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Topics on Calculus of variations

Augusto C. Ponce
Université François Rabelais
LMPT - UMR - CNRS 6083
Tours, France

TOPICS ON CALCULUS OF VARIATIONS

AUGUSTO C. PONCE

1. The Laplace equation

Let $\Omega \subset \mathbb{R}^N$ be a body made of some uniform material; on the boundary of Ω, we prescribe some fixed temperature f. Consider the following

Question. What is the equilibrium temperature inside Ω?

Mathematically, if u denotes the temperature in Ω, then we want to solve the equation

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

We shall always assume that $f : \partial\Omega \to \mathbb{R}$ is a given smooth function and $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a smooth, bounded, connected open set. We say that u is a classical solution of (1.1) if $u \in C^2(\overline{\Omega})$ and u satisfies (1.1) at every point of $\overline{\Omega}$.

This problem was studied by some of the greatest mathematicians of the 19th century: Fourier, Green, Dirichlet, Lord Kelvin, Riemann, Weierstrass, Schwarz, Neumann, Poincaré,... We shall devote ourselves to the variational approach called the *Dirichlet Principle*.

1.1. The Dirichlet Principle.

The Dirichlet Principle consists in replacing problem (1.1) by the following minimization problem:

$$I = \inf\left\{ \int_\Omega |\nabla v|^2 : v \in C^2(\overline{\Omega}) \text{ such that } v = f \text{ on } \partial\Omega \right\}. \tag{1.2}$$

Any function $v \in C^2(\overline{\Omega})$ such that $v = f$ on $\partial\Omega$ is called an admissible function.

We first note that problems (1.1) and (1.2) are equivalent by Theorem 1.1 below. We then say that $\Delta u = 0$ is the Euler-Lagrange equation associated to the functional

$$J(v) = \int_\Omega |\nabla v|^2.$$

Theorem 1.1 (Riemann). *u solves* (1.1) *if and only if u minimizes* (1.2).

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Proof. ($\Rightarrow$) Let u be a solution of (1.1). Given any admissible function v, we have to show that

$$\int_\Omega |\nabla u|^2 \le \int_\Omega |\nabla v|^2. \tag{1.3}$$

Note that $u = v$ on $\partial\Omega$, thus an integration by parts gives

$$\int_\Omega \nabla u \cdot \nabla(v-u) = -\int_\Omega \operatorname{div}(\nabla u)\,(v-u) = -\int_\Omega \Delta u\,(v-u) = 0. \tag{1.4}$$

We deduce that

$$\begin{aligned}
\int_\Omega |\nabla v|^2 &= \int_\Omega \big|\nabla\big(u + (v-u)\big)\big|^2 \\
&= \int_\Omega |\nabla u|^2 + 2\int_\Omega \nabla u \cdot \nabla(v-u) + \int_\Omega \big|\nabla(v-u)\big|^2 \\
\text{(by (1.4))} \quad &= \int_\Omega |\nabla u|^2 + \int_\Omega \big|\nabla(v-u)\big|^2 \\
&\ge \int_\Omega |\nabla u|^2.
\end{aligned}$$

This is precisely (1.3).

($\Rightarrow$) Assume u minimizes the Dirichlet integral. We want to show that $\Delta u = 0$ in Ω. Given $\varphi \in C_0^\infty(\Omega)$ (i.e., φ is a smooth function with compact support in Ω), let

$$F(t) = \int_\Omega \big|\nabla(u + t\varphi)\big|^2 - \int_\Omega |\nabla u|^2 \quad \forall t \in \mathbb{R}.$$

Since u is a minimizer, we have $F \ge 0$ and $F(0) = 0$. Thus, $F'(0) = 0$. On the other hand,

$$F'(t) = 2\int_\Omega \nabla u \cdot \nabla\varphi + 2t\int_\Omega |\nabla\varphi|^2.$$

We conclude that

$$0 = F'(0) = 2\int_\Omega \nabla u \cdot \nabla\varphi = -2\int_\Omega \Delta u\,\varphi.$$

Since φ is arbitrary, we have $\Delta u = 0$ in Ω. $\square$

The Dirichlet Principle was used in the 19th century in order to assure existence of solutions of (1.1) via the minimization problem (1.2). At that time, however, the existence of a solution of (1.2) was taken for granted. But in 1870 Weierstrass came out with the following example in dimension $N = 1$:

$$\tilde{I} = \inf\left\{\int_{-1}^1 \big(xv'(x)\big)^2\,dx : v \in C^2[-1,1],\ v(-1) = -1 \text{ and } v(1) = 1\right\}. \tag{1.5}$$

Proposition 1.1 (Weierstrass)**.** *Problem* (1.5) *has no solution.*

Proof. We first show that $\tilde{I} = 0$. Clearly, $\tilde{I} \geq 0$. It remains to show that $\tilde{I} \leq 0$. Indeed, let

$$v_n(x) = \frac{\arctan nx}{\arctan n} \quad \forall x \in [-1, 1].$$

An easy computation shows that

$$\int_{-1}^{1} \big(x v_n'(x)\big)^2 \, dx \to 0 \quad \text{as } n \to \infty.$$

Thus, $\tilde{I} = 0$ as claimed. In order to establish the proposition, assume by contradiction that (1.5) has a solution u. Since $\tilde{I} = 0$, this implies

$$u'(x) = 0 \quad \forall x \in (-1, 1) \setminus \{0\}.$$

By the continuity of u', we deduce that $u' \equiv 0$. Thus, u is constant. Since, by assumption, $u(-1) = -1$ and $u(1) = 1$, we have a contradiction. □

After Weierstrass' example, the Dirichlet Principle was regarded as a historical curiosity. Several mathematicians, notably Poincaré, eventually solved problem (1.1) using other methods: integral equations, conformal representation, balayage, etc. By the end of the 19th century, Hilbert would enter in the scene...

1.2. **The dawn of the Direct Methods.** In 1897, Hilbert tried to re-establish the Dirichlet Principle in rigorous terms. The task was to find a solution of the minimization problem (1.2), without making use of (1.1). We could summarize his idea as follows.

Let (v_n) be a sequence of admissible functions such that

$$\int_\Omega |\nabla v_n|^2 \to I;$$

(v_n) is called a minimizing sequence. One should try to construct out of (v_n) a new minimizing sequence (w_n) such that

$$w_n \to u,$$

the convergence taking place in some suitable sense. Then, u should be a solution of (1.2).

In the beginning, Hilbert's idea turned out to be quite difficult to implement. His argument in [5] was sketchy; immediately after, he came out with a rigorous proof, but it was quite long (see [6]). The reason is that he was trying to minimize the Dirichlet integral

$$\int_\Omega |\nabla v|^2$$

in the wrong space, namely $C^2(\overline{\Omega})$. Actually, Hilbert did not realize that. Nevertheless, his words were somewhat prophetical: "Every problem in the Calculus of Variations has a solution, provided the word 'solution' is suitably understood".

