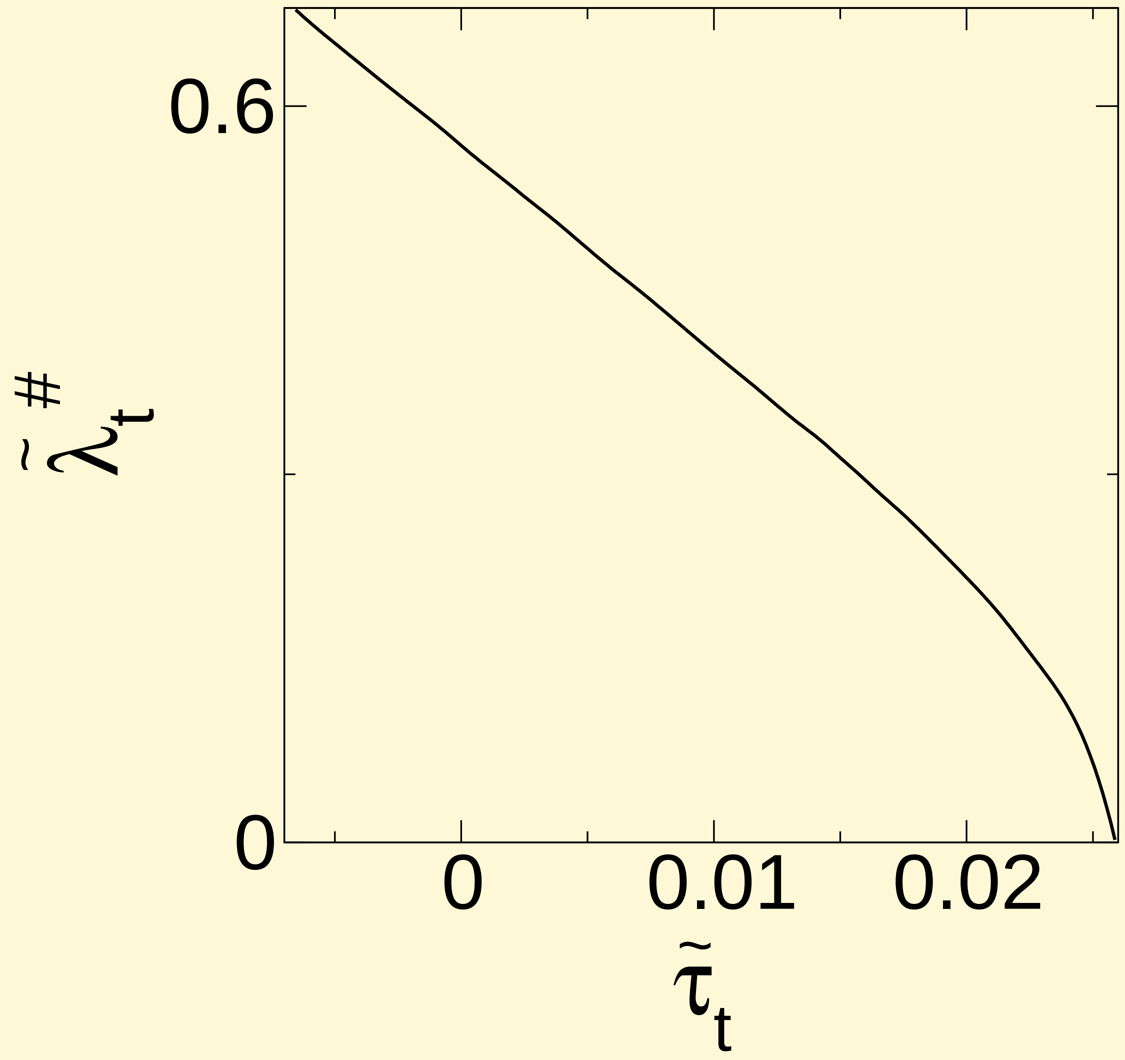
Λ_{sing} : singular part



Heat balance:

$$\tilde{\Lambda}\,\tilde{\Theta}' \;=\; \tilde{Q} - \langle \tilde{u}_z \tilde{\Theta} \rangle$$

