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Topological methods and differential equations II

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Contents

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1 Topological degree of LERAY-SCHAUDER.

Let X be a real Banach space, let Ω be a bounded, open subset of X and let $\Phi = I - T$, where I is the inclusion map of $\overline{\Omega}$ into X and $T : \overline{\Omega} \longrightarrow X$ is compact.

If $b \notin \Phi(\partial\Omega)$, then there exists a map of finite range $T_1 : \overline{\Omega} \longrightarrow X_1$ (finite range means that $\dim X_1 < \infty$) such that

$$\sup_{u\in\overline{\Omega}} \|T_1 u - Tu\| < \operatorname{dist}(b, \Phi(\partial\Omega)).$$

In addition, the integer given by the Brouwer degree $\deg(\,(I - T_1)|_{\Omega\cap X_1}, \Omega_1, b)$ is independent on T_1. Therefore we can define the **topological degree of Leray-Schauder**

$$\deg(\Phi, \Omega, b) = \deg(\,(I - T_1)|_{\Omega\cap X_1}, \Omega_1, b).$$

It satisfies the following basic properties.

i) **Normalization property**.

$$\deg(I, \Omega, b) = 1, \qquad \text{if } b \in \Omega.$$

ii) **Additivity property**.

Assume that Ω_1 and Ω_2 are open, bounded, disjoint subsets of Ω. If $b \notin \Phi(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$ then

$$\deg(\Phi, \Omega, b) = \deg(\Phi, \Omega_1, b) + \deg(\Phi, \Omega_2, b).$$

iii) **Homotopy property**.

Let $S \in C([0,1] \times \overline{\Omega}, X)$ be a compact map and define $H(t,u) = u - S(t,u)$. If $b : [0,1] \longrightarrow X$ is continuous and $b(t) \notin H([0,1] \times \partial\Omega)$, then

$$\deg(H(t,.), \Omega, b(t)) = \text{const.} \quad \forall t \in [0,1].$$

From the above properties it is easy to prove [9] the following ones.

iv) $\deg(\Phi, \emptyset, b) = 0$.

v) **Existence property**.

If $\deg(\Phi, \Omega, b) \neq 0$, then there exists $u \in \Omega$ such that $\Phi(u) = b$.

vi) **Excision property**.

If $K \subset \Omega$ is closed and $b \notin \Phi(K)$, then

$$\deg(\Phi, \Omega, b) = \deg(\Phi, \Omega - K, b).$$

vii)

$$S|_{\partial\Omega} = T|_{\partial\Omega} \Longrightarrow \deg((I - S), \Omega, b) = \deg((I - T), \Omega, b).$$

viii) **General homotopy property**

Let $\mathcal{O}$ be a bounded, open subset of $\mathbb{R} \times X$ and let $H : \overline{\mathcal{O}} \to X$ be a compact map. For every $\lambda \in \mathbb{R}$ we consider the λ-slice

$$\mathcal{O}_\lambda = \{u \in X : (\lambda, u) \in \mathcal{O}\}$$

and the map $H_\lambda : \overline{\mathcal{O}}_\lambda \to X$ given by

$$H_\lambda(u) = H(\lambda, u).$$

If

$$u - H_\lambda(u) \neq b, \quad \forall u \in \partial\mathcal{O}_\lambda, \quad \forall \lambda \in [a, b],$$

then the topological degree $\deg(I - H_\lambda, \mathcal{O}_\lambda, b)$ is well-defined and independent of λ.

ix) **Continuity**

a) **Continuity with respect to** b.

The degree is constant in each connected component of $X - \Phi(\partial\Omega)$.

b) **Continuity with respect to** T.

There exists a neighborhood V of T in the space[1] $Q(\Omega, X)$ of the compact operators from $\overline{\Omega}$ in X such that[2]

$$\deg\left(I - S, \Omega, b\right) = \text{const.}(= \deg\left(\Phi, \Omega, b\right)), \quad \forall S \in V.$$

[1] $Q(\Omega, X)$ with the norm $\|T\| = \sup_{x \in \overline{\Omega}} \|T(x)\|$.

[2] Note that V can be chosen to verify

$$b \notin (I - S)(\partial\Omega), \quad \forall S \in V.$$

2 A theorem of Leray and Schauder

Let X be a real Banach space, let Ω be a bounded and open subset of X, let $a < b$ be and let $T : [a,b] \times \overline{\Omega} \to X$ be a compact map. For $\lambda \in [a,b]$, consider the equation

$$\Phi(\lambda, u) = u - T(\lambda, u) = 0, \quad u \in X. \tag{1}$$

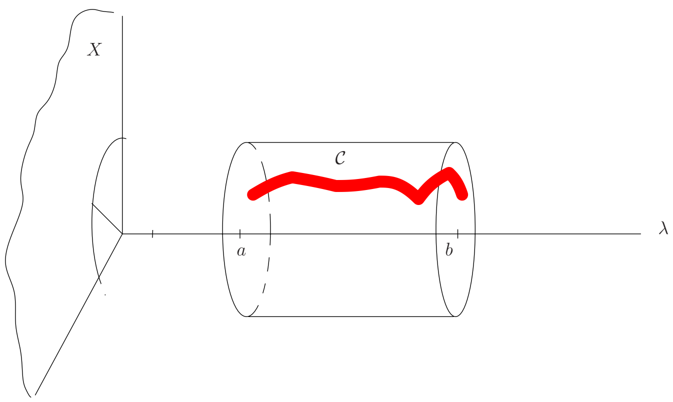
Sometimes, to put in evidence the dependence of (1) on λ, we refer it as $(1)_\lambda$. Observe that T can be seen as a family of compact operators

$$T_\lambda(u) := T(\lambda, u), \quad u \in X.$$

Similarly, we denote $\Phi_\lambda = I - T_\lambda$. Define

$$\Sigma = \{(\lambda, u) \in [a,b] \times \overline{\Omega} : \Phi(\lambda, u) = 0\}.$$

We use the notation Σ_λ for the λ-slice, i.e. $\Sigma_\lambda = \{u \in \overline{\Omega} : (\lambda, u) \in \Sigma\}$.



Theorem 1 (Leray-Schauder, 1934 (see also [13])). *Assume that X is a real Banach space, Ω is a bounded, open subset of X and $\Phi : [a,b] \times \overline{\Omega} \longrightarrow X$ is given by $\Phi(\lambda, u) = u - T(\lambda, u)$ with T a compact map. Suppose also that*

$$\Phi(\lambda, u) = u - T(\lambda, u) \neq 0, \quad \forall (\lambda, u) \in [a,b] \times \partial\Omega.$$

If

$$\deg(\Phi_a, \Omega, 0) \neq 0, \tag{2}$$

then

1. $(1)_\lambda$ *has a solution in* Ω *for every* $a \le \lambda \le b$.

2. Furthermore, there exists a compact connected set $\mathcal{C} \subset \Sigma$ *such that*

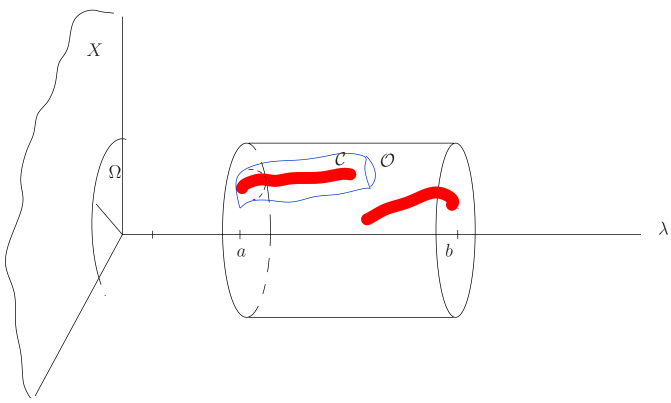
$$\mathcal{C} \cap \Sigma_a \neq \emptyset \text{ and } \mathcal{C} \cap \Sigma_b \neq \emptyset.$$

Proof. 1. First, observe that the homotopy property of the degree implies that

$$\deg(\Phi_\lambda, \Omega, 0) = \text{ const.}, \quad \lambda \in [a, b].$$

Therefore, by (2), $\deg(\Phi_\lambda, \Omega, 0) \neq 0$, for every $\lambda \in [a, b]$ and, in particular, from the existence property, $(1)_\lambda$ has a solution u_λ.