1.3. **The modern formulation of the Calculus of Variations.** The modern theory of the Calculus of Variations is based on the following strategy:

Step 1 Enlarge the set of admissible functions, where it is easier to find a solution (existence step);

Step 2 Show that the solution found in Step 1 actually belongs to the initial set; thus it is a solution of the original minimization problem (regularity step).

We shall illustrate how to implement Steps 1 and 2 in the study of (1.2).

1.4. **The space $H_0^1(\Omega)$.** In this section we introduce the underlying space where Step 1 will be performed. Let

$$H^1(\Omega) = \left\{ v \in L^2(\Omega) \;\middle|\; \begin{array}{l} \text{for every } i = 1, \ldots, N \\ \text{there exist } w_i \in L^2(\Omega) \text{ such that} \\ \displaystyle\int_\Omega v \frac{\partial \varphi}{\partial x_i} = -\int_\Omega w_i \varphi, \ \forall \varphi \in C_0^\infty(\Omega) \end{array} \right\}$$

We recall that $C_0^\infty(\Omega)$ denotes the set of smooth functions with compact support in Ω.

The functions w_i, whenever exist, are uniquely determined a.e. (see Exercise 1.1); these functions w_i are called weak derivatives of v; they will be denoted by $\frac{\partial v}{\partial x_i}$. For instance, if $u \in C^1(\overline{\Omega})$, then $u \in H^1(\Omega)$ (in which case the weak and classical derivatives coincide), but the converse is false (see Exercise 1.4).

The space $H^1(\Omega)$, equipped with the norm

$$\|v\|_{H_1} := \|v\|_{L^2} + \sum_{i=1}^{n} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^2},$$

is a Hilbert space (see Exercise 1.3). We denote by $\langle \ , \ \rangle_{H^1}$ the inner product associated with this norm.

Exercise 1.1.

(1) Prove the Fundamental Theorem of the Calculus of Variations:
Let $v \in L^1(\Omega)$ be such that

$$\int_\Omega v\varphi = 0 \quad \forall \varphi \in C_0^\infty(\Omega), \tag{1.6}$$

then $v = 0$ a.e.

Hint: First show that (1.6) holds for every $\varphi \in L^\infty(\Omega)$; then take $\varphi = \operatorname{sign} u$.

(2) Deduce that if $u \in H^1(\Omega)$, then the weak derivatives of u are well-defined a.e.

Exercise 1.2. Let $(v_n)_{n \geq 1}$ be a sequence in $H^1(\Omega)$ such that

$$v_n \to v \ \text{ in } L^2(\Omega) \quad \text{and} \quad \frac{\partial v_n}{\partial x_i} \to w_i \ \text{ in } L^2(\Omega) \quad \forall i = 1, \ldots, N.$$

Show that $v \in H^1(\Omega)$ and $\frac{\partial v}{\partial x_i} = w_i$ for every i.

Exercise 1.3. Prove that $H^1(\Omega)$ is a Hilbert space.

Exercise 1.4. Let $\Omega := B_{1/2}$ be the ball of radius 1/2 centered at 0, and

$$v(x) := \log\log\frac{1}{|x|} \quad \forall x \in \Omega.$$

Show that $v \in H^1(\Omega)$; in particular, $H^1(\Omega) \not\subset C(\overline{\Omega})$.

In view of Exercise 1.4, functions in $H^1(\Omega)$ need not be continuous. However, every element in $H^1(\Omega)$ can be approached by C^∞-functions. This fact is fundamental in the study of properties of $H^1(\Omega)$:

Theorem 1.2. *$C^\infty(\overline{\Omega})$ is dense in $H^1(\Omega)$.*

By definition, $H^1(\Omega)$ is a vector subspace of $L^2(\Omega)$. Surprisingly, functions in H^1 have some better integrability:

Theorem 1.3 (Sobolev)**.** *$H^1(\Omega) \subset L^{\frac{2N}{N-2}}(\Omega)$ and we have*

$$\|v\|_{L^{\frac{2N}{N-2}}} \le C\|v\|_{H^1} \quad \forall v \in H^1(\Omega),$$

for some constant $C > 0$ independent of v.

Since $H^1(\Omega)$ is an infinite dimensional vector space, closed bounded subsets of H^1 need not be compact with respect to the metric induced by the H^1-norm (Riesz's theorem). However,

Theorem 1.4 (Rellich-Kondrachov)**.** *If K is a bounded closed subset of $H^1(\Omega)$, then K is compact in $L^p(\Omega)$, for every $1 \le p < \frac{2N}{N-2}$.*

Exercise 1.5. For any smooth function $\varphi \in C_0^\infty(B_1)$, consider the sequence (φ_n) given by $\varphi_n(x) = n^{\frac{N-2}{2}}\varphi(nx)$, $\forall x \in B_1$.

(1) Show that

$$\int_{B_1} |\varphi_n|^{\frac{2N}{N-2}} = \int_{B_1} |\varphi|^{\frac{2N}{N-2}} \quad \text{and} \quad \int_{B_1} |\nabla\varphi_n|^2 = \int_{B_1} |\nabla\varphi|^2 \quad \forall n \ge 1.$$

(2) Deduce that Theorem 1.4 above fails in the critical case of the Sobolev imbedding, namely $p = \frac{2N}{N-2}$.

We now introduce the following

Definition 1.1. *Given a sequence of functions $(v_n) \subset H^1(\Omega)$, we say that (v_n) converges weakly to v if*

$$\langle v_n, w\rangle_{H^1} \to \langle v, w\rangle_{H^1} \quad \forall w \in H^1(\Omega).$$

This will be denoted by

$$v_n \rightharpoonup v \quad \textit{weakly in } H^1(\Omega).$$

Clearly, if $v_n \to v$ strongly in $H^1(\Omega)$ (i.e., $\|v_n - v\|_{H^1} \to 0$), then $v_n \rightharpoonup v$ weakly, but the converse is false (see Exercise 1.6). The weak convergence induces a topology in $H^1(\Omega)$ which has less open and closed sets than the topology induced by the H^1-norm. For instance, the sphere

$$\mathcal{S} = \Big\{ v \in H^1(\Omega) : \|v\|_{H^1} = 1 \Big\} \tag{1.7}$$

is not closed with respect to the weak topology (see Exercise 1.6). However, the following holds:

Proposition 1.2. *Let $F \subset H^1(\Omega)$ be a vector space. Assume that F is closed with respect to the H^1-norm. Then, F is closed with respect to weak convergence. In other words, if $(v_n) \subset F$ and $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$, then $v \in F$.*

