The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR1777/4

School on Nonlinear Differential Equations

(9 - 27 October 2006)

Introduction to critical point theory

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Chapter 1

The Mountain Pass Theorem

1. Function Spaces

We recall the definition and some properties of some function spaces.

Definition 1.1. Let $\Omega \subset \mathbb{R}^N$, Ω open. We denote by $C_c^\infty(\Omega)$ the set of all C^∞ real valued function with compact support in Ω. We will also assume that the reader is familiar with the Sobolev Spaces $H^1(\Omega)$, $H_0^1(\Omega)$. We recall that, for $N \geq 3$ and $2^* = 2N/(N-2)$, the space

$$\mathcal{D}^{1,2}(\mathbb{R}^N) = \left\{ u \in L^{2^*}(\mathbb{R}^N) \mid \nabla u \in L^2(\mathbb{R}^N) \right\}$$

with scalar product and norm

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v, \qquad \left(\int_{\mathbb{R}^N} |\nabla u|^2 \right)^{1/2}$$

is a Hilbert space. We also let $\mathcal{D}_0^{1,2}(\Omega)$ be the closure of $C_c^\infty(\Omega)$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. It follows that $H_0^1(\Omega) \subset \mathcal{D}_0^{1,2}(\Omega)$. Poincaré inequality (see theorem 1.4 below) implies that $H_0^1(\Omega) = \mathcal{D}_0^{1,2}(\Omega)$ if $|\Omega| < +\infty$.

Theorem 1.2 (Sobolev imbedding theorem)**.** *The following imbeddings are continuous:*

$$H^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N), \qquad 2 \leq p < \infty,\ N = 1, 2, \tag{1.1}$$

$$H^1(\mathbb{R}^N) \subset L^p(\mathbb{R}^N), \qquad 2 \leq p \leq 2^*,\ N \geq 3, \tag{1.2}$$

$$\mathcal{D}^{1,2}(\mathbb{R}^N) \subset L^{2^*}(\mathbb{R}^N), \qquad 2 \leq p \leq 2^*,\ N \geq 3. \tag{1.3}$$

In particular we have the following:

$$S = \inf\Big\{ \int_{\mathbb{R}^N} |\nabla u|^2 \;\Big|\; u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} u^{2^*} = 1 \Big\} > 0.$$

Theorem 1.3 (Rellich theorem)**.** *If $|\Omega| < +\infty$ the imbeddings*

$$H_0^1(\Omega) \subset L^p(\Omega), \qquad 1 \le p < \infty,\ N = 1, 2,$$
$$H_0^1(\Omega) \subset L^p(\Omega), \qquad 1 \le p < 2^*,\ N \ge 3,$$

are compact.

Theorem 1.4 (Poincaré inequality)**.** *Assume $|\Omega| < +\infty$. Then*

$$\lambda_1(\Omega) = \inf\Big\{ \int_\Omega |\nabla u|^2 \;\Big|\; u \in H_0^1(\Omega),\ \int_\Omega u^2 = 1 \Big\} > 0 \tag{1.4}$$

2. Differentiability

Let us recall some notion in differential calculus in Banach spaces (see [**1, 12, 13**]).

Definition 2.1. Let $f\colon U \to \mathbb{R}$, U open in the Banach space V. We say that f is Gateaux-differentiable at $x_0 \in U$ if there exist $g \in V'$ (the dual of V) such that for all $h \in V$

$$\lim_{t\to 0} \frac{f(x_0 + th) - f(x_0) - t\langle g, h\rangle}{t} = 0. \tag{2.1}$$

We also say that f is Frechet-differentiable at x_0 if there exist $g \in V'$ such that

$$\lim_{\|h\|\to 0} \frac{f(x_0 + h) - f(x_0) - \langle g, h\rangle}{\|h\|} = 0$$

If f is Gateaux or Frechet-differentiable at x_0 we write $df(x_0) = g$. If f is Frechet-differentiable at all points $x \in U$, and the map $x \mapsto df(x)$ is continuous, we write $f \in C^1(U)$.

Clearly Frechet-differentiability implies Gateaux-differentiability.

If f is Gateaux differentiable at x_0 and $df(x_0) = 0$ we say that x_0 is a critical point or stationary point and that $c = f(x_0)$ is a critical value.

If V is a Hilbert space and f is Gateaux differentiable at x_0, we define the gradient $\nabla f(x_0) \in V$ of f as the element such that

$$(\nabla f(x_0) \,|\, h) = \langle df(x_0), h\rangle \qquad \text{for all } h \in V.$$

Proposition 2.2. *$f \in C^1(U)$ if f is Gateaux-differentiable in U and $x \mapsto df(x)$ is continuous.*

Definition 2.3. Let $\Omega \subset \mathbb{R}^N$. We say that $g\colon \Omega \times \mathbb{R}$ satisfy the Carathéodory condition if

- For all $s \in \mathbb{R}$ the function $x \mapsto g(x,s)$ is measurable;
- for almost all $x \in \Omega$ the function $s \mapsto g(s,x)$ is continuos.

Proposition 2.4. *If $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory condition, then $x \mapsto g(x,u(x))$ is measurable for all measurable $u\colon \Omega \to \mathbb{R}$.*

Proof. It is clear if u is a simple, measurable function. The general case follows taking a sequence u_n of simple, measurable functions which converge to u almost everywhere. $\square$

Proposition 2.5. *Let g satisfy the Carathéodory condition and, for some $p,q \geq 1$ and $a(x) \in L^q(\Omega)$,*

$$|g(s,x)| \leq a(x) + c|s|^{p/q}.$$

Then the Nemitskii operator

$$g_{\#}\colon L^p(\Omega) \to L^q(\Omega) \qquad (g_{\#}u)(x) = g(x,u(x))$$

is continuous.

Proof. From $|g(x,u(x))|^q \leq |a(x)+c|u(x)|^{p/q}|^q \leq 2^{q-1}|a(x)|^q + 2^{q-1}c|u(x)|^p \in L^1(\Omega)$ we deduce that $g(x,u(x)) \in L^q(\Omega)$. Take $u_n \to u$ in $L^p(\Omega)$. There is a subsequence u_{n_k} and a function $\bar{g} \in L^p(\Omega)$ such that, almost everywhere, $u_{n_k}(x)$ converges to $u(x)$ and $|u_{n_k}(x)| \leq \bar{g}(x)$ (see, for example [**3, 13**]).

Then $g(x,u_{n_k}(x)) \to g(x,u(x))$ almost everywhere and, since

$$|g(x,u_{n_k}(x)) - g(x,u(x))|^q \leq 2^q(|a(x)| + c\bar{g}(x)^{p/q})^q \in L^1(\Omega)$$

the result follows by dominated convergence. $\square$

Proposition 2.6. *Let $1 < p < \infty$ and $\Omega \subset \mathbb{R}^N$. Assume $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the Carathéodory condition and*

$$|g(x,s)| \leq a(x) + c|s|^{p/q} \tag{2.2}$$

for some $a \in L^q(\Omega)$ and $1 = \frac{1}{q} + \frac{1}{p}$. Let

$$G(x,s) = \int_0^s g(x,t)\,dt \qquad (x,s) \in \Omega \times \mathbb{R}.$$

Then the functionals $\psi\colon L^p(\Omega) \to \mathbb{R}$ defined as

$$\psi(u) = \int_\Omega G(x,u(x))\,dx,$$

is of class $C^1(L^p(\Omega),\mathbb{R})$ and

$$\langle d\psi(u), v\rangle = \int_\Omega g(x,u(x))v(x)\,dx.$$

Proof. We have to show that (2.1) holds. For all $v \in L^p(\Omega)$, for all $x \in \Omega$ and for all $t \in [-1,1]$ by the mean value theorem there is $\lambda \in (0,1)$ such that

$$G(x, u(x) + tv(x)) - G(x, u(x)) = g(x, u(x) + \lambda t v(x)) t v(x).$$

Since

$$\begin{aligned} |g(x, u(x) + \lambda t v(x))||tv(x)| &\leq \big(a(x) + c|u(x) + \lambda t v(x)|^{p/q}\big)|v(x)| \\ &\leq \big(a(x) + c(|u(x)| + |v(x)|)^{p/q}\big)|v(x)| \end{aligned}$$

and

$$\begin{aligned} &\big(a(x) + c(|u(x)| + |v(x)|)^{p/q}\big)^q \\ &\qquad \leq 2^{q-1}\big(a(x)^q + 2^{p-1}c|u(x)|^p + 2^{p-1}c|v(x)|^p\big) \in L^1(\Omega) \end{aligned}$$

we have, by Hölder's inequality, that

$$\begin{aligned} &|g(x, u(x) + \lambda t v(x))||tv(x)| \\ &\qquad \leq 2^{q-1}\big(a(x)^q + 2^{p-1}c|u(x)|^p + 2^{p-1}c|v(x)|^p\big)|v(x)|. \in L^1(\Omega) \end{aligned}$$

From dominated convergence we then have

$$\lim_{t \to 0} \int_\Omega \frac{G(x, u(x) + tv(x)) - G(x, u(x))}{t} = \int_\Omega g(x, u)v$$

and the Gateaux-differentiability follows.

To prove that ψ is Frechet-differentiable, let us show that $u \mapsto g(x, u(x))$ is continuous from $L^p(\Omega)$ to $L^q(\Omega)$. This is a simple consequence of proposition 2.5.

□

Remark 2.7. It is possible to prove that ψ is of class C^2 in L^p with $p > 2$ when $g(x,s)$ is differentiable with respect to s and $g_s(x,s)$ satisfies the Carathéodory condition and the growth condition

$$|g_s(x,s)| \leq a(x) + b|s|^{p-2}$$

for some $a \in L^{p/(p-2)}$ and $b > 0$. ψ is *not* C^2 in L^2 unless $g(x,s)$ is linear in s.

A direct consequence of Sobolev imbedding 1.2 is

Corollary 2.8. *Let* $1 < p \leq 2^*$ *if* $N \geq 3$ *(*$1 < p < +\infty$ *if* $N = 1, 2$*). Then* $\psi \in C^1(H_0^1(\Omega); \mathbb{R})$*.*

If $N \geq 3$ *and* $p = 2^*$*, then* $\psi \in C^1(\mathcal{D}_0^{1,2}(\Omega); \mathbb{R})$*.*

3. Minimization

Let us recall some results on minimization.