2. In the sequel, we see that the solutions u_λ can be chosen in a connected set of Σ. We argue by contradiction: suppose there does not exist a compact, connected set $\mathcal{C} \subset \Sigma$ such that $\mathcal{C} \cap \Sigma_a \neq \emptyset$ and $\mathcal{C} \cap \Sigma_b \neq \emptyset$.



We apply the following lemma (see [11, 24]).

Lemma 2. *Let* (M, d) *be a compact metric space, let* A *be a connected component of* M *and let* B *be a closed subset of* M *such that* $A \cap B = \emptyset$. *Then there exists compact sets* M_A *and* M_B *satisfying*

- $A \subset M_A$, $B \subset M_B$.
- $M = M_A \cup M_B$ *and* $M_A \cap M_B = \emptyset$.

□

Remark 3. As a consequence, *if A and B are closed subsets of the compact metric space M, then, either there exists a closed, connected set of M that connects A and B or $M = M_A \cup M_B$, where M_A and M_B are compact subsets of M containing, respectively, to A and B.*

Using this remark with $A = \Sigma_a$, and $B = \Sigma_b$ we deduce that there exist disjoint compact sets $M_A \supset A$ and $M_B \supset B$ such that $\Sigma = M_A \cup M_B$. It follows that there exists a bounded open set $\mathcal{O}$ in $[a,b] \times X$ such that $\Sigma_a \subset \mathcal{O}$, $\Sigma_b \cap \mathcal{O} = \emptyset$ and $T(\lambda, u) \neq u$ for $u \in \partial\mathcal{O}_\lambda$, with $\lambda \in [a,b]$. (We are denoting by $\mathcal{O}_\lambda$ the λ-slice, i.e. $\mathcal{O}_\lambda = \{u : (\lambda, u) \in \Sigma\}$ (we allow $\mathcal{O}_\lambda$ to be empty)).

The general homotopy property of the degree implies that

$$\deg(\Phi_\lambda, \mathcal{O}_\lambda, 0) = \deg(\Phi_a, \mathcal{O}_a, 0)$$

for $a \leq \lambda \leq b$. However, we know that $\deg(\Phi_b, \mathcal{O}_b, 0) = 0$, because Φ_b has no zeroes in $\mathcal{O}_b$. Since we assume (2), we have a contradiction. $\square$

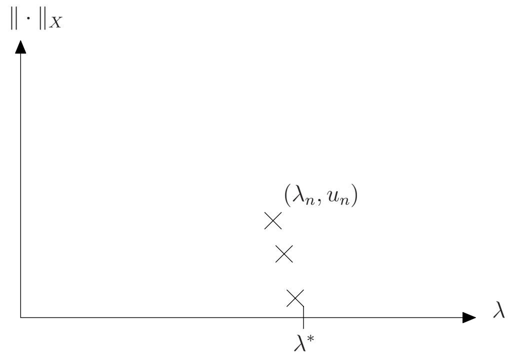
3 Bifurcation from the zero solution

Let X be a real Banach space and let $\Phi : \mathbb{R} \times X \to X$ be a compact map satisfying

$$\Phi(\lambda, 0) = 0, \quad \forall \lambda \in \mathbb{R}.$$

Denote by Σ^* the closure in $\mathbb{R} \times X$ of the set of pairs $(\lambda, u) \in \mathbb{R} \times X$ with u a nontrivial solution of the equation (1).

In next definition we give the notion of bifurcation from zero for this problem.



Definition 4. *We say that $\lambda^* \in \mathbb{R}$ is a bifurcation point from the trivial solution for* (1) *if* $(\lambda^*, 0) \in \overline{\Sigma^*}$. *Equivalently, if there exists a sequence* (λ_n, u_n) *in* $\mathbb{R} \times X$ *such that* $\lambda_n \to \lambda^*$, $\|u_n\| \to 0$ *and* $\Phi(\lambda, u_n) = 0$.

Remarks 5. *1. Observe that it can occur $\lambda_n = \lambda$ (vertical bifurcation).*

2. If $\Phi(\lambda, u) = u - \lambda L u$, with L a linear, compact operator, then $\lambda^* \in \mathbb{R} \setminus \{0\}$ is a bifurcation point if and only if $\frac{1}{\lambda^*}$ is a eigenvalue of L.

3. More generally, suppose that $\Phi(\lambda, u) = u - \lambda L u + N(\lambda, u)$, where L is a linear, compact operator and N is a compact operator satisfying

$$\lim_{\|u\| \to 0} \frac{N(\lambda, u)}{\|u\|} = 0,$$

uniformly in bounded sets of values λ. If $\lambda^* \in \mathbb{R} \setminus \{0\}$ is a bifurcation point, then $1/\lambda^*$ is an eigenvalue of L.

3.1 A necessary condition

A necessary condition to be a bifurcation point is easily deduced from the implicit theorem. It is given in the following proposition.

Proposition 6. *Assume that Φ is Fréchet differentiable and that λ^* is a bifurcation point from zero. Then the derivative $\Phi_u(\lambda^*, 0)$ of Φ with respect to u at $(\lambda^*, 0)$ is not invertible:*

$$\Phi_u(\lambda^*, 0) \notin \mathrm{Inv}\,(X).$$

$\square$

Example 7. For a bounded open subset D of $\mathbb{R}^N$ and $f \in C^1(\mathbb{R})$ such that $f(0) = 0$ and $f'(0) \neq 0$, let us consider the nonlinear boundary value problem

$$\begin{aligned} -\Delta u &= \lambda f(u), \quad x \in D, \\ u &= 0, \quad x \in \partial D. \end{aligned} \tag{3}$$

Observe that to find solutions of this problem is just to look for zeroes of an operator Φ. Indeed, if we consider the linear operator $(-\Delta)^{-1} : C(\overline{\Omega}) \longrightarrow C(\overline{\Omega})$ defined by taking $(-\Delta)^{-1}(v) = w$ as the unique solution of

$$\begin{aligned} -\Delta w &= v, \quad x \in D, \\ w &= 0, \quad x \in \partial D, \end{aligned}$$

and we denote with the same letter f the Nemistki operator $f : C(\overline{\Omega}) \longrightarrow C(\overline{\Omega})$ given by

$$f(v)(x) = f(v(x)), \quad \forall x \in \Omega, \quad \forall v \in C(\overline{\Omega}),$$

then the problem (3) is equivalent to the following one

$$\Phi(\lambda, u) = u - (-\Delta)^{-1}[f(u)] = 0, \quad u \in C(\overline{\Omega}).$$

The above necessary condition implies that if λ^* is a bifurcation point for (3), i.e. for $\Phi(\lambda, u) = 0$, then the derivative

$$\Phi_u(\lambda^*, 0)(v) = v - \lambda^*(-\Delta)^{-1}[f'(0)v], \quad v \in C(\overline{\Omega}),$$

is not invertible. This means that necessarily, λ^* is an eigenvalue of the Laplacian operator with zero Dirichlet boundary conditions.

Remark 8. *The necessary condition given by Proposition 6 is not sufficient as it is proved by the following counterexample.*

Example 9. Take $X = \mathbb{R}^2$ and $\Phi : \mathbb{R} \times X \longrightarrow X$ given by $\Phi(\lambda, x, y) = (\lambda x - y^3, \lambda y + x^3)$, for $\lambda \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2$. It is easy to check that the equation $\Phi(\lambda, x, y) = 0$ have not bifurcations points. Furthermore, $\Phi_u(0, 0, 0) = 0$ and hence $\lambda = 0$ satisfies the necessary condition.

Another way to prove that the necessary condition is not sufficient is given by the following example.

Example 10. Assume that $\Phi(\lambda, u) = \lambda u - Lu$ with L an bounded linear operator in X. Then a value λ^* is a bifurcation point for $\Phi(\lambda, u) = 0$ if and only if λ^* is an accumulation point of eigenvalues of L.

Observe that if $\lambda \in \mathbb{R}$ is not a bifurcation point, then $(\lambda, 0)$ is an isolated solution of (1). For isolated solutions it is useful the following definition.