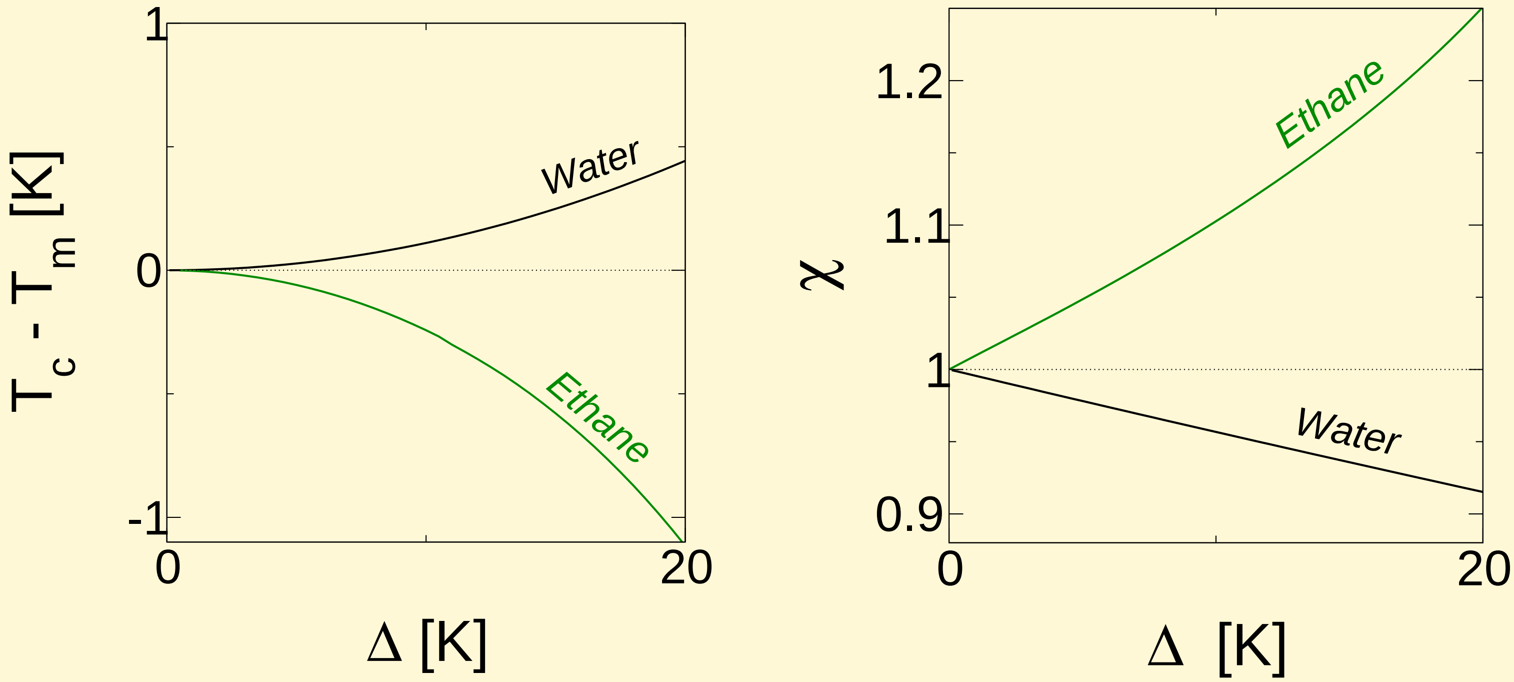
Thickness of the thermal sublayer



$T_m = 40^o\,C, \qquad P_m = 0.920 \cdot P_*, \qquad \tilde{\tau}_t = (T_t - T_*)/T_*$