Theorem 3.1. *Let V be a Hausdorff topological space, and $f\colon V \to \mathbb{R} \cup \{+\infty\}$ be such that*

$$\text{(BC)} \qquad \begin{array}{l} \textit{for all } \alpha \in \mathbb{R} \textit{ the set } K_\alpha = \Big\{ u \in V \,\Big|\, f(u) \le \alpha \Big\} \\ \textit{is compact (or sequentially compact).} \end{array}$$

Then

(a) $\beta = \inf_V f > -\infty$;

(b) *Exists* $x_0 \in V$ *such that* $f(x_0) = \beta$.

Remark 3.2. Let us note that

(1) (BC) implies that f is lower semi-continuous (sequentially lower semi-continuous).

(2) the conclusion of theorem 3.1 hold if f is lower semi-continuous (sequentially lower semi-continuous) and K_α is compact (sequentially compact) for some $\alpha \in \mathbb{R}$.

In case we are dealing with functionals defined in a Banach space, the following consequence of 3.1 is useful

Theorem 3.3. *Let V be a reflexive Banach space, and let X be a weakly closed subset of V. Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be such that*

(1) *f is coercive, that is $f(u_n) \to +\infty$ whenever $u_n \in X$, $\|u_n\| \to +\infty$;*

(2) *f is sequentially weakly lower semicontinuous, that is $u_n \in X$, $u_n \rightharpoonup u$ implies $f(u) \le \liminf_{n\to\infty} f(u_n)$.*

Then

(a) $\beta = \inf_X f > -\infty$;

(b) *Exists* $x_0 \in X$ *such that* $f(x_0) = \beta$.

Whenever f is differentiable, local minima are critical points of f. Indeed

Theorem 3.4. *Let V be a Banach space, and assume*

(1) *x_0 is a local minimum for f;*

(2) *f is Gateaux differentiable in x_0.*

Then $df(x_0) = 0$.

In the assumptions of theorem 3.1, every minimizing sequence converges. The following theorem, and the related corollary, shows that it is possible to find minimizing sequences with additional properties (and also gives information about minimizing sequences of bounded, non-coercive functionals).

Theorem 3.5 (Ekeland's Variational Principle, see, for example, [**12**])**.** *Let $f\colon M \to \mathbb{R} \cup \{+\infty\}$, where M is a complete metric space, and assume*

(1) *f is lower semicontinuous;*
(2) *f is bounded from below: $\beta = \inf_M f > -\infty$;*
(3) *$f \not\equiv +\infty$.*

Then for all ϵ, $\delta > 0$ and for all $u \in M$ such that $f(u) \leq \beta + \epsilon$ there exist $v \in M$ such that

(a) *$f(v) \leq f(u)$;*
(b) *$\operatorname{dist}(u, v) \leq \delta$;*
(c) *$f(v) < f(w) + \frac{\epsilon}{\delta} \operatorname{dist}(v, w)$ for all $w \in M$, $w \neq v$.*

A rather direct consequence of the above theorem is

Theorem 3.6. *Let V be a Banach space, $f \in C^1(V)$, $\beta = \inf_M f > -\infty$. Then there exist a sequence $u_n \in V$ such that*

(a) *$f(u_n) \to \beta$;*
(b) *$df(u_n) \to 0$ in V'.*

as $n \to +\infty$.

We will see later, in section 4, that sequences satisfying (a) and (b) of theorem 3.6 play an important rôle in critical point theory.

4. The Palais-Smale condition

Definition 4.1. Let V be a Banach space, $f \in C^1(V)$. We say that $u_n \in V$ is a *Palais-Smale* sequence at level β (shortly a $(\mathrm{PS})_\beta$ sequence), if

(1) $f(u_n) \to \beta$;
(2) $df(u_n) \to 0$.

We say that f satisfies the $(\mathrm{PS})_\beta$ condition if every $(\mathrm{PS})_\beta$ sequence has a converging subsequence.

We say that f satisfies the (PS) condition if it satisfies the $(\mathrm{PS})_\beta$ condition for all $\beta \in \mathbb{R}$.

Remark 4.2. If V is a finite dimensional space, $(\mathrm{PS})_\beta$ follows from boundedness of $(\mathrm{PS})_\beta$ sequences. In particular (PS) follows from coerciveness. More in general, (PS) follows (always in the finite dimensional case) from coerciveness of $x \mapsto f(x) + \|df(x)\|$.

If $f(x) = \sum_{i,j=1}^{N} a_{ij}x_i x_j + \sum_{i=1}^{N} b_i x_i + c$, then f satisfies (PS) if $A = [a_{ij}]$ is an invertible matrix.

For the infinite dimensional case, will be useful the following result:

Theorem 4.3. *Let $f \in C^1(V)$, V Banach, be such that*

(1) *Any $(PS)_\beta$ sequence is bounded;*
(2) *For all $u \in V$*
$$df(u) = Lu + K(u)$$
where L is an invertible linear operator and K is a compact operator.

Then f satisfies $(PS)_\beta$.

Proof. Take a $(\mathrm{PS})_\beta$ sequence. Then it is bounded by (1) and $Lu_n + K(u_n) = df(u_n) \to 0$. Let $y_n = K(u_n)$. By (2) $y_{n_k} \to y$ in V'. We deduce that
$$u_{n_k} = L^{-1}(df(u_{n_k}) - y_{n_k}) \to -L^{-1}y$$
□

From (PS) we deduce, in particular

Proposition 4.4. *Let $f \in C^1(V)$, V Banach. Assume that f satisfies $(PS)_\beta$. Then*

(a) $K_\beta = \left\{ u \in V \mid f(u) = \beta \text{ and } df(u) = 0 \right\}$ *is compact;*
(b) *If $K_\beta = \emptyset$ then exists $\delta > 0$ such that $\|df(u)\| \geq \delta$ for all u such that $|f(u) - \beta| < \delta$.*
(c) *For all neighborhoods U of K_β there exists $\delta > 0$ such that*
$$K_\beta \subset N_{\beta,\delta} := \left\{ u \in V \mid |f(u) - \beta| < \delta \text{ and } \|df(u)\| < \delta \right\} \subset U.$$

Proof. (a) $u_n \in K_\beta$ is clearly a $(\mathrm{PS})_\beta$ sequence. Hence it has a converging subsequence.
(b) Assume, by contradiction, that there exists u_n such that $f(u_n) \to \beta$ and $df(u_n) \to 0$. Then u_n is a $(\mathrm{PS})_\beta$ sequence which then converges to a critical point u at level β.
(c) Assume, by contradiction, that there exists $u_n \notin U$ such that $f(u_n) \to \beta$ and $df(u_n) \to 0$. By $(\mathrm{PS})_\beta$ u_n converges (up to a subsequence), to a critical point in K_β, contradiction.

□

Recalling Ekeland's variational principle 3.6, we get that

Theorem 4.5. *Suppose $f \in C^1(V)$, V Banach, $f \in C^1(V)$, $\beta = \inf_M f > -\infty$. If (PS)$_\beta$ holds, then the infimum is achieved.*

Remark 4.6. Actually it can be shown (see [**5**]) that any differentiable function bounded below which satisfies (PS) is coercive.

5. The deformation lemma

Definition 5.1. Let $f \in C^1(V)$. We let $\tilde{V} = \{ u \in V \mid df(u) \neq 0 \}$. We say that a map

$$v \colon \tilde{V} \to V$$

is a pseudo-gradient vector field for f if

(1) v is lipschitz-continuous;
(2) $\|v(u)\| \leq 2\|df(u)\|$;
(3) $\langle df(u), v(u)\rangle \geq \|df(u)\|^2$.

Remark 5.2. If $V = H$ is an Hilbert space and $\nabla f(u)$ is the gradient of f, we have that $\frac{3}{2}\nabla f(u)$ is a continuous pseudo-gradient vector field. The pseudo-gradient vector field is a something that "looks like" a gradient vector field in case $\nabla f(u)$ does not exists (is the case of the Banach space case) or is not regular enough.

Theorem 5.3. *Let $f \in C^1(V)$, V Banach. Then there exists a pseudo-gradient vector field.*

Proof. Fix $u \in \tilde{V}$. Since $df(u) \neq 0$, there exist $\bar{w}(u) \in V$ such that $\|\bar{w}(u)\| = 1$ and

$$\langle df(u), \bar{w}(u)\rangle > \frac{2}{3}\|df(u)\|.$$

Let $w(u) = \frac{3}{2}\|df(u)\|\bar{w}(u)$. Then

$$\|w(u)\| = \tfrac{3}{2}\|df(u)\| < 2\|df(u)\|$$
$$\langle df(u), w(u)\rangle > \|df(u)\|^2.$$

Since $f \in C^1(V)$, for any given $u \in \tilde{V}$ there exist a neighborhood U_u of u such that, for all $v \in U_u$

$$\|w(u)\| < 2\|df(v)\| \qquad \langle df(v), w(u)\rangle > \|df(v)\|^2. \tag{5.1}$$

$\{U_u\}_{u\in\tilde{V}}$ is an open cover of the metric (and hence paracompact) space $\tilde{V}$. Then there exist a locally finite refinement $\{V_\alpha\}_{\alpha\in A}$ of $\{U_u\}_{u\in\tilde{V}}$, that is

(1) $\{V_\alpha\}_{\alpha\in A}$ is an open cover of $\tilde{V}$;
(2) for all $\alpha \in A$ we have $V_\alpha \subset U_{u_\alpha}$ for some $u_\alpha \in \tilde{V}$.
(3) for a given $\alpha \in A$, $V_\alpha \cap V_\beta \neq \emptyset$ only for finitely many $\beta \in A$.