Proof. By contradiction, assume there exists a sequence $(v_n) \subset F$ such that $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$ but $v \notin F$. Since F is closed, by the Hahn-Banach Theorem there exists a continuous linear functional $f : H^1(\Omega) \to \mathbb{R}$ such that

$$f(v) = 1 \quad \text{and} \quad f(w) = 0 \quad \forall w \in F.$$

By the Riesz Representation Theorem, there exists $u_0 \in H^1(\Omega)$ such that

$$f(z) = \langle u_0, z \rangle_{H^1} \quad \forall z \in H^1(\Omega).$$

In particular,

$$0 = f(v_n) = \langle u_0, v_n \rangle_{H^1} \to \langle u_0, v \rangle_{H^1} = f(v) = 1.$$

This is a contradiction. Thus, F is closed under weak convergence. □

Here are some other properties satisfied by the weak convergence:

Theorem 1.5. *Let (v_n) be a sequence in $H^1(\Omega)$.*

(*i*) *If $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$, then (v_n) is bounded in $H^1(\Omega)$ and we have*

$$\|\nabla v\|_{L^2} \le \liminf_{n\to\infty} \|\nabla v_n\|_{L^2}.$$

(*ii*) *If (v_n) is bounded in $H^1(\Omega)$, then we can extract a subsequence (v_{n_k}) such that*

$$v_{n_k} \rightharpoonup v \quad \textit{weakly in } H^1(\Omega)$$

for some $v \in H^1(\Omega)$.

Combining Theorems 1.4 and 1.5, one deduces the following

Corollary 1.1. *Let (v_n) be a bounded sequence in $H^1(\Omega)$. Then, we can extract a subsequence (v_{n_k}) such that*

$$\begin{aligned} &v_{n_k} \rightharpoonup v \quad \textit{weakly in } H^1(\Omega),\\ &v_{n_k} \to v \quad \textit{strongly in } L^p(\Omega), \quad \textit{for every } p \in \big[1, \tfrac{2N}{N-2}\big),\\ &v_{n_k} \to v \quad \textit{a.e.} \end{aligned}$$

Exercise 1.6. Let (φ_n) be the sequence given in Exercise 1.5. Show that $\varphi_n \rightharpoonup 0$ weakly in $H^1(\Omega)$.

Consider the following vector subspace of $H^1(\Omega)$:

$$H_0^1(\Omega) := \overline{C_0^\infty(\Omega)}^{H^1}.$$

In other words, $H_0^1(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the H^1-norm topology. We warn the reader that $H_0^1(\Omega) \neq H^1(\Omega)$. As we shall see, this space is useful in applications. Functions in $H_0^1(\Omega)$ satisfy the following important inequality:

Theorem 1.6 (Poincaré)**.** *There exists $C > 0$ such that*

$$\|v\|_{L^2} \leq C\, \|\nabla v\|_{L^2} \quad \forall v \in H_0^1(\Omega). \tag{1.8}$$

Note that (1.8) cannot hold for every function $v \in H^1(\Omega)$. In fact, if v is a constant, then $\nabla v = 0$ and this would imply that v is identically zero.

To each element in $H^1(\Omega)$ it is possible to associate a notion of trace (= boundary value on $\partial\Omega$), even though these functions are only defined a.e.:

Theorem 1.7 (Trace theorem)**.** *There exists a (unique) continuous linear operator* $\mathrm{Tr} : H^1(\Omega) \to L^2(\partial\Omega)$ *such that*

$$\mathrm{Tr}\,(v) = v|_{\partial\Omega} \quad \forall v \in C^\infty(\overline{\Omega}).$$

Moreover,

$$\ker \mathrm{Tr} = H_0^1(\Omega).$$

In view of the theorem above, $H_0^1(\Omega)$ can be seen as the set of elements in $H^1(\Omega)$ which vanish on $\partial\Omega$.

Below, the reader will find some properties satisfied by functions in $H^1(\Omega)$ which will be used in the sequel:

Exercise 1.7. Let $\Phi \in C^1(\mathbb{R})$ be such that Φ' is bounded in $\mathbb{R}$. Prove the chain rule for functions in H^1:
If $v \in H^1(\Omega)$, then $\Phi(v) \in H^1(\Omega)$ and

$$\nabla\Phi(v) = \Phi'(v)\nabla v \quad \text{a.e.}$$

Hint: Apply Theorem 1.2.

Exercise 1.8. Show that if $u \in H^1(\Omega)$, then $u^+ \in H^1(\Omega)$ and

$$\nabla u^+ = \begin{cases} \nabla u & \text{if } u \geq 0, \\ 0 & \text{if } u < 0. \end{cases}$$

What is the analog for u^-?
Hint: Apply Exercise 1.7 to a suitable sequence of smooth functions $\Phi_k : \mathbb{R} \to \mathbb{R}$ such that $\Phi_k(t) \to t^+$ uniformly in $\mathbb{R}$ and in C^1 outside $t = 0$.

1.5. **The weak formulation of (1.2).** We now implement Step 1 described in Section 1.3 above. We first replace (1.2) by the following minimization problem in $H^1(\Omega)$:

$$I_w = \inf \left\{ \int_\Omega |\nabla v|^2 : v \in \bar{f} + H_0^1(\Omega) \right\}, \tag{1.9}$$

where $\bar{f} \in C^\infty(\overline{\Omega})$ is a fixed function such that $\bar{f} = f$ on $\partial\Omega$.

Exercise 1.9. Check that (1.9) does not depend on the extension $\bar{f}$. In other words, if $\tilde{f}$ is another extension of f, then

$$\bar{f} + H_0^1(\Omega) = \tilde{f} + H_0^1(\Omega).$$

Theorem 1.8 (Hilbert). *Problem* (1.9) *has a unique solution* $u \in H^1(\Omega)$. *Moreover,* u *satisfies*

$$\int_\Omega \nabla u \cdot \nabla \varphi = 0 \quad \forall \varphi \in C_0^\infty(\Omega). \tag{1.10}$$

We say that any function u satisfying (1.10) is a *weak solution* of the Laplace equation $\Delta u = 0$. Of course, if u is smooth and $\Delta u = 0$ in Ω, then (1.10) holds. As we will see in the next section, if u satisfies (1.10), then $u \in C^\infty(\Omega)$; thus, $\Delta u = 0$ in Ω in the classical sense. In other words, the notions of weak and classical solutions turn out to be the same. This is quite surprising!

Theorem 1.8 will be proved using Hilbert's strategy:

Proof of Theorem 1.8. Let (u_n) be a minimizing sequence. We first show that (u_n) is bounded in $H^1(\Omega)$. Since

$$\int_\Omega |\nabla u_n|^2 \to I_w,$$

we already know that (∇u_n) is bounded in $L^2(\Omega)$. It remains to show that (u_n) is bounded in $L^2(\Omega)$. Note that $u_n - \bar{f}$ belongs to $H_0^1(\Omega)$; thus, by Poincaré's inequality,

$$\|u_n - \bar{f}\|_{L^2} \le C\, \|\nabla u_n - \nabla \bar{f}\|_{L^2} \le C \quad \forall n \ge 1.$$

We conclude that (u_n) is bounded in $H^1(\Omega)$.