3.2 Index of an isolated zero

Let $\Psi = I - T$ be with $T : \overline{\Omega} \longrightarrow X$ a compact operator. If $u_0 \in \Omega$ is an isolated solution of the equation $\Psi(u) = 0$, i.e. a unique solution of this equation in a neighborhood of u_0, then, for $r_0 > 0$ sufficiently small, we deduce from the excision property that

$$\deg(\Psi, B_r(u_0), 0) = \deg(\Psi, B_{r_0}(u_0), 0), \quad \forall r \in (0, r_0),$$

where $B_r(u_0) = \{u \in \Omega : \|u - u_0\| < r\}$. Therefore, the index of Ψ relative to u_0 is well-defined by

$$\mathrm{i}\,(\Psi, u_0) = \lim_{r\to 0} \deg\,(\Psi, B_r(u_0), 0).$$

3.3 The Global bifurcation theorem of Krasnoselskii and Rabinowitz

If $\Phi(\lambda, u)$ verifies the conditions of Remark 5-3, Krasnoselskii [18] showed that each characteristic value[3] λ^* of odd algebraic multiplicity of L is a bifurcation point. Furthermore, Rabinowitz [21] proved in 1971 that there exists a continuum $\mathcal{C}_{\lambda^*}$ of Σ^* that either is unbounded, or $(\lambda^\sharp, 0) \in \mathcal{C}_{\lambda^*}$ for some characteristic value $\lambda^\sharp \neq \lambda^*$. To emphasize the role of the degree in this result we give first a more general version of the Krasnoselskii-Rabinowitz theorem.

Theorem 11 (Krasnoselskii-Rabinowitz)**.** *Let $\lambda^* \in \mathbb{R}$ and $\varepsilon_0 > 0$ be such that the set $(\lambda^* - \varepsilon_0, \lambda^* + \varepsilon_0) \setminus \{\lambda^*\}$ does not contain bifurcation points of* (1)*. Assume also that for every $\underline{\lambda} \in (\lambda^* - \varepsilon_0, \lambda^*)$ and $\overline{\lambda} \in (\lambda^*, \lambda^* + \varepsilon_0)$ it holds*

$$\mathrm{i}(\Phi_{\underline{\lambda}}, 0) \neq \mathrm{i}(\Phi_{\overline{\lambda}}, 0). \tag{4}$$

Then

1. *(Krasnoselskii) The value λ^* is a bifurcation point of* (1)*.*
2. *(Rabinowitz) The connected component, $\mathcal{C}_{\lambda^*}$, of $\overline{\Sigma^*}$ that contains to $(\lambda^*, 0)$ satisfies al least one of the following conditions:*
   (i) *$\mathcal{C}_{\lambda^*}$ is not bounded in $\mathbb{R} \times X$,*
   (ii) *there exists a bifurcation point $\lambda^\sharp \in \mathbb{R}\backslash\{\lambda^*\}$ such that $(\lambda^\sharp, 0) \in \mathcal{C}_{\lambda^*}$.*

Remark 12. Observe that, since $(\lambda^* - \varepsilon_0, \lambda^* + \varepsilon_0) \setminus \{\lambda^*\}$ does not contains bifurcations points, the homotopy property implies that

$$\mathrm{i}(\Phi_{\underline{\lambda}}, 0) = \text{constant}, \quad \forall \underline{\lambda} \in (\lambda^* - \varepsilon_0, \lambda^*)$$

and

$$\mathrm{i}(\Phi_{\overline{\lambda}}, 0) = \text{constant}, \quad \forall \overline{\lambda} \in (\lambda^*, \lambda^* + \varepsilon_0).$$

[3] inverse of a nonzero eigenvalue

Proof. 1. We make the proof by contrapositive. Assume that λ^* is not a bifurcation point. This means that there exists $\rho > 0$ such that

$$\Phi(\lambda, u) \neq 0, \quad \forall \|u\| = \rho, \quad \forall \lambda \in (\lambda^* - \varepsilon_0, \lambda^* + \varepsilon_0).$$

Then the degree $\deg(\Phi_\lambda, B_\rho(0), 0)$ is well-defined and, by the homotopy property, it is independent on λ, i.e.

$$\mathrm{i}(\Phi_\lambda, 0) = \text{constant}, \quad \forall \lambda \in (\lambda^* - \varepsilon_0, \lambda^* + \varepsilon_0),$$

and condition (4) fails.

2. By 1., $(\lambda^*, 0) \in \overline{\Sigma^*}$. Let $\mathcal{C}_{\lambda^*}$ be the connected component of $\overline{\Sigma^*}$. We argue by contrapositive and assume that $\mathcal{C}_{\lambda^*}$ does not verify neither (i) nor (ii). This means that $\mathcal{C}_{\lambda^*}$ is bounded and that for every $\lambda \neq \lambda^*$ there exists $\rho(\lambda) > 0$ such that

$$\mathcal{C}_{\lambda^*} \cap B_{\rho(\lambda)}(0) = \emptyset.$$

Now we use a similar argument to that of the Leray-Schauder theorem to deduce that there exists a set $\mathcal{O} \subset \mathbb{R} \times X$ satisfying

$$\partial\mathcal{O} \cap \Sigma^* = \emptyset, \tag{5}$$

$$(\lambda^*, 0) \in \mathcal{O} \tag{6}$$

and

$$\mathcal{O} \cap (\mathbb{R} \times X) \subset (\lambda^* - \varepsilon_0, \lambda^* + \varepsilon_0) \times X. \tag{7}$$

The general homotopy property allows to deduce that

$$\deg(\Phi_\lambda, \mathcal{O}_\lambda, 0) = \text{ constant}, \quad \forall \lambda \in \mathbb{R}. \tag{8}$$

Now, we are to compute this degree. To do it, fix $\overline{\lambda} \in (\lambda^*, \lambda^* + \varepsilon_0)$ such that $(\overline{\lambda}, 0) \in \mathcal{O}$. We can choose $\rho > 0$ such that

a) For every $\lambda \in [\overline{\lambda}, \overline{\lambda} + \varepsilon_0]$, the problem $(1)_\lambda$ has no nontrivial solutions in $\overline{B_\rho(0)}$, i.e.

$$\Sigma^* \cap B_\rho(0) = \emptyset.$$

b) For every $\lambda \geq \overline{\lambda + \varepsilon_0}$, the λ-slice $\mathcal{O}_\lambda$ of $\mathcal{O}$ does not contains points of the closed ball $\overline{B_\rho(0)}$, i.e.

$$\mathcal{O}_\lambda \cap \overline{B_\rho(0)} = \emptyset.$$

Take
$$\mathcal{U} = \mathcal{O} \cap \left[[\overline{\lambda}, +\infty) \times (X \setminus \overline{B_\rho(0)}) \right].$$
Observe that the λ-slice $\mathcal{U}_\lambda$ of U_λ is given by
$$\mathcal{U}_\lambda = \mathcal{O}_\lambda \setminus \overline{B_\rho(0)},$$
for every $\lambda \geq \overline{\lambda}$.

By a) and b), the general homotopy property of the degree implies that
$$\deg(\Phi_\lambda, \mathcal{U}_\lambda, 0) = \text{ constant}, \quad \forall \lambda \geq \overline{\lambda}.$$
But, since $\mathcal{O}$ is bounded, $\mathcal{U}_\lambda = \mathcal{O}_\lambda \setminus \overline{B_\rho(0)} = \mathcal{O}_\lambda = \emptyset$ provided that $\lambda >> \overline{\lambda}$. We obtain as a consequence that the above degree is zero. In particular, $\deg(\Phi_{\overline{\lambda}}, \mathcal{U}_{\overline{\lambda}}, 0) = 0$, that is,
$$\deg(\Phi_{\overline{\lambda}}, \mathcal{O}_{\overline{\lambda}} \setminus \overline{B_\rho(0)}, 0) = 0.$$
By the additivity property we conclude that
$$\begin{aligned} \deg(\Phi_{\overline{\lambda}}, \mathcal{O}_{\overline{\lambda}}, 0) &= \deg(\Phi_{\overline{\lambda}}, \mathcal{O}_{\overline{\lambda}} \setminus \overline{B_\rho(0)}, 0) + \deg(\Phi_{\overline{\lambda}}, B_\rho(0), 0) \\ &= \mathrm{i}\,(\Phi_{\overline{\lambda}}, 0) \end{aligned}$$

Similarly, if we fix $\underline{\lambda} \in (\lambda^* - \varepsilon_0, \lambda^*)$ such that $(\underline{\lambda}, 0) \in \mathcal{O}$, we can prove that
$$\deg(\Phi_{\underline{\lambda}}, \mathcal{O}_{\underline{\lambda}}, 0) = \mathrm{i}\,(\Phi_{\underline{\lambda}}, 0).$$
Consequently, taking into account (8) we conclude that
$$\mathrm{i}(\Phi_{\underline{\lambda}}, 0) = \mathrm{i}(\Phi_{\overline{\lambda}}, 0),$$
and (4) fails. □

Now, as a consequence of the above theorem, we are ready to prove the classical result by Krasnoselskii and Rabinowitz.