Define on $\tilde{V}$, the Lipschitz continuous functions

$$\rho_\alpha(v) = \operatorname{dist}(v, \tilde{V} \setminus V_\alpha)$$

and

$$W(v) = \sum_{\alpha \in A} \frac{\rho_\alpha(v) w(u_\alpha)}{\sum_{\beta \in A} \rho_\beta(v)}$$

Remark that the above series is actually a finite sum. It is clearly Lipschitz continuous and, using the convexity of the norm, the linearity of $df(u)$ together with (5.1) the result follows. $\square$

We now state and prove a rather general version of the deformation lemma.

Theorem 5.4 (Deformation Lemma). *Let $f \in C^1(V)$ and assume (PS)$_\beta$ holds.*

Then for all $\bar{\epsilon} > 0$ and for all U neighborhood of K_β there exist $\epsilon > 0$ and $\eta \in C(V \times \mathbb{R}; V)$ such that

(a) *$\eta(u, 0) = u$ for all u;*

(b) *$df(u) = 0$ implies $\eta(u, t) = u$ for all t;*

(c) *$|f(u) - \beta| > \bar{\epsilon}$ implies $\eta(u, t) = u$ for all t;*

(d) *$t \mapsto f(\eta(u, t))$ is nonincreasing in t;*

(e) *$u \in f^{\beta+\epsilon} \setminus U$ implies $\eta(u, 1) \in f^{\beta-\epsilon}$*

(f) *$u \in f^{\beta+\epsilon}$ implies $\eta(u, 1) \in f^{\beta-\epsilon} \cup U$*

(g) *$\eta(\eta(u, t), s) = \eta(u, t+s)$ (which implies that, for fixed t, $u \mapsto \eta(u, t)$ is a homeomorphism).*

Proof. The idea is to construct η as the solution of the Cauchy problem $\eta' = f'(\eta)$, $\eta(0) = u$. But in order to do this we will need to use the pseudo-gradient vector field (which is Lipschitz-continuous) and to truncate it.

Given $\bar{\epsilon}$ and U, take ρ, $\delta > 0$ such that (see (c) of Proposition 4.4)

$$K_\beta \subset N_{\beta,\delta} \subset U_\rho \subset U_{2\rho} \subset U$$

where

$$U_\rho = \left\{ u \in V \;\middle|\; \operatorname{dist}(u, K_\beta) < \rho \right\}.$$

We also take ρ, $\delta \leq 1$.

We first define two cutoff functions in order to achieve (b) and (c).

We let $\chi \colon V \to \mathbb{R}$ be a Lipschitz-continuous function such that $0 \leq \chi(u) \leq 1$, $\chi(u) = 0$ if $u \in N_{\beta,\delta/2}$, $\chi(u) = 1$ if $u \notin N_{\beta,\delta}$.

We then define $\phi\colon \mathbb{R} \to \mathbb{R}$ be a Lipschitz-continuous function such that $0 \le \phi(s) \le 1$, $\phi(s) = 0$ if $|s-\beta| \ge \min\{\bar\epsilon, \frac{\delta}{4}\}$, $\phi(s) = 1$ if $|s-\beta| \le \min\{\frac{\bar\epsilon}{2}, \frac{\delta}{8}\}$.

Finally we let $\xi(s) = \min\{1, \frac{1}{|s|}\}$. We will need this to get a bounded vector field.

Let $v\colon \tilde V \to V$ be a pseudo-gradient vector field for f. Define

$$e(u) = \begin{cases} -\chi(u)\phi(f(u))\xi(\frac{\|v(u)\|}{2})v(u) & u \in \tilde V \\ 0 & \text{if } df(u) = 0 \end{cases}$$

Then $e\colon V \to V$ is a Lipschitz-continuous vector field (remark that $\chi = 0$ near the critical points having value in $[\beta-\delta, \beta+\delta]$ while $\phi = 0$ close to the other critical points). We also have that $\|e(u)\| \le 2$. Hence the Cauchy problem

$$\begin{cases} \frac{\partial\eta}{\partial t} = e(\eta) \\ \eta(u,0) = u \end{cases} \tag{5.2}$$

has a solution defined for all $t \in \mathbb{R}$, continuous in t and u. Then (a) and (g) follow.

The definition of the cutoff functions χ and ϕ immediately give (b) and (c).

To prove (d), it is enough to compute

$$\begin{aligned} \frac{d}{dt} f(\eta(t,u)) &= \langle df(\eta), \frac{\partial\eta}{\partial t}\rangle \\ &= \langle df(\eta), e(\eta)\rangle \\ &= -\chi(\eta)\phi(f(\eta))\xi(\|\frac{\|v(u)\|}{2}\|)\langle df(\eta), v(\eta)\rangle \\ &\le -\chi(\eta)\phi(f(\eta))\xi(\|\frac{\|v(u)\|}{2}\|)\|df(\eta)\|^2 \le 0. \end{aligned}$$

We now take $\epsilon > 0$ such that $\epsilon < \frac{\bar\epsilon}{2}$, $\epsilon < \frac{\delta}{8}$, $\epsilon < \frac{\delta^2\rho}{4}$.

We assume

$$f(u) < \beta + \epsilon \tag{5.3}$$

$$\eta(u,t) \notin N_{\beta,\delta} \qquad \forall t \in [0,1] \tag{5.4}$$

and claim that in such a case $f(\eta(u,1)) < \beta - \epsilon$. We can, by (d), assume $f(\eta(u,t)) \ge \beta - \epsilon$ for all $t \in [0,1]$. Then it is clear that

$$\begin{aligned} &\chi(\eta(t,u)) = 1 && \forall t \in [0,1] \\ &\phi(f(\eta(t,u))) = 1 && \forall t \in [0,1] \\ &\|df(\eta(t,u))\| \ge \delta && \forall t \in [0,1] \end{aligned}$$

and
$$\frac{d}{dt} f(\eta(u,t)) \le -\xi(\|df(\eta)\|)\|df(\eta)\|^2 \le -\delta^2$$
so that
$$f(\eta(u,1)) \le f(u) - \delta^2 < \beta - \epsilon$$
by the assumptions on ϵ.

We can now prove (e). Assume $u \in f^{\beta+\epsilon} \setminus U$, $f(\eta(u,1)) \ge \beta - \epsilon$. We show that $\eta(u,t) \notin N_{\beta,\delta}$ for all $t \in [0,1]$. Then (e) will follow from the claim above.

If exists $\bar{t} \in [0,1]$ such that $\eta(u,\bar{t}) \in N_{\beta,\delta}$, then $\operatorname{dist}(u, \eta(u,\bar{t})) > \rho$ and we find an interval $[0,t_1] \subset [0,\bar{t}]$ such that, for all $t \in [0,t_1]$
$$\begin{aligned} &\beta - \epsilon \le f(\eta(u,t)) \le \beta + \epsilon \\ &\eta(u,t) \notin N_{\beta,\delta} \\ &\|u - \eta(u,t_1)\| > \rho. \end{aligned}$$
Then
$$\rho \le \|u - \eta(u,t_1)\| \le \int_0^{t_1} \|e(u)\| \le 2t_1$$
that is, $t_1 \ge \frac{\rho}{2}$. As before, we get
$$f(\eta(u,t_1)) \le f(u) - t_1\delta^2 < \beta + \epsilon - \frac{\rho\delta^2}{2}$$
and our choice of ϵ shows that $f(\eta(u,t_1)) < \beta - \epsilon$.

The proof of (f) is similar. Assume $u \in f^{\beta+\epsilon}$, $\eta(u,1) \notin f^{\beta-\epsilon}$. Then, by (e), $u \in U$. We want to show that $\eta(u,1) \in U$. Assume not. If $\eta(u,t) \notin N_{\beta,\delta}$ for all $t \in [0,1]$ we can use the claim to prove that $f(\eta(u,1)) \le \beta - \epsilon$. So for some $t_1 \in [0,1]$ $\eta(u,t_1) \in N_{\beta,\delta}$. But then $\|\eta(u,t_1) - \eta(u,1)\| > \rho$ and one can proceed as in proving (e). □

6. The Mountain Pass Theorem

Theorem 6.1. *Suppose $f \in C^1(V)$, V Banach space. Assume*

(1) $f(0) = 0$;
(2) *Exist $r > 0$ such that $f(u) > \alpha > 0$ for all $\|u\| = r$;*
(3) *Exist $\bar{u} \in V$ such that $\|\bar{u}\| > r$ and $f(\bar{u}) < 0$.*

Then, setting
$$\Gamma = \left\{ \gamma \in C([0,1];V) \;\middle|\; \gamma(0) = 0,\, \gamma(1) = \bar{u} \right\}$$
and
$$\beta := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t))$$

we have that $\beta \geq \alpha > 0$, there exist a (PS)$_\beta$ sequence and, if (PS)$_\beta$ holds, β is a critical level for f.

Proof. The original proof of the theorem can be found in [**2**].

One immediately checks that $\beta \geq \max_{\|u\|=r} f(u) \geq \alpha > 0$.

Let us assume (PS)$_\beta$ holds, and let us show that it is a critical level. By contradiction, assume $K_\beta = \emptyset$. Take $\bar{\epsilon} = \beta$, $U = \emptyset$ and find, using the deformation lemma 5.4, $\epsilon > 0$ and a deformation η.

Take $\gamma \in \Gamma$ such that $\max_{t\in[0,1]} f(\gamma(t)) < \beta + \epsilon$. Let $\bar{\gamma}(t) = \eta(\gamma(t), 1)$. Using the properties of the flow η (in particular point (c) of theorem 5.4 implies that $\eta(0,1) = 0$ and $\eta(\bar{u}, 1) = \bar{u}$) we see immediately that $\bar{\gamma} \in \Gamma$. But then point (e) of theorem 5.4 implies that $\max f(\bar{\gamma}(t)) < \beta - \epsilon$, contradiction which proves that β is a critical level.

Suppose now that there is no (PS)$_\beta$ sequence. Then (PS)$_\beta$ holds; from what we have seen β should then be a critical level. But this implies that a (PS)$_\beta$ sequence exist. Contradiction. □

Remark 6.2. It is easy to see that in the Mountain Pass theorem one can replace the class Γ of paths with the class

$$\Gamma = \left\{ \gamma \in C([0,1], E) \;\middle|\; \gamma(0) = 0,\ f(\gamma(1)) < 0 \right\}$$

The same proof given here allows us to prove a "general" minimax theorem (already stated in a similar form in [**9**]).