Conclusion/Outlook

Conclusion

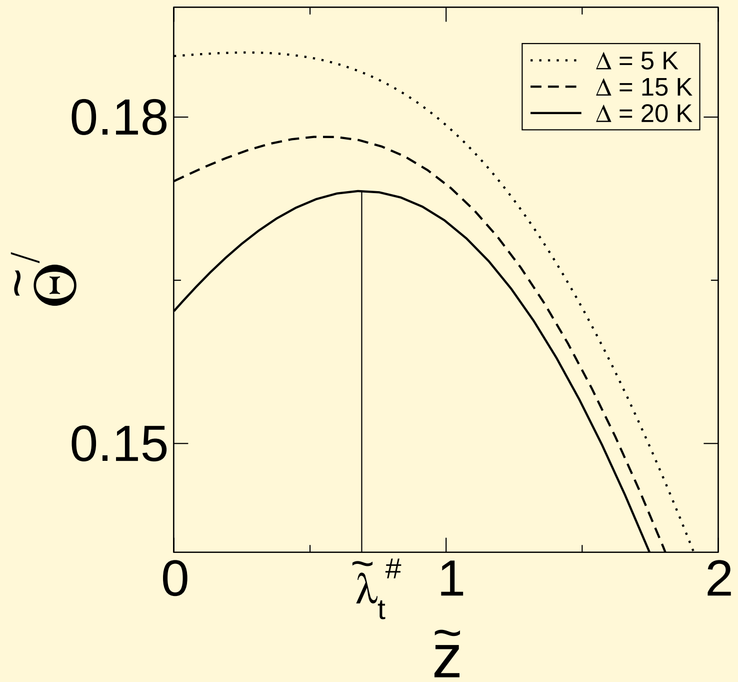


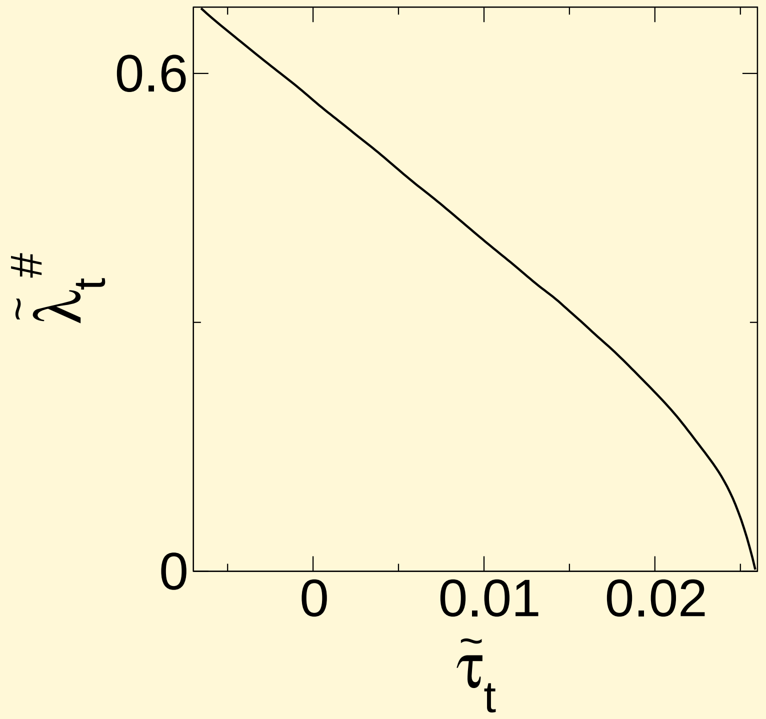
The symmetry breaking

$$T_c < T_m$$

in gaseous ethane is opposite to the case of liquids...

... the bottom BL becomes thicker than top one!



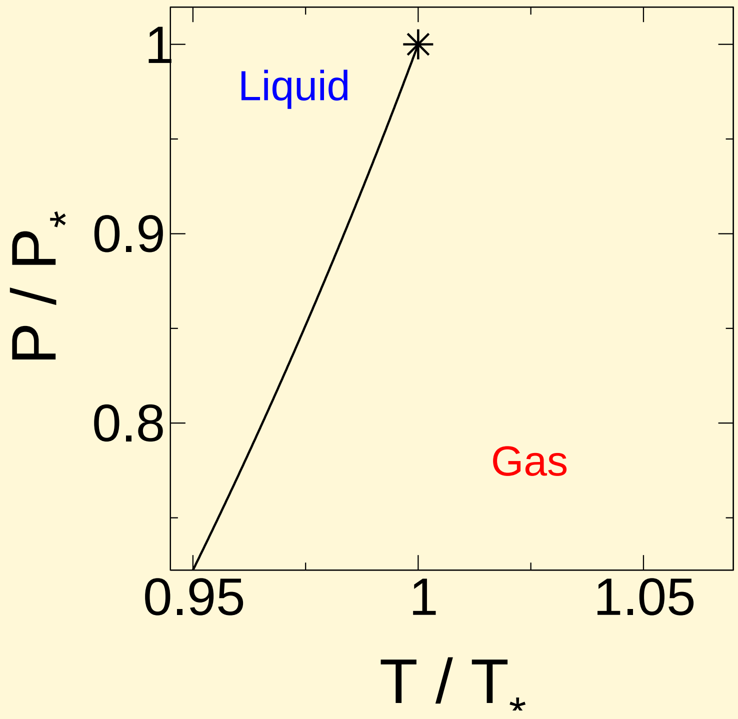


Near the critical point, the enhancement in thermal conductivity induces an inversion of the temperature gradient.