Corollary 13. *Assume that $\Phi(\lambda, u) = u - \lambda L u + N(\lambda, u)$, where L is a linear, compact operator and N is a compact operator satisfying*
$$\lim_{\|u\| \to 0} \frac{N(\lambda, u)}{\|u\|} = 0,$$
uniformly in bounded sets of values λ. Then each characteristic value λ^* of odd algebraic multiplicity of L is a bifurcation point. Furthermore, there exists a continuum $\mathcal{C}_{\lambda^*}$ of Σ^* that either is unbounded, or $(\lambda^\sharp, 0) \in \mathcal{C}_{\lambda^*}$ for some characteristic value $\lambda^\sharp \neq \lambda^*$.

Proof. The proof is based on the computation of the index by linearization.

Lemma 14. *Assume that $\Phi = I - T$, with $T \in Q(\overline{\Omega}, X)$, where Ω is a neighborhood of zero. Suppose also that $T(0) = 0$ and that T is Fréchet differentiable in 0. We have:*

1. *The derivative $T'(0)$ of T at zero is compact.*

2. *If $I - T'(0)$ is invertible*[4] *in X, then 0 is the unique solution of $\Phi(u) = 0$ and*
$$\mathrm{i}\,(\Phi, 0) = \mathrm{i}\,(\Phi'(0), 0) = (-1)^\beta,$$
where
$$\beta = \sum_{\substack{\lambda \in \mu(T'(0)) \\ 0<\lambda<1}} \mathrm{mult}\,(\lambda),$$
with $\mu(T'(0))$ is denoting the set all characteristic values of $T'(0)$ and $\mathrm{mult}\,(\lambda)$ *is denoting the multiplicity of λ.* □

By applying the preceding lemma, we obtain for every $\lambda \in \mathbb{R}$ that
$$\mathrm{i}\,(\Phi_\lambda, 0) = \mathrm{i}\,(\Phi'_\lambda(0), 0) = \mathrm{i}\,(I - \lambda L), 0) = (-1)^\beta,$$
with
$$\beta = \sum_{\substack{\beta \in \mu(\lambda L) \\ 0<\beta<1}} \mathrm{mult}\,(\beta) = \sum_{\substack{\mu \in \mu(L) \\ 0<\mu<\lambda}} \mathrm{mult}\,(\mu).$$

In particular, if we fix $\underline{\lambda} < \lambda^* < \overline{\lambda}$ such that the unique characteristic value in the interval $(\underline{\lambda}, \overline{\lambda})$ is λ^*, we deduce from the oddness of the multiplicity of λ^* that
$$\mathrm{i}\,(\Phi_{\overline{\lambda}}, 0) = (-1)^{\mathrm{mult}\,(\lambda^*) + \sum_{\substack{\mu\in\mu(L) \\ 0<\mu<\lambda^*}} \mathrm{mult}\,(\mu)} = (-1)^{\mathrm{mult}\,(\lambda^*)}\,\mathrm{i}\,(\Phi_{\underline{\lambda}}, 0) = -\mathrm{i}\,(\Phi_{\underline{\lambda}}, 0)$$
and the Theorem 11 applies. □

[4] i.e., if 1 is not a characteristic value of $T'(0)$

4 Asymptotically linear problems

We give here some of the existence results in the work by Ambrosetti and Hess [1]. Specifically we study the existence of positive solutions of the boundary value problem

$$\begin{aligned} -\Delta u = \lambda f(u), \quad x \in D \\ u = 0, \quad x \in \partial D, \end{aligned} \tag{9}$$

where D is a bounded open subset of $\mathbb{R}^N$, $\lambda > 0$ and $f \in C^1([0, +\infty)$, with $f(0) = 0$ and with positive right derivative $f'_+(0) = m_0 > 0$.

First we reduce the study of the existence of positive solutions to this one of the existence of solutions of an extended problem. Indeed, we extend f to $(-\infty, 0)$ by defining $f(s) = f(0)$ for $s < 0$. With this extension, the maximum principle implies that every nontrivial solution of (9) is positive.

Now, take $X = C(\overline{\Omega})$, and consider the operator $\Phi : [0, \infty) \times X \longrightarrow X$ given by $\Phi(\lambda, u) = u - \lambda(-\Delta)^{-1}[f(u)]$, for every $\lambda \geq 0$ and $u \in X$. Observe that we can rewrite the extended problem (9) as the zeros of Φ, i.e.

$$\Phi(\lambda, u) = 0.$$

In the next theorem we denote by $\lambda_1 > 0$ the first eigenfunction of the Laplace operator with zero Dirichlet boundary condition. We also denote by $\varphi_1 > 0$ an eigenfunction associated to λ_1 with $\|\varphi_1\| = 1$.

Theorem 15. *If $f(0) = 0$ and $f'_+(0) = m_0 > 0$, then $\lambda_0 = \lambda_1/m_0$ is the unique bifurcation point from zero of positive solutions of* (9)*. In addition, the continuum emanating from $(\lambda_0, 0)$ is unbounded.*

Proof. To apply the Theorem 11 we just have to prove the change of index to cross $\lambda = \lambda_0$. We divide the proof of it into two steps:

Step 1 There exists $\lambda_0 > 0$ such that for every interval $\Lambda \subset [0, +\infty) \setminus \{\lambda_0\}$ there is $\varepsilon > 0$ satisfying

$$\Phi(\lambda, u) \neq 0, \quad \forall \lambda \in \Lambda, \quad \forall 0 < \|u\| < \varepsilon.$$

Step 2 For every $\lambda > \lambda_0$ there exists $\delta > 0$ such that

$$\Phi(\lambda, u) \neq \tau\varphi_1, \quad \forall 0 < \|u\| < \delta, \quad \forall \tau \geq 0.$$

To prove the Step 1, we argue by contradiction that there exists a sequence $(\lambda_n, u_n) \in \Lambda \times X$ satisfying

$$\lambda_n \longrightarrow \lambda \neq \lambda_0, \quad \|u_n\| \longrightarrow 0,$$

$$\Phi(\lambda_n, u_n) = 0, \quad u_n \geq 0.$$

Since $(-\Delta)^{-1}$ is compact, dividing by $\|u_n\|$ the equation $u_n = \lambda_n(-\Delta)^{-1}[f(u_n)]$, we deduce that, up to a subsequence,

$$\frac{u_n}{\|u_n\|} \longrightarrow v,$$

with $v \in X$ an eigenfunction of norm one associated to λ, i.e. it satisfies

$$v = \lambda(-\Delta)^{-1}[f'(0)v], \quad \|v\| = 1.$$

Using φ_1 as test function in this eigenvalue problem we obtain

$$\lambda_1 \int v\varphi_1 = \lambda f'(0) \int v\varphi_1$$

and we conclude that $\lambda_1 = \lambda f'(0)$ which is a contradiction and the proof of Step 1 is finished.

