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Introduction to Optimization and Calculus of Variations

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Chapter 1

Direct Methods

1.1 An easy example

Let a and b be real numbers such that $a < b$ and let $\mathbf{f}\ \ [a\,, b] \ \rightarrow\ \mathbf{R}$ be differentiable. The problem $\min\limits_{x \in [a\,,\,b]} \mathbf{f}\,(x)$ has at least one solution $\underline{x} \in [a\,, b]$ by the Wierstrass's theorem. Moreover, the point $\underline{x}$ satisfies the following :

$$\forall\, x \;\in\; [a\,, b] \quad (x \;-\; \underline{x})\ \mathbf{f}^{'}\,(\underline{x}) \;\geq\; 0\ .$$

This relation is called the **Euler equation** of the problem $\min\limits_{x \in [a\,,\,b]} \mathbf{f}\,(x)$.
To prove this relation, we consider the three cases:

- If $\underline{x} \;=\; a$ then $\mathbf{f}^{'}\,(a) = \mathbf{f}^{'}_{d}\,(a) \;\geq\; 0$.
- If $\underline{x} \;=\; b$ then $\mathbf{f}^{'}\,(b) = \mathbf{f}^{'}_{g}\,(b) \;\leq\; 0$.
- If $\underline{x} \in \,]a\,, b[$, we have $\mathbf{f}^{'}\,(\underline{x}) \;=\; \mathbf{f}^{'}_{d}\,(\underline{x}) \;\geq\; 0$ and $\mathbf{f}^{'}\,(\underline{x}) \;=\; \mathbf{f}^{'}_{g}\,(\underline{x}) \;\leq\; 0$ thus $\mathbf{f}^{'}\,(\underline{x}) \;=\; 0$.

The compactness of $[a\,, b]$ permits us to prove the existence of a minimum and the derivation leads to the relation verified by the point $\underline{x}$ where $\mathbf{f}$ reaches its minimum. In this relation one should note that the properties are different depending on whether $\underline{x}$ is an interior point or a boundary point.

1.2 Minimum and maximum

1.2.1 Minimum

Definitions

Let $\mathbf{X}$ is a non void (non empty) set and $\mathbf{f}$ a map from $\mathbf{X}$ to $\mathbf{R} \cup \{+\infty\}$.

Definition 1 *Let $\underline{x} \in \mathbf{X}$, the function $\mathbf{f}$ has a minimum over $\mathbf{X}$ at the point $\underline{x} \in \mathbf{X}$, if we have :*

$$\forall x \in \mathbf{X} \quad \mathbf{f}(\underline{x}) \ \leq \ \mathbf{f}(x) \ .$$

One notes :

$$\mathbf{f}(\underline{x}) \ = \ \min_{x \in \mathbf{X}} \ \mathbf{f}(x) \ .$$

Definition 2 *Let $\underline{x} \in \mathbf{X}$, we say that $\mathbf{f}$ has a strict minimum over $\mathbf{X}$ at the point $\underline{x} \in \mathbf{X}$, if we have :*

$$\forall x \in \mathbf{X} \quad \mathbf{f}(\underline{x}) \ < \ \mathbf{f}(x) \ .$$

1.2.2 Maximum

Definitions

Let $\mathbf{X}$ a non void set and let $\mathbf{f}$ be a map from $\mathbf{X}$ to $\mathbf{R} \cup \{+\infty\}$.

Definition 3 *Let $\underline{x} \in \mathbf{X}$, we say that $\mathbf{f}$ has a maximum over $\mathbf{X}$ at the point $\underline{x} \in \mathbf{X}$, if we have :*

$$\forall x \in \mathbf{X} \quad \mathbf{f}(x) \ \leq \ \mathbf{f}(\overline{x}) \ .$$

One notes:

$$\mathbf{f}(\overline{x}) \ = \ \max_{x \in \mathbf{X}} \ \mathbf{f}(x) \ .$$

Definition 4 *Let $\underline{x} \in \mathbf{X}$, we say that $\mathbf{f}$ has a strict maximum over $\mathbf{X}$ at the point $\underline{x} \in \mathbf{X}$, if we have :*

$$\forall x \in \mathbf{X} \quad \mathbf{f}(x) \ < \ \mathbf{f}(\overline{x}) \ .$$

Remark 1 *:*

- *The map $\mathbf{f}$ has a maximum at the point $\overline{x}$ if and only if the map $-\mathbf{f}$ has a minimum at the point $\overline{x}$.*
- *The map $\mathbf{f}$ has a strict maximum at the point $\overline{x}$ if and only if the map $-\mathbf{f}$ has a strict minimum at the point $\overline{x}$.*

Remark 1 shows that the problem of finding the maximum may be posed as a minimization problem. We, therefore, restrict ourselves to the study of the problem of finding a minimum in this note only .

1.3 Lower semi continuity and upper semi continuity

Let $\mathcal{T}$ be a topology on $\mathbf{X}$. We denote the set of all neighbourhoods of a by $\mathcal{V}(a)$.

1.3.1 Lower semi continuity

Definition 5 *A function* $\mathbf{f}$ *of the topological space* $(\mathbf{X}\ ,\ \mathcal{T})$ *which takes its values in* $\boldsymbol{R} \cup \{+\infty\}$ *is lower semi continuous* (**lsc**) *at the point* $a \in \boldsymbol{X}$, *if one has :*

$$\forall \lambda \ \in \ \mathbf{R} \mid \lambda < \mathbf{f}(a) \ \ \exists V \in \mathcal{V}(a) \mid \forall x \in V \ \ \Rightarrow \ \ \mathbf{f}(x) > \ \lambda \ .$$

Exercice 1 *Show that if* $\mathbf{f}(a) \ \in \ \mathbf{R}$ *then* $\mathbf{f}$ *is* **lsc** *at the point* a *if and only if :*

$$\forall \epsilon \ > \ 0 \ \exists V \in \mathcal{V}(a) \mid \forall x \in V \ \ \Rightarrow \ \ \mathbf{f}(x) > \ \mathbf{f}(a) \ - \ \epsilon \ .$$

Deduce that if $\mathbf{f}$ *is continuous at a point* $a \ \in \ \mathbf{X}$ *then* $\mathbf{f}$ *is* **lsc** *at the point* a.

Exercice 2 *Let* $\mathbf{f}$ *be a map from the topological space* $(\mathbf{X}\ ,\ \mathcal{T})$ *which takes its values in* $\boldsymbol{R} \cup \{+\infty\}$.
Show that :

$$\mathbf{f}(a) \ \geq \ \sup_{V \in \mathcal{V}(a)} \ \inf_{y \in V} \ \mathbf{f}(y) \ \ .$$

Deduce that $\mathbf{f}$ *is* **lsc** *at a point* $a \in \boldsymbol{X}$ *if and only if :*

$$\mathbf{f}(a) \ = \ \sup_{V \in \mathcal{V}(a)} \ \inf_{y \in V} \ \mathbf{f}(y) \ \ .$$

Exercice 3 *Let* $\mathbf{f}$ *and* $\mathbf{g}$ *are maps from the topological space* $(\mathbf{X}\ ,\ \mathcal{T})$ *which take there values in* $\boldsymbol{R} \cup \{+\infty\}$ **lsc**, *show if* α *and* β *are real positive numbers then* $\alpha \mathbf{f} + \beta \mathbf{g}$ *is* **lsc**.

Definition 6 *A function* $\mathbf{f}$ *from the topological space* $(\mathbf{X}\ ,\ \mathcal{T})$ *which takes its values in* $\mathbf{R} \cup \{+\infty\}$ *is lower semi continuous, if it is* **lsc** *at every point of* $\mathbf{X}$.

Remark 2 *A continuous function is* **lsc***.*

We have the following properties :

Proposition 1 *A function* $\mathbf{f}$ *from the topological space* $(\mathbf{X}\,,\ \mathcal{T})$ *which takes its values in* $\mathbf{R} \cup \{+\infty\}$ *is* **lsc***, if and only if :*

$$\forall \lambda \,\in\, \mathbf{R},\ \ \mathbf{f}^{-1}\left(]\lambda\,,\,+\infty]\right) \in \mathcal{T}\ .$$

Proof :
Suppose that $\mathbf{f}$ is **lsc**. One sets :

$$\mathcal{O} \,=\, \mathbf{f}^{-1}\left(]\lambda\,,\,+\infty]\right)\ .$$

Let $a \in \mathcal{O}$, then $\mathbf{f}\,(a) \,>\, \lambda$. Since the function $\mathbf{f}$ is **lsc**,there exists $V \in \mathcal{V}\,(a)$ such that :

$$\forall x \in V \ \Rightarrow\ \mathbf{f}\,(x) > \,\lambda\ .$$

This implies that $V \,\subset\, \mathcal{O}$, thus $\mathcal{O} \,\in\, \mathcal{T}$.
Next, we prove the reverse. Let $a \,\in\, \mathbf{X}$ and let $\lambda \,\in\, \mathbf{R}$ such that $\mathbf{f}\,(a) \,>\, \lambda$ then

$$\mathcal{O} \,=\, \mathbf{f}^{-1}\left(]\lambda\,,\,+\infty]\right) \,\in\, \mathcal{V}\,(a)\ .$$

Thus $\mathbf{f}$ is **lsc**.

Definition 7 *If* $\mathbf{f}$ *is a map from* $\mathbf{X}$ *to* $\mathbf{R} \cup \{+\infty\}$*, the subset of* $\mathbf{X} \,\mathrm{x}\, \mathbf{R}$ *defined by :*

$$\mathbf{epi}\,(\mathbf{f}) \,=\, \{\ (x\,,\lambda) \in \mathbf{X} \,\mathrm{x}\, \mathbf{R} \,|\, \mathbf{f}\,(x) \leq\, \lambda\}\ .$$

is called the ***epigraph*** *of* $\mathbf{f}$*.*

The following proposition gives a characterisation of the lower semi continuity of a function by the properties of its epigrah.

Proposition 2 *A function* $\mathbf{f}$ *defined from the topological space* $(\mathbf{X}\,,\ \mathcal{T})$ *which takes its values in* $\mathbf{R} \cup \{+\infty\}$ *is* **lsc** *if and only if* $\mathbf{epi}\,(\mathbf{f})$ *is closed in* $\mathbf{X} \,\mathrm{x}\, \mathbf{R}$.

Proof : Suppose that $\mathbf{f}$ is **lsc**. Let $(a\,,\lambda_0) \in\, \mathbf{X} \,\mathrm{x}\, \mathbf{R}$ such that $(a\,,\lambda_0) \,\notin\, \mathbf{epi}\,(\mathbf{f})$ then $\mathbf{f}\,(a) \,>\, \lambda_0$. Let $\epsilon \,>\, 0$ be such that $\lambda_0 + \epsilon \,<\, \mathbf{f}\,(a)$. There exists $V \,\in\, \mathcal{V}\,(a)$ such that

$$\forall x \,\in\, V \ \ \mathbf{f}\,(x) \,>\, \lambda_0 + \epsilon\ .$$

Therefore $(V \text{ x }]\lambda_0 - \epsilon \, ; \lambda_0 + \epsilon[\,) \bigcap \mathbf{epi}\,(\mathbf{f})$ is void (empty) ; but $V \text{x}]\lambda_0 - \epsilon \, ; \lambda_0 + \epsilon[$ is a neighbourhood of $(a \, , \lambda_0)$ thus, $\mathbf{epi}\,(\mathbf{f})$ is closed.
Conversely, suppose that $\mathbf{epi}\,(\mathbf{f})$ is closed in $\mathbf{X}\,\text{x}\,\mathbf{R}$. Let $a \in \mathbf{X}$ and $\lambda \in \mathbf{R}$ such that $\mathbf{f}\,(a) > \lambda$, then $(\mathbf{f}\,(a) \, , \lambda) \notin \mathbf{epi}\,(\mathbf{f})$ thus, there exists $V \in \mathcal{V}\,(a)$ and $\epsilon > 0$ such that $(V \text{ x }]\lambda - \epsilon \, ; \lambda + \epsilon[\,) \bigcap \mathbf{epi}\,(\mathbf{f})$ is void. Let $x \in V$,

$$(x \, , \lambda) \notin \mathbf{epi}\,(\mathbf{f}) \ .$$

therefore $\mathbf{f}\,(x) > \lambda$. Then $\mathbf{f}$ is **lsc**.

For a family of **lsc** functions , we have :

Proposition 3 *If* $(\mathbf{f}_i)_{i \in \mathbf{I}}$ *is a family of* **lsc** *functions from the topological space* $(\mathbf{X} \, , \, \mathcal{T})$ *which take its values in* $\mathbf{R} \cup \{+\infty\}$ *then* $\sup_{i \in \mathbf{I}} \mathbf{f}_i$ *is a* **lsc** *function.*

Proof : It is enough to remark that :

$$\mathbf{epi}\,(\mathbf{f}) \ = \ \bigcap_{i \in \mathbf{I}} \mathbf{epi}\,(\mathbf{f}_i) \ .$$

Proposition 4 *If a function* $\mathbf{f}$ *from a topological space* $(\mathbf{X} \, , \, \mathcal{T})$ *to* $\mathbf{R} \cup \{+\infty\}$ *is* **lsc** *at a point* $a \in \mathbf{X}$*, if* $(x_n)_{n \in \mathbf{N}}$ *is a sequence in* $\mathbf{X}$ *such that* $\lim_{n \mapsto +\infty} x_n = a$*, then* $\liminf_{n \mapsto +\infty} \mathbf{f}\,(x_n) \ \geq \ \mathbf{f}\,(a)$

Proof : Let $\lambda < \mathbf{f}\,(a)$, there exists $V \in \mathcal{V}\,(a)$ such that :

$$\forall x \in V \quad \mathbf{f}\,(x) > \ \lambda \ ;$$

and since $\lim_{n \mapsto +\infty} x_n \ = \ a$, there exists $N \in \mathbf{N}$ such that

$$\forall n \in \mathbf{N}, \, n \geq N \ \Rightarrow \ x_n \in V \ .$$

Thus if $n \in \mathbf{N}$ is such that $n \geq N$ then

$$\inf_{p \geq n} \mathbf{f}\,(x_p) \ \geq \ \lambda \ ,$$

therefore

$$\sup_{n \in \mathbf{N}} \inf_{p \geq n} \mathbf{f}\,(x_p) \ \geq \ \lambda \ .$$

Conequently,

$$\liminf_{n \mapsto +\infty} \mathbf{f}\,(x_n) \ = \ \sup_{n \in \mathbf{N}} \inf_{p \geq n} \mathbf{f}\,(x_p) \ \geq \ \mathbf{f}\,(a) \ .$$

Exercice 4 *Prove that if* $(\mathbf{X}\ ,\ d)$ *is a metric space, a function* $\mathbf{f}$ *is* **lsc** *if and only if for all sequences* $(x_n)_{n\in\mathbf{N}}$ *such that* $\lim\limits_{n\mapsto+\infty} x_n = a$ *implies* $\lim\limits_{n\mapsto+\infty} \mathbf{f}(x_n) \geq \mathbf{f}(a)$.

Definition 8 *Let* $\lambda \in \mathbf{R}$*, a subset of* $\mathbf{X}$ *denoted by* $S_\lambda(\mathbf{f})$ *where :*

$$S_\lambda(\mathbf{f}) = \{x \in \mathbf{X} \mid \mathbf{f}(x) \leq \lambda\}$$

is called a section of $\mathbf{f}$.

Remark 3 *If* $\mathbf{f}$ *is* **lsc** *then* $S_\lambda(\mathbf{f})$ *is closed.*

1.3.2 Upper semi continuity

Definition 9 *A function* $\mathbf{f}$ *from a topological space* $(\mathbf{X}\ ,\ \mathcal{T})$ *which takes its values in* $\boldsymbol{R} \cup \{-\infty\}$ *is upper semi continuous* (**usc**) *at the point* $a \in \boldsymbol{X}$*, if one has :*

$$\forall\lambda \in \mathbf{R} \mid \mathbf{f}(a) < \lambda\ \exists V \in \mathcal{V}(a) \mid \forall x \in V \Rightarrow \mathbf{f}(x) < \lambda\ .$$

Remark 4 *The map* $\mathbf{f}$ *is* **usc** *if and only if* $-\mathbf{f}$ *is* **lsc**.

1.4 Wierstrass's theorem

Definition 10 *If* $\mathbf{f}$ *is a function from* $\mathbf{X}$ *to* $\mathbf{R} \cup \{+\infty\}$*, the* ***domain of*** $\mathbf{f}$ *is the subset denoted by* $\mathbf{dom}(\mathbf{f})$ *and defined by :*

$$\mathbf{dom}(\mathbf{f}) = \{x \in \mathbf{X} \mid \mathbf{f}(x) < +\infty\}\ .$$

If $\mathbf{dom}(\mathbf{f})$ *is non void, one says that* $\mathbf{f}$ *is* ***proper***.

Theorem 1 ***(Wierstrass)***
If $(\mathbf{X}\ ,\ \mathcal{T})$ *is a* ***compact*** *topological space and if* $\mathbf{f}$ *is a* ***proper*** *map and* **lsc** *from* $(\mathbf{X}\ ,\ \mathcal{T})$ *which takes its values in* $\mathbf{R} \cup \{+\infty\}$ *then there exists* $\underline{x} \in \mathbf{X}$ *such that*

$$\forall x \in \mathbf{X}\quad \mathbf{f}(\underline{x}) \leq \mathbf{f}(x)\ .$$

Proof : Let $m = \inf\limits_{x\in\mathbf{X}} f(x)$.
Suppose that $m = -\infty$. There exists a sequence $(x_n)_{n\in\mathbf{N}}$ of elements of $\mathbf{X}$ such that

$$\forall n \in \mathbf{N}\quad \mathbf{f}(x_n) < -n\ .$$

Let $\lambda \in \mathbf{R}$, then there exists $N \in \mathbf{N}$ such that

$$\forall\, n \in \mathbf{N} \mid n \geq N \Rightarrow \mathbf{f}(x_n) < \lambda$$

The sequence $(x_n)_{n\in\mathbf{N}}$ has a cluster point $\underline{x} \in \mathbf{X}$. The function $\mathbf{f}$ is lsc at $\underline{x}$ then there exists $V \in \mathcal{V}(\underline{x})$ such that

$$\forall x \in V \quad \mathbf{f}(x) > \lambda$$

There exists also $p \in \mathbf{N}$ such that $p > N$ and $x_p \in V$ then $\mathbf{f}(x_p) > \lambda$ and $\mathbf{f}(x_p) < \lambda$. It is impossible. Thus $m \in \mathbf{R}$.
Let $\underline{x} \in \mathbf{X}$ be the cluster point of the minimizing sequence $(x_n)_{n\in\mathbf{N}}$. Suppose that $\mathbf{f}(\underline{x}) > m$. Then there exists $\delta > 0$ such that $\mathbf{f}(\underline{x}) > m + \delta$. But the function $\mathbf{f}$ is lsc at $\underline{x}$, thus there exists $V \in \mathcal{V}(\underline{x})$ such that

$$\forall x \in V \quad \mathbf{f}(x) > m + \delta$$

There exists also $N \in \mathbf{N}$ such that

$$\forall\, n \in \mathbf{N} \mid n \geq N \Rightarrow m \leq \mathbf{f}(x_n) < m + \delta$$

There exists $p \in \mathbf{N}$ such that $p > N$ and $x_p \in V$ then $\mathbf{f}(x_p) > m + \delta$ and $\mathbf{f}(x_p) < m + \delta$. It is impossible. Thus $m = \mathbf{f}(\underline{x})$.