Outlook:

Characterize $\tilde{\lambda}_t^{\#}$ in terms of the critical exponents.

Outlook



Scan the phase diagram closer to the critical point.

... Compare our results with experimental measurements by Ahlers, Brown, and Funfschilling (UCSB)...

Acknowledgments

Stichting FOM

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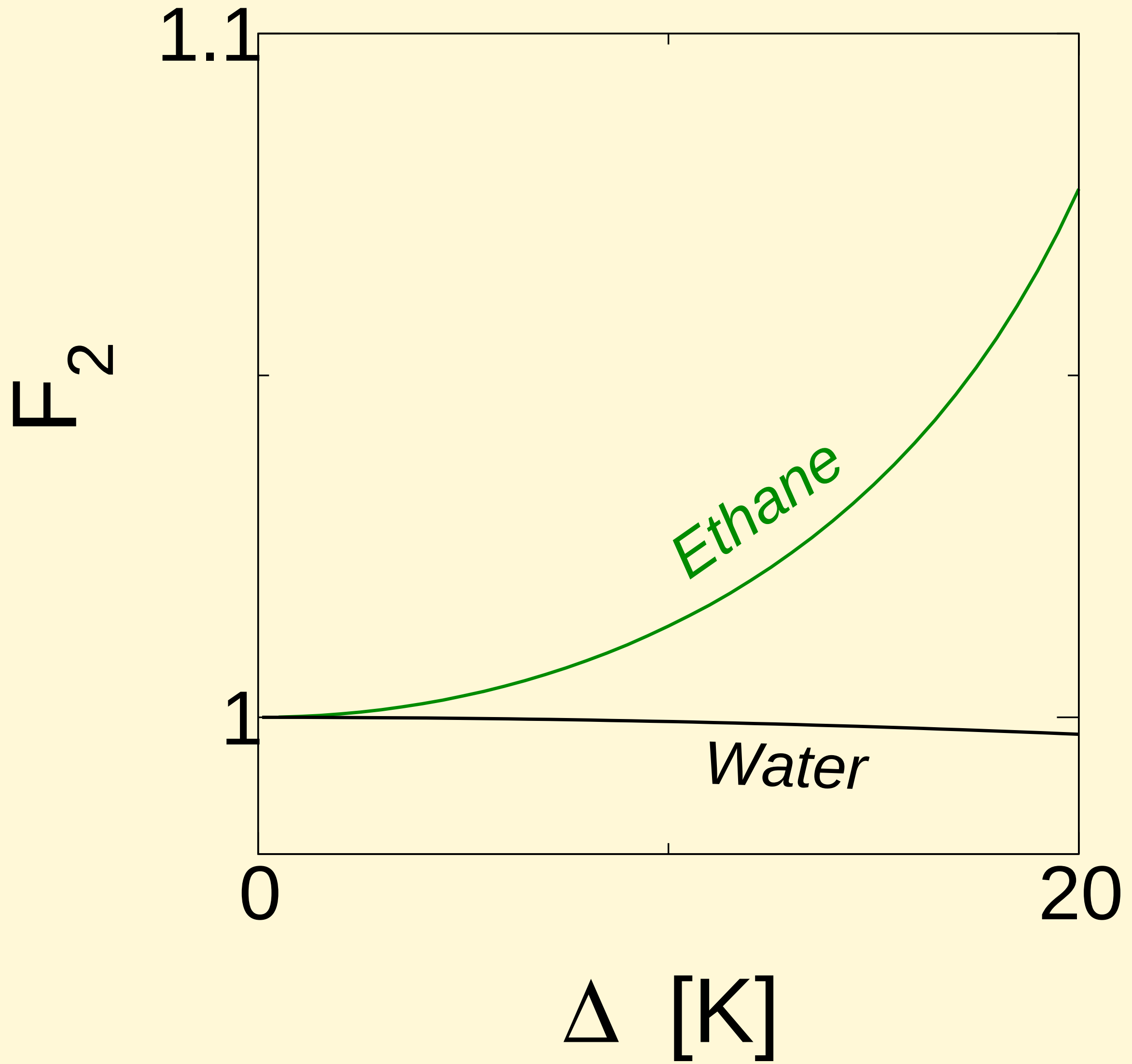
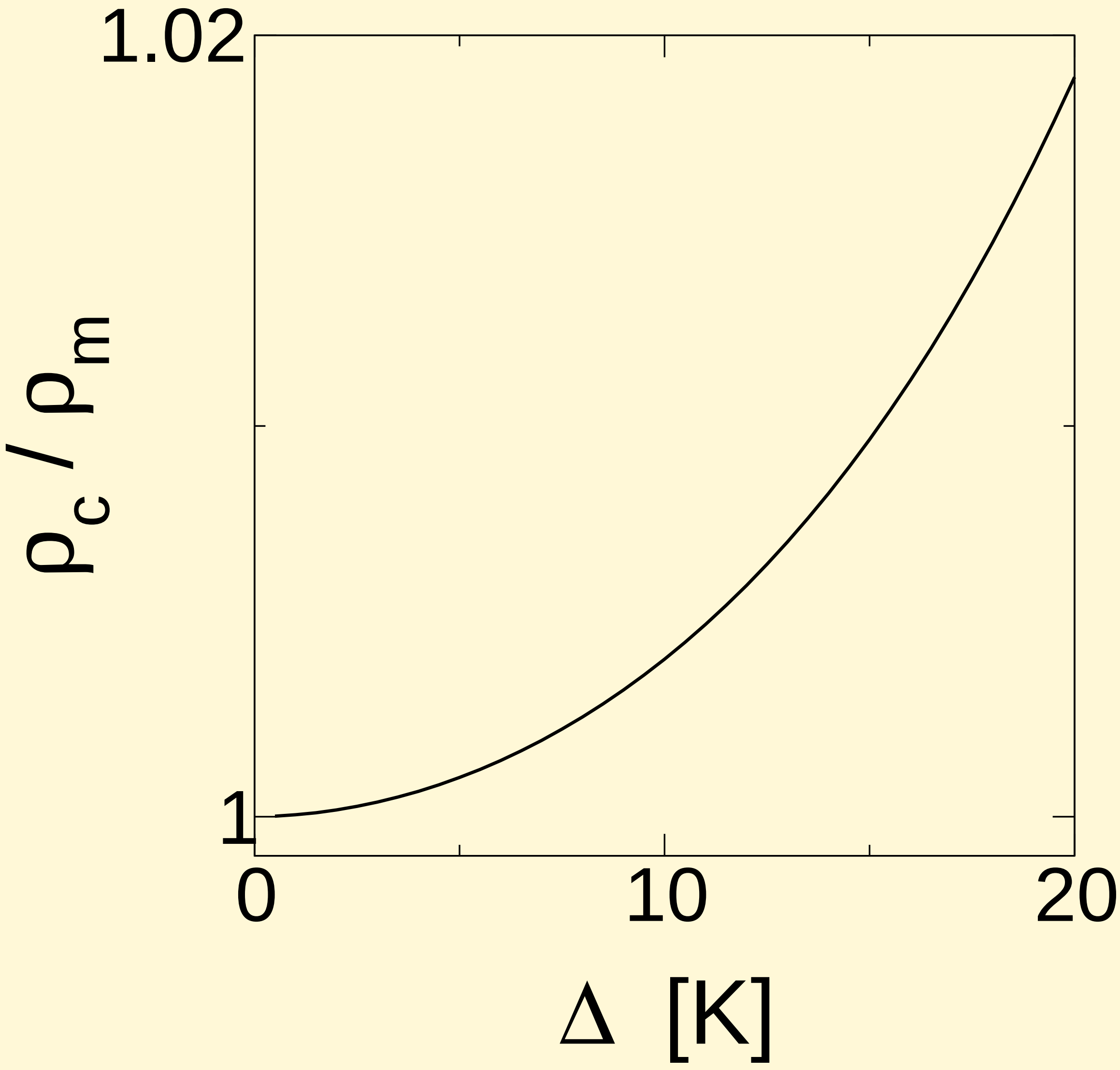
Appendix