Consequences of Step 1:

a) The unique possible bifurcation point of positive solutions is $\lambda = \lambda_0$.

b) If $\lambda < \lambda_0$ and we take $\Lambda = [0, \lambda]$ then

$$\mathrm{i}\,(\Phi_\lambda, 0) = \mathrm{i}\,(\Phi_0, 0) = \mathrm{i}\,(I, 0) = 1.$$

With respect to the proof of Step 2, we fix $\lambda > \lambda_0$ and we assume, by contradiction, that there exist sequences $u_n \in X$ and $\tau_n \geq 0$ satisfying $u_n > 0 \quad \text{in } D$, $\|u_n\| \longrightarrow 0$ and

$$\Phi(\lambda, u_n) = \tau\varphi_1,$$

or, equivalently,

$$u_n = \lambda(-\Delta)^{-1}[f(u_n)] + \tau_n\varphi_1.$$

Dividing this equation by $\|u_n\|$ and using the compactness of $(-\Delta)^{-1}$, we deduce that, up to a subsequence, $(-\Delta)^{-1}[\frac{f(u_n)}{\|u_n\|}]$ is convergent and hence $\frac{\tau_n}{\|u_n\|}$

is bounded. Passing again to a subsequence, if necessary, we can assume that $\frac{\tau_n}{\|u_n\|} \longrightarrow \tau \geq 0$ and $\frac{u_n}{\|u_n\|} \longrightarrow v$ with $v \in X$ satisfying

$$\begin{aligned} -\Delta v = \lambda f'(0) v + \tau \lambda_1 \varphi_1, \quad x \in D \\ v = 0, \quad x \in \partial D \\ \|v\| = 1. \end{aligned}$$

As in the Step 1, we deduce then that $\lambda f'(0) = \lambda_1$, a contradiction.
Consequence of Step 2: For every $\lambda > \lambda_0$, we have from Step 2 that

$$\mathrm{i}\,(\Phi_\lambda, 0) = \mathrm{i}\,(\Phi_\lambda - \tau\varphi_1, 0), \quad \forall \tau > 0.$$

Using again the Step 2, the problem

$$\begin{aligned} -\Delta w = \lambda f(w) + \tau \varphi_1, \quad x \in D \\ w = 0, \quad x \in \partial D \end{aligned}$$

has not any nontrivial solutions. Since, $w = 0$ is not a solution provided that $\tau > 0$, we deduce that the last index is zero, i.e.

$$\mathrm{i}\,(\Phi_\lambda, 0) = 0.$$

□

5 Bifurcation from infinity

Definition 16. *λ_∞ is a bifurcation point from infinity of (1) if there exists a sequence $(\lambda_n, u_n) \in \mathbb{R} \times X$ satisfying*

$$\lambda_n \longrightarrow \lambda_\infty, \quad \|u_n\| \longrightarrow +\infty, \quad \Phi(\lambda_n, u_n) = 0.$$

Assume that

$$\Phi(\lambda, u) = u - T(\lambda, u),$$

with T a compact operator. Following [22], if we make the Kelvin transform

$$z = \frac{u}{\|u\|^2}, \quad u \neq 0,$$

we derive that

$$\left.\begin{array}{c} \Phi(\lambda, u) = 0 \\ u \neq 0 \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{c} z - \|z\|^2 T\left(\lambda, \dfrac{z}{\|z\|^2}\right) = 0 \\ z \neq 0. \end{array}\right.$$

Therefore, if we define

$$\widetilde{\Phi}(\lambda, z) = \begin{cases} z - \|z\|^2 T\left(\lambda, \dfrac{z}{\|z\|^2}\right), & \text{if } z \neq 0, \\ 0, & \text{if } z = 0, \end{cases}$$

we deduce that λ_∞ is a bifurcation point from infinity for $\Phi(\lambda, u) = 0$ if and only if λ_∞ is a bifurcation point from zero for $\widetilde{\Phi}(\lambda, z) = 0$.

Theorem 17. *Let $D \subset \mathbb{R}^N$ be bounded and open and let f be a C^1-function in $[0, +\infty)$ such that*

$$f(s) = m_\infty s + g(s)$$

where g satisfies

$$\lim_{s \to +\infty} g(s)/s = 0.$$

Then $\lambda_\infty = \lambda_1/m_\infty$ is the unique bifurcation point from infinity of positive solutions of (9). Moreover, there exists a subset S_∞ in $\mathbb{R} \times C(\overline{\Omega})$ of positive solutions of (9) such that if $\widetilde{S_\infty} = \{(\lambda, z) \,/\, (\lambda, z/\|z\|) \in S_\infty\}$ then $\widetilde{S_\infty} \cup \{(\lambda^*, 0)\}$ is connected and unbounded.

Proof. Apply the same ideas of the proof of Theorem 15 to verify that $\widetilde{\Phi}$ satisfies the claims of Steps 1 and 2 of this proof. □

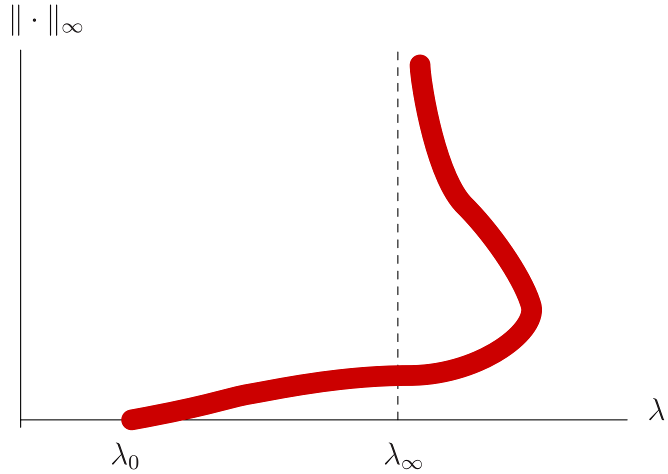


Figure 1: Bifurcation diagram for the case i) of Remark 18.

Remark 18. Assume that the hypotheses of Theorems 15 and 17 are satisfied.

1. Let α be a positive number. If $f(s) > \alpha s$ for every $s > 0$ then it is easy to show (taking φ_1 as test function) that the problem (9) has not any solution for $\lambda >> \lambda^*$. In this case, the continuum bifurcating from $(\lambda, 0)$ is the same that emanates from infinity at λ_∞.

2. In the case that there exists $0 < \theta_1 < \theta_2$ such that $f(s) \leq 0$, for every $s \in (\theta_1, \theta_2)$, then the reader can verify that the problem (9) has not any solution (λ, u) in the strip of $\mathbb{R} \times C(\overline{\Omega})$ given by $\theta_1 \leq \|u\| \leq \theta_2$.

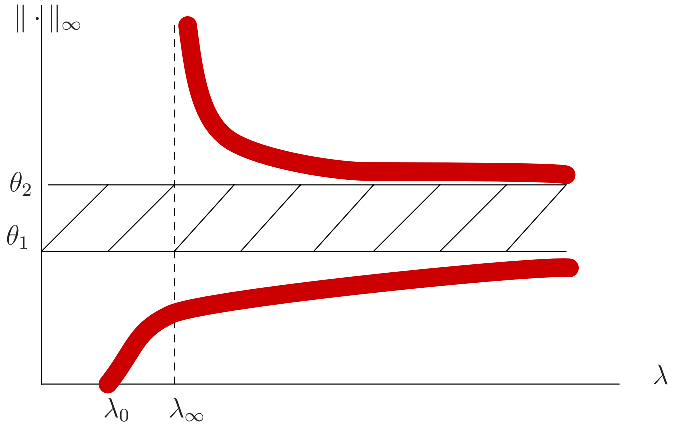


Figure 2: Bifurcation diagram for the case ii) of Remark 18.

6 On the side of the bifurcations from infinity

Let $D \subset \mathbb{R}^N$ be bounded and open and let g be a C^1-function in $[0, +\infty)$ satisfying

$$\lim_{s \to +\infty} g(s)/s = 0.$$

Consider the boundary value problem

$$\begin{aligned} -\Delta u = \lambda u + g(u), \quad & x \in D, \\ u = 0, \quad & x \in \partial D. \end{aligned} \tag{10}$$

In a similar way to the preceding results, it is possible to prove the following result.