Theorem 6.3. *Suppose $f \in C^1(V)$ and that (PS) holds. Assume Γ is a class of subsets of V such that*

(1) $\beta = \inf_{A\in\Gamma} \sup_{u\in A} f(u) \in \mathbb{R}$;

(2) *for all $A \in \Gamma$ and for all maps $\eta \in C(V \times \mathbb{R}; \mathbb{R})$ satisfying, for some $\bar{\epsilon}$,* (a), (b), (c), (d) *and* (g) *of the deformation lemma 5.4 we have that $\eta(A, 1) \in \Gamma$.*

Then β is a critical level for f.

Chapter 2

Applications to elliptic equations

1. Application I: superlinear elliptic equations

In this section we will apply some of the above abstract results to study the boundary value problem

$$\text{(BVP)} \qquad \begin{cases} -\Delta u + u = g(x,u) & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases}$$

Here $\Omega \subset \mathbb{R}^N$ ($n \geq 3$ for simplicity) is an open set. If Ω is not bounded, the boundary condition must be understood as $u \in H_0^1(\Omega)$.

We will assume

(g1) $g \in C^1(\mathbb{R}^N \times \mathbb{R}; \mathbb{R})$, $g(x,0) = \frac{\partial g}{\partial u}(x,0) = 0$ for all $x \in \Omega$;

(g2) $c_1|s|^{p-1} \leq |g(x,s)| \leq c_2|s|^{p-1}$ for some $p \in (2, 2^*]$, $2^* = \frac{2n}{n-2}$;

(g3) Exist $\mu > 2$ such that $0 < \mu G(x,s) \leq g(x,s)s$ for all $x \in \Omega$ and $s \neq 0$, where $G(x,s) = \int_0^s g(x,t)\,dt$.

The above assumptions are satisfied if $g(x,s) = |s|^{p-2}s$. We also remark that the theory which follows can be applied (with minor changes) also if g satisfies (g1), $g(x,s) = 0$ for all $s \leq 0$ and (g2) and (g3) hold for all $s > 0$, in particular if $g(x,s) = (s^+)^{p-1}$.

Let us remark that one can take much weaker assumptions, in particular if Ω is bounded. See, for example [**12**, Theorem 8.5, pp. 128] and the reference in this book, or [**2, 10**]. For the case $\Omega = \mathbb{R}^N$, see, for example, [**11**]. Here we just want to indicate how the abstract results can be applied.

Let $H = H_0^1(\Omega)$ with norm $\|u\| = \int_\Omega [|\nabla u|^2 + u^2]\,dx$ and scalar product $\langle u, v\rangle = \int_\Omega [\nabla u \nabla v + uv]\,dx$ and define, for all $u \in H$,

$$f(u) = \frac{1}{2}\int_\Omega [|\nabla u|^2 + u^2]\,dx - \int_\Omega G(x,u)\,dx = \frac{1}{2}\|u\|^2 - \int_\Omega G(x,u)\,dx.$$

Since $u \mapsto \|u\|^2$ is differentiable, it follows from corollary 2.8 that f is differentiable in $H_0^1(\Omega)$, more precisely that $f \in C^1(H_0^1(\Omega))$.

Lemma 1.1. *f satisfies the geometric assumptions of the Mountain Pass Theorem 6.1, that is*

(a) *Exists $\rho > 0$ such that $f(u) \geq \alpha > 0$ for all $\|u\| = \rho$;*
(b) *For all $\bar{u} \not\equiv 0$ $f(\lambda\bar{u}) \to -\infty$ as $\lambda \to +\infty$.*

Proof. (a) From $G(x,u) \leq \frac{c_2}{\mu}|u|^p$ we deduce that

$$\int G(x,u) \leq \frac{c_2}{\mu}\int |u|^p \leq C\|u\|^p.$$

Hence

$$\begin{aligned} f(u) &= \frac{1}{2}\|u\|^2 - \int G(x,u) \\ &\geq \frac{1}{2}\|u\|^2 - C\|u\|^p \geq \frac{1}{4}\|u\|^2 \end{aligned}$$

for $\|u\| = \rho$ small.

(b) Remark that (g3) implies that $g(x,u) > 0$ if $u > 0$, and hence, for $u > 0$,

$$G(x,u) = \int_0^u g(x,s)\,ds \geq c_1 \int_0^u |u|^{p-1} = Cu^p = C|u|^p.$$

Similarly one gets $G(x,u) \geq C|u|^p$ for all u. Hence, for all $\bar{u} \not\equiv 0$

$$f(\lambda\bar{u}) \leq \frac{\lambda^2}{2}\|\bar{u}\| - \lambda^p C\int |\bar{u}|^p \to -\infty.$$

□

In order to apply the Mountain Pass theorem 6.1, we have to study (PS) sequences.

Lemma 1.2. *Palais Smale sequences are bounded.*

Proof. Take $u_n \in H$ such that $f(u_n) \to \beta$, $df(u_n) \to 0$. Then

$$\begin{aligned}\beta + 1 + \frac{\epsilon_n \|u_n\|}{\mu} &\geq f(u_n) - \frac{1}{\mu}\langle df(u_n), u_n\rangle \\ &= \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u_n\|^2 - \int \big[G(x, u_n) - \frac{1}{\mu} g(x, u_n)u_n\big] \\ &\geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\|u_n\|^2\end{aligned}$$

and the boundedness of (PS) sequences follow. □

We can now prove that (PS) holds whenever $|\Omega| < +\infty$.

Lemma 1.3. *Suppose $|\Omega| < +\infty$ and $p \in (2, 2^*)$. Then (PS) holds.*

Proof. The lemma will follow from lemma 1.2 and Theorem 4.3. Indeed, let us remark that the gradient $\nabla f(u) \in H_0^1(\Omega)$ is defined by

$$\langle df(u), v\rangle = (\nabla f(u) \,|\, v) = (u \,|\, v) - \int_\Omega g(x, u)v.$$

Hence $\nabla f(u) = u - K(u)$, where $K\colon H_0^1(\Omega) \to H_0^1(\Omega)$ is defined by

$$(K(u) \,|\, v) = \int_\Omega g(x, u)v.$$

Let us show that K is compact. Assume that $u_n \in H_0^1(\Omega)$ is bounded and fix $p \in [2, 2^*)$. By Rellich theorem 1.3, there exists a subsequence (still denoted u_n), such that $u_n \to u$ in $L^p(\Omega)$, $u_n(x) \to u(x)$ a.e. and such that $|u_n(x)|^p \leq h(x) \in L^1(\Omega)$. It is then a consequence of Lebesgue dominated convergence, together with the growth conditions on g, that $g(x, u_n(x)) \to g(x, u(x))$ in $L^q(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$. This implies that

$$\begin{aligned}\|K(u_n) - K(u))\| &= \sup_{\|v\| \leq 1} (K(u_n) - K(u) \,|\, u_n - u) \\ &\leq \|g(x, u_n) - g(x, u)\|_q \|v\|_p \to 0,\end{aligned}$$

showing that K is a compact operator. □

Putting all together, we have proved:

Theorem 1.4. *Suppose g satisfies (g1–3) with $p \in (2, 2^*)$ in an open set $\Omega \subset \mathbb{R}^N$, $|\Omega| < +\infty$.*

Then (BVP) *has a nontrivial solution.*

Proof. The theorem follows from an application of the Mountain Pass theorem 6.1, which can be applied thanks to the lemmas 1.1 and 1.3. The solution is not trivial since we know that the MP level $\beta > 0$ is critical, while $f(0) = 0$. □

A particular case of the above theorem is the following:

Theorem 1.5. *Suppose* $|\Omega| < +\infty$ *and* $2 < p < 2^*$*. Then the problem*

$$(\mathcal{MP}) \qquad \begin{cases} -\Delta u + \lambda u = |u|^{p-2}u \\ u > 0, \\ u \in H_0^1(\Omega) \end{cases}$$

has a (nontrivial) solution if and only if $\lambda > -\lambda_1(\Omega)$*.*

Proof. The proof is essentially the same as that of Theorem 1.4. One has just to apply the Mountain Pass theorem to the functional

$$f(u) = \frac{1}{2}\int_\Omega [|\nabla u|^2 + \lambda u^2]\,dx - \frac{1}{p}\int_\Omega (u^+)^p\,dx$$

and notice that 0 is a (strict) local minimum, thanks to the Poincaré inequality (1.4), for all $\lambda > -\lambda_1(\Omega)$. One deduce the existence of a critical point, which is a solution of the problem

$$-\Delta u + \lambda u = (u^+)^{p-1}. \qquad u \in H_0^1(\Omega).$$

Multiplying the equation by u^- and integrating in Ω we obtain

$$0 = \int_\Omega |\nabla u^-|^2 + \lambda |u^-|^2,$$

which, by Poincaré inequality, implies $u^- = 0$ and $u \geq 0$. From the strong maximum principle we finally deduce that $u > 0$ hence is a solution of $(\mathcal{MP})$,

To prove that no positive solution exist if $\lambda \leq \lambda_1(\Omega)$, we assume u is a solution and multiply equation $(\mathcal{MP})$ by the eigenfunction ϕ_1 of $-\Delta$ corresponding to the first eigenvalue $\lambda_1(\Omega)$. Then we have

$$\lambda \int_\Omega \phi_1 u = \int_\Omega \phi_1(u^{p-1} + \Delta u) > -\lambda_1(\Omega)\int_\Omega \phi_1 u$$

and hence $\lambda > -\lambda_1(\Omega)$. □

2. Application II: the case Ω unbounded

With the same notations and under the same assumptions of the preceding section, we know, using the Mountain pass theorem 6.1, that the functional f has a $(\mathrm{PS})_\beta$ sequence even when Ω is unbounded. But we do not no, in general, if such a (PS) sequence (bounded by lemma 1.2), has a subsequence (strongly) convergent to a (non-trivial) critical point of f. In this section we will analyze more closely the situation, and prove, under some additional assumption, that a solution exists.