Applying Theorem 1.5, we can extract a subsequence (u_{n_k}) such that $u_{n_k} \rightharpoonup u$ weakly in $H^1(\Omega)$. Moreover, by Theorem 1.5 (i),

$$\int_\Omega |\nabla u|^2 \le \varliminf_{k\to\infty} \int_\Omega |\nabla u_{n_k}|^2 = I_w.$$

It remains to show that $u \in \bar{f} + H_0^1(\Omega)$. To see this, note that $H_0^1(\Omega)$ is a closed vector subspace of $H_0^1(\Omega)$ and the sequence $(u_n - \bar{f})$ is contained in $H_0^1(\Omega)$. Thus, by Proposition 1.2, we deduce that

$$u - \bar{f} \in H_0^1(\Omega).$$

We conclude that u is an admissible function of problem (1.9). In particular,

$$I_w \leq \int_\Omega |\nabla u|^2.$$

Therefore,

$$I_w = \int_\Omega |\nabla u|^2.$$

The proof of (1.10) will be left as an exercise (the reader can proceed as in Proposition 1.1). $\square$

1.6. **Weyl's lemma.** We now turn ourselves to Step 2 of the modern approach. More precisely, we want to prove that for any function $u \in H^1(\Omega)$ satisfying

$$\int_\Omega \nabla u \cdot \nabla \varphi = 0 \quad \forall \varphi \in C_0^\infty(\Omega),$$

then $u \in C^2(\Omega)$ and $\Delta u = 0$ in Ω.

In 1940, Weyl showed that this is indeed the case. It is one of the first regularity results in the context of the Calculus of Variations:

Theorem 1.9 (Weyl)**.** *Let $u \in H^1(\Omega)$ be such that*

$$\int_\Omega \nabla u \cdot \nabla \varphi = 0 \quad \forall \varphi \in C_0^\infty(\Omega). \tag{1.11}$$

Then, $u \in C^\infty(\Omega)$ and $\Delta u = 0$ in Ω.

Proof. By the definition of weak derivative, (1.11) gives

$$\int_\Omega u \Delta \varphi = 0 \quad \forall \varphi \in C_0^\infty(\Omega). \tag{1.12}$$

For simplicity, we shall now assume that $\Omega = \mathbb{R}^N$.
Let $\rho \in C_0^\infty(B_1)$ be a radial function such that $\rho \geq 0$ and $\int_{B_1} \rho = 1$. For any $\varepsilon > 0$, let

$$\rho_\varepsilon(x) = \frac{1}{\varepsilon^N} \rho\Big(\frac{x}{\varepsilon}\Big).$$

Given $\varphi \in C_0^\infty(\mathbb{R}^N)$, we choose $\rho_\varepsilon * \varphi$ as a test function in (1.12). By Fubini's theorem, we have

$$\int_{\mathbb{R}^N} (\rho_\varepsilon * u)\Delta\varphi = \int_{\mathbb{R}^N} u\,(\rho_\varepsilon * \Delta\varphi) = \int_{\mathbb{R}^N} u\Delta(\rho_\varepsilon * \varphi) = 0.$$

On the other hand, integrating by parts twice, we get

$$\int_{\mathbb{R}^N} (\rho_\varepsilon * u)\Delta\varphi = -\int_{\mathbb{R}^N} \nabla(\rho_\varepsilon * u) \cdot \nabla\varphi = \int_{\mathbb{R}^N} \Delta(\rho_\varepsilon * u)\,\varphi$$

Thus,

$$\int_{\mathbb{R}^N} \Delta(\rho_\varepsilon * u)\,\varphi = 0$$

for every test function $\varphi \in C_0^\infty(\mathbb{R}^N)$. Since $\rho_n * u$ is smooth, we deduce that

$$\Delta(\rho_\varepsilon * u) = 0 \quad \text{in } \mathbb{R}^N.$$

In other words, $\rho_\varepsilon * u$ is a harmonic function. In particular, it satisfies the Mean Value Formula:

$$\rho_\varepsilon * u(x) = \fint_{B_r(x)} \rho_\varepsilon * u \quad \forall x \in \mathbb{R}^n, \quad \forall r > 0.$$

Here, "–" denotes the average over $B_r(x)$, i.e., $\fint_{B_r} = \frac{1}{|B_r|} \int_{B_r}$. Since $\rho_\varepsilon * u \to u$ in $L^1(\mathbb{R}^N)$, we conclude that

$$u(x) = \fint_{B_r(x)} u \quad \text{for a.e. } x \in \mathbb{R}^n, \quad \text{for a.e. } r > 0. \tag{1.13}$$

Note that for $r > 0$ fixed, the right-hand side is a continuous function of x (apply the Lebesgue Dominated Convergence Theorem). Thus, u coincides a.e. with a continuous function. Replacing u by this continuous representative if necessary, we may assume that u itself is continuous. In particular, identity (1.13) holds for *every* $x \in \mathbb{R}^N$ and $r > 0$.

Multiply both sides of (1.13) by r^N and differentiate with respect to r. We deduce that

$$u(x) = \fint_{\partial B_r(x)} u \quad \forall x \in \mathbb{R}^n, \quad \forall r > 0. \tag{1.14}$$

(The average in this case is computed over ∂B_r.) In order to conclude the proof of Theorem 1.9, we show that

$$\rho * u(x) = u(x) \quad \forall x \in \mathbb{R}^N. \tag{1.15}$$

For simplicity, let us prove this equality for $x = 0$. Since ρ is radial, we have

$$\rho * u(0) = \int_{\mathbb{R}^N} \rho(y) u(y)\, dy = \int_0^\infty \rho(r)\, dr \int_{\partial B_r(0)} u(r, \sigma)\, d\sigma.$$

Applying (1.14), we then get

$$\rho * u(0) = \int_0^\infty \rho(r)\, |\partial B_r|\, u(0)\, dr = \left(\int_{\mathbb{R}^N} \rho \right) u(0) = u(0).$$

Thus, (1.15) holds. Since $\rho * u$ is smooth, we conclude that u is smooth as well. $\square$

Remark 1.1. Theorem 1.9 shows that the solution u we got in Theorem 1.8 via minimization in $H_0^1(\Omega)$ actually belongs to $C^\infty(\Omega)$. This takes care of the regularity in the interior of our domain Ω. It remains to show that u is smooth up to the boundary; in other words, $u \in C^\infty(\overline{\Omega})$. This is more difficult to prove and it will not be done here (see [4]).

2. The Poisson equation

Let $\Omega \subset \mathbb{R}^N$ be such that Ω has electric density $g(x)$ at each point $x \in \Omega$. Assume that $\partial\Omega$ is connected to the earth (its electric potential is thus identically zero). Consider the following

Question. What is the electric potential inside Ω generated by g?