1.4.1 Sequentially compact set

Let $(\mathbf{X}\,,\ \mathcal{T})$ a topological space.

Definition 11 *A subset* $\mathbf{K}$ *of* $\mathbf{X}$ *is said to be* ***sequentially compact*** *if every sequence of elements of* $\mathbf{K}$ *has a subsequence which converges to an element of* $\mathbf{K}$*.*

Following the proof of the **Wierstrass's theorem** we have:

Theorem 2 *If* $(\mathbf{X}\,,\ \mathcal{T})$ *is a* ***sequentially compact*** *topological space and if* $\mathbf{f}$ *is a* ***proper*** *map and* **lsc** *from* $(\mathbf{X}\,,\ \mathcal{T})$ *which takes its values in* $\mathbf{R}\cup\{+\infty\}$ *then there exists* $\underline{x} \in \mathbf{X}$ *such that*

$$\forall x \in \mathbf{X} \quad \mathbf{f}(\underline{x}) \leq \mathbf{f}(x) \ .$$

1.5 Coercivity property

1.5.1 Coercivity

Definition 12 *A function* $\mathbf{f}$ *from the topological space* $(\mathbf{X}\,,\ \mathcal{T})$ *which takes its values in* $\mathbf{R}\cup\{+\infty\}$ *is said* ***coercive*** *if the closure of every section*

$$S_\lambda(\mathbf{f}) = \{x\in\mathbf{X} \mid \mathbf{f}(x)\le\lambda\}$$

is compact in $\mathbf{X}$.

Definition 13 *A map* $\mathbf{f}$ *from the topological space* $(\mathbf{X}\,,\ \mathcal{T})$ *to* $\mathbf{R}\cup\{+\infty\}$ *is said to be* ***sequentially coercive*** *if the closure of every section*

$$S_\lambda(\mathbf{f}) = \{x\in\mathbf{X} \mid \mathbf{f}(x)\le\lambda\}$$

is sequentially compact in $\mathbf{X}$*.*

If $(\mathbf{X}\,,\ \|\ \|_{\mathbf{X}})$ is a reflexive Banach space, one defines, in general, the coercivity of $\mathbf{f}$ by the property :

$$\lim_{\|x\|_{\mathbf{X}}\mapsto+\infty}\mathbf{f}(x) \;=\; +\infty\ .$$

In fact we have :

Proposition 5 *If* $(\mathbf{X}\,,\ \|\ \|_{\mathbf{X}})$ *is a reflexive Banach space then the map* $\mathbf{f}$ *is weakly sequentially coercive if and only if :*

$$\lim_{\|x\|_{\mathbf{x}}\mapsto+\infty}\mathbf{f}(x) \;=\; +\infty\ .$$

Proof : One supposes that $\mathbf{f}$ is weakly sequentially coercive . If $\mathbf{f}$ does not tend to $+\infty$ when $\|x\|_{\mathbf{x}}\mapsto+\infty$, there exists a sequence $(x_n)_{n\in\mathbf{N}}$ such that : $\lim_{n\mapsto+\infty}\|x_n\|_{\mathbf{x}} \;=\; +\infty$ and $(\mathbf{f}(x_n))_{n\in\mathbf{N}}$ is bounded. Let $\lambda\in\mathbf{R}$ such that :

$$\forall n\in\mathbf{N},\ |\mathbf{f}(x_n)|\le\lambda\ .$$

As $S_\lambda(\mathbf{f})$ is weakly sequentially compact, the sequence $(x_n)_{n\in\mathbf{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbf{N}}$ which converges weakly to an element $\underline{x}$ of $S_\lambda(\mathbf{f})$. Then $(x_{n_k})_{k\in\mathbf{N}}$ is bounded . This is impossible .
Now, we prove the converse. Let $\lambda\in\mathbf{R}$ and $(x_n)_{n\in\mathbf{N}}$ a sequence of elements of $S_\lambda(\mathbf{f})$ then $(x_n)_{n\in\mathbf{N}}$ is a bounded sequence ; then it has a weakly convergent subsequence $(x_{n_k})_{k\in\mathbf{N}}$ in the closure of $S_\lambda(\mathbf{f})$. Thus $\mathbf{f}$ is sequentially coercive.

Theorem 3 ***(Tonelli's Theorem)*** *Let* $\mathbf{f}$ *from* $(\mathbf{X}\ ,\ \mathcal{T})$ *to* $\mathbf{R}\cup\{+\infty\}$*, be a* ***proper****,* ***coercive****,* **lsc** *function. Then* $\mathbf{f}$ *has a minimum in* $\mathbf{X}$*.*

Proof : Let $a\in\mathbf{X}$ be such that $\mathbf{f}(a)<+\infty$, the subset $S_{\mathbf{f}(a)}(\mathbf{f})$ is relatively compact (that is, the closure is compact), but $\mathbf{f}$ is **lsc**, since $S_{\mathbf{f}(a)}(\mathbf{f})$ is closed, it is compact. Thus by **Wierstrass's theorem**, there exists $\underline{x}\in S_{\mathbf{f}(a)}(\mathbf{f})$ such that :

$$\forall x\in S_{\mathbf{f}(a)}(\mathbf{f})\ \ \mathbf{f}(\underline{x})\leq\mathbf{f}(x)\ \ .$$

As a result :

$$\forall x\in\mathbf{X}\ \mathbf{f}(\underline{x})\leq\mathbf{f}(x)\ \ .$$

The following theorem is easy to prove :

Theorem 4 ***(Tonelli's Theorem)*** *A map* $\mathbf{f}$ *from the topological space* $(\mathbf{X}\ ,\ \mathcal{T})$ *to* $\mathbf{R}\cup\{+\infty\}$*,* ***proper****,* **lsc** *and* ***sequentially coercive*** *has at least a minimum in* $\mathbf{X}$*.*

1.6 Minimizing sequences

Let $\mathbf{f}$ be a map from $\mathbf{X}$ to $\mathbf{R}\cup\{+\infty\}$ such that

$$m=\inf_{x\in\mathbf{X}}\mathbf{f}(x)\ \ .$$

Definition 14 *We say that a sequence* $(x_n)_{n\in\ \mathbf{N}}$ *is a* ***minimizing sequence*** *of* $\mathbf{f}$*, if it verifies :*

$$\lim_{n\mapsto+\infty}\mathbf{f}(x_n)\ =\ m\ \ .$$

Exercice 5 *Prove that every proper map* $\mathbf{f}$ *has a minimizing sequence.*

Remark 5 *As consequences of the* ***Tonelli's theorems****, if* $\mathbf{f}$ *is a map from the topological space* $(\mathbf{X}\ ,\ \mathcal{T})$ *to* $\mathbf{R}\cup\{+\infty\}$ *is* ***proper*** *and* **lsc***, one has :*

- *if* $\mathbf{f}$ *is coercive, every minimizing sequence* $(x_n)_{n\in\mathbf{N}}$ *of* $\mathbf{f}$ *has a cluster point* $\underline{x}\in\mathbf{X}$ *where* $\underline{x}$ *is the minimum point of* $\mathbf{f}$ *:* $\mathbf{f}(\underline{x})\ =\ m$*;*
- *if* $\mathbf{f}$ *is sequentially coercive, every minimizing sequence* $(x_n)_{n\in\mathbf{N}}$ *of* $\mathbf{f}$ *has a subsequence* $(x_{n_k})_{k\in\mathbf{N}}$ *which converges to a point* $\underline{x}\in\mathbf{X}$ *where* $\underline{x}$ *is the minimum point of* $\mathbf{f}$ *:* $\mathbf{f}(\underline{x})\ =\ m$*.*

1.7 Convexity

In this part of the note, $\mathbf{X}$ is a real linear space with the norm $\| \|_{\mathbf{X}}$. One denotes by $\mathbf{X}^*$, the topological dual space of $\mathbf{X}$, that is $\mathbf{X}^*$ is the real linear space of the continuous linear forms on the normed space $(\mathbf{X}\,,\,\| \|_{\mathbf{X}})$. One denotes the bilinear pairing in the duality between $\mathbf{X}$ and $\mathbf{X}^*$ by $<\,,\,>$ then :

$$\forall x^* \in \mathbf{X}^* \ \forall x \in \mathbf{X}, \ \ x^*(x) = < x^*\,, x > \ .$$

If we set for every $x^* \in \mathbf{X}^*$,

$$\|x^*\|_{\mathbf{X}^*} = \sup_{\|x\|_{\mathbf{X}}=1} < x^*\,, x > \ ,$$

then $(\mathbf{X}^*\,,\,\| \|_{\mathbf{X}^*})$ is a normed linear space.

1.7.1 Convex sets

Definition 15 *: A subset* $\mathbf{C}$ *of* $\mathbf{X}$ *is said to be* ***convex*** *if :*

$$\forall t \in [0\,,1] \ \ \forall x \in \mathbf{C} \ \forall y \in \mathbf{C} \ \ tx + (1-t)y \ \in \mathbf{C} \ .$$

Definition 16 *Let x and y belong to $\mathbf{X}$, a subset of $\mathbf{X}$ denoted by $[x\,,y]$ is called the geometrical segment with extremal points x and y and it is given by :*

$$[x\,,y] \ = \ \{\, tx + (1-t)y \mid t \in [0\,,1]\} \ .$$

A subset $\mathbf{C}$ of $\mathbf{X}$ is **convex** if and only if every geometrical segment with extremal points in $\mathbf{C}$ is included in $\mathbf{C}$.

We now have the following :

Proposition 6 *A subset* $\mathbf{C}$ *of* $\mathbf{X}$ *is convex if and only if :*

$$\forall \ x_1,\dots,x_p \ \in \ \mathbf{C} \ \ \forall \ \alpha_1,\dots,\alpha_p \ \in \ \mathbf{R}_+ \mid \sum_{i=1}^{p} \alpha_i \ = \ 1 \ \Rightarrow \ \sum_{i=1}^{p} \alpha_i\, x_i \ \in \ \mathbf{C} \ .$$

Proof : The proof is given by induction . Suppose that $\mathbf{C}$ is a convex subset . Then the case $p = 2$ is obvious . Suppose that the hypothesis is true for an integer p greater than 2, we prove that it is also true foe $p+1$. Now, let $x_1,\dots,x_{p+1}$ be points of $\mathbf{C}$. Let $\alpha_1,\dots,\alpha_{p+1}$ any positive real numbers such that $\sum_{i=1}^{i=p+1} \alpha_i \ = \ 1$. If $\alpha_{p+1} = 0$, we are in the case of p points. So, by induction hypothesis, we are done. If $\alpha_{p+1} \neq 0$ then $y = \frac{1}{1-\alpha_{p+1}} \sum_{i=1}^{p} \alpha_i \, x_i$ belong **C** then

$$\sum_{i=1}^{p+1} \alpha_i \, x_i \;=\; (1-\alpha_{p+1})\, y + \alpha_{p+1} x_{p+1} \;\in\; \mathbf{C} \; .$$

Exemples :

- The linear space $\mathbf{X}$, every sublinear space of $\mathbf{X}$ and every affine subspace of $\mathbf{X}$ are convex .
- Let $\mathbf{f} \ \mathbf{X} \ \rightarrow \ \mathbf{R}$ be a linear map such that $\mathbf{f}$ is not identically zero and $\alpha \in \mathbf{R}$, the subset denoted by $H_{\mathbf{f}\ \alpha} = \{x \in \mathbf{X} \mid \mathbf{f}\,(x) = \alpha\}$, called a hyperplane is convex .
- Let $\mathbf{f} \ \mathbf{X} \ \rightarrow \ \mathbf{R}$ be a linear map such that $\mathbf{f}$ is not identically zero and $\alpha \in \mathbf{R}$, the following subsets $D^{+}_{\mathbf{f}\ \alpha} = \{x \in \mathbf{X} \mid \mathbf{f}\,(x) \geq \alpha\}$ and $D^{-}_{\mathbf{f}\ \alpha} = \{x \in \mathbf{X} \mid \mathbf{f}\,(x) \leq \alpha\}$ called closed half spaces are convex .
- Let $\mathbf{f} \ \mathbf{X} \ \rightarrow \ \mathbf{R}$ be a linear map such that $\mathbf{f}$ is not identically zero and $\alpha \in \mathbf{R}$, the following subsets $D^{*\,+}_{\mathbf{f}\ \alpha} = \{x \in \mathbf{X} \mid \mathbf{f}\,(x) > \alpha\}$ and $D^{*\,-}_{\mathbf{f}\ \alpha} = \{x \in \mathbf{X} \mid \mathbf{f}\,(x) < \alpha\}$ called open half spaces are convex .

Exercice 6 *Let* $\mathbf{X}$ *be a real linear space endowed with the norm* $\| \ \|_{\mathbf{X}}$. *Let* $\mathbf{f} \ \mathbf{X} \ \rightarrow \ \mathbf{R}$ *be a linear map such that* $\mathbf{f}$ *is not identically zero and* $\alpha \in \mathbf{R}$.

- *Prove that* $\mathbf{f}$ *is continuous if and only if* $H_{\mathbf{f}\ \alpha}$ *is a closed subset .*
- *Prove that if* $\mathbf{f}$ *is continuous then* $D^{+}_{\mathbf{f}\ \alpha}$ *and* $D^{-}_{\mathbf{f}\ \alpha}$ *are closed subsets .*
- *Prove that if* $\mathbf{f}$ *is continuous then* $D^{*\,+}_{\mathbf{f}\ \alpha}$ *and* $D^{*\,-}_{\mathbf{f}\ \alpha}$ *are open subsets .*

The convex subsets have the following proprties :

- If $(\mathbf{C}_i)_{i \in \mathbf{I}}$ is a family of convex subsets of $\mathbf{X}$ then $\bigcap_{i \in \mathbf{I}} \mathbf{C}_i$ is convex .
- If $(\mathbf{C}_i)_{1 \leq i \leq n}$ is a finite family of convex subsets and if $(\lambda_i)_{1 \leq i \leq n}$ are real numbers then $\sum_{i=1}^{n} \lambda_i \mathbf{C}_i$ is a convex subset .
- The closure of a convex subset is convex .

1.7.2 The Convex Functions

Definition 17 *A function* $\mathbf{f}$ *defined on a subset* $\mathbf{C}$ *of* $\mathbf{X}$ *which takes its values in* $\mathbf{R} \cup \{+\infty\}$*, is called convex if* $\mathbf{C}$ *is convex and if*

$$\forall t \in [0\,,1]\ \ \forall x \in \mathbf{C}\ \forall y \in \mathbf{C},\ \ \mathbf{f}\,(t\,x{+}(1-t)\ y)\ \leq\ t\,\mathbf{f}\,(x)\ +\ (1-t)\ \mathbf{f}\,(y)\ \ .$$

Remark 6 *If* $\mathbf{f}$ *is a convex function from the convex subset* $\mathbf{C}$ *of* $\mathbf{X}$ *to* $\mathbf{R} \cup \{+\infty\}$*, one defines* ***the convex extention*** *of* $\mathbf{f}$ *as the function* $\mathbf{f}_{co}$ *from* $\mathbf{X}$ *to* $\mathbf{R} \cup \{+\infty\}$ *such that*

- $\forall x \in \mathbf{C}\ \mathbf{f}_{co}\,(x) = \mathbf{f}\,(x)\ \ ,$
- $\forall x \notin \mathbf{C}\ \mathbf{f}_{co}\,(x) = +\infty\ \ .$

The function $\mathbf{f}_{co}$ is convex on $\mathbf{X}$ if and only if $\mathbf{f}$ is convex on $\mathbf{C}$.
The function $\mathbf{f}_{co}$ and $\mathbf{f}$ are proper at the same time .
This extension does not change the minimization problem. Then if $\mathbf{f}$ is proper, it has a minimum at a point $\underline{x} \in \mathbf{C}$ if and only if $\mathbf{f}_{co}$ has a minimum on $\mathbf{X}$ at a point $\underline{x}$.

We shall consider in what follows the functions defined on $\mathbf{X}$ and which takes its values in $\mathbf{R} \cup \{+\infty\}$.

Definition 18 *A function* $\mathbf{f}$ *defined on a subset* $\mathbf{C}$ *of* $\mathbf{X}$ *which takes its values in* $\mathbf{R} \cup \{+\infty\}$*, is said to be stricly convex if* $\mathbf{C}$ *is convex and if*

$$\forall t \in]0\,,1[\ \ \forall x \in \mathbf{C}\ \forall y \in \mathbf{C}\ \text{and}\ x \neq y,\ \ \mathbf{f}\,(t\,x{+}(1-t)\ y) < t\mathbf{f}\,(x){+}(1-t)\,\mathbf{f}\,(y)\ \ .$$

A stricly convex function is a convex function .

Proposition 7 *A function* $\mathbf{f}$ *from* $\mathbf{X}$ *to* $\mathbf{R} \cup \{+\infty\}$ *is convex if and only if* $\mathbf{epi}\,(\mathbf{f})$ *is a convex subset of* $\mathbf{X}\,\mathrm{x}\,\mathbf{R}$*.*

Proof : Suppose that $\mathbf{f}$ is convex .
Let $(x\,,\gamma_1) \in \mathbf{epi}\,(\mathbf{f})$, $(y\,,\gamma_2) \in \mathbf{epi}\,(\mathbf{f})$, let $t \in [0\,,1]$, One has :

$$\mathbf{f}\,(tx + (1-t)\,y) \leq t\mathbf{f}\,(x) + (1-t)\,\mathbf{f}\,(y) \leq t\gamma_1 + (1-t)\,\gamma_2\ .$$

Thus

$$t\,(y\,,\gamma_1) + (1-t)\,(y\,,\gamma_2) = (tx + (1-t)\,y\,, t\gamma_1 + (1-t)\,\gamma_2) \in \mathbf{epi}\,(\mathbf{f})\ \ .$$

So, $\mathbf{epi}(\mathbf{f})$ is convex .
Conversely we suppose $\mathbf{epi}(\mathbf{f})$ to be convex .
Let $x \in \mathbf{dom}(\mathbf{f})$ and $y \in \mathbf{dom}(\mathbf{f})$, then $(x \ , \mathbf{f}(x)) \in \mathbf{epi}(\mathbf{f})$ and $(y \ , \mathbf{f}(y)) \in \mathbf{epi}(\mathbf{f})$ thus if $t \in [0\, , 1]$ then

$$t\,(x \ , \mathbf{f}(x)) + (1-t)\,(y \ , \mathbf{f}(y)) \in \mathbf{epi}(\mathbf{f}) \ ,$$

therefore

$$\mathbf{f}(t\,x + (1-t)\ y) \ \leq \ t\,\mathbf{f}(x) \ + \ (1-t)\ \mathbf{f}(y) \ .$$

So, the function $\mathbf{f}$ is convex.