Let us start by showing that

Lemma 2.1. *Suppose $u_n \in H_0^1(\Omega)$ is a (PS)$_\beta$ sequence for f. Then it is bounded. Let $u \in H$ be such that*

(1) *$u_n \rightharpoonup u$ weakly in H;*
(2) *$u_n \to u$ in L^p_{loc} for $p \in [2, 2^*)$, and a.e..*

Then u is a critical point for f with $f(u) \leq \beta$.

Proof. Take $\phi \in C_c^\infty(\Omega)$. We have that

$$\langle df(u_n), \phi\rangle = \langle u_n, \phi\rangle - \int g(x, u_n)\phi.$$

Since $\langle u_n, \phi\rangle \to \langle u, \phi\rangle$ by weak convergence and

$$\int g(x, u_n)\phi \to \int g(x, u)\phi$$

since the L^p_{loc} convergence of u_n implies that $g(x, u_n)$ converges in L^q_{loc} ($\frac{1}{p} + \frac{1}{q} = 1$). We deduce that

$$\begin{aligned}\langle df(u), \phi\rangle &= \langle u, \phi\rangle - \int g(x, u)\phi \\ &= \lim_{n\to+\infty}\Big(\langle u_n, \phi\rangle - \int g(x, u_n)\phi\Big) = \lim_{n\to+\infty}\langle df(u_n), \phi\rangle = 0.\end{aligned}$$

And hence u is a critical point for f by the density of C_c^∞ in $H_0^1(\Omega)$. □

Remark 2.2. The above lemma states that the weak limit of a $(PS)_\beta$ sequence is a solution. We know that $u = 0$ is a solution. So, how can we prove that u is non trivial? In the proof of theorem 1.4, the solution we found was non trivial since we found it at the positive MP level $\beta > 0$. In this case, since we only know that $u_n \rightharpoonup u$ we cannot deduce that $f(u) > 0$ (the fact that $f(v) \geq 0$ for all critical points v is an easy consequence of the superquadraticity assumption (g3), we will see this later).

In order to continue our analysis, we need a general result about bounded sequences in $H^1(\mathbb{R}^N)$.

Lemma 2.3 (P.L. Lions [**8**])**.** *Let $u_m \in H^1(\mathbb{R}^N)$ be a sequence such that*

(1) *$\|u_m\| \leq C$;*
(2) *There exist $R > 0$ such that*

$$\limsup_{m\to+\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}|u_m|^2 = 0$$

Then $u_m \to 0$ in $L^q(\mathbb{R}^N)$ for all $q \in (2, 2^*)$.

Proof. Recall that, by Sobolev embedding,

$$\|u\|_{L^q(B_R(y))} \le C\|u\|^{1-\theta}_{L^2(B_R(y))}\|u\|^{\theta}_{H^1(B_R(y))},$$

where $\theta = \frac{q-2}{2q}n$. Then

$$\int_{B_R(y)} |u|^q \le C\|u\|^{(1-\theta)q}_{L^2(B_R(y))}\|u\|^{\theta q}_{H^1(B_R(y))}.$$

Let us distinguish two cases:

THE CASE $\theta q \ge 2$. In this case $q \ge 2 + \frac{4}{n}$ and

$$\begin{aligned}\int_{B_R(y)} |u|^q &\le C\|u\|^{(1-\theta)q}_{L^2(B_R(y))}\|u\|^{\theta q-2}_{H^1(B_R(y))}\|u\|^{2}_{H^1(B_R(y))}\\ &\le C\Big(\sup_{y\in\mathbb{R}^N}\|u\|^{(1-\theta)q}_{L^2(B_R(y))}\Big)\|u\|^{\theta q-2}_{H^1(\mathbb{R}^N)}\int_{B_R(y)}(|\nabla u|^2+u^2).\end{aligned}$$

Take now $y_k \in \mathbb{R}^N$ such that

(1) $\mathbb{R}^N \subset \cup_k B_R(y_k)$;
(2) There exist $\ell \in \mathbb{N}$ such that every point $x \in \mathbb{R}^N$ belongs to at most ℓ sets $B_R(y_k)$.

Then

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^q &\le \sum_k \int_{B_R(y_k)} |u|^q\\ &\le C\Big(\sup_{y\in\mathbb{R}^N}\|u\|^{(1-\theta)q}_{L^2(B_R(y))}\Big)\|u\|^{\theta q-2}_{H^1(\mathbb{R}^N)}\sum_k\int_{B_R(y_k)}(|\nabla u|^2+u^2)\\ &\le \ell C\|u\|^{\theta q}_{H^1_{\mathbb{R}^N}}\Big(\sup_{y\in\mathbb{R}^N}\|u\|^{(1-\theta)q}_{L^2(B_R(y))}\Big)\end{aligned}$$

and assumption 2 implies that

$$\int_{\mathbb{R}^N} |u_m|^q \to 0.$$

THE CASE $\theta q < 2$. In this case $2 < q < 2 + \frac{4}{n} = \bar{q}$. Let us remark that, since $\theta\bar{q} \ge 2$, $\|u_m\|_{L^{\bar{q}}(\mathbb{R}^N)} \to 0$ (we are in the first case). Let $q = \lambda 2 + (1-\lambda)\bar{q}$. Then

$$\|u_m\|_{L^q(\mathbb{R}^N)} \le \|u_m\|^{\lambda}_{L^2(\mathbb{R}^N)}\|u_m\|^{(1-\lambda)}_{L^{\bar{q}}(\mathbb{R}^N)} \to 0$$

□

Remark 2.4. A general way to analyze the behavior of sequences of functions (even measures) bounded in some norm (as it is the case for the (PS) sequence u_m we found via the MP theorem), has been developed by P.L. Lions. It is called Concentration-compactness methods. See [**8**], [**12**] or [**13**].

In terms of this theory, the above lemma shows that if the sequence u_n vanish in the L^2 norm, then it converges to 0 strongly in L^p, $p \in (2, 2^*)$. We will now see that this is not possible for (PS) sequences.

Lemma 2.5. *Assume $\Omega = \mathbb{R}^N$, and let u_m be a (PS)$_\beta$ sequence with $\beta > 0$. Then there exists a sequence $y_m \in \mathbb{R}^N$ such that the sequence $v_m(x) = u_m(x - y_m)$ converges $v \in H^1(\mathbb{R}^N)$, $v \not\equiv 0$, weakly in H^1.*

Proof. We know, from lemma 1.2, that u_m is bounded in H^1. Suppose, by contradiction, that exists $R > 0$ such that

$$\limsup_{m\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_m|^2 = 0.$$

Then we can apply lemma 2.3 to deduce that $\|u_m\|_{L^q(\mathbb{R}^N)} \to 0$ for all $q \in (2, 2^*)$. Hence, from $\langle df(u_m), u_m\rangle = \|u_m\|^2 - \int_{\mathbb{R}^N} g(x, u_m)u_m$ we deduce that

$$\|u_m\|^2 \le \|df(u_m)\|\|u_m\| + C\|u_m\|^p_{L^p(\mathbb{R}^N)} \to 0.$$

Hence

$$f(u_m) = \frac{1}{2}\|u_m\|^2 - \int_{\mathbb{R}^N} G(x, u_m) \to 0,$$

contradiction (since $\beta > 0$) which shows that for all $R > 0$ exist $y_m \in \mathbb{R}^N$ such that

$$\int_{B_R(y_m)} |u_m|^2 \ge \delta > 0.$$

Let $v_m(x) = u_m(x - y_m)$. Then $\|v_m\| = \|u_m\| \le C$ and

$$\int_{B_R(0)} |v_m|^2 \ge \delta > 0.$$

Since v_m is bounded, it converges to some $v \in H^1(\mathbb{R}^N)$, weakly in H^1 and strongly in L^p_{loc} for all $p \in [2, 2^*)$. Then

$$\int_{B_R(0)} |u|^2 \ge \delta > 0,$$

showing that $v \not\equiv 0$. □

A rather direct consequence of the this lemma is the following existence theorem:

Theorem 2.6. *Suppose $g \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ satisfies (g1-3) with $\Omega = \mathbb{R}^N$. Assume also*

(g4) *$g(x, s)$ is periodic in $x \in \mathbb{R}^N$, that is*

$$g(x_1 + k_1, \ldots, x_n + k_n, s) = g(x_1, \ldots, x_n, s) \qquad \forall (k_1, \ldots, k_n) \in \mathbb{Z}^N.$$

Then the boundary value problem

$$\begin{cases} -\Delta u + u = g(x,u) \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

has a non trivial solution w.

Proof. We know that there exists u_m, a $(\mathrm{PS})_\beta$ sequence, where β is the Mountain Pass critical level.

The lemma 2.5 then shows that there exists a sequence $y_m \in \mathbb{R}^N$ such that $u_m(\cdot + y_m)$ converges weakly to a nontrivial function $v \in H^1$.

We let $\tilde{y}_m$ be the point of $\mathbb{Z}^N$ closer to y_m. Then $\|y_m\| - \tilde{y}_m \le \sqrt{n}$. We claim that $w_m(x) = u_m(x - \tilde{y}_m)$ is a (PS) sequence for f which converges weakly to a nontrivial $w \in H^1$. Then the theorem follow from 2.1.

To prove the claim it is enough to remark that, thanks to the symmetry assumption (g4), $f(u_m) = f(w_m)$ and $\|df(u_m)\| = \|df(w_m)\|$. To show that the weak limit of w_m is non trivial, it is enough to remark that

$$\int_{B_{R+\sqrt{n}}(0)} w^2 \ge \int_{B_R(0)} v^2 > 0$$

for some $R > 0$ since v is non trivial. □

Remark 2.7. Let us point out that in general we can only prove that $0 < f(w) \le \beta$. Indeed a more careful analysis (see for example [**6**]) shows that

(a) if u_n is a $(\mathrm{PS})_\alpha$ sequence for f, then $\alpha \ge 0$ (in particular $f(u) > 0$ if u is a nontrivial critical point);

(b) if u_n is a $(\mathrm{PS})_\alpha$ sequence for f, having a weak limit u, then $u_n - u$ is a $(\mathrm{PS})_{\alpha - f(u)}$ sequence;

In particular we cannot say, even if we are able to find a solution, that the functional satisfies (PS) at level β.