Theorem 19. *The value $\lambda = \lambda_1$ is the unique bifurcation point from infinity of positive solutions of* (10)*. Moreover, if $g(0) = 0$, then $\lambda = \lambda_1 - g'(0)$ is the unique bifurcation point from zero of positive solutions of* (10)*. In addition, it emanates from it a continuum "connecting" $(\lambda_1 - g'(0), 0)$ with (λ_1, ∞).*

Proof. The bifurcation from zero at $\lambda_1 - g'(0)$ and the bifurcation from infinity at λ_1 are deduced as in the preceding theorems. On the other hand, since g is C^1, there exists $\alpha > 0$ such that $\alpha u > g(u) > -\alpha u$ for $u > 0$. Then the problem (10) has no solution provided that $|\lambda| >> 0$ and, therefore, the continuum emanating from zero at $\lambda_1 - g'(0)$ is also bifurcating from infinity at λ_1. □

Remark 20. *In particular, there exists a solution of* (10) *for every λ in the interval of extrema λ_1 and $\lambda_1 - g'(0)$. However, in the cases that we be able to decide the side of the bifurcations from infinity and from zero, we will improve this existence result.*

The side of the bifurcation from zero is completely described by the following theorem.

Theorem 21. *If there exists $\varepsilon > 0$ such that*

$$g(u) \geq 0, \quad \forall u \in (0, \varepsilon), \tag{11}$$

then the bifurcation from zero of the Theorem 19 is to the left. Similarly, if the inequality in (11) *is reversed, then the bifurcation from zero is to the right.*

Proof. If $(\lambda_n, u_n) \in \mathbb{R} \times X$ are solutions of (9) with $\lambda_n \to \lambda_1 - g'(0)$ and $\|u_n\| \to 0$, then, as we have seen before, up to a subsequence $u_n/\|u_n\|$ converges to φ_1. Using this eigenfunction as test function in the equation satisfied by u_n, we obtain

$$(\lambda_1 - \lambda_n) \int_D u_n \varphi_1 = \int_D g(u_n) \varphi_1.$$

Since $0 < u_n$ is uniformly convergent to zero, we deduce by (11) that $g(u_n(x)) \geq 0$, for every $x \in D$ and hence that $\lambda_n \leq \lambda_1$.

Similarly it is proved the result for the reversed inequalities. □

With respect to the bifurcation from infinity we deduce the following result.

Theorem 22 ([6]). *If there exists $\varepsilon > 0$ such that*

$$g(u)u^2 \geq \varepsilon, \quad \forall u >> 0, \tag{12}$$

then the bifurcation from infinity of the preceding theorem is to the left. Similarly, if the inequality in (12) *is reversed, then the bifurcation from infinity is to the right.*

Proof. If $(\lambda_n, u_n) \in \mathbb{R} \times X$ are solutions of (9) with $\lambda_n \to \lambda_1$ and $\|u_n\| \to \infty$, then, up to a subsequence $u_n/\|u_n\|$ converges to φ_1. Using this eigenfunction as test function in the equation satisfied by u_n and dividing by $\|u_n\|$, we obtain

$$(\lambda_1 - \lambda_n) \int_D \frac{u_n}{\|u_n\|}\varphi_1 = \frac{1}{\|u_n\|} \int_D g(u_n)\varphi_1.$$

Hence, taking into account that $\int_D \frac{u_n}{\|u_n\|}\varphi_1$ converges to $\int_D \varphi_1^2 > 0$, we deduce

$$\operatorname{sgn}[\lambda_1 - \lambda_n] = \operatorname{sgn}\left[\int_D g(u_n)\varphi_1\right].$$

To conclude the proof, we just have to show that the sign of the right hand is positive. This is deduced from the Fatou lemma. Indeed, by (12), we have

$$\begin{aligned} \liminf_{n\to+\infty} \int_D g(u_n)\varphi_1 &= \liminf_{n\to+\infty} \frac{1}{\|u_n\|^2} \int_D g(u_n)u_n^2 \left(\frac{u_n}{\|u_n\|}\right)^{-2} \varphi_1 \\ &\geq \varepsilon \int_D \frac{1}{\varphi_1} > 0. \end{aligned}$$

□

Remark 23. In [6], some counterexamples show that, in general, if the nonlinearity g is below any quadratic hyperbola c/s^2, then the side of the bifurcation from infinity can not be decided. The case of quasilinear operators in divergence form (instead of the Laplacian operator) is studied in [5]. More recent results can be found in [15, 16].

7 The local anti-maximum principle

As a consequence of the preceding results, we also point out the bifurcation nature of some classical results like the (local) Antimaximum Principle of

Clement and Peletier and the Landesman-Lazer theorem for resonant problems. We devote this section to give either the local anti-maximum principle or a local maximum principle.

Theorem 24. *Let $r > N$. For every $h \in L^r(D)$, there exists $\varepsilon = \varepsilon(h) > 0$ such that*

1. *If $\int_D h\varphi_1 < 0$, then every solution (λ, u) of*

$$\begin{array}{ll} -\Delta u = \lambda u + h(x), & \text{if } x \in D, \\ u(x) = 0, & \text{if } x \in \partial D, \end{array} \tag{13}$$

   satisfies

   (a) $u > 0$ in D provided that $\lambda_1 < \lambda < \lambda_1 + \varepsilon$,

   (b) $u < 0$ in D provided that $\lambda_1 - \varepsilon < \lambda < \lambda_1$.

2. *If $\int_D h\varphi_1 > 0$, then every solution (λ, u) of (13) satisfies*

   (a) $u < 0$ in D provided that $\lambda_1 < \lambda < \lambda_1 + \varepsilon$,

   (b) $u > 0$ in D provided that $\lambda_1 - \varepsilon < \lambda < \lambda_1$.

3. *If $\int_D h\varphi_1 = 0$, then every solution (λ, u) of (13) with $\lambda \neq \lambda_1$ changes its sign in D.*

Remark 25. In [10, Theorem 2] Clement and Peletier proved a slightly less general version of this theorem. Indeed, these authors substituted the condition of the sign of the integral of $u\varphi$ by a condition on the sign of h in all D.

Proof. We start with the case *1.* Note that by the Fredholm Alternative, the linear problem (13) has no solutions for $\lambda = \lambda_1$, and a unique solution if λ is not an eigenvalue of the Laplacian operator. In addition, for $X = W^{2,r}(D)$, the value $\lambda = \lambda_1$ is a bifurcation point "from $+\infty$" in the sense that there are solutions (λ, u) emanating from λ_1 at infinity satisfying that $u/\|u\|$ is converging in $W^{2,r}(D) \subset C^1(\overline{D})$ to $+\varphi_1$ as λ tends to λ_1. Also, there is a bifurcation "from $-\infty$", i.e. solutions (λ, u) emanating from λ_1 at infinity satisfying that $u/\|u\|$ is converging to $-\varphi_1$ as λ tends to λ_1. Now it is immediate to conclude from the preceding section that the bifurcation from $+\infty$ (of positive solutions at the beginning) is to the right, while the bifurcation from $-\infty$ is to the left. The proof of *1* is thus concluded. The argument for *2* is similar.

Finally, to prove *3*, it suffices to take φ_1 as test function in (13) to conclude that every solution (λ, u) of this problem satisfies $(\lambda_1 - \lambda) \int_D u\varphi_1 = 0$ and u changes of sign. $\square$

Remarks 26. 1. The choice of $r > N$ allows to apply our bifurcation results which involve the space $X = W^{2,r}(D)$, continuously embedded in $C^1(\overline{D})$. As we pointed out in the introduction, this fact allows to ensure that the normalized solutions converge to φ_1 (or to $-\varphi_1$) in the C^1-topology. Since φ_1 lies in the interior of the cone of positive functions of $C^1(\overline{D})$, then the positivity (or negativity) of the solutions near the bifurcation point easily follows. On the contrary, if we consider $r \leq N$, such an argument does not work, and in fact the result is not true, as it is proved in [23].

2. A related result for elliptic problems with nonlinear boundary conditions is given in [7].

8 The Landesman-Lazer condition

In this section we study the problem

$$\begin{array}{ll} -\Delta u = \lambda_1 u + g(u), & \text{if } x \in D, \\ u(x) = 0, & \text{if } x \in \partial D, \end{array} \tag{14}$$

where g is a continuous function for which

$$\exists g(+\infty) = \lim_{s \to +\infty} g(s) \quad \text{(pointwise limit)}, \tag{15}$$

$$\exists g(-\infty) = \lim_{s \to -\infty} g(s) \quad \text{(pointwise limit)}. \tag{16}$$

Recall the classical result by Landesman and Lazer [19] related to resonance at the principal eigenvalue λ_1.