Remark 7 *If* $\mathbf{f}$ *is convex then the sections* $\mathbf{S}_\alpha(\mathbf{f})$ *are convex .*

Exemples :

- If $\mathbf{f}$ and $\mathbf{g}$ are convex functions from $\mathbf{X}$ to $\mathbf{R} \cup \{+\infty\}$, if $\lambda \in \mathbf{R}_+$ and if $\mu \in \mathbf{R}_+$ then $\lambda\mathbf{f} + \mu\mathbf{g}$ is convex.
- If $\mathbf{f}$ is convex and $\mathbf{g}$ is strictly convex from $\mathbf{X}$ to $\mathbf{R} \cup \{+\infty\}$ then $\mathbf{f} + \mathbf{g}$ is strictly convex .
- If $\mathbf{f}$ and $\mathbf{g}$ are strictly convex functions from $\mathbf{X}$ to $\mathbf{R} \cup \{+\infty\}$, if $\lambda \in \mathbf{R}_+^*$ and if $\mu \in \mathbf{R}_+^*$ then $\lambda\mathbf{f} + \mu\mathbf{g}$ is strictly convex .
- If $(\mathbf{f}_i)_{i\in\mathbf{I}}$ is a family of convex functions from $\mathbf{X}$ to $\mathbf{R} \cup \{+\infty\}$ then $\sup\limits_{i\in\mathbf{I}} \mathbf{f}_i$ is convex .

1.7.3 Continuity of the convex functions

Proposition 8 *If* $\mathbf{f}$ *is a convex function from* $\mathbf{X}$ *to* $\mathbf{R} \cup \{+\infty\}$ *which is bounded above on a neighbourhood of a point a belonging to its domain, then* $\mathbf{f}$ *is continuous at the point a and, moreover,* $\mathbf{f}$ *is locally lipschitz in the interior of its domain .*

Proof : Define the function $\mathbf{g}$ by :

$$\mathbf{g}(y) = \mathbf{f}(a + y) - \mathbf{f}(a) \ .$$

Then 0 is in the domain of $\mathbf{g}$, $\mathbf{g}(0) = 0$ and $\mathbf{g}$ is bounded above in a neighbourhood of 0. The function $\mathbf{f}$ is continuous at the point a if and only if $\mathbf{g}$ is continuous at 0. Let $M > 0$ and $r > 0$ such that:

$$\forall y \in \mathbf{X} \mid \ \|y\|_{\mathbf{X}} \leq r \Rightarrow \ \mathbf{g}(y) \leq M \ .$$

Let $r > 0$, if $x \in \mathbf{X} \setminus \{0\}$ and if $\|x\|_{\mathbf{X}} < r$ then because, $\mathbf{g}$ is convex, we have :

$$\mathbf{g}(x) = \mathbf{g}\left(\left(1 - \frac{\|x\|_{\mathbf{X}}}{r}\right)0 + \frac{\|x\|_{\mathbf{X}}}{r}\,\frac{r}{\|x\|_{\mathbf{X}}}x\right) \leq \left(1 - \frac{\|x\|_{\mathbf{X}}}{r}\right)\mathbf{g}(0) + \frac{\|x\|_{\mathbf{X}}}{r}\mathbf{g}\left(\frac{r}{\|x\|_{\mathbf{X}}}\,x\right) .$$

As $\left\|\frac{r}{\|x\|_{\mathbf{X}}}\,x\right\|_{\mathbf{X}} = r$, one has :

$$\mathbf{g}(x) \leq \frac{\|x\|_{\mathbf{X}}}{r}M .$$

In addition, one has :

$$0 = \frac{r}{r + \|x\|_{\mathbf{X}}}\,x + \left(1 - \frac{r}{r + \|x\|_{\mathbf{X}}}\right)\left(-\frac{r}{\|x\|_{\mathbf{X}}}x\right)$$

then

$$0 \leq \frac{r}{r + \|x\|_{\mathbf{X}}}\mathbf{g}(x) + \left(1 - \frac{r}{r + \|x\|_{\mathbf{X}}}\right)\mathbf{g}\left(-\frac{r}{\|x\|_{\mathbf{X}}}x\right)$$

thus

$$-\frac{\|x\|_{\mathbf{X}}}{r}\,\mathbf{g}\left(-\frac{r}{\|x\|_{\mathbf{X}}}x\right) \;\leq\; \mathbf{g}(x)$$

therefore

$$-\frac{\|x\|_{\mathbf{X}}}{r}\,M \;\leq\; \mathbf{g}(x) .$$

At the end, we have

$$|\mathbf{g}(x)| \;\leq\; \frac{\|x\|_{\mathbf{X}}}{r}\,M .$$

Let $\epsilon > 0$, we set $\eta = \mathtt{min}\left(\frac{r}{M}\epsilon\,,\,r\right)$, if $x \in \mathbf{X}$ and $\|x\|_{\mathbf{X}} < \eta$ then we have :

$$|\mathbf{g}(x)| \;< \epsilon .$$

Now we prove that $\mathbf{g}$ is continuous in the interior of its domain . It is enough to prove that $\mathbf{g}$ is bounded above a neighbourhood of every point of $\overbrace{\mathbf{dom}(\mathbf{g})}^{0}$.

Let x belong to $\mathbf{dom}(\mathbf{g})$. The function of the segment $[0\,,1]$ to $\mathbf{X}$ which associates t with $(1+t)\,x$ is continuous, then there exists $t_0 \in\,]0\,,1[$ such that

$$\forall t \in [0\,,t_0] \;\Rightarrow\; (1+t)\,x \;\in \overbrace{\mathbf{dom}(\mathbf{g})}^{0} .$$

We set $x_0 = (1+t_0)\,x$ and $r_1 = \frac{t_0}{1+t_0} r$. Let $y \in \mathbf{X}$ be such that $\|y-x\|_{\mathbf{X}} < r_1$, then we have : $\left\| \frac{1+t_0}{t_0}\,(y-x) \right\|_{\mathbf{X}} < r$. However, $y = \frac{t_0}{1+t_0}\left(\frac{1+t_0}{t_0}\,(y-x)\right) + \frac{1}{1+t_0}\,((1+t_0)\,x)$ then $y = \frac{t_0}{1+t_0}\left(\frac{1+t_0}{t_0}\,(y-x)\right) + \frac{1}{1+t_0} x_0$ thus $\mathbf{g}\,(y) \le \frac{t_0}{1+t_0}\mathbf{g}\left(\frac{1+t_0}{t_0}\,(y-x)\right) + \frac{1}{1+t_0}\mathbf{g}\,(x_0)$ d'où $\mathbf{g}\,(y) \le \frac{t_0}{1+t_0} M + \frac{1}{1+t_0}\mathbf{g}\,(x_0)$.
We set $M_1 = \mathtt{max}\,(M\ ,\mathbf{g}\,(x_0))$.
Then one has :

$$\forall y \in \mathbf{X} \mid \ \|y-x\|_{\mathbf{X}} < r_1 \ \Rightarrow \ \mathbf{g}\,(y) \le M_1 \ .$$

Therefore $\mathbf{g}$ is continuous at the point x. To complete the proof we show that $\mathbf{g}$ is locally lipschitz on $\overbrace{\mathbf{dom}\,(\mathbf{g})}^{0}$. It is enough to prove it at the point 0 . Let $\delta > 0$ such that $\delta < r$ and let $u \in B\,(0,\delta)$, $v \in B\,(0,\delta)$.
Let $n \in \mathbf{N}$ such that $n > \frac{\|u-v\|_{\mathbf{X}}}{r-\delta}$, we set $\forall i \in \{1,...,n\}\ \ x_{i+1} = x_i + \frac{1}{n}\,(v-u)$ with $x_1 = u$. Then $x_{n+1} = v$ and $\forall i \in \{1,...,n-1\}$, one has : $x_{i+1} \in B\,(x_i,r-\delta)$ thus $x_{i+1} \in B\,(0,r)$. The first part of the proof give us :

$$|\mathbf{g}\,(x_{i+1}) - \mathbf{g}\,(x_i)| \le \frac{M}{r-\delta}\,\|x_{i+1} - x_i\|_{\mathbf{X}} \ .$$

then

$$|\mathbf{g}\,(x_{i+1}) - \mathbf{g}\,(x_i)| \le \frac{M}{r-\delta}\frac{1}{n}\,\|v-u\|_{\mathbf{X}} \ .$$

thus

$$|\mathbf{g}\,(v) - \mathbf{g}\,(u)| = |\mathbf{g}\,(x_n) - \mathbf{g}\,(x_1)| \le \sum_{i=1}^{n-1} |(\mathbf{g}\,(x_{i+1}) - \mathbf{g}\,(x_i))| \le \frac{M}{r-\delta}\,\|v-u\|_{\mathbf{X}} \ .$$

1.7.4 Lsc convex functions

Proposition 9 *A function* $\mathbf{f}$ *from* $\mathbf{X}$ *to* $\mathbf{R}\cup\{+\infty\}$ *is convex and* **lsc** *if and only if it is* ***weakly lsc****.*

Proof : It is enough to remark that **epi** (**f**) is closed convex if and only if **epi** (**f**) is weakly closed convex .

Exemples :

- A continuous convex function is a weakly lsc convex function . In particular the function $\|\ \|_{\mathbf{X}}$ is a weakly lsc convex function on $\mathbf{X}$; every continuous linear form on $\mathbf{X}$ is weakly lsc convex function and every continuous affine form on $\mathbf{X}$ is weakly lsc convex function.

- Let $\mathbf{a}$ be a positive bilinear form on $\mathbf{X}$ then the map $\mathbf{q}$ defined by $\mathbf{q}(x) = \mathbf{a}(x, x)$ is convex . Let $x \in \mathbf{X}$, $y \in \mathbf{X}$ and $t \in [0, 1]$, one has :

  $$\mathbf{q}(t\,x + (1-t)\ y) = \mathbf{a}((t\,x + (1-t)\ y)\ ,\ (t\,x + (1-t)\ y))\ .$$

  thus

  $$\mathbf{q}(t\,x + (1-t)\ y) = t^2\mathbf{a}(x, x) + t(1-t)\left[\mathbf{a}(x, y) + \mathbf{a}(y, x)\right] + (1-t)^2\,\mathbf{a}(y, y)\ .$$

  Because $\mathbf{a}$ is positive, developing $\mathbf{a}(x - y, x - y) \geq 0$, one obtains $\mathbf{a}(x, y) + \mathbf{a}(y, x) \leq \mathbf{a}(x, x) + \mathbf{a}(y, y)$. Finally one has :

  $$\mathbf{q}(t\,x + (1-t)\ y) \leq t\mathbf{a}(x, x) + (1-t)\,\mathbf{a}(y, y) \leq t\mathbf{q}(x) + (1-t)\,\mathbf{q}(y)\ .$$

  Thus, $\mathbf{q}$ is convex .
  If $\mathbf{a}$ is positive definite, one verifies by the same method that $\mathbf{q}$ is strictly convex . In particular $\mathbf{a}$ is positive definite if it satisfies the following coercivity condition :

  $$\exists \alpha > 0 \mid \forall x \in \mathbf{X}\ \ \mathbf{a}(x, x) \geq \alpha\ \|x\|^2_{\mathbf{X}}\ .$$

  If $\mathbf{a}$ is positive continuous bilinear then $\mathbf{a}$ is convex lsc .

One sets :

$$\mathcal{E}(\mathbf{f}) = \{(x^*, \alpha) \in \mathbf{X}^*\mathrm{x}\mathbf{R} \mid \forall x \in \mathbf{X}\ \ < x^*, x > +\alpha \leq \mathbf{f}(x)\}\ .$$

Proposition 10 *If* $\mathbf{f}$ *is a lsc convex proper function from* $\mathbf{X}$ *to* $\mathbf{R} \cup \{+\infty\}$ *then :*

$$\mathbf{f}(x)\ =\ \sup_{(x^*, \alpha) \in \mathcal{E}(\mathbf{f})} < x^*, x > +\alpha\ .$$

Proof : We have according to the definition of $\mathcal{E}(\mathbf{f})$:

$$\mathbf{f}(x)\ \geq\ \sup_{(x^*, \alpha) \in \mathcal{E}(\mathbf{f})} < x^*, x > +\alpha\ .$$

Let $x_0 \in \mathbf{dom}(\mathbf{f})$ and let $\epsilon > 0$ then $(x_0, \mathbf{f}(x_0) - \epsilon) \notin \mathbf{epi}(\mathbf{f})$. Because $\mathbf{epi}(\mathbf{f})$ is closed convex subset of $\mathbf{X}\mathrm{x}\mathbf{R}$, there exists $x^* \in \mathbf{X}^*$, $\alpha \in \mathbf{R}$ and $\gamma \in \mathbf{R}$ such that :

$$\forall (x, \lambda) \in \mathbf{epi}(\mathbf{f})\quad < x^*, x_0 > +\alpha(\mathbf{f}(x_0) - \epsilon) < \gamma \leq < x^*, x > +\alpha\lambda\ .$$

One verifies that α is strictly positive. If $\alpha = 0$, when we set $x = x_0$ in the two members, we have $< x^* , x_0 >< \gamma \leq < x^* , x_0 >$ this is impossible . If we suppose that $\alpha < 0$, we take λ such that $\mathbf{f}(x_0) \leq \lambda$, one has

$$< x^* , x_0 > + \alpha (\mathbf{f}(x_0) - \epsilon) < \gamma \leq < x^* , x_0 > + \alpha\lambda$$

and as λ tends to $+\infty$ then :

$$< x^* , x_0 > + \alpha (\mathbf{f}(x_0) - \epsilon) < \gamma \leq -\infty \ .$$

It is impossible . Then we have :

$$\forall x \in \mathbf{dom}(\mathbf{f}) \quad < \frac{1}{\alpha} x^* , x_0 > + \mathbf{f}(x_0) - \epsilon < \gamma \leq < \frac{1}{\alpha} x^* , x > + \mathbf{f}(x) \ ,$$

thus

$$\forall x \in \mathbf{dom}(\mathbf{f}) \quad < \frac{-1}{\alpha} x^* , x > + < \frac{1}{\alpha} x^* , x_0 > + \mathbf{f}(x_0) - \epsilon \leq \mathbf{f}(x) \ .$$

Therefore we have :

$$\forall x \in \mathbf{dom}(\mathbf{f}) \quad < \frac{-1}{\alpha} x^*, x > + < \frac{1}{\alpha} x^*, x_0 > + \mathbf{f}(x_0) - \epsilon \leq \sup_{(x^* , \alpha) \in \mathcal{E}(\mathbf{f})} < x^*, x > + \alpha \leq \mathbf{f}(x) \ .$$

Finally ,

$$\mathbf{f}(x_0) - \epsilon \leq \sup_{(x^* , \alpha) \in \mathcal{E}(\mathbf{f})} < x^* , x_0 > + \alpha \leq \mathbf{f}(x_0) \ .$$

Since $\epsilon > 0$ is arbitrary, we conclude that :

$$\sup_{(x^* , \alpha) \in \mathcal{E}(\mathbf{f})} < x^* , x_0 > + \alpha = \mathbf{f}(x_0) \, .$$

1.7.5 Minimization of convex functions

We have the following proposition :

Proposition 11 *If* $\mathbf{f}$ *is a strictly proper convex function from* $\mathbf{X}$ *to* $\mathbf{R} \cup \{+\infty\}$ *then if it has a minimum at a point, this point is unique .*

Proof : We suppose that $a \in \mathbf{X}$ and $b \in \mathbf{X}$ are such that $a \neq b$ and

$$\forall x \in \mathbf{X} \ \mathbf{f}(a) = \mathbf{f}(b) \leq \mathbf{f}(x) \ ,$$

then

$$\mathbf{f}\left(a\right) \leq \mathbf{f}\left(\frac{1}{2}a + \frac{1}{2}b\right) < \frac{1}{2}\mathbf{f}\left(a\right) + \frac{1}{2}\mathbf{f}\left(b\right) = \mathbf{f}\left(a\right) \ .$$

It is impossible .

We have below a theorem which is very useful .

Theorem 5 *If* $\mathbf{X}$ *is a reflexive Banach space, if* $\mathbf{f}$ *is a lsc proper convex function from* $\mathbf{X}$ *to* $\mathbf{R} \cup \{+\infty\}$ *and if*

$$\lim_{\|x\| \mapsto +\infty} \mathbf{f}\left(x\right) = +\infty$$

then $\mathbf{f}$ *has a minimum at a point of* $\mathbf{X}$.

Proof : The sections of $\mathbf{f}$ are closed, convex and bounded;thus they are weakly compact . Since $\mathbf{f}$ is proper and weakly lsc, we apply the **Tonelli theorem** .

Usual particular cases : Let $\mathbf{H}$ a real Hilbert space endowed with its scalar product $< \ , \ >_{\mathbf{H}}$ and with the associated norm $\| \ \|_{\mathbf{H}}$. It is well known that $\mathbf{H}$ is a reflexive Banach space : the **Riez's theorem** permits us to establish an isometric isomorphism between $\mathbf{H}$ and its topological dual $\mathbf{H}^*$.