We now present a situation in which we are able to prove that β is a critical level. We first prove the following lemma.

Lemma 2.8. *Assume g satisfies (g1–3) and*

(g5) $\quad \dfrac{g(x,u)}{u}$ *is non decreasing for $u \ge 0$, non increasing for $u \le 0$.*

Then, if $\Gamma = \{ \gamma \in C([0,1],E) \mid \gamma(0) = 0,\ f(\gamma(1)) < 0 \}$ the mountain pass level

$$\beta = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} f_\infty(\gamma(t))$$

is such that

$$\beta \le \inf\Big\{ f(u) \;\Big|\; df(u) = 0,\, u \neq 0 \Big\}.$$

Proof. Take u such that $df(u) = 0$, $u \neq 0$. Consider the line $\gamma\colon \mathbb{R} \to H^1$ defined by $\gamma(\lambda) = \lambda\lambda_0 u$. We have that $\gamma \in \Gamma$ if λ_0 is large enough and

$$f(\gamma(\lambda)) = \frac{\lambda^2\lambda_0^2}{2}\|u\|^2 - \int G(x, \lambda\lambda_0 u)$$
$$\frac{d}{d\lambda} f(\gamma(\lambda)) = \lambda\lambda_0^2\|u\|^2 - \int g(x, \lambda\lambda_0 u)u.$$

Since $\|u\|^2 = \int g(x,u)u$, we find that

$$\begin{aligned}\frac{d}{d\lambda} f(\gamma(\lambda)) &= \int \Big[g(x,u)\lambda\lambda_0^2 u - g(x, \lambda\lambda_0 u)\lambda_0 u\Big] \\ &= \lambda\lambda_0^2 \int \Big[\frac{g(x,u)}{u} - \frac{g(x,\lambda\lambda_0 u)}{\lambda\lambda_0 u}\Big] u^2.\end{aligned}$$

Since

$$\frac{g(x,u)}{u} - \frac{g(x,\lambda\lambda_0 u)}{\lambda\lambda_0 u}$$

is nonnegative for $\lambda\lambda_0 \in [0,1]$, nonpositive for $\lambda\lambda_0 \ge 1$ (from assumption (g3)), we get that $\frac{df}{d\lambda} \ge 0$ for $\lambda\lambda_0 \in [0,1]$, $\frac{df}{d\lambda} \le 0$ for $\lambda\lambda_0 \ge 1$. So we find that $\max_{\lambda \ge 0} f(\lambda\lambda_0 u) = f(u)$ if u is a critical point. Since the path γ is an element of Γ, one gets that

$$\beta = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)) \le \inf_{u \text{ critical}} \max_{\lambda\ge 0} f(\lambda u) = \inf_{u \text{ critical}} f(u),$$

and the lemma follows. □

The following Theorem, due to Brezis and Lieb, allows us to better describe sequences that do not converge.

Theorem 2.9 (Brezis and Lieb [**4**]). *Let u_m be a sequence of measurable functions such that, for some $0 < p < +\infty$ and $C > 0$*

(1) $u_m(x) \to u(x)$ *almost everywhere;*
(2) $\int_\Omega |u_m|^p \le C$.

Then

$$\lim_{m\to\infty} \int_\Omega \Big| |u_m(x)|^p - |u_m(x) - u(x)|^p - |u(x)|^p \Big| = 0. \tag{2.1}$$

Proof. We will follow the proof in [**7**]. Let us assume the following inequalities: for all $\epsilon > 0$ there is $C_\epsilon \in \mathbb{R}$ such that for all $a, b \in \mathbb{C}$

$$\Big| |a+b|^p - |b|^p \Big| \le \epsilon |b|^p + C_\epsilon |a|^p. \tag{2.2}$$

We first remark that, by Fatou lemma, $\int_\Omega |u|^p \le C$.

Let $u_m = u + v_m$. It follows that $v_m \to 0$ almost everywhere. We claim that

$$w_{m,\epsilon} = \left(\left||u+v_m|^p - |v_m|^p - |u|^p\right| - \epsilon|v_m|^p\right)^+ \tag{2.3}$$

is such that $\lim_{m\to\infty} \int_\Omega w_{m,\epsilon} = 0$. We first remark that

$$\begin{aligned}\left||u+v_m|^p - |v_m|^p - |u|^p\right| &\le \left||u+v_m|^p - |v_m|^p\right| + |u|^p \\ &\le \epsilon|v_m|^p + (1+C_\epsilon)|u|^p\end{aligned}$$

so that $w_{m,\epsilon} \le (1+C_\epsilon)|u|^p$. Moreover $w_{m,\epsilon} \to 0$ almost everywhere, so that the claim follows from Lebesgue dominated convergence. Then

$$\int_\Omega \left||u+v_m|^p - |v_m|^p - |u|^p\right| \le \epsilon \int_\Omega |v_m|^p + \int_\Omega w_{m,\epsilon}.$$

So we only have to show that $\int_\Omega |v_m|$ is uniformly bounded. This follows from

$$\int_\Omega |v_m|^p = \int_\Omega |u-u_m|^p \le 2^p \int_\Omega (|u|^p + |u_m|^p) \le 2^{p+1}C.$$

Hence

$$\limsup_{m\to+\infty} \int_\Omega \left||u+v_m|^p - |v_m|^p - |u|^p\right| \le \epsilon D,$$

and the theorem follows.

To prove (2.2) we first remark that the function $t \mapsto |t|^p$ is convex for $p > 1$. Hence

$$|a+b|^p \le (|a|+|b|)^p = ((1-\lambda)\frac{|a|}{1-\lambda} + \lambda\frac{|b|}{\lambda})^p \le (1-\lambda)^{1-p}|a|^p + \lambda^{1-p}|b|^p$$

for all $0 < \lambda < 1$. Taking $\lambda = (1+\epsilon)^{-1/(p-1)}$ we get (2.2) when $p > 1$. If $0 < p \le 1$ we have the simple inequality $|a+b|^p - |b|^p \le |a|^p$. □

Remark 2.10. From (2.1) we deduce that

$$\int_\Omega |u_m|^p = \int_\Omega |u|^p + \int_\Omega |u-u_m|^p + o(1), \tag{2.4}$$

where $o(1) \to 0$ as $m \to \infty$. We also deduce from (2.4), that $\int |u_m|^p \to \int |u|^p$ and $u_m \to u$ a.e., imply

$$\int |u-u_m|^p \to 0.$$

As a consequence of Brezis-Lieb theorem and lemma 2.8, we have:

Theorem 2.11. *Suppose g satisfies (g1–5) in $\Omega = \mathbb{R}^N$. Then the Mountain pass level β is critical.*

Proof. Let us give the proof in the case $G(x,s) = |s|^p$.

We first remark that, whenever (g3) holds, $\liminf_{m\to+\infty} f(w_m) \geq 0$ along any bounded (PS) sequence w_m. Indeed

$$\begin{aligned}
f(w_m) &= \frac{1}{2}\|w_m\|^2 - \int_\Omega G(x,w_m) \\
&= \frac{1}{2}\langle df(w_m), w_m\rangle + \int_\Omega \big(\frac{1}{2}g(x,w_m)w_m - G(x,w_m)\big) \\
&\geq \big(\frac{\mu}{2} - 1\big)\int_\Omega G(x,w_m) + \frac{1}{2}\langle df(w_m), w_m\rangle \\
&\geq \big(\frac{\mu}{2} - 1\big)\int_\Omega C|w_m|^p + \frac{1}{2}\langle df(w_m), w_m\rangle \geq \frac{1}{2}\langle df(w_m), w_m\rangle \to 0.
\end{aligned}$$

If $f(w_m) \to 0$, we deduce from above that $w_m \to 0$ in L^p and also that $\|w_m\|^2 = 2f(w_m) + \int_\Omega G(x,w_m) \to 0$.

So we have that $w_m \to 0$ if w_m is a bounded (PS) sequence such that $f(u_m) \to 0$.

We know that there is a bounded $(\mathrm{PS})_\beta$ (here β is the (MP) level) sequence whose weak limit u is a non zero critical point for f. We want to show that $f(u) \leq \beta$. Indeed we have from theorem 2.9 that

$$\begin{aligned}
f(u_m) &= \frac{1}{2}\|u_m\|^2 - \frac{1}{p}\int_\Omega |u_m|^p \\
&= \frac{1}{2}\|u_m - u\|^2 + \frac{1}{2}\|u\|^2 + (u_m - u \,|\, u) - \\
&\qquad - \frac{1}{p}\int_\Omega |u|^p - \frac{1}{p}\int_\Omega |u - u_m|^p + o(1) \\
&= f(u) + f(u_m - u) + o(1)
\end{aligned}$$

We can also easily check that $u_m - u$ is a (PS) sequence: taking any $h \in C^\infty_{\mathrm{c}}(\Omega)$ we have that

$$\langle df(u - u_m), h\rangle = (u_m - u \,|\, h) - \int_\Omega g(x, u_m - u)h \to 0$$

since $u_m \to u$ weakly in H^1_0 and strongly in L^p_{loc}. Then $\liminf_{m\to\infty} f(u_m - u) \geq 0$ and

$$f(u) = f(u_m) - f(u_m - u) + o(1) \leq \beta + o(1).$$

Since $f(u) \geq \beta$ (by lemma 2.8) we get that $f(u) = \beta$. □

We now consider a non periodic case.

Theorem 2.12. *Assume $p \in (2, 2^*)$, $a(x) \geq 1$, $a(x) \to 1$ as $|x| \to +\infty$, .*

Then equation

$$(\mathcal{PU}) \qquad \begin{cases} -\Delta u + u = a(x)|u|^{p-2}u \\ u \in H_0^1(\mathbb{R}^N) \end{cases}$$

has a nontrivial solution.

Proof of Theorem 2.12. We consider the functional

$$f(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2) - \frac{1}{p}\int_{\mathbb{R}^N} a(x)|u|^p$$

in $H^1(\mathbb{R}^N)$, whose critical points give rise to solutions of $(\mathcal{PU})$. Remark that the nonlinearity $g(x,s) = a(x)|s|^{p-1}$ satisfies (g1), (g2), (g3), (g5).