Mathematically, given $g \in C^\infty(\overline{\Omega})$, we wish to solve

$$\begin{cases} -\Delta u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Note that (2.1) is the Euler-Lagrange equation associated to the functional

$$J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega gv.$$

Instead of repeating the same strategy as before, we can reduce (2.1) to a problem of the form (1.1).

2.1. The Newtonian potential.

Taking an extension of g to $\mathbb{R}^N$ if necessary, we may always assume that $g \in C_0^\infty(\mathbb{R}^N)$. The Newtonian potential of g is defined as

$$G(x) = \int_{\mathbb{R}^N} \Phi(x-y)g(y)\,dy \quad \forall x \in \mathbb{R}^N,$$

where Φ is the fundamental solution of $-\Delta$:

$$\Phi(z) = \frac{1}{N(N-2)|B_1|}\frac{1}{|z|^{N-2}};$$

here, $|B_1|$ denotes the volume of the unit ball in $\mathbb{R}^N$. Note that Φ is smooth outside the origin and $\Delta\Phi = 0$ in $\mathbb{R}^N \setminus \{0\}$. The constant factor in front of $\frac{1}{|z|^{N-2}}$ is chosen so that

$$-\Delta\Phi = \delta_0$$

in the sense of distributions, i.e., (see [3])

$$-\int_{\mathbb{R}^N} \Phi\Delta\varphi = \varphi(0) \quad \forall \varphi \in C_0^\infty(\mathbb{R}^N). \tag{2.2}$$

We now establish the following

Proposition 2.1. *Let $g \in C_0^\infty(\mathbb{R}^N)$. Then, $G \in C^\infty(\mathbb{R}^N)$ and*

$$-\Delta G = g \quad \textit{in } \mathbb{R}^N. \tag{2.3}$$

Proof. Changing variables, we get

$$G(x) = \int_{\mathbb{R}^N} g(x-y)\Phi(y)\,dy.$$

Since $g \in C_0^\infty(\mathbb{R}^N)$, we conclude that $G \in C^\infty(\mathbb{R}^N)$.
We now prove (2.3). For every $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \Delta G\, \varphi = \int_{\mathbb{R}^N} G \Delta\varphi.$$

On the other hand, by Fubini's theorem and (2.2),

$$\begin{aligned}\int_{\mathbb{R}^N} G\Delta\varphi &= \int_{\mathbb{R}^N} \Delta\varphi(x)\, dx \int_{\mathbb{R}^N} \Phi(x-y) g(y)\, dy\\ &= \int_{\mathbb{R}^N} g(y)\, dy \int_{\mathbb{R}^N} \Phi(x-y)\Delta\varphi(x)\, dx\\ &= \int_{\mathbb{R}^N} g(y)\, dy \int_{\mathbb{R}^N} \Phi(z)\Delta\varphi(z+y)\, dz = -\int_{\mathbb{R}^N} g(y)\varphi(y)\, dy.\end{aligned}$$

Thus,

$$-\int_{\mathbb{R}^N} \Delta G\, \varphi = \int_{\mathbb{R}^N} g\varphi.$$

Since this holds for every $\varphi \in C_0^\infty(\mathbb{R}^N)$, we conclude that

$$-\Delta G = g \quad \text{in } \mathbb{R}^N.$$

□

2.2. **Solving (2.1).** We now prove the following

Theorem 2.1. *Equation* (2.1) *has a solution for every* $g \in C^\infty(\overline{\Omega})$.

Proof. Let G be the Newtonian potential of a C_0^∞-extension of g in $\mathbb{R}^N$. Set $U = u - G$. Then, equation (2.1), in terms of U, becomes

$$\begin{cases} -\Delta U = 0 & \text{in } \Omega,\\ U = -G & \text{on } \partial\Omega. \end{cases}$$

In Section 1 we saw that this equation has a solution. We conclude that (2.1) also has a solution. □

3. The eigenvalue problem

We wish to find $u \neq 0$ satisfying

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

for some $\lambda \in \mathbb{R}$. In other words, u is an eigenfunction of $-\Delta$ with eigenvalue λ.

The strategy to obtain the existence of such u and λ will be to consider the following minimization problem under constraint:

$$\lambda_1 = \inf\left\{\int_\Omega |\nabla v|^2 : v \in H_0^1(\Omega) \text{ and } \int_\Omega v^2 = 1\right\}. \tag{3.2}$$

In particular, λ_1 satisfies

$$\lambda_1 \int_\Omega v^2 \le \int_\Omega |\nabla v|^2 \quad \forall v \in H_0^1(\Omega). \tag{3.3}$$

Note that, by definition, $\frac{1}{\lambda_1}$ is the smallest constant for which Poincaré's inequality holds.

We summarize the main properties satisfied by λ_1 in the next

Theorem 3.1.
(*i*) *λ_1 is the smallest eigenvalue of* (3.1)*;*
(*ii*) *$\lambda_1 > 0$ and λ_1 is simple;*
(*iii*) *If u is an eigenfunction of* (3.1) *associated to λ_1, then $u \in C^\infty(\overline{\Omega})$ and u does not change sign.*

The proof of Theorem 3.1 is presented below.

3.1. **Existence step.** The proof of existence of a minimizer of (3.2) is similar to the Dirichlet Principle:

Theorem 3.2. *The minimization problem* (3.2) *has a solution.*

Proof. Let (u_n) be a minimizing sequence. By Corollary 1.1, one can find a subsequence (u_{n_k}) such that

$$\begin{aligned} u_{n_k} &\rightharpoonup u \quad \text{weakly in } H_0^1(\Omega) \\ u_{n_k} &\to u \quad \text{in } L^2(\Omega). \end{aligned}$$

Thus,

$$\int_\Omega u^2 = 1$$

and, by Theorem 1.5 (*i*),

$$\int_\Omega |\nabla u|^2 \le \lim_{k\to\infty} \int_\Omega |\nabla u_{n_k}|^2 = \lambda_1.$$

Since $H_0^1(\Omega)$ is a closed vector subspace of $H^1(\Omega)$, we have $u \in H_0^1(\Omega)$. Thus, u is a minimizer of (3.2). $\square$

We now show that u is a weak solution of (3.1):

Proposition 3.1. *If u is a solution of* (3.2)*, then u satisfies*

$$\int_\Omega \nabla u \cdot \nabla v = \lambda_1 \int_\Omega uv \quad \forall v \in H_0^1(\Omega). \tag{3.4}$$

Proof. Let $\Phi, \Psi : H_0^1(\Omega) \to \mathbb{R}$ be given by

$$\Phi(v) = \int_\Omega |\nabla v|^2 \quad \text{and} \quad \Psi(v) = \int_\Omega v^2.$$