- **Projection on a convex closed subset**
  Let $\mathbf{C}$ be a non void convex closed subset of $\mathbf{X}$. Let $x \in \mathbf{X}$ and let $\mathbf{F}$, the function from $\mathbf{X}$ to $\mathbf{R} \cup \{+\infty\}$ which is defined as follows :

  $$\text{if } y \in \mathbf{C} \text{ then } \mathbf{F}\left(y\right) = \|y - x\|_{\mathbf{X}}$$

  and

  $$\text{if } y \notin \mathbf{C} \ \text{ then } \quad \mathbf{F}\left(y\right) = +\infty$$

  . The function $\mathbf{F}$ is convex lsc and it satisfies :

  $$\lim_{\|x\| \mapsto +\infty} \mathbf{F}\left(x\right) = +\infty \ .$$

  Thus there exists $\underline{y} \in \mathbf{C}$ such that

  $$\mathbf{F}\left(\underline{y}\right) = \min_{y \in \mathbf{C}} \mathbf{F}\left(y\right) \ .$$

- **Quadratic optimization** Let a be a bilinear form on $\mathbf{X}$ which is continuous and coercive. These properties mean :

  $$\textbf{continuity} : \exists M > 0 \mid \forall x \in \mathbf{X} \forall y \in \mathbf{X} \quad |a(x\,,y)| \leq M \ \|x\|_{\mathbf{X}} \ \|y\|_{\mathbf{X}} \ ,$$

  $$\textbf{coercivity} : \exists \alpha > 0 \mid \forall x \in \mathbf{X} \quad \alpha \ \|x\|_{\mathbf{X}}^2 \ \leq \ a(x\,,x) \ .$$

  Let ℓ be a continuous linear form on $\mathbf{X}$, there exists $L > 0$ such that

  $$\forall x \in \mathbf{X} \ |\ell(x)| \ \leq \ L \ \|x\|_{\mathbf{X}}$$

  and let $k \in \mathbf{R}$. One defines on $\mathbf{X}$, the function denoted $\mathbf{J}$ by :

  $$\forall x \in \mathbf{X} \ \mathbf{J}(x) = \frac{1}{2} a(x\,,x) \ - \ \ell(x\,,x) \ + \ k \ .$$

  The function $\mathbf{J}$ is convex lsc proper and verifies :

  $$\lim_{\|x\| \mapsto +\infty} \mathbf{J}(x) \ = \ +\infty \ .$$

  because

  $$\forall x \in \mathbf{X} \ \ \mathbf{J}(x) \geq \alpha \ \|x\|_{\mathbf{X}}^2 \ - L \ \|x\|_{\mathbf{X}} \ + \ k \ .$$

  Then there exists $\underline{x} \in \mathbf{X}$ such that

  $$\mathbf{J}(\underline{x}) \ = \ \min_{x \in \mathbf{X}} \mathbf{J}(x) \ .$$

1.8 Duality

In this section, $\mathbf{X}$ is a real linear space with the norm $\| \ \|_{\mathbf{X}}$. One denotes $\mathbf{X}^*$, the topological dual space of $\mathbf{X}$, it means the real linear space of continuous linear forms on the normed space $(\mathbf{X}\ ,\ \| \ \|_{\mathbf{X}})$. One denotes the bilinear pairing by $< \ , \ >$ then :

$$\forall x^* \in \mathbf{X}^* \ \forall x \in \mathbf{X} \ \ x^*(x) = < x^*\,, x > \ .$$

If we set for every $x^* \in \mathbf{X}^*$,

$$\|x^*\|_{\mathbf{X}^*} = \sup_{\|x\|_{\mathbf{X}}=1} < x^*\,, x >$$

then $(\mathbf{X}^*\ ,\ \| \ \|_{\mathbf{X}^*})$ is a normed linear space .

Definition 19 ***conjugate or polar function***
Let $\mathbf{f}$ *from* $\mathbf{X}$ *to* $\mathbf{R}\cup\{+\infty\}$, ***the conjugate function or the polar function*** *of* $\mathbf{f}$ *denoted by* $\mathbf{f}^*$ *is the function from* $\mathbf{X}^*$ *to* $\mathbf{R}\cup\{-\infty\ ,\ +\infty\}$ *which is defined by :*

$$\forall x^* \in \mathbf{X}^* \quad \mathbf{f}^*(x^*) = \sup_{x\in\mathbf{X}} (< x^*\ , x > -\mathbf{f}(x)) \ .$$

One has :

$$\forall x^* \in \mathbf{X}^* \quad \mathbf{f}^*(x^*) = \sup_{x\in\mathbf{dom}(\mathbf{f})} (< x^*\ , x > -\mathbf{f}(x)) \ .$$

Remark 8 *The function* $\mathbf{f}^*$ *from* $\mathbf{X}^*$ *to* $\mathbf{R}\cup\{-\infty\ ,\ +\infty\}$ *is convex and lsc .*

Proof : It enough to remark that the function $\mathbf{f}^*_x$, which is defined by

$$\forall x^* \in \mathbf{X}^* \quad \mathbf{f}^*_x(x^*) =< x^*\ , x > -\mathbf{f}(x)$$

is lsc, convex and moreover $\mathbf{f}^*(x^*) = \sup_{x\in\mathbf{X}} \mathbf{f}^*_x(x^*)$.

Then we have :

$$\mathbf{f}^*(0) = -\inf_{x\in\mathbf{dom}(\mathbf{f})} \mathbf{f}(x) \ .$$

Proposition 12 *If* $\mathbf{f}$ *is an application from* $\mathbf{X}$ *to* $\mathbf{R}\cup\{+\infty\}$ *convex and proper then* $\mathbf{f}^*$ *is convex, lsc and proper .*

Proof : We have to prove that $\mathbf{f}^*$ is proper . There exists $(x_0^*\ , \alpha) \in \mathbf{X}^*\mathbf{x}\mathbf{R}$ such that :

$$\forall x \in \mathbf{X} \quad < x_0^*\ , x > -\alpha \ \leq\ \mathbf{f}(x)$$

then

$$\forall x \in \mathbf{X} \quad < x_0^*\ , x > -\mathbf{f}(x) \ \leq\ \alpha$$

thus $x_0^* \in \mathbf{dom}(\mathbf{f}^*)$.

Exercice 7

- *If* $\mathbf{f}$ *and* $\mathbf{g}$ *are functions from* $\mathbf{X}$ *to* $\mathbf{R}\cup\{+\infty\}$ *such that* $\mathbf{f}\leq\mathbf{g}$, *prove that* $\mathbf{f}^*\geq\mathbf{g}^*$.
- *Let* $\mathbf{f}$ *be a function from* $\mathbf{X}$ *to* $\mathbf{R}\cup\{+\infty\}$. *Let* $\lambda\in\mathbf{R}\setminus\{0\}$, *and suppose that* $\forall x \in \mathbf{X} \quad \mathbf{f}_\lambda(x) = \mathbf{f}(\lambda\, x)$.
  Prove that $\mathbf{f}^*_\lambda(x^*) = \mathbf{f}^*\left(\frac{1}{\lambda}x^*\right)$.
  Prove that $(\mathbf{f}+\lambda)^* = \mathbf{f}^* - \lambda$.

- *Let* $\mathbf{f}$ *be a function from* $\mathbf{X}$ *to* $\mathbf{R}\cup\{+\infty\}$*. Let* $a\in\mathbf{X}$*, one denotes* $\tau_a\mathbf{f}$*, the function defined by* $\forall x\in\mathbf{X}\ \ \tau_a\mathbf{f}(x)=\mathbf{f}(x+a)$*.*
  Prove that $\forall x^*\ \ \tau_a\mathbf{f}^*(x^*)=\mathbf{f}^*(x^*)-<x^*,a>$*.*

We have :

Proposition 13 ***Young Inequality***
If $\mathbf{f}$ *is a function from* $\mathbf{X}$ *to* $\mathbf{R}\cup\{+\infty\}$ *then :*

$$\forall x^*\in\mathbf{X}^*\ \forall x\in\mathbf{X}\quad <x^*,x>\ \leq\ \mathbf{f}^*(x^*)\ +\ \mathbf{f}(x)\ \ .$$

1.8.1 Bidual

Definition 20 *If* $\mathbf{f}$ *is a function from* $\mathbf{X}$ *to* $\mathbf{R}\cup\{+\infty\}$*, the* ***bipolar of*** $\mathbf{f}$ *is the map denoted by* $\mathbf{f}^{**}$ *from* $\mathbf{X}$ *to* $\mathbf{R}\cup\{-\infty\,,\,+\infty\}$ *and which is defined by :*

$$\forall x\in\mathbf{X}\ \ \mathbf{f}^{**}(x)=(\mathbf{f}^*)^*(x)=\sup_{x^*\in\mathbf{X}^*}\left(<x^*,x>-\mathbf{f}^*(x^*)\right)\ \ .$$

By the **Young's inequality** , we have that :

$$\forall x\in\mathbf{X}\ \ \mathbf{f}^{**}(x)\leq\mathbf{f}(x)\ \ .$$

In addition $\mathbf{f}^{**}$ is convex and lsc .

Theorem 6 *If* $\mathbf{f}$ *is an application from* $\mathbf{X}$ *to* $\mathbf{R}\cup\{+\infty\}$*, convex, lsc and proper then*

$$\mathbf{f}^{**}=\mathbf{f}\ \ .$$

Proof : It is enough to prove that $\forall x\in\mathbf{X}\ \ \mathbf{f}^{**}(x)\geq\mathbf{f}(x)$. Suppose that there exists $x_0\in\mathbf{X}$ such that $\mathbf{f}^{**}(x_0)<\mathbf{f}(x_0)$.
Then $(x_0\,,\mathbf{f}^{**}(x_0))\notin\mathbf{epi}\,(\mathbf{f})$, there exists $(x^*\,,\alpha)\in\mathbf{X}^*\mathrm{x}\mathbf{R}$ which verifies :

$$\mathbf{f}^{**}(x_0)\ <\ <x^*,x_0>+\alpha\ \leq\ \mathbf{f}(x_0)$$

and

$$\forall x\in\mathbf{X}\quad <x^*,x>+\alpha\ \leq\ \mathbf{f}(x)\ \ .$$

The second inequality :

$$\forall x\in\mathbf{X}\quad <x^*,x>-\mathbf{f}(x)\ \leq\ -\alpha\ ,$$

let

$$\mathbf{f}^*(x^*) \leq -\alpha \ .$$

The inequality gives :

$$< x^*, x_0 > -\mathbf{f}^*(x^*) << x^*, x_0 > +\alpha \ ,$$

then

$$-\alpha < \mathbf{f}^*(x^*) \ .$$

It is impossible .

1.9 Applications to some problems of calculus of variations

Let Ω a non void open subset of $\mathbf{R}^N$. On $\mathbf{R}^N$, we use the Lebesgue measure .

Definition 21 *A function* $\mathbf{F}$ *from* $\Omega \text{x} \mathbf{R}^p$ *to* $\mathbf{R} \cup \{-\infty \ , +\infty\}$ *is said to be of* ***Caratheodory*** *if it satisfies :*

- *for every* $x \in \Omega$, *the function* $u \mapsto \mathbf{F}(x, u)$ *is* ***continuous***,
- *for every* $u \in \mathbf{R}^p$, *the function* $x \mapsto \mathbf{F}(x,u)$ *is measurable .*

Proposition 14 *Suppose a function* $\mathbf{F}$ *from* $\Omega \text{x} \mathbf{R}^p$ *to* $\mathbf{R} \cup \{-\infty \ , +\infty\}$ *is of* ***Caratheodory***, *if* $\mathbf{u} \ \Omega \rightarrow \mathbf{R}^p$ *is measurable then the map*

$$x \mapsto \mathbf{F}(x,\mathbf{u}(x))$$

is measurable .

Proof : It is enough to remark that a measurable function is the limit almost everywhere of a sequence of simple functions . However if u is a simple function then the function $x \mapsto \mathbf{F}(x, \mathbf{u}(x))$ is obviosly measurable.

To obtain integrability in the spaces of type $\mathbf{L}^p$ for $p \geq 1$ some kind of growth controls on $\mathbf{F}$ are used . One gives here an exemple of this type of estimations.

Proposition 15 *If* $p_1 \geq 1$ *and* $p_2 \geq 1$, *if* $a > 0$ *and if* $\mathbf{b} \in \mathbf{L}^{p_2}(\Omega)$ *then if* $\mathbf{F}$ *is of Caratheodory and verifies*

$$\forall x \in \Omega \ \forall \xi \in \mathbf{R}^p \quad |\mathbf{F}(x, \xi)| \leq \mathbf{b}(x) + a \ \|\xi\|_{\mathbf{R}^p}$$

then the map Φ *of* $\mathbf{L}^{p_1}(\Omega, \mathbf{R}^p)$ *to* $\mathbf{L}^{p_2}(\Omega, \mathbf{R})$ *which associates* $\mathbf{u}$ *to* $\Phi(\mathbf{u}) \quad x \rightarrow \Phi(\mathbf{u})(x) = \mathbf{F}(x,\mathbf{u}(x))$ *is a continuous map and transforms the bounded subsets to bounded subsets .*

Proof : Let $\mathbf{u} \in \mathbf{L}^{p_1}(\Omega, \mathbf{R}^p)$ the map $\Phi(\mathbf{u}) \quad x \rightarrow \mathbf{F}(x,\mathbf{u}(x))$ is measurable. And we have :

$$|\mathbf{F}(x, \mathbf{u}(x))| \leq \mathbf{b}(x) + a \|\mathbf{u}(x)\|_{\mathbf{R}^p}^{\frac{p_1}{p_2}} .$$

The second part of the inequality belong $\mathbf{L}^{p_2}(\Omega, \mathbf{R})$ then by the dominated convergence theorem of Lebesgue the map $\Phi(\mathbf{u}) \quad x \rightarrow \mathbf{F}(x,\mathbf{u}(x))$ belong to $\mathbf{L}^{p_1}(\Omega, \mathbf{R}^p)$.
In the other hand we have

$$\|\Phi(\mathbf{u})\|_{\mathbf{L}^{p_2}(\Omega, \mathbf{R}^p)} \leq \|\mathbf{b}\|_{\mathbf{L}^{p_2}(\Omega, \mathbf{R})} + a \|\mathbf{u}\|_{\mathbf{L}^{p_2}(\Omega, \mathbf{R})}^{\frac{p_1}{p_2}} .$$

Let $(\mathbf{u}_n)_{n\in\mathbf{N}}$ a sequence of functions of $\mathbf{L}^{p_1}(\Omega, \mathbf{R}^p)$ which converges to $\mathbf{u} \in \mathbf{L}^{p_1}(\Omega, \mathbf{R}^p)$. It has a subsequence $(\mathbf{u}_{n_k})_{k\in\mathbf{N}}$ converges almost everywhere to $\mathbf{u}$ and such that :

$$\forall i \in \mathbf{N} \quad \|\mathbf{u}_{n_{k+1}} - \mathbf{u}_{n_k}\|_{\mathbf{L}^{p_1}(\Omega, \mathbf{R}^p)} < \frac{1}{2^i} .$$

Then

$$\forall x \in \Omega \quad \mathbf{K}(x) = \|\mathbf{u}_{n_1}(x)\|_{\mathbf{R}^*} + \sum_{i=1}^{+\infty} \|\mathbf{u}_{n_{i+1}}(x) - \mathbf{u}_{n_i}(x)\|_{\mathbf{R}^*} .$$

The function $\mathbf{K}$ is measurable positive and we have :

$$\|\mathbf{K}\|_{\mathbf{L}^{p_1}(\Omega, \mathbf{R}^p)} \leq \|\mathbf{u}_{n_1}\|_{\mathbf{L}^{p_1}(\Omega, \mathbf{R}^p)} + \sum_{i=1}^{+\infty} \|\mathbf{u}_{n_{i+1}} - \mathbf{u}_{n_i}\|_{\mathbf{L}^{p_1}(\Omega, \mathbf{R}^p)} .$$

The function $\mathbf{K}$ is then in $\mathbf{L}^{p_1}(\Omega, \mathbf{R}^p)$. Because

$$\forall i \in \mathbf{N}^* \quad |u_{n_i}| \leq \mathbf{K} ,$$

then

$$\forall i \in \mathbf{N}^* \quad |\Phi(u_{n_i})| \leq \mathbf{b} + (\mathbf{K})^{\frac{p_1}{p_2}} \in \mathbf{L}^{p_2}(\Omega, \mathbf{R}) .$$

By the Lebesgue's dominated convergence theorem , the sequence $(\Phi(\mathbf{u}_{n_k}))_{k\in\mathbf{N}}$ converges in $\mathbf{L}^{p_2}(\Omega, \mathbf{R})$ to $\Phi(\mathbf{u})$. Finally Φ is continuous .

Chapter 2

Optimality Conditions

In this chapter, we give some methods, when it is possible, to determine the equations or the inequalities satisfied by the solutions of the minimization problems .

In this chapter, $\mathbf{X}$ is a real linear space with the norm $\| \|_{\mathbf{X}}$. One denotes $\mathbf{X}^*$, the topological dual space $\mathbf{X}$, it means the linear space of the continuous forms on the normed space $(\mathbf{X}\ ,\ \| \|_{\mathbf{X}})$. One denote the duality pairing by $< \ , \ >$ then :

$$\forall x^* \in \mathbf{X}^* \ \forall x \in \mathbf{X} \quad x^*(x) = < x^* \, , x > \ .$$

If we set, for every $x^* \in \mathbf{X}^*$,

$$\|x^*\|_{\mathbf{X}^*} = \sup_{\|x\|_{\mathbf{X}}=1} < x^* \, , x > \ .$$

then $(\mathbf{X}^*\ ,\ \| \|_{\mathbf{X}^*})$ is a linear normed space . If $(\mathbf{X}\ ,\ \| \|_{\mathbf{X}})$ is a Banach space then $(\mathbf{X}^*\ ,\ \| \|_{\mathbf{X}^*})$ is also a Banach space .

If $\mathbf{f}$, a function from $\mathbf{X}$ to $\mathbf{R} \cup \{+\infty\}$. One denotes by $(P_{\mathtt{min}})$ the following problem :

$$\text{Find } \underline{x} \in \mathbf{X}, \text{ a solution of } \min_{x \in \mathbf{X}} \mathbf{f}(x) \ .$$

One says that $\underline{x}$ is solution of the problem $(P_{\mathtt{min}})$ if $\mathbf{f}$ has a minimum on $\mathbf{X}$ at the point $\underline{x}$, then :

$$\forall x \in \mathbf{X} \quad \mathbf{f}(\underline{x}) \ \leq \ \mathbf{f}(x) \ .$$

.

2.1 Different concepts of derivatives

Let $(\mathbf{Y}\,,\,\|\,\|_{\mathbf{Y}})$ be a linear normed space over $\mathbf{R}$. One denotes by $\mathcal{L}(\mathbf{X}\,,\mathbf{Y})$ the space of linear continuous functions from $\mathbf{X}$ to $\mathbf{Y}$.