We also introduce the functional

$$f_\infty(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2) - \frac{1}{p}\int_{\mathbb{R}^N}|u|^p$$

For f_∞ also (g4) holds and we can apply Theorem 2.11 to deduce the existence of a nontrivial critical point $\bar{u} \in H^1(\mathbb{R}^N)$ at the min-max level

$$f_\infty(\bar{u}) = \beta_\infty = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} f_\infty(\gamma(t)) = \inf_{u\neq 0}\max_{t>0} f_\infty(tu) = \max_{t>0} f_\infty(t\bar{u}).$$

An easy calculation then shows that

$$\beta_\infty = (\frac{1}{2} - \frac{1}{p})\left(\frac{\|\bar{u}\|}{|\bar{u}|_p}\right)^{2p/p-2}.$$

If $a(x) \equiv 1$, the above shows that the theorem holds. In case $a(x) \not\equiv 1$, we have the the mountain pass level

$$\beta = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} f(\gamma(t)) < \beta_\infty.$$

Indeed we have that

$$\beta \le \max_{t>0} f_\infty(t\bar{u}) = (\frac{1}{2} - \frac{1}{p})\left(\frac{\|\bar{u}\|}{\left(\int_\Omega a(x)|\bar{u}|^p\right)^{1/p}}\right)^{2p/p-2}.$$

We also know that the Palais-Smale sequences are bounded (lemma 1.2), that for all Palais-Smale sequences u_m there is a sequence $y_m \in \mathbb{R}^N$ such that the sequence $v_m(x) = u_m(x - y_m)$ converges weakly in $H^1(\mathbb{R}^N)$ to a nontrivial function $v \in H^1(\mathbb{R}^N)$ (lemma 2.5). Moreover also (g5) holds, so that lemma 2.8 implies that the MP level is smaller or equal than the least (nontrivial) critical level.

Assume first that y_m is bounded, so that u_m converges weakly to $u \neq 0$. Since (g5) holds, $f(u) \ge \beta$. Arguing as in the proof of theorem 2.11, we have that

$$f(u_m) = \frac{1}{2}\|u_m - u\|^2 + \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_\Omega a(x)|u|^p - \frac{1}{p}\int_\Omega a(x)|u - u_m|^p + o(1).$$

We know that u_m converges weakly to u. We can assume that it also converges strongly in L^p_{loc}, and hence

$$\int_\Omega a(x)|u - u_m|^p \to \int_\Omega |u - u_m|^p$$

so that

$$f(u_m) = f(u) + f_\infty(u - u_m) + o(1).$$

It can be easily shown that $u - u_m$ is a (PS) sequence for f_∞. We also know that

$$o(1) \leq f_\infty(u - u_m) = f(u_m) - f(u) + o(1) \leq \beta - \beta \leq o(1).$$

so that $f_\infty(u - u_m) \to 0$ and we deduce that $u - u_m \to 0$ and $u_m \to u$ strongly.

If y_m is unbounded and u_m converges weakly to 0, we can consider $v_m(x) = u_m(x - y_m)$. In this case we deduce that

$$\begin{aligned} f(u_m) &= \frac{1}{2}\|v_m\|^2 - \frac{1}{p}\int_\Omega a(x + y_m)|v_m|^p \\ &= \frac{1}{2}\|v_m\|^2 - \frac{1}{p}\int_\Omega |v_m|^p + o(1) \\ &= \frac{1}{2}\|v_m - v\|^2 + \frac{1}{2}\|v\|^2 \\ &\qquad - \frac{1}{p}\int_\Omega |v|^p - \frac{1}{p}\int_\Omega |v - v_m|^p + o(1) \\ &= f_\infty(v) + f_\infty(v - v_m) \geq \beta_\infty, \end{aligned}$$

contradicting the fact that $f(u_m) \to \beta < \beta_\infty$.

□

Chapter 3

A linking theorem

1. The abstract result

In this section we will present a theorem which is a generalization of the Mountain Pass theorem.

We start by giving some abstract definitions.

Definition 1.1. Let S be a closed subset of a Banach space V, and Q a submanifold of V. We say that S and ∂Q link if

(1) $S \cap \partial Q = \emptyset$;
(2) for all $h \in C(V;V)$ such that $h_{|\partial Q} = id$ we have that $h(Q) \cap S \neq \emptyset$

In practical applications, one needs to know when some particular ∂Q and S link. The following examples shows two such situations.

Proposition 1.2. *Let V be a Banach space, $V = V_1 \oplus V_2$, V_1 and V_2 closed, $\dim V_2 < +\infty$. Let $S = V_1$, $Q = B_R(0, V_2) = \{ u \in V_2 \mid \|u\| \leq R \}$ so that $\partial Q = \{ u \in V_2 \mid \|u\| = R \}$ (see figure 1).*

Then S and ∂Q link.

Proof. Let $\pi\colon V \to V_2$ be the projection onto V_2, and take $h\colon V \to V$ such that $h_{|\partial Q} = id$. Let us show that $0 \in \pi(h(Q))$.

We take $t \in [0,1]$, $u \in V_2$ and define

$$h_t(u) = t\pi(h(u)) + (1-t)u.$$

We have that $h_t \in C([0,1] \times V_2, V_2)$ is a homotopy between $h_0(u) = u$ and $h_1(u) = \pi(h(u))$. We want to apply degree theory. Let us observe that, for

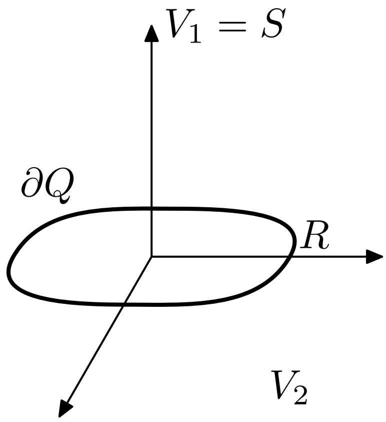

Figure 1. a linking

all $u \in \partial Q$,

$$h_t(u) = t\pi(h(u)) + (1-t)u = t\pi(u) + (1-t)u = u.$$

Hence it is well defined, for all $t \in [0,1]$, $\deg(h_t, Q, 0)$ and by homotopy invariance of the degree

$$\deg(\pi \circ h, Q, 0) = \deg(h_1, Q, 0) = \deg(h_0, Q, 0) = \deg(id, Q, 0) = 1.$$

Hence $\pi(h(u)) = 0$ has a solution $u \in Q$. □

Proposition 1.3. *Let V be a Banach space, $V = V_1 \oplus V_2$, V_1 and V_2 closed, $\dim V_2 < +\infty$. Let $\bar{u} \in V_1$, $\|\bar{u}\| = 1$ and ρ, R_1, $R_2 \in \mathbb{R}$ be such that $0 < \rho < R_1$, $0 < R_2$.*

Let $S = \left\{ u \in V_1 \mid \|u\| = \rho \right\}$ and $Q = \left\{ s\bar{u} + u_2 \mid 0 \le s \le R_1,\ u_2 \in V_2,\ \|u_2\| \le R_2 \right\}$, so that $\partial Q = \left\{ s\bar{u} + u_2 \in Q \mid s \in \{0, R_1\} \text{ or } \|u_2\| = R_2 \right\}$ (see figure 2).

Then S and ∂Q link.

Proof. Let $\pi \colon V \to V_2$ be the projection onto V_2, and take $h \colon V \to V$ such that $h_{|\partial Q} = id$. Let us show that exists $u \in Q$ such that $h(u) \in S$, that is such that $\pi(h(u)) = 0$ and $\|h(u)\| = \rho$.

We take $u_2 \in V_2$, $s \in \mathbb{R}$, $t \in [0,1]$ and let $u = s\bar{u} + u_2$ and

$$h_t(s, u_2) = (t(\|h(u)\| - \rho) + (1-t)(s-\rho), t\pi(h(u)) + (1-t)u_2).$$

Hence $h_t \colon \mathbb{R} \times V_2 \to \mathbb{R} \times V_2$ is a homotopy between $h_0(u) = (s - \rho, u_2)$ and $h_1(u) = (\|h(u)\| - \rho, \pi(h(u)))$. We want to find $u \in Q$ such that $h_1(u) = (0,0)$.

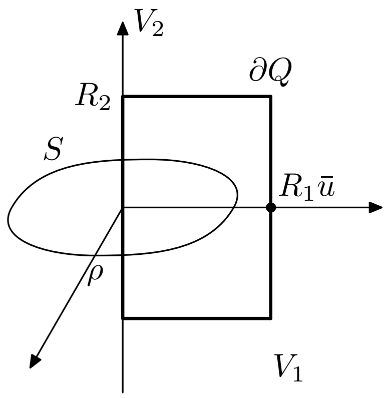

Figure 2. another linking

Let us observe that, for all $u \in \partial Q$,

$$\begin{aligned} h_t(u) &= (t(\|h(u)\| - \rho) + (1-t)(s-\rho), t\pi(h(u)) + (1-t)u_2) \\ &= (t(\|u\| - \rho) + (1-t)(s-\rho), t\pi(u) + (1-t)u_2) \\ &= (t(\|u\| - \rho) + (1-t)(s-\rho), tu_2 + (1-t)u_2) \\ &= (t(\|u\| - \rho) + (1-t)(s-\rho), u_2) \end{aligned}$$

Since $u = s\bar{u} + u_2 \in \partial Q$ and $u_2 = 0$ implies $s \in \{0, R_2\}$, we have that $h_t(u) \neq 0$ for all $t \in [0,1]$. Hence by the invariance under homotopy of the degree, we get that $\deg(h_0, Q, 0) = \deg(h_1, Q, 0) = 1$ and the existence of the required $u \in Q$ follows. $\square$

Remark 1.4. Let us note that, if $\bar{u} \in V$ is such that $\bar{u} \neq 0$, then $Q = \{\, \lambda\bar{u} \mid \lambda \in [0,1] \,\}$ and $S = \{\, u \in V \mid \|u\| = \alpha \,\}$, $0 < \alpha < \|\bar{u}\|$, then S and $\partial Q = \{0, \bar{u}\}$ link.