By assumption, u is a critical point of Φ restricted to the set $[\Psi = 1]$. By the Lagrange Multiplier Rule, there exists $\mu \in \mathbb{R}$ such that

$$\Phi'(u) = \mu \Psi'(u).$$

Note that

$$\Phi'(v)w = 2 \int_\Omega \nabla v \cdot \nabla w \quad \text{and} \quad \Psi'(v)w = 2 \int_\Omega vw \quad \forall v, w \in H_0^1(\Omega).$$

Taking in particular $w = v = u$, we deduce that $\mu = \lambda_1$. This establishes the proposition. □

The next lemma will be important in the proof of parts (*ii*) and (*iii*) of Theorem 3.1:

Lemma 3.1. *If u is a solution of* (3.2)*, then u^+ and u^- also satisfy* (3.4)*.*

Proof. Let

$$a = \int_\Omega (u^+)^2 \quad \text{and} \quad b = \int_\Omega (u^-)^2.$$

If $a = 0$ or $b = 0$, then we are done. We may thus assume that $a, b > 0$. Note that

$$a + b = \int_\Omega u^2 = 1.$$

Moreover, in view of (3.3) and Exercise 1.8,

$$\lambda_1 = \int_\Omega |\nabla u|^2 = \int_\Omega |\nabla u^+|^2 + \int_\Omega |\nabla u^-|^2 \geq \lambda_1 (a + b) = \lambda_1.$$

We must have equality everywhere; in particular,

$$\int_\Omega |\nabla u^+|^2 = \lambda_1 \int_\Omega (u^+)^2 \quad \text{and} \quad \int_\Omega |\nabla u^-|^2 = \lambda_1 \int_\Omega (u^-)^2.$$

Therefore, $\frac{u^+}{a^{1/2}}$ and $\frac{u^-}{b^{1/2}}$ are also solutions of the minimization problem (3.2). In view of the previous proposition, they must satisfy (3.4). Simplifying the resulting equations, we get the result. □

3.2. **Regularity step.** As we have seen above, problem (3.1) has weak solutions. Our goal in this section is to establish the next

Theorem 3.3. *Let $u \in H_0^1(\Omega)$ be a weak solution of* (3.1)*. Then, $u \in C^\infty(\Omega)$.*

The strategy to prove Theorem 3.3 is to proceed inductively as follows:

(1) We first show that if $u \in H^k$, then $u \in H^{k+2}$;
(2) If $u \in H^k$ for some $k \geq 1$ sufficiently large, then u is continuous.

We say that $v \in H^2(\Omega)$ if $v, Dv \in H^1(\Omega)$. By induction, given any $k \geq 2$, we say that $v \in H^{k+1}(\Omega)$ if $v, Dv \in H^k(\Omega)$.

The basic ingredient to prove (1) is given by the next (see [3])

Theorem 3.4. *Let $u \in H^1_{\mathrm{loc}}(\Omega)$ and $f \in L^2_{\mathrm{loc}}(\Omega)$ be such that*

$$-\Delta u = f \quad \text{in } \Omega. \tag{3.5}$$

Then, $u \in H^2_{\mathrm{loc}}(\Omega)$.

Whenever we say that $u \in H^1_{\mathrm{loc}}(\Omega)$ verifies (3.5), it is implicitly understood in the weak sense; in other words,

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega f\varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

Assertion (2) is provided by the following (see [3])

Theorem 3.5 (Morrey). *If $u \in H^k(\Omega)$ for some $k > \frac{N}{2}$, then u is continuous. Moreover,*

$$\|u\|_{C^0} \leq C\|u\|_{H^k},$$

where the constant $C > 0$ is independent of u.

We can now prove Theorem 3.3:

Proof of Theorem 3.3. Let us first show that $u \in H^k_{\mathrm{loc}}(\Omega)$, $\forall k \geq 2$. For $k = 2$, this follows directly from Theorem 3.4. Assume by induction the assertion is true for some $k \geq 2$. Given a multi-index $(j_1, \dots, j_{k-1})$, we will prove that $w = D^{j_1,\dots,j_{k-1}}u \in H^2_{\mathrm{loc}}(\Omega)$. Indeed, note that by linearity w also satisfies the equation

$$\Delta w = \lambda w \quad \text{in } \Omega.$$

By assumption, $w \in H^1_{\mathrm{loc}}(\Omega)$. It then follows from Theorem 3.4 that $w \in H^2_{\mathrm{loc}}(\Omega)$. Since this is true for every multi-index $(j_1, \dots, j_{k-1})$, we have $u \in H^{k+1}_{\mathrm{loc}}(\Omega)$. By induction, we get the result.
Thus, by Theorem 3.5, u and $D^j u$ are continuous in Ω for every $j \geq 1$. We deduce that $u \in C^\infty(\Omega)$. We refer the reader to [3] for the proof that $u \in C^\infty(\overline{\Omega})$. ☐

3.3. **Proof of Theorem 3.1.** By Theorems 3.2 and 3.3, (3.2) has a minimizer $u \in C^\infty(\overline{\Omega})$. In addition, λ_1 is an eigenvalue of $-\Delta$ and

$$\lambda_1 = \int_\Omega |\nabla u|^2 \geq 0.$$

Note that $\lambda_1 > 0$, for otherwise u would be a constant; since $u = 0$ on $\partial\Omega$, then we would have $u = 0$ in Ω. Moreover, if λ is another eigenvalue of (3.1) with eigenfunction v, then

$$\lambda \int_\Omega v^2 = -\int_\Omega v\Delta v = \int_\Omega |\nabla v|^2 \geq \lambda_1 \int_\Omega v^2.$$

Therefore, $\lambda \geq \lambda_1$.
We next prove the following

Claim. If u is any eigenfunction associated to λ_1, then u does not vanish in Ω.

Assume by contradiction that $u(x_0) = 0$ for some $x_0 \in \Omega$. In particular, $u^+(x_0) = 0$. By Lemma 3.1, u^+ also satisfies (3.4). Thus, u^+ is also smooth and

$$-\Delta u^+ = \lambda_1 u^+ \geq 0 \quad \text{in } \Omega.$$

Since u^+ vanishes in Ω, it follows from the strong maximum principle that $u^+ = 0$ in Ω. Similarly, $u^- = 0$ in Ω. We conclude that $u = 0$ in Ω, which is a contradiction, since u is a non-trivial solution of (3.1).