Theorem 27. *Assume one of the following two conditions:*

$$\int_D g(+\infty)\varphi_1 < 0 < \int_D g(-\infty)\varphi_1, \tag{17}$$

or

$$\int_D g(+\infty)\varphi_1 > 0 > \int_D g(-\infty)\varphi_1. \tag{18}$$

Then the problem (14) *admits at least one solution.*

Proof. We approach the problem (14) by imbedding it into a one parameter family of problems as follows

$$\begin{array}{ll} -\Delta u = \lambda u + g(u), & \text{if } x \in D, \\ u(x) = 0, & \text{if } x \in \partial D, \end{array} \tag{19}$$

with $\lambda \in \mathbb{R}$. Observe that the boundedness of the function g ensures that bifurcation from infinity occurs for problem (19) at λ_1. In addition, by taking φ_1 as test function in (19), it is easily deduced that if the condition (17) holds then the bifurcation from infinity is to the right. Similarly, if (18) holds, the bifurcation from infinity is to the left.

As we will see, the behavior of the bifurcations from infinity at λ_1 for problem (19) determines the existence of solution for the resonant problem (14). The key to relate these two problems is to interpretate the concepts of bifurcations to the left and to the right in the sense of *a priori* bounds for the norms of the solutions. From this point of view, observe that *every possible bifurcation from ∞ at λ_1 is to the left (respectively to the right) if and only if there exist $\varepsilon > 0$ and $M > 0$ such that for every solution* (λ, u) *of* (19) *one has*

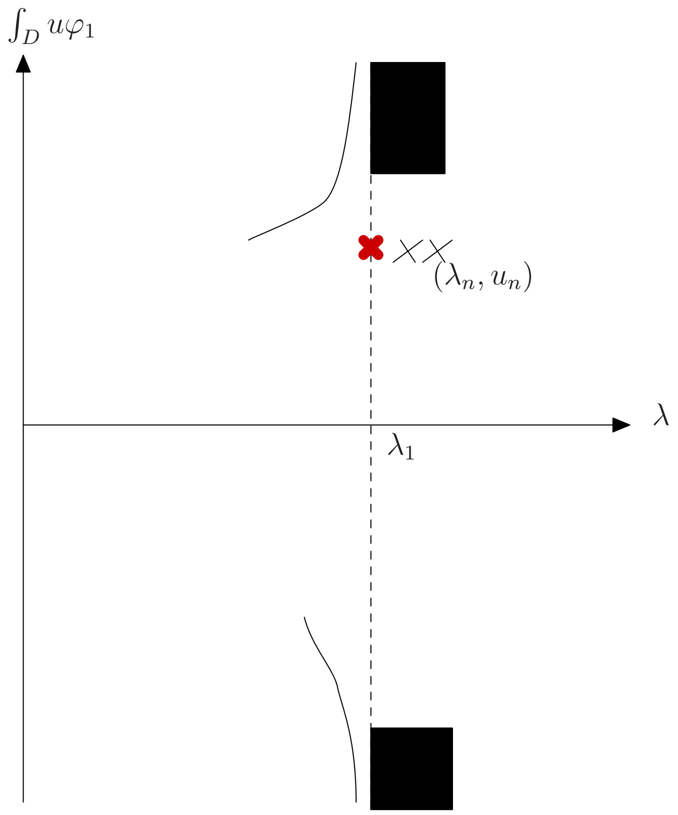
$$\lambda \in [\lambda_1, \lambda_1 + \varepsilon] \Rightarrow \|u\| \leq M$$

$$\textit{(respectively } \lambda \in [\lambda_1 - \varepsilon, \lambda_1] \Rightarrow \|u\| \leq M\textit{)}.$$

We just complete here the proof in case that condition (18). Our assumption implies that there exists $\varepsilon > 0$ and $M > 0$ such that

$$\|u\| \leq M$$

for every solution (λ, u) of (19) with $\lambda_1 \leq \lambda \leq \lambda_1 + \varepsilon$.



For every λ which is not an eigenvalue, there exists at least a solution (λ, u) of (19). Hence, if we take a sequence (λ_n, u_n) of solutions with $\lambda_n \to \lambda_1$, $\lambda_n > \lambda_1$, $\forall n \in \mathbb{N}$, then $\|u_n\| \leq M$ for n large. Standard arguments prove that a subsequence of u_n must converge to a solution of the resonant problem (14). $\square$

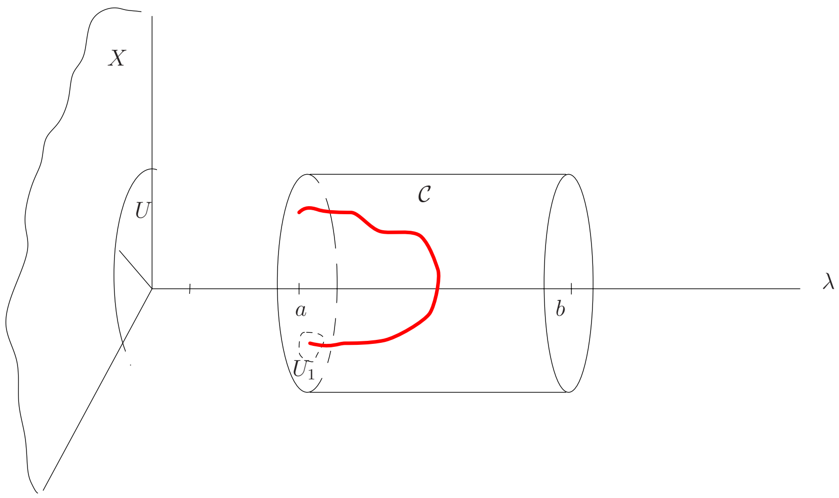
Remark 28. The previous general results of the side of bifurcations from infinity can be applied to obtain an improvement of the above classical existence result (see [6]).

9 Continuum with the shape $\supset$ and the Ambrosetti-Prodi problem

Let X be a real Banach space and consider a compact map $T : \mathbb{R} \times X \to X$. We denote again by Σ the closed set of the pairs $(\lambda, u) \in \mathbb{R} \times X$ with u a solution (not necessarily non-trivial) of $(1)_\lambda$. Next result is useful to prove existence of a continuum with the shape $\supset$.

Theorem 29. *Let $U \subset X$ be bounded, open and let $a, b \in \mathbb{R}$ be such that $(1)_\lambda$ has no solution in ∂U, provided that $\lambda \in [a,b]$, and that $(1)_b$ has no solution in $\overline{U}$. Let $U_1 \subset U$ be open such that $(1)_a$ has no solution in ∂U_1 and* $\deg(I - T_a, U_1, 0) \neq 0$. *Then there exists a continuum C in $\Sigma = \{(\lambda, u) \in [a,b] \times X : u \text{ is a solution of } (1)_\lambda\}$, such that*

$$C \cap (\{a\} \times U_1) \neq \emptyset, \quad C \cap (\{a\} \times (U \setminus U_1)) \neq \emptyset.$$



Proof. We use the following notation

$$K = ([a,b] \times U) \cap \Sigma,$$

$$A = (\{a\} \times \overline{U}_1) \cap K,$$
$$B = (\{a\} \times \overline{(U \setminus U_1)}) \cap K.$$

Since $(1)_b$ has no solution in $\overline{U}$ and K is compact, we can consider $K \subset [a,s] \times \overline{U}$ for some $s \in (a,b)$.