The following theorem is a generalization of the Mountain Pass Theorem.

Theorem 1.5 (Linking Theorem)**.** *Suppose $f \in C^1(V)$ satisfies the (PS) condition. Let $S \subset V$ be a closed set and $Q \subset V$ be a submanifold with relative boundary ∂Q. Suppose*

(1) *S and ∂Q link;*

(2) *$\alpha = \inf_{u\in S} f(u) > \sup_{u\in\partial Q} f(u) = \alpha_0$.*

Let

$$\Gamma = \left\{ h \in C(V,V) \;\middle|\; h_{|\partial Q} = id \right\}.$$

Then, setting

$$\beta = \inf_{\gamma\in\Gamma} \sup_{u\in Q} f(h(u))$$

we have that $\beta \geq \alpha$, and β is a critical value for f.

Proof. It is clear from the assumptions that $\beta \geq \alpha$.

Suppose $K_\beta = \{ u \in V \mid df(u) = 0 \text{ and } f(u) = \beta \} = \emptyset$. Let $\bar{\epsilon} = \alpha - \alpha_0$, $U = \emptyset$. Then, by theorem 5.4, there exist $\epsilon > 0$ and a deformation η. From the properties of η, it follows that $\eta(u,t) = u$ for all $t \in [0,1]$ if $u \in \partial Q$. Indeed $u \in \partial Q$ implies that $f(u) \leq \alpha_0 = \alpha_0 - (\alpha - \alpha_0) \leq \beta - \bar{\epsilon}$.

Take $h \in \Gamma$ such that

$$\sup_{u \in Q} f(h(u)) \leq \beta + \epsilon,$$

and let $\bar{h}(u) = \eta(h(u), 1)$. Then $\bar{h} \in \Gamma$ and $\sup_{u \in Q} f(\bar{h}(u)) \leq \beta - \epsilon$, contradiction which proves the theorem. □

2. Application III

In this section we will show how to apply the Linking theorem to study a problem a semilinear elliptic problem similar to the ones already seen in sections 1 and 2.

Let Ω be an open and bounded subset of $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, and assume g satisfies (g1-3). Consider the semilinear elliptic problem

$$(\text{BVP}_\lambda) \qquad \begin{cases} -\Delta u - \lambda u = g(x,u) & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

Remark 2.1. We will denote by $\lambda_1 = \lambda_1(\Omega) > 0$ the smallest eigenvalue of the linear eigenvalue problem

$$(\text{EVP}) \qquad \begin{cases} -\Delta u = \lambda u & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

and by $\phi_1(x) > 0$ the corresponding eigenfunction, normalized by $\int_\Omega \phi_1^2 = 1$. Then it holds that

$$(2.1) \qquad \int_\Omega |\nabla u|^2 \geq \lambda_1 \int_\Omega u^2 \qquad \forall u \in H_0^1(\Omega).$$

Similarly λ_k will denote the k-th eigenvalue of (EVP), with corresponding normalized eigenfunction ϕ_k. Recall that $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \to +\infty$.

Let $f \colon H_0^1(\Omega) \to \mathbb{R}$ be defined by

$$f(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \frac{\lambda}{2} \int_\Omega u^2 - \int_\Omega G(x,u).$$

where G is defined as in (g3). When working in $H_0^1(\Omega)$, Ω bounded, we will take as a norm $\|u\| = \int_\Omega |\nabla u|^2$, which, as a consequence of (2.1), is a norm equivalent to the usual one. Then our functional can be written as

$$f(u) = \frac{1}{2} \|u\|^2 - \frac{\lambda}{2} \int_\Omega u^2 - \int_\Omega G(x,u).$$

We already know from the results of section 1 that $f \in C^1$.

Remark 2.2. If $\lambda < \lambda_1$, one immediately sees that

$$f(u) \geq \frac{1}{2}\Big(1 - \frac{\lambda}{\lambda_1}\Big) \int |\nabla u|^2 - \int G(x,u)$$

and also in this situation $u = 0$ is a strict local minimum; moreover, since G is superquadratic, our functional has the Mountain Pass geometry. Since it can be proved (exactly as in section 1, see also lemma 2.3 below) that (PS) holds, the Mountain Pass theorem applies to prove existence of a nontrivial critical point.

Let us prove that (PS) holds for f regardless of the value of λ. (We remark here that this is true thanks to the rather strong assumptions (g1–g3). Under less restrictive assumptions (PS) would fail for $\lambda = \lambda_k$). Since the case $\lambda < 0$ is the one already seen in section 1, we will restrict to $\lambda > 0$.

Lemma 2.3. *Assume (g1–3) hold. Then, for all $\lambda \geq 0$, (PS) holds.*

Proof. Let us show that (PS) sequences are bounded. We have that

$$\begin{aligned}
\langle df(u_n), u_n \rangle &= \|u_n\| - \lambda \int u_n^2 - \int g(x,u_n)u_n \\
f(u_n) &= \frac{1}{2}\|u_n\| - \frac{\lambda}{2} \int u_n^2 - \int G(x,u_n)u_n.
\end{aligned}$$

Hence

$$\begin{aligned}
f(u_n) - \frac{1}{2}\langle df(u_n), u_n \rangle &= \int \big[\frac{1}{2} g(x,u_n)u_n - G(x,u_n)\big] \\
&\geq \Big(\frac{\mu}{2} - 1\Big) \int G(x,u_n) \\
&\geq C|u_n|_p^p \geq C|u_n|_2^p
\end{aligned}$$

and

$$\begin{aligned}
f(u_n) - \frac{1}{\mu}\langle df(u_n), u_n \rangle = \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\Big(\|u_n\|^2 - \lambda \int u_n^2\Big) \\
+ \int \big[\frac{1}{\mu} g(x,u_n)u_n - G(x,u_n)\big] \\
\geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big)\Big(\|u_n\|^2 - \lambda |u_n|_2^2\Big)
\end{aligned}$$

From the above equations we get that

$$\begin{aligned}\Big(\frac{1}{2}-\frac{1}{\mu}\Big)\|u_n\|^2 &\le f(u_n)-\frac{1}{\mu}\langle df(u_n),u_n\rangle+\lambda\Big(\frac{1}{2}-\frac{1}{\mu}\Big)|u_n|_2^2\\ &\le C+\epsilon_n\|u_n\|+C\big(f(u_n)-\frac{1}{2}\langle df(u_n),u_n\rangle\big)^{2/p}\\ &\le C+\epsilon_n\|u_n\|+\epsilon_n'\|u_n\|^{2/p}\end{aligned}$$

and the boundedness of the sequence $\{u_n\}$ follows.

To prove that (PS) holds we can then use theorem 4.3 as in lemma 1.3. □

We can now prove that the boundary value problem (BVP$_\lambda$) has a solution for all $\lambda \in \mathbb{R}$. Since we have already remarked that a solution exist whenever $\lambda < \lambda_1$ (see remark 2.2), we will only study the case $\lambda \ge \lambda_1$.

Theorem 2.4. *Suppose (g1–3) hold, $\lambda \ge \lambda_1$ and Ω is an open and bounded subset of $\mathbb{R}^N$. Then there exist a nontrivial solution of* (BVP$_\lambda$).

Proof. We want to apply theorem 1.5, taking S and Q as in proposition 1.3.

Suppose that $\lambda_{k_0} \le \lambda < \lambda_{k_0+1}$, and let $H_2 = \operatorname{span}\{\phi_1,\dots,\phi_{k_0}\}$, $H_1 = H_2^\perp = \operatorname{span}\{\phi_{k_0+1},\dots\}$. Then $\dim H_2 = k_0 < +\infty$, $H_0^1 = H_1 \oplus H_2$. Moreover

$$\int |\nabla u|^2 \le \lambda_{k_0}\int u^2 \qquad \forall u \in H_2 \tag{2.2}$$

$$\int |\nabla u|^2 \ge \lambda_{k_0+1}\int u^2 \qquad \forall u \in H_1 \tag{2.3}$$

It is then immediate to find that for $u \in H_1$, $\|u\| = \rho$ we have that

$$\begin{aligned} f(u) &\ge \frac{1}{2}\Big(1-\frac{\lambda}{\lambda_{k_0+1}}\Big)\int u^2 - C\int |u|^p\\ &\ge \frac{1}{2}\Big(1-\frac{\lambda}{\lambda_{k_0+1}}\Big)\|u\|^2 - C\|u\|^p \ge \alpha > 0\end{aligned}$$

provided $\rho > 0$ and small. We let $S = \big\{\, u \in H_1 \mid \|u\| = \rho \,\big\}$.

We also have that

$$f(u) \le \frac{1}{2}(\lambda_{k_0}-\lambda)\int u^2 - \int G(x,u) \qquad \forall u \in H_2. \tag{2.4}$$

Let us now fix $\bar u \in H_1$, $\bar u \not\equiv 0$. Take $u = s\bar u + u_2$, $s \in \mathbb{R}$, $u_2 \in H_2$. Since such a u belongs to the finite dimensional space $H_2 \oplus \mathbb{R}\bar u$, there exist a constant $C > 1$ such that

$$\frac{1}{C}\int u^2 \le \int |\nabla u|^2 \le C\int u^2.$$

As a consequence we have that

$$\begin{aligned} f(u) &= \frac{1}{2}\int |\nabla u|^2 - \lambda \int u^2 - \int G(x,u) \\ &\leq \Big(\frac{C}{2} - \frac{\lambda}{2}\Big) \int u^2 - C \int |u|^p \\ &\leq \Big(\frac{C}{2} - \frac{\lambda}{2}\Big) - C\Big(\int u^2\Big)^{p/2} \to -\infty \end{aligned}$$

as $\|u\| \to +\infty$. It is then clear that, letting, as in proposition 1.3, $Q = \big\{\, s\bar{u} + u_2 \mid 0 \leq s \leq R_1,\, u_2 \in H_2,\, \|u_2\| \leq R_2 \,\big\}$, $f_{|\partial Q} < 0$ provided $R_1 > \rho$ and R_2 are large enough. This is enough to apply theorem 1.5 and to finish the proof. $\square$

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