This claim immediately implies assertion (iii). We are left to show that λ_1 is simple. Assume $\tilde{u}$ is any eigenfunction associated to λ_1. Let $c \in \mathbb{R}$ be such that

$$\int_\Omega (\tilde{u} - cu) = 0. \tag{3.6}$$

Since $\tilde{u} - cu$ is also an eigenfunction associated to λ_1, it follows from the claim above that $\tilde{u} - cu$ does not change sign. In view of (3.6), we conclude that $\tilde{u} = cu$. Thus, λ_1 is a simple eigenvalue. □

4. Semilinear elliptic equations

We consider the Dirichlet problem

$$\begin{cases} -\Delta u = h(u) & \text{in } \Omega, \\ u = 0 & \text{on } \Omega, \end{cases} \tag{4.1}$$

where $h : \mathbb{R} \to \mathbb{R}$ is smooth. We say that u is a classical solution of (4.1) if $u \in C^2(\overline{\Omega})$ verifies (4.1) in the usual sense; u is a weak solution of (4.1) if $u \in H_0^1(\Omega)$, $h(u) \in L^1(\Omega)$ and

$$\int_\Omega \nabla u \cdot \nabla \varphi = \int_\Omega h(u)\varphi \quad \forall \varphi \in C_0^\infty(\Omega). \tag{4.2}$$

In this section, we devote ourselves to the case where $h(s) = \lambda s + (s^+)^q$, with $\lambda \in \mathbb{R}$ and $q > 1$. We shall look for non-trivial solutions of the equation

$$\begin{cases} -\Delta u = \lambda u + u^q & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Omega. \end{cases} \tag{4.3}$$

The main result is the following

Theorem 4.1. *Assume* $1 < q < \frac{N+2}{N-2}$. *Then,* (4.3) *has a non-trivial solution* $u \in C^2(\overline{\Omega})$ *if and only if* $\lambda < \lambda_1$.

4.1. Existence step.

Two main difficulties arise in this case

- (i) $u = 0$ is a solution of (4.3), thus one has to make sure to avoid it;
- (ii) The functional associated to (4.3), namely

$$J(v) = \frac{1}{2}\int_\Omega \left(|\nabla v|^2 - \lambda v^2\right) - \frac{1}{q+1}\int_\Omega (v^+)^{q+1}$$

is *not* bounded from below in $H_0^1(\Omega)$.

Exercise 4.1. Let $v \in H_0^1(\Omega)$ with $v^+ \neq 0$. Show that

$$\lim_{t\to\infty} J(tv) = -\infty.$$

One possible approach to study (4.3) is to look for solutions using the Mountain Pass Theorem of Ambrosetti-Rabinowitz (see [7]), but we shall not pursue this direction. Instead, we consider the following minimization problem with constraint:

$$I_\lambda = \inf\left\{\int_\Omega \left(|\nabla v|^2 - \lambda v^2\right) : v \in H_0^1(\Omega) \text{ and } \int_\Omega (v^+)^{q+1} = 1\right\}. \tag{4.4}$$

We have the following

Theorem 4.2. *If $1 < q < \frac{N+2}{N-2}$ and $\lambda < \lambda_1$, then $I_\lambda > 0$ and (4.4) has a solution $u \in H_0^1(\Omega)$. Moreover, $u \geq 0$ a.e. and satisfies*

$$\int_\Omega \nabla u \cdot \nabla \varphi - \lambda \int_\Omega u\varphi = I_\lambda \int_\Omega u^q \varphi \quad \forall \varphi \in C_0^\infty(\Omega). \tag{4.5}$$

Proof. Since $\lambda < \lambda_1$, we have $I_\lambda \geq 0$ (apply inequality (3.3)). In particular, $I_\lambda > -\infty$. The proof of the existence of a minimizer u, and that u satisfies

$$\int_\Omega \nabla u \cdot \nabla \varphi - \lambda \int_\Omega u\varphi = I_\lambda \int_\Omega (u^+)^q \varphi \quad \forall \varphi \in C_0^\infty(\Omega) \tag{4.6}$$

then follows along the same lines of Theorem 3.2 and Proposition 3.1. We leave the details to the reader.
Let us prove that $I_\lambda > 0$. Assume by contradiction that $I_\lambda = 0$. Then, u satisfies

$$\int_\Omega \nabla u \cdot \nabla \varphi = \lambda \int_\Omega u\varphi \quad \forall \varphi \in C_0^\infty(\Omega).$$

Since $u \neq 0$, we deduce that $\lambda < \lambda_1$ is an eigenvalue of $-\Delta$. But this contradicts the fact that λ_1 is the smallest eigenvalue (see Theorem 3.1). Thus, $I_\lambda > 0$.
Finally, we prove that $u \geq 0$ a.e. In order to do that, we first note that (4.6) still holds for functions $\varphi \in H_0^1(\Omega)$ (it suffices to apply a density argument). We then apply (4.6) with $\varphi = u^-$. A simple computation shows that

$$uu^- = -(u^-)^2 \quad \text{and} \quad (u^+)^q u^- = 0.$$

Moreover, by Exercise 1.8, we have

$$\nabla u \cdot \nabla u^- = -|\nabla u^-|^2.$$

It then follows from (3.3) and (4.6) that

$$0 = \int_\Omega |\nabla u^-|^2 - \lambda \int_\Omega (u^-)^2 \geq (\lambda_1 - \lambda) \int_\Omega (u^-)^2.$$

Since $\lambda < \lambda_1$, we conclude that $u^- = 0$ a.e. Hence, $u \geq 0$ a.e. $\square$

Exercise 4.2. The goal of this exercise is to study the minimization problem (4.4) when $\lambda \geq \lambda_1$. Show that

(*i*) If $\lambda > \lambda_1$, then $I_\lambda = -\infty$.

(*ii*) If $\lambda = \lambda_1$, then $I_\lambda = 0$ and the minimum is attained by $u = t\zeta_1$, where ζ_1 is an eigenfunction of $-\Delta$ corresponding to λ_1 and $t = \frac{1}{\|\zeta_1\|_{L^{q+1}}}$.

4.2. **Regularity step.** We must keep in mind that our goal is to find classical solutions of (4.1). We now show that any weak solution of (4.1) is classical:

Theorem 4.3. *Let* $1 < q < \frac{N+2}{N-2}$*. If* u *is a weak solution of* (4.1)*, then* $u \in C^2(\overline{\Omega})$*.*

Before proving Theorem 4.3, let us describe two ways of improving the regularity of weak solutions of

$$-\Delta u = g \quad \text{in } \Omega. \tag{4.7}$$

Firstly, there are $C^{2,\alpha}$-estimates, also known in the literature as Schauder estimates:

Theorem 4.4 (Schauder)**.** *Let* $u \in H_0^1(\Omega)$ *and* $g \in L^1(\Omega)$ *satisfying* (4.7)*. If* $f \in C^\alpha(\overline{\Omega})$ *for some* $0 < \alpha < 1$*, then* $u \in C^{2,\alpha}(\overline{\Omega})$*.*

Here, we denote by $C^\alpha(\overline{\Omega})$ the space of Hölder-continuous functions with exponent α; $C^{2,\alpha}(\overline{\Omega})$ is the subspace of C^2-functions u such that $D^2u \in C^\alpha(\overline{\Omega})$.