2.1.1 Derivatives following a direction

Definition 22 *Let $\mathbf{f}\ \mathbf{X} \to \mathbf{Y}$; let $x \in \mathbf{X}$ and let $h \in \mathbf{X}$, one says that $\mathbf{f}$ has a derivative at the point x in the direction h if*

$$\lim_{t \mapsto 0_+} \frac{\mathbf{f}(x+th) - \mathbf{f}(x)}{t} \quad \textit{exists.}$$

In this case, we name this limit, the derivative of $\mathbf{f}$ at the point x following the direction h and we set :

$$\mathbf{f}'(x;h) = \lim_{t \mapsto 0_+} \frac{\mathbf{f}(x+th) - \mathbf{f}(x)}{t} \ .$$

Remark 9 *If $\mathbf{f}\ \mathbf{X} \to \mathbf{R} \cup \{+\infty\}$, we have the same definition if we take $x \in \mathbf{dom}(\mathbf{f})$ and let $h \in \mathbf{X}$.*

If $x \in \mathbf{dom}(\mathbf{f})$, One has : $\mathbf{f}'(x;0_{\mathbf{X}}) = 0$
If we set $\mathbf{g}_{x\,,\,h}(t) = \mathbf{f}(x+th)$, if $\mathbf{g}_{x\,,\,h}$ is defined on a segment $[0\,,\delta]$ for a real number $\delta > 0$ then $\mathbf{f}$ has a derivative at the point x in the direction h if and only if $\mathbf{g}_{x\,,\,h}$ has a derivative at the right at 0. moreover we have :

$$\mathbf{f}'(x;h) = \left(\mathbf{g}'_{x\,,\,h}\right)_d(0) \ .$$

Remark 10 *If $\mathbf{f}^{'}(x;h)$ is defined then :*

$$\forall \lambda \geq 0 \quad \mathbf{f}^{'}(x;\lambda h) = \lambda \mathbf{f}^{'}(x;h) \ .$$

Exemple : Let a be a bilinear continuous form on $\mathbf{X}$, let $x^* \in \mathbf{X}^*$ and $k \in \mathbf{R}$, if we set :

$$\mathbf{J}(x) = \frac{1}{2}a(x\,,x) - < x^*\,,x > +k \ ,$$

we obtain

$$\begin{aligned}\mathbf{J}(x+th) - \mathbf{J}(x) \ &= \ \frac{1}{2}\left[a(x\,,x) + ta(x\,,h) + ta(h\,,x) + t^2 a(h\,,h)\right] \\ &\quad - < x^*\,,x > -t < x^*\,,h > \ .\end{aligned}$$

Then

$$\frac{\mathbf{J}(x+th)-\mathbf{J}(x)}{t}=\frac{1}{2}\left[a(x\,,h)+a(h\,,x)-2<x^*\,,h>\right]\;+\;t\,a(h\,,h)\ \ .$$

Thus

$$\forall x\in\mathbf{X}\ \forall h\in\mathbf{X}\ \ \mathbf{J}^{'}(x;h)=\frac{1}{2}\left[a(x\,,h)+a(h\,,x)-2<x^*\,,h>\right]\ \ .$$

In the particular case where a is symmetric, that is if :

$$\forall x\in\mathbf{X}\ \forall y\in\mathbf{X},\ \ a(x\,,y)=a(y\,,x)\ \ .$$

We have :

$$\forall x\in\mathbf{X}\ \forall h\in\mathbf{X}\ \ \mathbf{J}^{'}(x;h)\ =\ a(x\,,h)\ -\ <x^*\,,h>\ \ .$$

Exemple : Let $\mathbf{H}$ be a real Hilbert space with its associated scalar product $<\ ,\ >_{\mathbf{H}}$ and the corresponding norm $\|\ \|_{\mathbf{H}}$. If :

$$\forall x\in\mathbf{H},\ \ \mathbf{J}(x)=\|x\|_{\mathbf{H}}\ \ .$$

We have :

$$\mathbf{g}_{x,h}(t)=\left(\|x+th\|_{\mathbf{H}}^2\right)^{\frac{1}{2}}=\left(<x+th\,,x+th>_{\mathbf{H}}\right)^{\frac{1}{2}}\ \ .$$

Thus :

$$\forall x\in\mathbf{H}\setminus\{0\}\ \ \forall h\in\mathbf{H}\ \ \mathbf{J}^{'}(x;h)\ =\ \left(\mathbf{g}^{'}_{x\,,\,h}\right)_d(0)\ =\ \frac{<x\,,h>_{\mathbf{H}}}{\|x\|_{\mathbf{H}}}\ .$$

$$\forall h\in\mathbf{H}\ \ \mathbf{J}^{'}(0_{\mathbf{X}};h)\ =\ \left(\mathbf{g}^{'}_{0_{\mathbf{X}}\,,\,h}\right)_d(0)\ =\ \|h\|_{\mathbf{H}}\ \ .$$

Exemple : Let $\mathbf{f}$ be a continuously differentiable map from $\mathbf{R}$ to $\mathbf{R}$. Let $\mathbf{K}$ be a compact subset of $\mathbf{R}^N$. One denotes by $\mathbf{X}=\mathbf{C}(\mathbf{K}\,,\mathbf{R})$, the linear space of continuous functions from $\mathbf{K}$ to $\mathbf{R}$ and if $u\in\mathbf{X}$, we set

$$\|u\|_{\mathbf{X}}=\max_{x\in\mathbf{K}}|u(x)|\ \ \ .$$

One defines the map $\Phi\ \mathbf{X}\ \rightarrow\ \mathbf{X}$ which associates $u\in\mathbf{X}$ to its image $\Phi(u)=\mathbf{f}\,\mathrm{o}\,u$. We determine the derivative of Φ in any direction as follows . Let $u\in\mathbf{X}$, $h\in\mathbf{X}$ and $t\in[0\,,1]$, there exists $\theta(x{,}t)\in[0\,,1]$ such that :

$$\mathbf{f}(u(x)\ +\ t\,h(x))\ -\ \mathbf{f}(u(x))\ =\ t\,\mathbf{f}'(u(x)\ +\ \theta(x{,}t)\ t\,h(x))\ h(x)\ \ .$$

The subset

$$\mathbf{K_1} \; = \; u\,(\mathbf{K}) + [-1\,,1]$$

is compact in $\mathbf{R}$. The map $\mathbf{f}'$ is thus uniformly continuous on $\mathbf{K_1}$. Let $\epsilon > 0$

$$\exists \eta > 0 \mid \forall p \in \mathbf{K_1} \; \forall q \in \mathbf{K_1} \quad |p-q| < \eta \;\Rightarrow\; \left|\mathbf{f}'\,(p) - \mathbf{f}'\,(q)\right| < \epsilon \; .$$

We set $\eta_1 = \min\,(\eta\,,1)$, if $h \in \mathbf{X}$ is such that

$$\forall x \in \mathbf{K} \;\; |h\,(x)| < \eta_1 \; ,$$

this implies that

$$\|h\|_{\mathbf{X}} < \eta_1 \; ,$$

then for every x belonging to $\mathbf{K}$, we set

$$\Delta\,(x) \; = \; \left| \frac{\mathbf{f}\,(u\,(x) \; + \; t\,h\,(x)) \; - \; \mathbf{f}\,(u\,(x))}{t} \; - \; h\,\mathbf{f}'\,(u\,(x)) \right| \; ,$$

then

$$\Delta\,(x) \; = \; |\,\mathbf{f}'\,(u\,(x) \; + \; \theta\,(x,t) \;\, t\,h\,(x)) \;\, h\,(x) \; - \; \mathbf{f}'\,(u\,(x)) \;\, h\,(x)| \; ,$$

hence, we obtain that

$$\forall x \in \mathbf{K}, \;\; \left| \frac{\mathbf{f}\,(u\,(x) \; + \; t\,h\,(x)) \; - \; \mathbf{f}\,(u\,(x))}{t} \; - \mathbf{f}'\,(u\,(x)) \right| \; < \; \epsilon \; |h\,(x)| \; .$$

Finally,

$$\left\| \frac{\Phi\,(u + th) - \Phi\,(u)}{t} \; - \; h\,\mathbf{f}' \mathrm{o}\, u \right\|_{\mathbf{X}} < \; \epsilon \; \|h\|_{\mathbf{X}} \; ,$$

Then

$$\Phi'\,(u\,;h) \; = \; h\,\mathbf{f}' \mathrm{o}\, u \; .$$

Exercice 8 *1. Compute the derivative of the absolute value function on* $\mathbf{R}$ *in every direction .*

2. Let $x \in \mathbf{R}$, one sets $x^+ = x$ si $x \geq 0$, $x^+ = 0$ si $x < 0$. Determine the derivatives following the directions of the function from $\mathbf{R}$ to $\mathbf{R}$ which associates x to x^+.

Exercice 9 *Let* $\mathbf{f}$ *be a continuous map having continuous partial derivatives from* $\mathbf{R}$x$\mathbf{R}^N$ *to* $\mathbf{R}$*. Let* $\mathbf{K}$ *be a compact subset of* $\mathbf{R}^N$*. Denote by* $\mathbf{X} = \mathbf{C}^1\,(\mathbf{K}\,,\mathbf{R})$*, the space of continuous functions having continuous partial* *derivatives from* $\mathbf{K}$ *to* $\mathbf{R}$ *and* $\mathbf{Y} = \mathbf{C}(\mathbf{K}, \mathbf{R})$*, the linear space of continuous functions from* $\mathbf{K}$ *to* $\mathbf{R}$*. If* $u \in \mathbf{X}$*, we define*

$$\|u\|_{\mathbf{X}} = \max_{x \in \mathbf{K}} |u(x)| + \sum_{i=1}^{N} \max_{x \in \mathbf{K}} \left| \frac{\partial u}{\partial x_i}(x) \right|$$

. If $v \in \mathbf{Y}$*, we define*

$$\|v\|_{\mathbf{Y}} = \max_{x \in \mathbf{K}} |v(x)| \ .$$

Define the map $\Psi\ \mathbf{X} \to \mathbf{Y}$ *which associates* $u \in \mathbf{X}$ *to* $\Psi(u)$ *by*

$$\Psi(u)(x) = \mathbf{f}(u(x), \nabla u(x)) \quad \forall x \in \mathbf{K} \ .$$

Determine the directional derivatives of Ψ *.*

2.1.2 Gâteaux Derivatives

Definition 23 *If* $\mathbf{f}\ \mathbf{X} \to \mathbf{R} \cup \{+\infty\}$*, if* $x \in \mathbf{dom}(\mathbf{f})$*, one says that* $\mathbf{f}$ *is* ***Gâteaux differentiable at the point*** $x \in \mathbf{X}$ *if it admits a derivative following every direction* $h \in \mathbf{X}$ *and if there exists* $\mathbf{L}_x \in \mathbf{X}^*$ *such that :*

$$\forall h \in \mathbf{X} \quad \mathbf{f}'(x\,;h) = <\mathbf{L}_x\,;h> \ .$$

The linear continuous form $\mathbf{L}_x$ *is called the* ***Gâteaux derivative of*** $\mathbf{f}$ *at the point* x*; it is also called the* ***gradient of*** $\mathbf{f}$ *at the point* x *and is denoted by* $\nabla \mathbf{f}(x)$*. Thus :*

$$\mathbf{L}_x = \nabla \mathbf{f}(x) \ .$$

Remark 11 *When* $\mathbf{X}$ *is a Hilbert space, the Riez theorem permits the identification of* $\nabla \mathbf{f}(x)$ *with an element of* $\mathbf{X}$*.*

Exemple : Let $\mathbf{f}$ be a continuous map having continuous partial derivatives from $\mathbf{R}^N$ to $\mathbf{R}$. Let $\mathbf{K}$ be a compact subset of $\mathbf{R}^N$. One denotes by $\mathbf{X} = \mathbf{C}^1(\mathbf{K}, \mathbf{R})$, the linear space of continuous functions having continuous partial derivatives from $\mathbf{K}$ to $\mathbf{R}$. If $u \in \mathbf{X}$, one sets

$$\|u\|_{\mathbf{X}} = \max_{x \in \mathbf{K}} |u(x)| + \sum_{i=1}^{N} \max_{x \in \mathbf{K}} \left| \frac{\partial u}{\partial x_i}(x) \right| \ .$$

Define the map $\mathbf{J}\ \mathbf{X} \to \mathbf{R}$ which associates $u \in \mathbf{X}$ to $\mathbf{J}(u)$ by

$$\mathbf{J}(u) = \int_{\mathbf{K}} \mathbf{f}(\nabla u(x))\ \mathtt{dx} \ .$$

Step by step computation as in the preceeding subsection gives :

$$\mathbf{J}'(u\,,h) \;=\; \int_{\mathbf{K}} \sum_{i=1}^{N} \frac{\partial \mathbf{f}}{\partial x_i}(\nabla u(x))\, \frac{\partial h}{\partial x_i}\, \mathtt{dx}\ .$$

The map $h \to \mathbf{J}'(u\,,h)$ is a linear form on $\mathbf{X}$ and there exits a constant C such that :

$$\left|\mathbf{J}'(u\,,h)\right| \;\le\; C\ \|h\|_{\mathbf{X}}\ .$$

Finally, $\mathbf{J}$ is Gâteaux differentiable at every point of $\mathbf{X}$.

Definition 24 *If* $\mathbf{f}$ *is Gâteaux differentiable at every point of* $\mathbf{X}$*, we say that* $\mathbf{f}$ *is Gâteaux differentiable .*

2.1.3 Relationship between Gâteaux differentiability and Fréchet differentiability

We recall the concept of **differentiability** or **Fréchet differentiability**.

Definition 25 *A map* $\mathbf{f}\ \mathbf{X} \to \mathbf{Y}$ *is said to be* ***differentiable or Fréchet differentiable at the point*** $x \in \mathbf{X}$ *if there exists* $\mathbf{L}_x \in \mathcal{L}(\mathbf{X}\,,\mathbf{Y})$ *such that :*

$$\lim_{h \mapsto 0_{\mathbf{X}}} \frac{\|\mathbf{f}(x+h) - \mathbf{f}(x) - \mathbf{L}_x(h)\|_{\mathbf{Y}}}{\|h\|_{\mathbf{X}}} \;=\; 0\ .$$

The linear continuous map $\mathbf{L}_x$ *is called the* ***derivative of*** $\mathbf{f}$ ***at the point*** x*, we also call* $\mathbf{L}_x$ *the Fréchet derivative of* $\mathbf{f}$ *at the point* x*.*

This definition is equivalent to the following property :
There exists $\mathbf{L}_x \in \mathcal{L}(\mathbf{X}\,,\mathbf{Y})$ and $\epsilon_x\ \mathbf{X} \to \mathbf{Y}$ such that :

- $\lim_{h \mapsto 0_{\mathbf{X}}} \epsilon_x(h) \;=\; 0_{\mathbf{Y}}$,
- $\forall h \in \mathbf{X},\ \ \mathbf{f}(x+h) \;=\; \mathbf{f}(x) \;+\; \mathbf{L}_x(h) + \;\|h\|_{\mathbf{X}}\ \epsilon_x(h)$.

If $\mathbf{f}\ \mathbf{X} \to \mathbf{R} \cup \{+\infty\}$, If $x \in \overbrace{\mathbf{dom}\,(\mathbf{f})}^{o}$, we say that $\mathbf{f}$ is **differentiable at a point** $x \in \mathbf{X}$ if there exists $\mathbf{L}_x \in \mathbf{X}^*$ such that :

$$\lim_{h \mapsto 0_{\mathbf{X}}} \frac{|\mathbf{f}(x+h) - \mathbf{f}(x) - \mathbf{L}_x(h)|}{\|h\|_{\mathbf{X}}} \;=\; 0\ .$$

The linear continuous form $\mathbf{L}_x$ is called the **derivative of f at the point** x, $\mathbf{L}_x$ is also called **the Fréchet derivative of f at the point** x.

We may characterise the differentiability of $\mathbf{f}$ at a point $x \in \overbrace{\mathbf{dom}\,(\mathbf{f})}^{o}$ by the following property : there exists $\mathbf{L}_x \in \mathcal{L}\,(\mathbf{X}\,,\,\mathbf{Y})$, $r > 0$ and $\epsilon_x \;\; \mathbf{B}\,(0_{\mathbf{X}}\,,\,r) \;\to\; \mathbf{R}$ such that :

- $\lim\limits_{h \,\mapsto\, 0_{\mathbf{X}}} \epsilon_x\,(h) \;=\; 0$.
- $\forall h \in \mathbf{B}\,(0_{\mathbf{X}}\,,\,r) \quad \mathbf{f}\,(x+h) \;=\; \mathbf{f}\,(x) \;+\; \mathbf{L}_x\,(h) + \;\|h\|_{\mathbf{X}} \;\epsilon_x\,(h)$.

Notation : We set : $\mathbf{f}'\,(x) = \mathbf{L}_x$.

The following proposition is obvious .

Proposition 16 *If* $\mathbf{f}$ *is differentiable at the point* x *then* $\mathbf{f}$ *is Gâteaux differentiable at the point* x*. Moreover,* $\quad \nabla\mathbf{f}\,(x) = \mathbf{f}'\,(x)$.

We have the converse in the following case :

Proposition 17 *If* $\mathbf{f} \;\; \mathbf{X} \;\to\; \mathbf{R} \cup \{+\infty\}$*, if* $x \in \overbrace{\mathbf{dom}\,(\mathbf{f})}^{o}$*, suppose that* $\mathbf{f}$ *is Gâteaux differentiable in a neighbourhood* $\mathbf{V}$ *of* x *and if* $\nabla\mathbf{f} \;\; \mathbf{V} \;\to\; \mathbf{X}^*$ *is continuous then* $\mathbf{f}$ *is Fréchet differentiable at the point* x*, in addition :*

$$\mathbf{f}'\,(x) = \nabla\mathbf{f}\,(x) \;\; .$$

Proof : If $\epsilon > 0$, there exists $\eta > 0$ such that $\mathbf{B}\,(x\,,\,\eta) \subset \mathbf{V}$ and :

$$\forall \xi \in \mathbf{B}\,(0_{\mathbf{X}}\,,\,\eta) \quad \|\nabla\mathbf{f}\,(x+\xi) - \nabla\mathbf{f}\,(x)\|_{\mathbf{X}^*} < \epsilon \; .$$

Let $h \;\in\; \mathbf{B}\,(x\,,\,\eta)$, we denote by $\mathbf{g} \quad [0\,,\,1] \;\to\; \mathbf{R}$, the map which associate $t \in [0\,,\,1]$ to $\mathbf{g}\,(t) \;=\; \mathbf{f}\,(x \;+\; t\,h)$. The function $\mathbf{g}$ is differentiable in $[0\,,\,1]$ and

$$\forall t \in [0\,,\,1] \quad \mathbf{g}'\,(t) \;= < \nabla\mathbf{f}\,(x \;+\; t\,h) \;, h > \;\; .$$

By the mean value theorem, there exits $\theta \in [0\,,\,1]$ such that :

$$\mathbf{f}\,(x+h) - \mathbf{f}\,(x) \;=\; \mathbf{g}\,(1) - \mathbf{g}\,(0) \;= < \nabla\mathbf{f}\,(x \;+\; \theta\,h) \;, h > \;\; .$$

One deduces that :

$$\mathbf{f}\,(x+h) - \mathbf{f}\,(x) - < \nabla\mathbf{f}\,(x) \;, h >= < \nabla\mathbf{f}\,(x \;+\; \theta\,h) - \nabla\mathbf{f}\,(x) \;, h > \;\; ,$$

so that

$$|\mathbf{f}\,(x+h) - \mathbf{f}\,(x) - < \nabla\mathbf{f}\,(x) \;, h >| \;\leq\; \|\nabla\mathbf{f}\,(x+\theta h) - \nabla\mathbf{f}\,(x)\|_{\mathbf{X}^*} \;\|h\|_{\mathbf{X}} \;\; ,$$

thus,

$$|\mathbf{f}(x+h)-\mathbf{f}(x)-<\nabla\mathbf{f}(x)\ ,h>|\ \le\ \epsilon\ \|h\|_{\mathbf{X}}\ .$$

The function $\mathbf{f}$ is then differentiable at the point x and we have :

$$\mathbf{f}'(x)\ =\ \nabla\mathbf{f}(x)\ .$$

Proposition 18 *Let* $\mathbf{f}\ \mathbf{X}\ \rightarrow\ \mathbf{R}\cup\{+\infty\}$ *be a convex function . If* $\mathbf{f}$ *is Gâteaux différentiable on* $\mathbf{X}$ *then* $\mathbf{f}$ *is a convex function if and only if :*

$$\forall x\in\mathbf{X}\ \forall y\in\mathbf{X}\ \ \mathbf{f}(x)\ +\ <\nabla\mathbf{f}(x)\ ,y-x>\le\ \mathbf{f}(y)$$

(Convexity Inequality).