Ethane: Nusselt ratio (F_2)

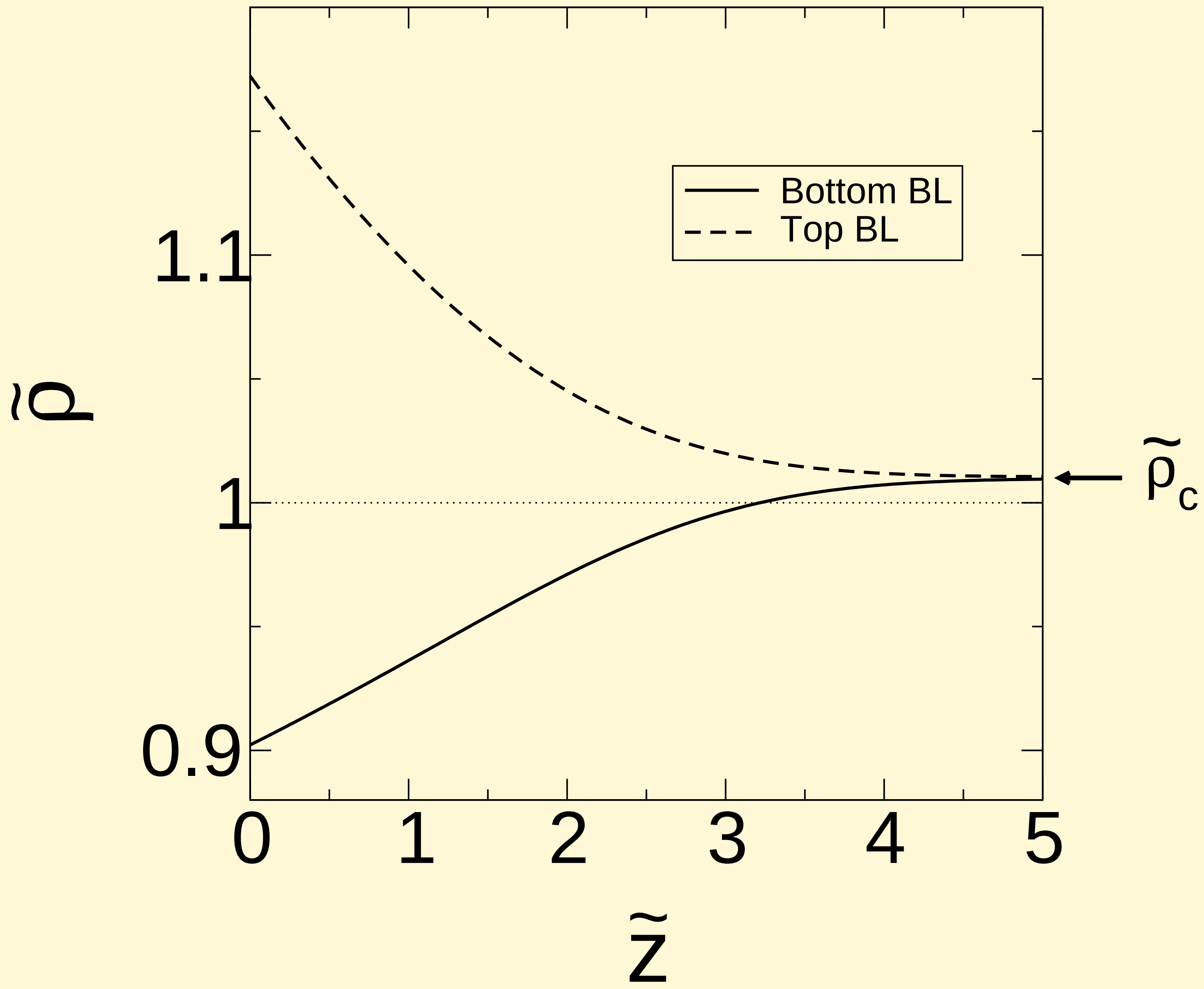


Center density



$T_m = 40^o\,C, \qquad P_m = 0.920 \cdot P_*, \qquad \Delta = 15^o\,C$

Density profile



$T_m = 40^o\,C, \qquad P_m = 0.920 \cdot P_*, \qquad \Delta = 15^o\,C$