Another way to improve the regularity of solutions of (4.7) is to apply $W^{2,p}$-regularity estimates due to Calderón-Zygmund:

Theorem 4.5 (Calderón-Zygmund)**.** *Let* $u \in H_0^1(\Omega)$ *and* $g \in L^1(\Omega)$ *satisfying* (4.7)*. If* $g \in L^p(\Omega)$ *for some* $1 < p < \infty$*, then* $u \in W^{2,p}(\Omega)$*.*

We say that $u \in W^{2,p}(\Omega)$ if u and its weak derivatives Du, D^2u belong to $L^p(\Omega)$.

In order to establish Theorem 4.3 we need to suitably combine Theorems 4.4 and 4.5 using a "bootstrap argument". Note that in our case we know that $g = \lambda u + u^q$. The idea is to start with Theorem 4.5 and to show that u is Hölder continuous, then this will imply that g is also Hölder continuous. By Theorem 4.4 we will conclude that u is a classical solution of (4.3).

In order to implement this, we will need the following two important results. The first is the counterpart of Theorem 1.3 for $W^{2,p}$-spaces:

Theorem 4.6 (Sobolev)**.** *If* $1 < p < \frac{N}{2}$*, then* $W^{2,p}(\Omega) \subset L^{\frac{pN}{N-2p}}(\Omega)$ *and we have*

$$\|v\|_{L^{\frac{pN}{N-2p}}} \leq C\|v\|_{W^{2,p}} \quad \forall v \in W^{2,p}(\Omega),$$

for some constant $C > 0$ *independent of* v*.*

The second is the analog of Theorem 3.5:

Theorem 4.7 (Morrey)**.** *If* $u \in W^{2,p}(\Omega)$ *for some* $p > \frac{N}{2}$*, then* $u \in C^\alpha(\overline{\Omega})$*, with* $\alpha = \min\left\{2 - \frac{N}{p}, 1\right\}$*. Moreover,*

$$\|u\|_{C^\alpha} \leq C\|u\|_{W^{2,p}},$$

where the constant $C > 0$ *is independent of* u*.*

We now present the

Proof of Theorem 4.3. We first show that $u \in W^{2,p}(\Omega)$ for every $1 < p < \infty$. Recall that

$$u \in H_0^1(\Omega) \quad \Longrightarrow \quad u \in L^{t_1},$$

where $t_1 := \frac{2N}{N-2}$, by the Sobolev imbedding. Thus, $g := \lambda u + u^q \in L^{r_1}(\Omega)$, where $r_1 := \frac{t_1}{q}$. Since $q < t_1$, we can apply Theorem 4.5 to conclude that $u \in W^{2,r_1}(\Omega)$. By Theorem 4.6,

$$u \in W^{2,r_1}(\Omega) \quad \Longrightarrow \quad u \in L^{t_2}(\Omega),$$

where $t_2 := \frac{r_1 N}{N-2r_1}$. It is important to realize that we have improved the integrability of u since $t_2 > t_1$. Repeating this process, one construct an increasing sequence (t_k) so that, from one step to the next one,

$$u \in L^{t_k}(\Omega) \quad \Longrightarrow \quad u \in L^{t_{k+1}}(\Omega).$$

Moreover, it is easy to see that $t_k \to \infty$. Thus, $u \in L^t(\Omega)$ for every $t < \infty$. This immediately implies that g also belong to $L^r(\Omega)$ for every $r < \infty$. Applying once again Theorem 4.5 we conclude that $u \in W^{2,p}(\Omega)$, for every $1 < p < \infty$, as claimed.
We can now apply Theorem 4.7 to deduce that u is Hölder continuous for every $0 < \alpha < 1$. Thus, g is also Hölder continuous. Finally, applying Theorem 4.4 we deduce in particular that $u \in C^2(\overline{\Omega})$. □

4.3. **Proof of Theorem 4.1.** ($\Rightarrow$). Assume (4.3) has a nontrivial solution u. Let ζ_1 be an eigenfunction of $-\Delta$ associated to λ_1. By Theorem 3.1, we may assume that $\zeta_1 > 0$ in Ω. Multiplying the equation in (4.3) by ζ_1 and integrating by parts we have

$$\int_\Omega \nabla u \cdot \nabla \zeta_1 = \lambda \int_\Omega u\zeta_1 + \int_\Omega u^q \zeta_1.$$

On the other hand, ζ_1 satisfies

$$\int_\Omega \nabla \zeta_1 \cdot \nabla u = \lambda_1 \int_\Omega \zeta_1 u.$$

Therefore,

$$(\lambda_1 - \lambda) \int_\Omega u\zeta_1 = \int_\Omega u^q \zeta_1 > 0.$$

This implies that $\lambda < \lambda_1$ as we wanted to prove.

($\Leftarrow$). By Theorem 4.2, equation (4.5) has a nontrivial solution $u \geq 0$ a.e. Note that (4.5) is not the weak form of (4.3) unless $I_\lambda = 1$, which need not be the case. However, since $I_\lambda > 0$, we can renormalize u to obtain a nontrivial weak solution of (4.3). In fact, the function $U := \alpha u$, where $\alpha = (I_\lambda)^{\frac{1}{q-1}}$, does the job. Applying Theorem 4.3, we have $U \in C^2(\overline{\Omega})$. Finally, since $\lambda < \lambda_1$, it follows from the Maximum Principle that $U > 0$ in Ω. The proof of Theorem 4.1 is complete. □

References

[1] Brezis, Haïm. *Analyse Fonctionnelle* : Théorie et Applications. 2nd ed. Paris : Dunod, 1999.

[2] Dacorogna, Bernard. *Introduction to the calculus of variations*, London : Imperial College Press, 2004.

[3] Evans, Lawrence C. *Partial Differential Equations.* 1st ed. Providence, RI : American Mathematical Society, 1998. (Graduate Studies in Mathematics, 19)

[4] Gilbarg, David e Trudinger, Neil S. *Elliptic partial differential equations of second order.* 2nd ed. Berlin : Springer-Verlag, 1998. (Grundlehren der mathematischen Wissenschaften, 224)

[5] Hilbert, David. "Über das Dirichlet'sche Princip". *Jahresber. Deutsch. Math.-Verein* **8** (1900), 184–188.

[6] ———. "Über das Dirichletsche Prinzip". *Math. Ann.* **59** (1904), 161–186.

[7] Rabinowitz, Paul H. *Minimax methods in critical point theory with applications to differential equations*, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986. (CBMS Regional Conference Series in Mathematics, 65)

[8] Willem, Michel. *Minimax methods.* Boston : Birkhäuser, 1996. (Progress in Nonlinear Differential Equations and their Applications, 24)

Laboratoire de Mathématiques et Physique Théorique (UMR CNRS 6083)
Fédération Denis Poisson
Université François Rabelais
37200 Tours
France