Proof : It is given in the proof of some following propositions .

We characterise the convexity of a function by the properties of the Gâteaux derivatives .
Let $\mathbf{C}$ be a non void convex closed subset of $\mathbf{X}$ and let $\mathbf{f}$ be a convex function from $\mathbf{C}$ to $\mathbf{R}$.

Proposition 19 *If* $\mathbf{f}$ *has a Gâteaux derivative on* $\mathbf{C}$ *then* $\mathbf{f}$ *is convex if and only if :*

$$\forall y\ \in\ \mathbf{C}\ \ \forall x\ \in\ \mathbf{C},\ \ <\nabla\mathbf{f}(y)\ -\ \nabla\mathbf{f}(x)\ ,y\ -\ x>\ge\ 0\ .$$

Remark 12 *If the inequality in proposition* 19 *holds, we say that* $\nabla\mathbf{f}$ *is* ***monotone****.*

Proof : We suppose that $\mathbf{f}$ is convex then one may apply the convexity inequality for x and y belonging to $\mathbf{C}$, thus :

$$\mathbf{f}(x)+<\nabla\mathbf{f}(x)\ ,y-x>\le\mathbf{f}(y)$$

and

$$\mathbf{f}(y)+<\nabla\mathbf{f}(y)\ ,x-y>\le\mathbf{f}(x)\ .$$

Adding these two inequalities, we obtain

$$<\nabla\mathbf{f}(x)\ -\ \nabla\mathbf{f}(y),y-x>\le\ 0$$

which implies

$$<\nabla\mathbf{f}(y)\ -\ \nabla\mathbf{f}(x),y-x>\ge\ 0$$

We prove now the converse .
Let $x \in \mathbf{C}$, $y \in \mathbf{C}$ and $t \in [0\,,1]$, we set :

$$\varphi(t) \ = \ \mathbf{f}((1-t)\ x \ + \ t\,y)$$

The function φ is continuously differentiable :

$$\varphi'(t) \ = < \nabla \mathbf{f}((1-t)\ x \ + \ t\,y)\,, y - x > \ .$$

As $\nabla \mathbf{f}$ is monotone then φ' is inceasing on $[0\,,1]$ thus φ is convex on $[0\,,1]$. Hence, we have that

$$\varphi(t) = \varphi((1-t)\ 0 \ + \ t\,1) \ \leq \ (1-t)\,\varphi(0) + t\varphi(1)$$

which implies

$$\mathbf{f}((1-t)\ x \ + \ t\,y) \ \leq \ (1-t)\ \mathbf{f}(x) \ + \ t\,\mathbf{f}(y) \ .$$

Thus $\mathbf{f}$ is convex .

2.1.4 Subdifferential

Definition 26 *Let* $\mathbf{f}\ \mathbf{X} \ \rightarrow \ \mathbf{R} \cup \{+\infty\}$, *if* $x \in \mathbf{dom}(\mathbf{f})$, *then* $\mathbf{f}$ *is* ***subdifferentiable at the point*** x *if there exits a linear continuous form* $x^* \in \mathbf{X}^*$ *such that :*

$$\forall y \in \mathbf{X} \ \ \mathbf{f}(x) + \ < x^*\,, y - x > \leq \ \mathbf{f}(y) \ .$$

We denote by $\partial \mathbf{f}(x)$ *the set of the linear continuous forms* $x^* \in \mathbf{X}^*$ *which satisfy the above property . The subset* $\partial \mathbf{f}(x)$ *is called* ***the subdifferential of*** $\mathbf{f}$ ***at the point*** x.

Remark 13 *The subset* $\partial \mathbf{f}(x)$ *is convex and closed in* $\mathbf{X}^*$

Proposition 20 *Let* $\mathbf{f}\ \mathbf{X} \ \rightarrow \ \mathbf{R} \cup \{+\infty\}$ *be a convex function . If* $\mathbf{f}$ *is Gâteaux differentiable at the point* $x \in \mathbf{dom}(\mathbf{f})$ *then* $\partial \mathbf{f}(x) = \{\nabla \mathbf{f}(x)\}$.

Proof : Let $y \in \mathbf{dom}(\mathbf{f})$ and $t \in]0\,,1]$, then by convexity of $\mathbf{f}$ we have :

$$\mathbf{f}((1-t)\ x \ + \ t\,y) \ \leq \ (1-t)\ \mathbf{f}(x) \ + \ t\,\mathbf{f}(y) \ .$$

this implies

$$\frac{1}{t}\ [\mathbf{f}(x \ + \ t\ (y-x)) \ - \ \mathbf{f}(x)] \ \leq \ \mathbf{f}(y) \ - \ \mathbf{f}(x) \ .$$

As t tends to $0+$, we obtain :

$$< \nabla \mathbf{f}(x), y - x > \leq \mathbf{f}(y) - \mathbf{f}(x) \ ,$$

thus,

$$\forall y \in \mathbf{X} \ \ \mathbf{f}(x) + < \nabla \mathbf{f}(x) \ , y - x > \leq \mathbf{f}(y) \ .$$

Let

$$\nabla \mathbf{f}(x) \in \partial \mathbf{f}(x) \ .$$

Let $x^* \in \partial \mathbf{f}(x)$, let $h \in \mathbf{X}$ if $t \in]0\,, 1]$, we have :

$$\mathbf{f}(x + t\,h) \geq \mathbf{f}(x) + < x^*, t\,h > \ .$$

this implies

$$\frac{1}{t} \left[\mathbf{f}(x + t\,h) - \mathbf{f}(x)\right] \geq < x^*, h > \ .$$

As t tends to $0+$, we obtain :

$$\forall h \in \mathbf{X} \quad < \nabla \mathbf{f}(x) \ , h > \geq < x^*, h > \ .$$

If we replace h by $-h$ in the preceeding inequality, we obtain :

$$\forall h \in \mathbf{X} \quad < \nabla \mathbf{f}(x) \ , h > \leq < x^* \ , h > \ .$$

Thus,

$$\forall h \in \mathbf{X} \quad < \nabla \mathbf{f}(x) \ , h > = < x^* \ , h > \ ,$$

hence,

$$\partial \mathbf{f}(x) = \{\nabla \mathbf{f}(x)\}$$

The converse of proposition 20 is given as follows :

Proposition 21 *Let* $\mathbf{f}$ $\mathbf{X} \rightarrow \mathbf{R} \cup \{+\infty\}$ *be a convex application . If* $\mathbf{f}$ *is continuous at a point* $x \in \mathbf{dom}(\mathbf{f})$ *and if* $\partial \mathbf{f}(x)$ *is a singleton then* $\mathbf{f}$ *is Gâteaux différentiable at the point* x*.*

Some computational properties of the usual derivitives also hold for sub-differentials .

Proposition 22 *Let* $\mathbf{f}$ *and* $\mathbf{g}$ *be convex functions from* $\mathbf{X}$ *to* $\mathbf{R} \cup \{+\infty\}$*, if* $\underline{x} \in \mathbf{dom}(\mathbf{f}) \cap \mathbf{dom}(\mathbf{g})$ *:*

- *If* $\lambda > 0$ *then*

$$\partial(\lambda\,\mathbf{f}) = \lambda\,\partial(\mathbf{f}) \ .$$

- $$\partial(\mathbf{f})(\underline{x}) + \partial(\mathbf{f})(\underline{x}) \subset \partial(\mathbf{f} + \mathbf{g})(\underline{x}) .$$

- *If* $\mathbf{f}$ *and* $\mathbf{g}$ *are lsc, proper and and if* $\mathbf{f}$ *is continuous at the point* $\underline{x}$ *then :*
$$\partial(\mathbf{f} + \mathbf{g})(\underline{x}) = \partial(\mathbf{f})(\underline{x}) + \partial(\mathbf{g})(\underline{x}) .$$

Proof : The first property is obvious . We prove now the inclusion .

$$\partial(\mathbf{f} + \mathbf{g})(\underline{x}) \subset \partial(\mathbf{f})(\underline{x}) + \partial(\mathbf{f})(\underline{x}) .$$

Let $x^* \in \partial(\mathbf{f} + \mathbf{g})(\underline{x})$ then

$$\forall y \in \mathbf{X} \;\; \mathbf{f}(\underline{x}) + \mathbf{g}(\underline{x}) + < x^*, y - \underline{x} > \leq \mathbf{f}(y) + \mathbf{g}(y) .$$

Set :

$$\mathbf{C_1} = \{(y,\lambda) \in \mathbf{X}\mathrm{x}\mathbf{R} \mid \mathbf{f}(y) - \mathbf{f}(\underline{x}) - < x^*, y - \underline{x} > \leq \lambda\}$$

and

$$\mathbf{C_2} = \{(y,\lambda) \in \mathbf{X}\mathrm{x}\mathbf{R} \mid \lambda \leq \mathbf{g}(\underline{x}) - \mathbf{g}(y)\} .$$

The preceeding inequality shows that the common points of $\mathbf{C_1}$ and $\mathbf{C_2}$ are the boundary points only . In addition the function $\mathbf{F}$ which is defined by :

$$\forall y \in \mathbf{X} \;\; \mathbf{F}(y) = \mathbf{f}(y) - \mathbf{f}(\underline{x}) - < x^*, y - \underline{x} > .$$

has $\mathbf{C_1}$ as its epigraph, also, $\mathbf{F}$ is continuous at the point $\underline{x}$ thus the interior of $\mathbf{C_1}$ is empty . As $\mathbf{C_2}$ is convex , there exists $(u^*,\alpha) \in \mathbf{X}^*\mathrm{x}\mathbf{R} \setminus \{(0_{\mathbf{X}},0)\}$ such that :

$$\forall y \in \mathbf{X} \;\; \mathbf{g}(\underline{x}) - \mathbf{g}(y) \leq < u^*, y > + \alpha \leq \mathbf{f}(y) - \mathbf{f}(\underline{x}) - < x^*, y - \underline{x} > .$$

Thus for $y = \underline{x}$, we obtain $< u^*,\underline{x} > + \alpha = 0$ and this implies $\alpha = < u^*, -\underline{x} >$. On the one hand, we have :

$$\forall y \in \mathbf{X} \;\; \mathbf{g}(\underline{x}) - \mathbf{g}(y) \leq < u^*, y - \underline{x} > .$$

That is

$$\forall y \in \mathbf{X} \;\; \mathbf{g}(\underline{x}) + < -u^*, y - \underline{x} > \leq \mathbf{g}(y) .$$

thus $-u^* \in \partial(\mathbf{g})(\underline{x})$.
In addition

$$\forall y \in \mathbf{X} \;\; < u^*, y - \underline{x} > \leq \mathbf{f}(y) - \mathbf{f}(\underline{x}) - < x^*, y - \underline{x} > ,$$

this implies

$$\forall y \in \mathbf{X} \ \ \mathbf{f}(\underline{x}) + < x^* + u^* , y - \underline{x} > \leq \mathbf{f}(y) - \mathbf{f}(\underline{x}) \ ,$$

thus, $x^* + u^* \in \partial(\mathbf{f})(\underline{x})$.
Hence $x^* = (x^* + u^*) + (-u^*)$ belongs to

$$\partial(\mathbf{f})(\underline{x}) + \partial(\mathbf{g})(\underline{x}) \ .$$

2.2 Euler Equations

With the help of the concept of differentiability, it is possible for us to write the the relation satisfied by a solution $\underline{x}$ of the problem :

$$\min_{x \in \mathbf{X}} \mathbf{f}(x)$$

2.2.1 Optimality conditions

Let $\mathbf{f} \ \mathbf{X} \rightarrow \mathbf{R} \cup \{+\infty\}$ be a function . Suppose that the problem $\min_{x \in \mathbf{X}} \mathbf{f}(x)$ has at least one solution $\underline{x} \in \mathbf{dom}(\mathbf{f})$.

Proposition 23 *If* $\mathbf{f}$ *has a derivative in the direction* $h \in \mathbf{X}$ *at the point* $\underline{x}$ *then :*

$$\mathbf{f}'(\underline{x}, h) \geq 0$$

Proof : We have :

$$\forall t > 0 \ \ \mathbf{f}(\underline{x} + th) \geq \mathbf{f}(\underline{x}) \ ,$$

this implies

$$\mathbf{f}'(\underline{x}, h) = \lim_{t \to 0+} \frac{\mathbf{f}(\underline{x} + th) - \mathbf{f}(\underline{x})}{t} \geq 0 \ .$$

Proposition 24 *If* $\mathbf{f}$ *admits a Gâteaux derivative at the point* $\underline{x}$ *and if* $\underline{x} \in \overbrace{\mathbf{dom}(\mathbf{f})}^{0}$ *then :*

$$\nabla \mathbf{f}(\underline{x}) = 0 \ .$$

Proof : This is obvious by proposition 23 .

We now examine important properties of convex functions . Let $\mathbf{C}$ be a non void closed convex subset of $\mathbf{X}$ and $\mathbf{f}$ a convex function of $\mathbf{C}$ to $\mathbf{R}$.

Proposition 25 *If* $\mathbf{f}$ *has a continuous Gâteaux derivative at the point* $\underline{x} \in \mathbf{C}$ *then the following properties are equivalent :*

- $$\underline{x} \text{ is the solution of the problem } \min_{x \in \mathbf{C}} \mathbf{f}(x) \; .$$
- $$\forall y \in \mathbf{C} \quad < \nabla \mathbf{f}(\underline{x}) \, , y - \underline{x} > \geq 0 \; .$$
- $$\forall y \in \mathbf{C} \quad < \nabla \mathbf{f}(y) \, , y - \underline{x} > \geq 0 \; .$$

Proof : We suppose that $\underline{x}$ is solution of the problem $\min_{x \in \mathbf{C}} \mathbf{f}(x)$.
Let $y \in \mathbf{C}$ and $t \in]0,1]$, then

$$\mathbf{f}((1-t)\underline{x} + ty) \geq \mathbf{f}(\underline{x}) \; ,$$

this gives

$$\frac{\mathbf{f}(\underline{x} + t(y - \underline{x})) - \mathbf{f}(\underline{x})}{t} \geq 0 \; ,$$

thus, taking the limit as t tends to $0+$ gives

$$< \nabla \mathbf{f}(\underline{x}) \, , y - \underline{x} > \geq 0 \; .$$

Now, we suppose that $\forall y \in \mathbf{C} \quad < \nabla \mathbf{f}(\underline{x}) \, , y - \underline{x} > \geq 0$
But the function $\mathbf{f}$ is convex and its Gâteaux derivative is monotone thus

$$\forall y \in \mathbf{C} \; \forall z \in \mathbf{C} \quad < \nabla \mathbf{f}(y) - \nabla \mathbf{f}(z) \, , y - z > \geq 0 \; .$$

If we put $z = \underline{x}$, we obtain :

$$\forall y \in \mathbf{C} \; \forall z \in \mathbf{C} \quad < \nabla \mathbf{f}(y) - \nabla \mathbf{f}(\underline{x}) \, , y - \underline{x} > \geq 0 \; .$$

Since

$$< \nabla \mathbf{f}(\underline{x}) \, , y - \underline{x} > \geq 0 \; .$$

We easily obtain :

$$< \nabla \mathbf{f}(y) \, , y - \underline{x} > \geq 0 \; .$$

Now suppose that :

$$\forall y \in \mathbf{C} \quad < \nabla \mathbf{f}(y) \, , y - \underline{x} > \geq 0 \; .$$

Let $y \in \mathbf{C}$, and $t \in [0,1]$, define

$$\varphi(t) = \mathbf{f}((1-t)\,\underline{x} + t\,y) \ .$$

The function φ is differentiable and

$$\varphi'(t) = < \nabla\mathbf{f}((1-t)\,\underline{x} + t\,y) \ , y - \underline{x} > \ ,$$

thus

$$\forall t \in [0,1] \quad \varphi'(t) \geq 0 \ ,$$

then $\varphi(1) \geq \varphi(0)$ that is $\mathbf{f}(y) \geq \mathbf{f}(\underline{x})$. Hence, $\underline{x}$ is solution of the problem $\min\limits_{x \in \mathbf{C}} \mathbf{f}(x)$.