We argue by contradiction and assume that the theorem is false. By Lemma 2, there exist disjoint, compact subsets K_A, K_B containing respectively to A and B, such that $K = K_A \cup K_B$. Let $\mathcal{O}$ be a δ-neighborhood of K_A such that $\mathrm{dist}(\mathcal{O}, K_B) > 0$. Hence the Leray-Schauder degree is well defined in $\mathcal{O}_\lambda = \{u \in \overline{U} : (\lambda, u) \in \mathcal{O}\}$ for every $\lambda \in [a, b]$. Furthermore, by the general homotopy property, we have

$$\deg(I - T_\lambda, \mathcal{O}_\lambda, 0) = \text{constant},$$

and consequently

$$\deg(I - T_a, \mathcal{O}_a, 0) = \deg(I - T_b, \mathcal{O}_b, 0). \tag{20}$$

On the other hand, since $\mathcal{O} \cap K_B = \emptyset$, there are no solutions of e $(1)_a$ in $\mathcal{O}_a \setminus \overline{U}_1$ and hence, by excision property, we deduce that

$$\deg(I - T_a, \mathcal{O}_a, 0) = \deg(I - T_a, U_1, 0) \neq 0.$$

However, by hypothesis we know that $\mathcal{O}_b = \emptyset$, and thus we conclude that $\deg(I - T_b, O_b, 0) = 0$. This is a contradiction with (20), proving the theorem. $\square$

We apply the preceding theorem to give a proof of the well-known theorem of Ambrosetti and Prodi [2] for the boundary value problem

$$\begin{array}{c} -\Delta u = f(x, u) + t\varphi, \quad x \in D, \\ u = 0, \quad x \in \partial D. \end{array} \tag{P_t}$$

where $\varphi \in L^\infty(D)$ is a positive function, f is a continuous function such that

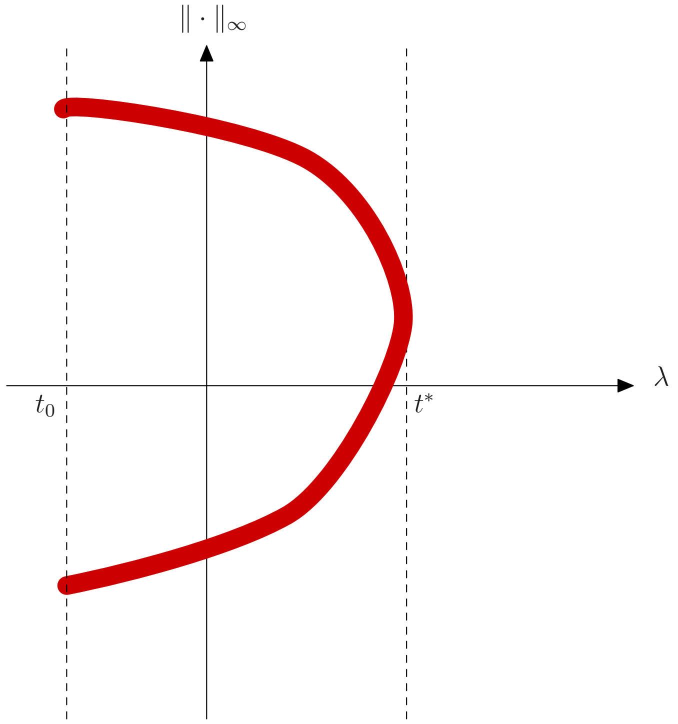
$$\lim_{s \to \pm\infty} \frac{f(s)}{s} = f'_{\pm\infty}. \tag{21}$$

Theorem 30. *Let $\varphi \in L^\infty(D)$ be a positive function and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying* (21) *and the condition*

$$f'_{-\infty} < \lambda_1 < f'_{+\infty} < \lambda_2. \tag{22}$$

Then $t^* \equiv \sup\{t \in \mathbb{R} : (P_t) \text{ admits a solution}\}$ is finite and for every $t_0 < t^*$ there exists a continuum $\mathcal{C}$ in $\Sigma \equiv \{(t, u) \in \mathbb{R} \times C_0^1(\overline{D}) : u \text{ solution of } (P_t)\}$ satisfying that

1. *$[t_0, t^*] \subset Proj_{\mathbb{R}}\, \mathcal{C}$.*
2. *For every $t \in [t_0, t^*)$, $Proj_{C_0^1(\overline{D})}\, \left[\mathcal{C} \cap (\{t\} \times C_0^1(\overline{D}))\right]$ contains two distinct solutions of (P_t) .*



Remark 31. We observe that, roughly speaking, the continuum $\mathcal{C}$ of solutions in $\mathbb{R} \times C_0^1(\overline{D})$ emanates from $\{t_0\} \times C_0^1(\overline{D})$, reaches $\{t^*\} \times C_0^1(\overline{D})$ and then, it turns left to meet a different solution in $\{t_0\} \times C_0^1(\overline{D})$ ($\supset$-shaped continuum). As a consequence,

1. (P_t) has, at least, two solutions for $t < t^*$,
2. (P_t) has, at least, one solution for $t \leq t^*$,
3. (P_t) has no solution for every $t > t^*$.

Proof. Let us denote $S \equiv \{t \in \mathbb{R} : (P_t) \text{ admits a solution}\}$. First we show that S is not the empty set. This relies upon two facts [12]:

- (P_t) has a supersolution for some $t \in \mathbb{R}$,
- given a supersolution, $\overline{u}$, of (P_t) for some $t \in \mathbb{R}$, there exists a subsolution $\underline{u}$ of it, such that $\underline{u} < \overline{u}$ in D.

Indeed, by means of the sub and super solution method, this implies that S is a closed interval unbounded from below. Moreover, the usual trick of multiplying by one positive eigenfunction associated to λ_1 leads to the nonexistence of solution for $t >> 0$ large enough and thus S is bounded from above. This means that the supremum of the closed interval S is attained. Denote

$$t^* \equiv \sup S = \max S.$$

We now prove the existence of the continuum of solutions. First it is easily deduced that if $t_0 < t^* < t_1$, then there exists $R > 0$ such that $\|u\|_{C^1} < R$ for each solution u of (P_t) with $t \in [t_0, t_1]$. Denote by Φ_t the map $\Phi_t(u) \equiv u - (-\Delta)^{-1}(f(x,u) + t\varphi)$. Using the homotopy invariance of Leray-Schauder degree and that problem (P_{t_1}) has no solution, we get

$$\deg(\Phi_t, B_R(0), 0) = \deg(\Phi_{t_1}, B_R(0), 0) = 0, \quad \forall t \in [t_0, t_1],$$

where $B_R(0)$ denotes the open ball in $C_0^1(\overline{D})$ of radius R centered at zero.

Let u^* be a solution of (P_{t^*}). Observe that u^* is a super-solution of (P_t) for every $t \in [t_0, t^*)$ and it is not a solution. Moreover, as it has been mentioned above, there exists a sub-solution $u_{t_0} < u^*$ of (P_{t_0}) which is not a solution. Clearly u_{t_0} is also a sub-solution and no solution for (P_t) if $t \in [t_0, t^*)$. Consider the set

$$U_1 = \{u \in C_0^1(\overline{D}) : u_{t_0} < u < u^* \text{ in } D,\ \frac{\partial u^*}{\partial n} < \frac{\partial u}{\partial n} < \frac{\partial u_{t_0}}{\partial n} \text{ on } \partial D\} \cap B_R(0).$$

The strong maximum principle implies the nonexistence of solutions of (P_t) on ∂U_1 if $t < t^*$ (see [14]). Hence, the degree of Φ_t is well defined in this set U_1. In addition, by using the results in [12],

$$\deg(\Phi_t, U_1, 0) = 1, \ \forall t \in [t_0, t^*).$$

Applying Lemma 29 with $E = C_0^1(\overline{D})$, $[a,b] = [t_0, t_1]$ and $U = B_R(0)$, we deduce the existence of a continuum $\mathcal{C}$ in Σ such that

$$\mathcal{C} \cap (\{t_0\} \times U_1) \neq \emptyset,$$

and

$$\mathcal{C} \cap (\{t_0\} \times [B_R(0) \setminus \overline{U_1}]) \neq \emptyset.$$

In particular, the continuum $\mathcal{C}$ crosses $\{t\} \times \partial U_1$, for some $t \in (t_0, t^*]$. It has been observed that this is possible if and only if $t = t^*$. This concludes the proof.

Remark 32. The above proof can cover the case of a nonlinearity f such that $f'_{+\infty} = +\infty$ (superlinear at $+\infty$) (see [4]). In this case the necessary a priori bound of the solutions of (P_t) is obtained by slightly improving the Gidas-Spruck technique [17]. More general (nonvariational) quasilinear elliptic operators than the Laplacian one can be considered [3, 8].

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