Proposition 26 *If $\mathbf{f}$ is a convex function from $\mathbf{X}$ to $\mathbf{R} \cup \{+\infty\}$ then $\mathbf{f}$ has a minimum at the point $\underline{x} \in \mathbf{X}$ if and only if :*

$$0_{\mathbf{X}^*} \in \partial\mathbf{f}(\underline{x}) \ .$$

2.2.2 Ekeland Variational Principle

Let $\mathbf{f}$ be a function from a Banach space $\mathbf{X}$ to $\mathbf{R} \cup \{+\infty\}$

Ekeland Variational Principle

Theorem 7 *Suppose that $\mathbf{f}$ is proper, bounded below and lsc such that there exits $\epsilon > 0$ and $x_\epsilon \in \mathbf{X}$ verifying $\mathbf{f}(x_\epsilon) \leq \inf\limits_{x \in \mathbf{X}} \mathbf{f}(x) + \epsilon$. Then there exists $y_\epsilon \in \mathbf{X}$ such that :*

- $$\mathbf{f}(y_\epsilon) \leq \mathbf{f}(x_\epsilon) \ .$$
- $$\|x_\epsilon - y_\epsilon\|_{\mathbf{X}} \leq 1 \ .$$
- $$\forall x \in \mathbf{X} \ | x \neq y_\epsilon \Rightarrow \mathbf{f}(x) > \mathbf{f}(y_\epsilon) - \epsilon\,\|x - y_\epsilon\|_{\mathbf{X}} \ .$$

Proof : Observe that the function $x \mapsto \mathbf{f}(x) - \epsilon\,\|x - y_\epsilon\|_{\mathbf{X}}$ has a strict minimum at the point y_ϵ. We construct a sequence $(z_n)_{n \in \mathbf{N}}$ to approximate y_ϵ. Put $z_0 = x_\epsilon$, suppose that we have defined z_1 to z_n; then we set :

$$\mathbf{S}_n = \{u \in \mathbf{X} \ |\mathbf{f}(u) \leq \mathbf{f}(z_n) - \epsilon\,\|u - z_n\|_{\mathbf{X}}\} \ .$$

Observe that $z_n \in \mathbf{S}_n$ thus, $\mathbf{S}_n \neq \emptyset$. As $\mathbf{f}(z_n) > \inf_{u \in \mathbf{S}_n} \mathbf{f}(u)$ we obtain

$$\inf_{u \in \mathbf{S}_n} \mathbf{f}(u) < \frac{1}{2} \inf_{u \in \mathbf{S}_n} \mathbf{f}(u) + \mathbf{f}(z_n) \ ,$$

thus there exists $z_{n+1} \in \mathbf{S}_n$ such that :

$$\mathbf{f}(z_{n+1}) \ \leq \ \frac{1}{2} \inf_{u \in \mathbf{S}_n} \mathbf{f}(u) \ + \ \frac{1}{2} \mathbf{f}(z_n) \ ,$$

this gives

$$\mathbf{f}(z_{n+1}) \ - \ \inf_{u \in \mathbf{S}_n} \mathbf{f}(u) \ \leq \ \frac{1}{2} \left[\mathbf{f}(z_n) \ - \ \inf_{u \in \mathbf{S}_n} \mathbf{f}(u) \right] \ .$$

We prove that the sequence $(z_n)_{n \in \mathbf{N}}$ is a Cauchy sequence . The sequence $(\mathbf{f}(z_n))_{n \in \mathbf{N}}$ is decreasing, as $\mathbf{f}$ is bounded below, it converges . If m and n are integers such that $m > n$, we have :

$$\epsilon \ \|z_m - z_n\|_{\mathbf{X}} \ \leq \ \mathbf{f}(z_n) \ - \ \mathbf{f}(z_m) \ .$$

Thus, the sequence $(z_n)_{n \in \mathbf{N}}$ is a Cauchy sequence, so there exists $z \in \mathbf{X}$ such that $z = \lim_{n \mapsto +\infty} z_n$. Since the function $\mathbf{f}$ is lsc , we obtain :

$$\mathbf{f}(z) \ \leq \ \liminf_{n \mapsto +\infty} \mathbf{f}(z_n) \ .$$

Then

$$\mathbf{f}(z) \ \leq \ \liminf_{n \mapsto +\infty} \mathbf{f}(z_n) \ \leq \ \liminf_{n \mapsto +\infty} \inf_{u \in \mathbf{S}_n} \mathbf{f}(u) \ .$$

As the sequence $(\mathbf{f}(z_n))_{n \in \mathbf{N}}$ is decreasing , we have $\mathbf{f}(z) \leq \mathbf{f}(z_0) = \mathbf{f}(x_\epsilon)$ then

$$\epsilon \ \|x_\epsilon - z\|_{\mathbf{X}} = \epsilon \ \|z_0 - z\|_{\mathbf{X}} \ ,$$

thus,

$$\epsilon \ \|x_\epsilon - z\|_{\mathbf{X}} \ \leq \ \mathbf{f}(x_\epsilon) \ - \ \mathbf{f}(z) \ \leq \ \mathbf{f}(x_\epsilon) \ - \ \inf_{x \in \mathbf{S}_n} \mathbf{f}(x) \ ,$$

therefore,

$$\epsilon \ \|x_\epsilon - z\|_{\mathbf{X}} \ < \ \epsilon \ ,$$

hence,

$$\|x_\epsilon - z\|_{\mathbf{X}} \ < \ 1 \ .$$

Finally, to verify the last part of the theorem, we assume that z does not satisfy it . Then there exists $v \neq z$ such that

$$\mathbf{f}(v) \leq \mathbf{f}(z) - \epsilon \ \|v - z\|_{\mathbf{X}} \ ,$$

thus

$$\mathbf{f}(v) \; < \; \mathbf{f}(z) \; .$$

We also have that :

$$\forall n \in \mathbf{N} \; \mathbf{f}(v) \leq \mathbf{f}(z_n) - \epsilon \; \|v - z_n\|_{\mathbf{X}} \; ,$$

then

$$v \in \mathbf{S}_n \; \forall n \in \mathbf{N} \; ,$$

thus

$$\mathbf{f}(z) \; \leq \; \mathbf{f}(v)$$

This is impossible.
To conclude it is enough to take $y_\epsilon = z$. This completes the proof .

From this theorem, we obtain the following obvious proposition :

Corollary 1 *If $\mathbf{X}$ is a Banach space and if $\mathbf{f}\,\mathbf{X} \to \mathbf{R} \cup \{+\infty\}$ is lsc, proper and bounded below then*

$$\forall \epsilon > 0 \; \exists x_\epsilon \in \mathbf{X} \mid \forall x \in \mathbf{X} \; x \neq x_\epsilon \; \mathbf{f}(x) \; > \; \mathbf{f}(x_\epsilon) \; - \; \|x - x_\epsilon\|_{\mathbf{X}} \; .$$

We also have the following corollary :

Corollary 2 *If $\mathbf{X}$ is a Banach space and if $\mathbf{f}\,\mathbf{X} \to \mathbf{R}$ is lsc, Gâteaux differentiable and such that there exists $\epsilon > 0$ and $x_\epsilon \in \mathbf{X}$ satisfying $\mathbf{f}(x_\epsilon) \leq \inf_{x \in \mathbf{X}} \mathbf{f}(x) + \epsilon$. Then there exists $y_\epsilon \in \mathbf{X}$ such that :*

- $$\mathbf{f}(y_\epsilon) \leq \mathbf{f}(x_\epsilon) \; .$$
- $$\|x_\epsilon - y_\epsilon\|_{\mathbf{X}} \leq \sqrt{\epsilon} \; .$$
- $$\|\mathbf{f}'(y_\epsilon)\|_{\mathbf{X}^*} \leq \sqrt{\epsilon} \; .$$

Proof : As in the proof of Ekeland variational Principle with the following equivalent norm

$$\| \; \|_1 = \frac{1}{\sqrt{\epsilon}} \| \; \|_{\mathbf{X}} \; .$$

One consequence of this result is :

Corollary 3 *If* $\mathbf{X}$ *is a Banach space and if* $\mathbf{f}\ \mathbf{X} \rightarrow \mathbf{R}$ *is lsc, Gâteaux differentiable, then there exits a sequence* $(x_n)_{n\in\mathbf{N}}$ *of elements belonging to* $\mathbf{X}$ *such that :*

- $$\inf_{x\in\mathbf{X}} \mathbf{f}(x) = \lim_{n\rightarrow+\infty} \mathbf{f}(x_n) \ .$$

- $$\lim_{n\rightarrow+\infty} \nabla\mathbf{f}(x_n) = 0_{\mathbf{X}^*} \ .$$

Remark 14 *This result is a generalization of the Euler equation .*

Palais Smale condition

Definition 27 *If* $\mathbf{X}$ *is a Banach space and if* $\mathbf{f}\ \mathbf{X} \rightarrow \mathbf{R}$ *is of class* $\mathbf{C}^1$*.*
One says that $\mathbf{f}$ *satisfies the* ***Palais Smale*** *conditions at the level* $c \in \mathbf{R}$*, i.e.,* $\mathbf{f}$ *satisfies* $(\mathbf{PS})_{\mathbf{c}}$*, if for every sequence* $(x_n)_{n\in\mathbf{N}}$ *of elements in* $\mathbf{X}$ *such that :*

- $$\lim_{n\rightarrow+\infty} \mathbf{f}(x_n) \ = \ c \ .$$

- $$\lim_{n\rightarrow+\infty} \nabla\mathbf{f}(x_n) = 0_{\mathbf{X}^*} \ .$$

then $(x_n)_{n\in\mathbf{N}}$ *converges to an element of* $\mathbf{X}$*.*
When $\mathbf{f}$ *satisfies* $(\mathbf{PS})_{\mathbf{c}}$ *for all* $c \in \mathbf{R}$*, we say that* $\mathbf{f}$ *satisfies* ***Palais Smale*** *conditions and we write* $\mathbf{f}$ *satisfies* $(\mathbf{PS})$*.*

Proposition 27 *If* $\mathbf{X}$ *is a Banach space, if* $\mathbf{f}\mathbf{X} \rightarrow \mathbf{R}$ *is of class* $\mathbf{C}^1$*, bounded below and satisfies the* ***Palais Smale*** *condition, then* $\mathbf{f}$ *has a minimum on* $\mathbf{X}$*.*

Remark 15 *The* ***Palais Smale*** *conditions are often used in the proof of the existence of critical points .*

2.2.3 Optimality conditions with constraints

Optimality conditions with equality constraints

Let $\mathbf{X}$ be a reflexive Banach space with the norm $\| \, \|_{\mathbf{X}}$. Let $\mathbf{f} \; \mathbf{X} \to \mathbf{R}$ and $\mathbf{g}_1 ,..., \mathbf{g}_p \; \mathbf{X} \to \mathbf{R}$. Define the map $\mathbf{g}$ from $\mathbf{X}$ to $\mathbf{R}^p$ by $\mathbf{g}(x) = (\mathbf{g}_1(x) ,..., \mathbf{g}_p(x))$, $\forall x \in \mathbf{X}$.
Put $\mathbf{S}_{\mathbf{g}} = \{x \in \mathbf{X} \mid \mathbf{g}(x) = 0_{\mathbf{R}^p}\}$.
We consider the following problem $\mathcal{P}_{\mathtt{min}}$:

$$\min_{x \in \mathbf{S}_{\mathbf{g}}} \mathbf{f}(x) \ .$$

Proposition 28 *If* $\mathbf{f}$ *and* $\mathbf{g}_1 ,..., \mathbf{g}_p$ *are continuously differentiable, if* $\underline{x}$ *is solution of the problem* $\mathcal{P}_{\mathtt{min}}$ *and if* $\mathbf{g}'(\underline{x})$ *is onto, then there exists real numbers* $\lambda_1 ,..., \lambda_p$ *such that :*

$$\mathbf{f}'(\underline{x}) + \sum_{i=1}^{p} \lambda_i \, \mathbf{g}_i'(\underline{x}) = 0_{\mathbf{X}^*} \ .$$

Proof : The linear function $\mathbf{g}'(\underline{x})$ is continuous and onto from $\mathbf{X}$ to $\mathbf{R}^p$. If we set $\mathbf{X}_1 = \mathrm{Ker}(\mathbf{g}'(\underline{x}))$ then $\mathbf{X}_1$ is a closed linear subspace of $\mathbf{X}$ with codimension p, with the norm $\| \, \|_{\mathbf{X}_1}$ which is restriction of $\| \, \|_{\mathbf{X}}$ on $\mathbf{X}_1$, thus, $\mathbf{X}_1$ is a Banach space . If $\mathbf{X}_2$ is the orthogonal complement of $\mathbf{X}_1$ with the norm $\| \, \|_{\mathbf{X}_2}$, then $(\mathbf{X} , \| \, \|_{\mathbf{X}})$ can be identified as the product space $\mathbf{X}_1 \, \mathrm{x} \, \mathbf{X}_2$ with the product norm . If $x \in \mathbf{X}$, and $x = (x_1 , x_2)$ where $x_1 \in \mathbf{X}_1$ and $x_2 \in \mathbf{X}_2$.
The hypothesis of the theorem are equivalent to : $\mathbf{f}$ and $\mathbf{g}_1 ,..., \mathbf{g}_p$ are continuously differentiable and the continuous linear application $\frac{\partial \mathbf{g}}{\partial x_2}(\underline{x})$ is an isomorphism from $\mathbf{X}_2$ to $\mathbf{R}^p$.

The tangent linear space to $\mathbf{S}_{\mathbf{g}}$ at the point $\underline{x}$ is given by:

$$\mathbf{T}_{\underline{x}} = \{h \in \mathbf{X} \mid \exists \delta > 0 \text{ and } \gamma_\delta \;]-\delta , -\delta[\; \to \mathbf{S}_{\mathbf{g}} \text{ differentiable and } \gamma_\delta'(0) = h\} \ .$$

Put :

$$\mathbf{E}_{\underline{x}} = \{h \in \mathbf{X} \mid \mathbf{g}'(\underline{x}).h = 0_{\mathbf{R}^p}\} \ .$$

Let $h \in \mathbf{T}_{\underline{x}}$, there exits $\delta > 0$ and a function

$$\gamma_\delta \;]-\delta , -\delta[\; \to \mathbf{S}_{\mathbf{g}} \ ,$$

differentiable such that

$$\gamma_\delta'(0) = h \ .$$

We have

$$\forall t \in]-\delta , -\delta[\quad \mathbf{g}(\gamma_\delta(t)) = 0_{\mathbf{R}^p} ,$$

thus,

$$\forall t \in]-\delta , \quad -\delta[\quad \mathbf{g}'(\gamma_\delta(t)) .\gamma_\delta'(t) = 0_{\mathbf{R}^p} .$$

In particular, for $t = 0$, we have

$$\mathbf{g}'(\gamma_\delta(0)) .\gamma_\delta'(0) = 0_{\mathbf{R}^p} ,$$

hence,

$$\mathbf{g}'(\underline{x}) .h = 0 ,$$

thus $h \in \mathbf{E}_{\underline{x}}$.

Let $h \in \mathbf{E}_{\underline{x}}$, suppose that $h = (h_1 , h_2)$ where $h_1 \in \mathbf{X}_1$ and $h_2 \in \mathbf{X}_2$; then:

$$\frac{\partial \mathbf{g}}{\partial x_1}(\underline{x}) \; h_1 + \frac{\partial \mathbf{g}}{\partial x_2}(\underline{x}) \; h_2 = 0_{\mathbf{R}^p} .$$

By implicit function theorem, there exists an open set $\mathbf{U}$ which contains $\underline{x}_1$, an open set $\mathbf{W}$ containing $\underline{x}$ and a function $\phi \; \mathbf{U} \to \mathbf{X}_2$ continuously differentiable such that :

$$(x_1 , x_2) \in \mathbf{W} \cap \mathbf{S}_{\mathbf{g}} \Leftrightarrow x_1 \in \mathbf{U} \text{ and } x_2 = \phi(x_1) .$$

We have :

$$\forall x_1 \in \mathbf{U} \quad \mathbf{g}(x_1,\phi(x_1)) = 0_{\mathbf{R}^*} .$$

Differentiating this relation gives :

$$\forall x_1 \in \mathbf{U} \quad \frac{\partial \mathbf{g}}{\partial x_1}(x_1 , \phi(x_1)) + \frac{\partial \mathbf{g}}{\partial x_2}(x_1 , \phi(x_1)) \; \phi'(x_1) = 0_{\mathbf{X}_1^*} .$$

Replacing x_1 by $\underline{x}_1$, and as $\underline{x}_2 = \phi(\underline{x}_1)$, we obtain :

$$\frac{\partial \mathbf{g}}{\partial x_1}(\underline{x}) + \frac{\partial \mathbf{g}}{\partial x_2}(\underline{x}) \; \phi'(\underline{x}_1) = 0_{\mathbf{X}_1^*} .$$

thus :

$$\phi'(\underline{x}_1) = - \left[\frac{\partial \mathbf{g}}{\partial x_2}(\underline{x}) \right]^{-1} \frac{\partial \mathbf{g}}{\partial x_1}(\underline{x}) .$$

Let $\mathbf{F}$ be the function defined by

$$\mathbf{F}(x_1) = \mathbf{f}(x_1 , \phi(x_1)) \; \forall x_1 \in \mathbf{U} ,$$

we have :

$$\mathbf{F}(x_1) \geq \mathbf{F}(\underline{x}_1) \; \forall x_1 \in \mathbf{U} ,$$

thus the fréchet derivative

$$\mathbf{F}'(\underline{x}_1) = 0_{\mathbf{X}_1^*},$$

that is

$$\frac{\partial \mathbf{f}}{\partial x_1}(\underline{x}) + \frac{\partial \mathbf{f}}{\partial x_2}(\underline{x})\ \phi'(\underline{x}_1) = 0_{\mathbf{X}_1^*}.$$

So for

$$h = (h_1, h_2) \in \mathbf{X}_1 \mathrm{x} \mathbf{X}_2,$$

such that

$$\mathbf{g}'.h = 0_{\mathbf{R}^p},$$

that is

$$\frac{\partial \mathbf{g}}{\partial x_1}(\underline{x})\ h_1 + \frac{\partial \mathbf{g}}{\partial x_2}(\underline{x})\ h_2 = 0_{\mathbf{R}^p},$$

we have

$$h_2 = -\left[\frac{\partial \mathbf{g}}{\partial x_2}(\underline{x})\right]^{-1} \frac{\partial \mathbf{g}}{\partial x_1}(\underline{x})\ h_1,$$

But :

$$\mathbf{f}'(\underline{x}).h = \frac{\partial \mathbf{f}}{\partial x_1}(\underline{x})\ h_1 + \frac{\partial \mathbf{f}}{\partial x_2}(\underline{x})\ h_2.$$

Substituting for h_2 gives

$$\mathbf{f}'(\underline{x}).h = \frac{\partial \mathbf{f}}{\partial x_1}(\underline{x})\ h_1 - \frac{\partial \mathbf{f}}{\partial x_2}(\underline{x}) \left[\frac{\partial \mathbf{g}}{\partial x_2}(\underline{x})\right]^{-1} \frac{\partial \mathbf{g}}{\partial x_1}(\underline{x})\ h_1.$$

thus,

$$\mathbf{f}'(\underline{x}).h = 0_{\mathbf{X}^*}.$$

Finally, we have :

$$\forall h \in \mathbf{X} \text{ if } \forall i \in \{1,...,p\}\ \mathbf{g}_i'(\underline{x}).h = 0 \Rightarrow \mathbf{f}'(\underline{x}).h = 0_{\mathbf{X}^*}.$$

To conclude we use the following proposition :

Proposition 29 *If $x_1^*,...,x_p^*$ and x^* are linear continuous forms on $\mathbf{X}$ such that :*

$$\forall h \in \mathbf{X}\ \ \textit{if}\ \forall i \in \{1,...,p\}\ < x_1^*, h >= 0 \ \Rightarrow < x^*, h >= 0.$$

Then there exist real numbers $\lambda_1,..., \lambda_p$ such that :

$$x^* = \sum_{i=1}^{p} \lambda_i\, x_i^*.$$

Proof : It is obvious from the following lemma :

Lemma 1 *If x_1^* ,..., x_p^* and x^* are continuous linear forms on* **X** *such that :*

$$\forall h \in \mathbf{X} \quad \text{if}\; \forall i \in \{1 ,..., p\} \;\; < x_i^* \,, h > \geq \; 0 \quad \Rightarrow < x^* \,, h > \geq \; 0 \; .$$

Then there exist real positive numbers μ_1,...,μ_p such that :

$$x^* \; = \; \sum_{i=1}^{p} \mu_i \, x_i^* \; .$$

Proof : One may suppose that $\left\{x_1^* ,..., x_p^*\right\}$ is linearly independant . Put :

$$\mathcal{C} = \left\{ u^* \; \in \; \mathbf{X}^* \quad u^* \; = \; \sum_{i=1}^{p} \mu_i \, x_i^* \right\} \; .$$

The subset $\mathcal{C}$ is convex and closed . We suppose that $x^* \notin \; \mathcal{C}$, there exists $(h \,, \alpha) \; \in \; \mathbf{X} \, \mathrm{x} \, \mathbf{R}$ such that :

- $< x^* \,, h > > \; \alpha$.
- $\forall u^* \; \in \; \mathcal{C} \quad < u^* \,, h > \leq \; \alpha$.

We have $\alpha \; \geq \; 0$ thus, $< x^* \,, h > > \; 0$. If there exists $i_0 \in \{1 ,..., p\}$ such that $< x_{i_0}^* \,, h > > \; 0$. As

$$\forall \mu_1 \geq 0,...,\forall \mu_p \geq 0 \quad \sum_{i=1}^{i=p} \mu_i \;\; < x_i^* \,, h > \leq \; \alpha \; .$$

As μ_{i_0} tends to $+\infty$, the preceeding property is false . Thus,

$$\forall i \in \{1 ,..., p\} \quad < x_i^* \,, h > \leq \; 0 \; .$$

then $< x^* \,, \, -h > \geq \; 0$ this implies $< x^* \,, h > \leq \; 0$ and this is impossible . Thus $x^* \in \; \mathcal{C}$.

Optimality conditions with inequality constraints

Let $\{\mathbf{g}_1 ,..., \mathbf{g}_p\}$ be functions from **X** to **R** and $\mathbf{f} \;\, \mathbf{X} \; \rightarrow \; \mathbf{R}$. Set $\mathbf{S}_{\mathbf{g}}^- \; = \{x \in \mathbf{X} \; | \forall i \in \{1,...,p\} \;\; \mathbf{g}_i \,(x) \; \leq \; 0\}$ We consider the following problem $\mathcal{P}_{\mathtt{min}}$:

$$\min_{x \in \mathbf{S}_{\mathbf{g}}^-} \; \mathbf{f} \,(x) \;\; .$$

. As in the proof of the preceeding theorem, we obtain :

Proposition 30 *If* $\mathbf{f}$ *and* $\mathbf{g}_1 ,..., \mathbf{g}_p$ *are continuously differentiable, if* $\underline{x}$ *is a solution of the problem* $\mathcal{P}_{\mathtt{min}}$ *and if* $\mathbf{g}'(\underline{x})$ *is onto, then there exists real positive numbers* $\mu_1 ,..., \mu_p$ *such that :*

$$\mathbf{f}(\underline{x}) + \sum_{i=1}^{p} \mu_i \, \mathbf{g}_i'(\underline{x}) = 0_{\mathbf{X}^*} \ .$$

and

$$\forall i \in \{1 ,..., p\} \quad \mu_i \, \mathbf{g}_i(\underline{x}) = 0_{\mathbf{X}^*}$$

2.2.4 Applications

2.2.5 Some examples in Hilbert spaces

Let $\mathbf{H}$ be a Hilbert space over the set of real numbers with the scalar product $< \ , \ >_{\mathbf{H}}$ and the associated norm $\| \ \|_{\mathbf{H}}$.

Projection Theorem

Theorem 8 *If* $\mathbf{C}$ *is a convex closed subset of* $\mathbf{H}$*, then for every* x *belonging to* $\mathbf{H}$ *there exits one and only one element of* $\mathbf{C}$ *denoted by* $\mathbf{P_C}(x)$ *such that :*

$$\|x - \mathbf{P_C}(x)\|_{\mathbf{H}} = \min_{y \in \mathbf{C}} \|x - y\|_{\mathbf{H}} \ .$$

In addition $\mathbf{P_C}(x)$ *is the unique element* $z \in \mathbf{C}$ *such that*

$$\forall y \in \mathbf{C} \quad < x - z \, , y - z >_{\mathbf{H}} \leq 0 \ .$$

Proof : We set

$$\forall y \in \mathbf{H} \ \ \mathbf{J}(y) = \frac{1}{2} \|x - y\|_{\mathbf{H}}^2 \ .$$

this gives :

$$\forall y \in \mathbf{H} \ \ \mathbf{J}(y) = \frac{1}{2} < y \, , y >_{\mathbf{H}} - < x \, , y >_{\mathbf{H}} + < x \, , x >_{\mathbf{H}} \ .$$

The problem $\min_{y \in \mathbf{C}} \mathbf{J}(y)$ is a quadratic optimization problem and the function $\mathbf{J}$ is convex and coercive. Thus, this problem has one and only one solution $\mathbf{P_C}(x) \in \mathbf{C}$. The function $\mathbf{J}$ is differentiable and

$$\forall h \in \mathbf{H} \quad < \mathbf{J}'(y) \, , h >_{\mathbf{H}} = < y \, , h >_{\mathbf{H}} - < x \, , h >_{\mathbf{H}} \ .$$

Thus $z \in \mathbf{H}$ is solution of $\min\limits_{y \in \mathbf{C}} \mathbf{J}(y)$ if and only if:

$$z \in \mathbf{C} \text{ and } \forall y \in \mathbf{C} \quad < \mathbf{J}'(z) \, , y - z >_{\mathbf{H}} = < z - x \, , y - z >_{\mathbf{H}} \geq 0 \, ,$$

then

$$\forall y \in \mathbf{C} \quad < x - z \, , y - z >_{\mathbf{H}} \leq 0 \, .$$

Exercice :

Let $\mathbf{C}$ be a non void convex closed subset of $\mathbf{H}$.

- Show that

$$\forall x \in \mathbf{H} \, \forall y \in \mathbf{H} \quad \|\mathbf{P_C}(x) - \mathbf{P_C}(y)\|_{\mathbf{H}} \leq \|x - y\|_{\mathbf{H}} \, .$$

- Show that if $\mathbf{C}$ is a closed linear subspace of $\mathbf{H}$ then :

$$z = \mathbf{P_C}(x) \text{ if and only if } z \in \mathbf{C} \text{ and } \forall y \in \mathbf{C} \quad < x - z \, , y >_{\mathbf{H}} = 0 \, .$$

- Show that if $\mathbf{C}$ is a closed linear subspace of $\mathbf{H}$ then $\mathbf{P_C}$ is linear and continuous and moreover,

$$\forall x \in \mathbf{H} \quad \|x\|^2_{\mathbf{H}} = \|\mathbf{P_C}(x)\|^2_{\mathbf{H}} + \|x - \mathbf{P_C}(x)\|^2_{\mathbf{H}} \, .$$

- Prove that if $\mathbf{C}$ is a closed linear subspace $\mathbf{H}$ and if

$$\mathbf{C}^{\perp} = \{x \in \mathbf{H} \mid \forall y \in \mathbf{C} \quad < x \, , y >_{\mathbf{H}} = 0\}$$

then

$$\mathbf{H} = \mathbf{C} \oplus \mathbf{C}^{\perp} \, .$$

Stampacchia Theorem : the symmetric case

Lett $\mathbf{a}$ be a bilinear form on $\mathbf{H}$ which is continuous, coercive and symmetric . We have:

- the continuity of $\mathbf{a}$ is equivalent to

$$\exists M > 0 \mid \forall x \in \mathbf{X} \, \forall y \in \mathbf{X} \; |\mathbf{a}(x \, , y)| \leq M \; \|x\|_{\mathbf{X}} \; \|y\|_{\mathbf{X}} \; ;$$

- the coercivity of $\mathbf{a}$ is equivalent to

$$\exists \alpha > 0 \mid \forall x \in \mathbf{X} \; \alpha \, \|x\|^2_{\mathbf{X}} \leq a(x \, , x) \; ;$$

- $\mathbf{a}$ is symmetric if and only if

$$\forall x \in \mathbf{H}\ \forall y \in \mathbf{H}\ \ \mathbf{a}(x, y) = \mathbf{a}(y, x) \ ;$$

Let ℓ be a linear continuous form on $\mathbf{H}$. There exists $L > 0$ such that

$$\forall x \in \mathbf{H}\ \ |\ell(x)| \leq M\ \ \|x\|_{\mathbf{H}}$$

We define on $\mathbf{H}$, the function denoted by $\mathbf{J}$ by :

$$\mathbf{J}(x) = \frac{1}{2}\mathbf{a}(x, x) - \ell(x, x)\ \forall x \in \mathbf{H}\ .$$

Theorem 9 *If* $\mathbf{C}$ *is a non void closed convex subset of the Hilbert space* $\mathbf{H}$ *, if* $\mathbf{a}$ *is a bilinear form on* $\mathbf{H}$ *which is symmetric, continuous and coercive; and if* ℓ *is a linear continuous form on* $\mathbf{H}$ *then the problem :*

$$\textit{Find} u \in \mathbf{C} \textit{ such that } \forall v \in \mathbf{C}\ \ \mathbf{a}(u, v-u) \geq \ell(v-u)$$

has one and only one solution .

Proof : Let $\mathbf{J}$ be the proper convex continuous and coercive function which is defined by :

$$\mathbf{J}(v) = \frac{1}{2}\mathbf{a}(v, v) - \ell(v)\ .$$

The problem

$$\min_{x \in \mathbf{C}} \mathbf{J}(x)$$

is a quadratic optimization problem which has one and only one solution $u \in \mathbf{C}$ and which is characterized by :

$$\forall v \in \mathbf{C}\ \ \mathbf{J}'(u).\ (v-u) \geq 0\ ;$$

or

$$\forall v \in \mathbf{C}\ \ \mathbf{a}(u, v-u) - \ell(v-u) \geq 0\ ,$$

thus $u \in \mathbf{C}$ verifies :

$$\forall v \in \mathbf{C}\ \ \mathbf{a}(u, v-u) \geq \ell(v-u)\ .$$

Remark 16 *The problem may be interpreted like a projection problem when we endow* $\mathbf{H}$ *with the scalar product defined by* $\mathbf{a}$ *.*
The theorem is also true if $\mathbf{a}$ *is not symmetric .*

Lax Milgram Theorem

Theorem 10 *If* $\mathbf{H}$ *is a Hilbert space, if* $\mathbf{a}$ *is a bilinear form on* $\mathbf{H}$ *which is symmetric, continuous and coercive and if* ℓ *is a linear form on* $\mathbf{H}$ *which is continuous then the problem :*

$$\textit{Find } u \in \mathbf{H} \textit{ such that } \forall v \in \mathbf{H} \;\; \mathbf{a}(u, v) = \ell(v)$$

admits one and only one solution .

Proof : It is enough to prove the equivalence of the problem $\mathcal{P}_1$

$$\text{Find } u \in \mathbf{H} \text{ such that } \forall v \in \mathbf{H} \;\; \mathbf{a}(u, v) = \ell(v)$$

with the problem $\mathcal{P}_2$

$$\text{Find } u \in \mathbf{H} \text{ such that } \forall v \in \mathbf{H} \;\; \mathbf{a}(u, v-u) \geq \ell(v-u) \;.$$

We suppose that $u \in \mathbf{H}$ is a solution of $\mathcal{P}_1$:
Let $v \in \mathbf{H}$, we have

$$\mathbf{a}(u, v-u) = \mathbf{a}(u, v) - \mathbf{a}(u, u) = \ell(v) - \ell(u) \;,$$

then

$$\mathbf{a}(u, v-u) = \ell(v-u) \;.$$

thus u is the solution of $\mathcal{P}_2$.
Conversely, let $u \in \mathbf{H}$ be a solution of $\mathcal{P}_2$:
Let $v \in \mathbf{H}$ then $w = u + v$ belong $\mathbf{H}$. Replacing v with w in $\mathcal{P}_2$ gives:

$$\mathbf{a}(u, v) \geq \ell(v) \;.$$

If we replace v by $-v$ in this relation, we obtain :

$$\mathbf{a}(u, v) \leq \ell(v) \;,$$

so

$$\mathbf{a}(u, v) = \ell(v) \;.$$

Thus u is a solution of $\mathcal{P}_1$.

An example of application in solving partial differential equation

Let Ω be a non void open domain of $\mathbf{R}^N$. Let $\mathbf{f} \in \mathbf{L}^2(\Omega)$, we want to solve the problem :
Find $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ such that

$$-\Delta\mathbf{u} = \mathbf{f} \text{ in } \Omega$$

This problem is equivalent to the minimization problem which is defined by the function $\mathbf{J}$ where:

$$\mathbf{J}(\mathbf{v}) = \frac{1}{2}\int_\Omega < \nabla\mathbf{v}, \nabla\mathbf{v} >_{\mathbf{R}^N} dx - \int_\Omega \mathbf{f}\,\mathbf{v}\,dx \;\mathbf{v} \in \mathbf{H}_0^1(\Omega)$$

and the minimization problem

$$\min_{\mathbf{v}\in \mathbf{H}_0^1(\Omega)} \mathbf{J}(\mathbf{v}) .$$

This problem is a quadratic optimization problem, it has one and only one solution .

2.2.6 An example in a Banach space

Let $p > 1$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\mathbf{f} \in \mathbf{L}^q(\Omega)$, where Ω a non void bounded domain of $\mathbf{R}^N$. We want to solve the problem :

Find $\mathbf{u} \in \mathbf{W}_0^{1\,p}(\Omega)$ such that

$$-\mathbf{div}\left(\|\nabla\mathbf{u}\|_{\mathbf{R}^N}^{p-2}\nabla\mathbf{u}\right) = \mathbf{f} \text{ in } \Omega .$$

This problem is equivalent to the minimization problem which is defined by the function $\mathbf{J}$ where:

$$\mathbf{J}(\mathbf{v}) = \frac{1}{p}\int_\Omega \|\nabla\mathbf{v}\|_{\mathbf{R}^N}^p dx - \int_\Omega \mathbf{f}\,\mathbf{v}\,dx \;\mathbf{v} \in \mathbf{W}_0^{1\,p}(\Omega)$$

and the minimization problem

$$\min_{\mathbf{v}\in \mathbf{W}_0^{1\,p}(\Omega)} \mathbf{J}(\mathbf{v}) .$$

The function $\mathbf{J}$ is strictly convex, lsc, proper and coercive on the Sobolev space $\mathbf{W}_0^{1\,p}$ which is a reflexive Banach space . The minimizing problem has one and only one solution .

2.2.7 An eigenvalue problem

Let Ω be a non void bounded domain of $\mathbf{R}^N$.
We want to find a function $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \setminus \{0\}$ such that there exists a real number λ with

$$-\Delta \mathbf{u} = \lambda \, \mathbf{u} \text{ on } \Omega .$$

This problem is an optimization problem with equality constraints

$$\min_{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \;\; \mathbf{g}(\mathbf{v}) = 0} \mathbf{J}(\mathbf{v})$$

$\mathbf{J}$ and $\mathbf{g}$ are defined as follows:

$$\mathbf{v} \in \mathbf{H}_0^1(\Omega) \quad \mathbf{J}(\mathbf{v}) = \frac{1}{2} \int_\Omega \|\nabla \mathbf{v}\|_{\mathbf{R}^N}^2 \, dx \; ;$$

$$\mathbf{v} \in \mathbf{H}_0^1(\Omega) \quad \mathbf{g}(\mathbf{v}) = \frac{1}{2} \int_\Omega |\mathbf{v}|^2 \, dx - \frac{1}{2} \, .$$

We remark that if $\{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \mathbf{g}(\mathbf{v}) = 0\} \neq \emptyset$ then

$$\forall h \in \mathbf{H}_0^1(\Omega) \quad \mathbf{g}'(\mathbf{v}).h = \int_\Omega \mathbf{v} \, h \, dx$$

and $\mathbf{g}'(\mathbf{v})$ is a continuous onto linear form on $\mathbf{H}_0^1(\Omega)$. The function $\mathbf{J}$ is differentiable and :

$$\forall h \in \mathbf{H}_0^1(\Omega) \quad \mathbf{J}'(\mathbf{v}).h = \int_\Omega < \nabla \mathbf{v} \, , \nabla h >_{\mathbf{R}^N} \, dx \, .$$

Thus, if the minimization problem has a solution $\mathbf{u}$, there exists $\lambda \in \mathbf{R}$ such that $\forall h \in \mathbf{H}_0^1(\Omega) \quad \mathbf{J}'(\mathbf{u}).h = \lambda \, \mathbf{g}'(\mathbf{u}).h$, then

$$\forall h \in \mathbf{H}_0^1(\Omega) \quad \int_\Omega < \nabla \mathbf{u} \, , \nabla h >_{\mathbf{R}^N} \, dx = \lambda \int_\Omega \mathbf{u} \, h \, dx \, .$$

We deduce that $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and $-\Delta \mathbf{u} = \lambda \, \mathbf{u}$.
Moreover, if $\lambda > 0$, it is enough to take $h = \mathbf{u}$. Now, we prove the existence of $\mathbf{u}$. The function $\mathbf{J}$ is bounded below by 0 thus, it has a finite infimum. There exists a minimizing sequence $(\mathbf{u}_n)_{n \in \mathbf{N}}$ of elements of $\mathbf{H}_0^1(\Omega)$ such that $\forall n \in \mathbf{N} \quad \|\mathbf{u}_n\|_{\mathbf{L}^2(\Omega)} = 1$.
The sequence $(\mathbf{u}_n)_{n \in \mathbf{N}}$ is bounded in $\mathbf{H}_0^1(\Omega)$ thus it has a subsequence $(\mathbf{u}_{n_k})_{k \in \mathbf{N}}$ which converges weekly in $\mathbf{H}_0^1(\Omega)$ to an element $\mathbf{u}$. The space $\mathbf{H}_0^1(\Omega)$ is included with compact inclusion in $\mathbf{L}^2(\Omega)$,then the sequence $(\mathbf{u}_{n_k})_{k \in \mathbf{N}}$

converges in $\mathbf{L}^2(\Omega)$ to $\mathbf{u}$. Thus, $\mathbf{g}(\mathbf{v}) = 0$ and $\|\mathbf{u}\|_{\mathbf{L}^2(\Omega)} = 1$, $\mathbf{u} \neq 0$. In addition, $\mathbf{J}$ is weakly lsc, so

$$\mathbf{J}(\mathbf{u}) \leq \liminf_{k \to +\infty} \mathbf{J}(\mathbf{u}_{n_k}) = \inf_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)\ \mathbf{g}(\mathbf{v}) = 0} \mathbf{J}(\mathbf{v}) .$$

The minimization problem has one solution .

2.3 A very short bibliography

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