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Degree theory and boundary value problems

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Degree theory and boundary value problems

Rafael Ortega

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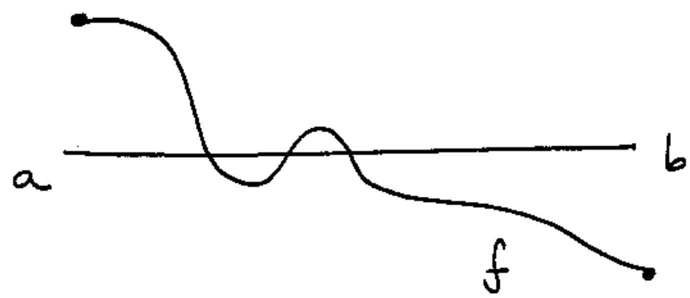
Bolzano's Theorem

$f : [a,b] \to \mathbb{R}$ continuous, $f(a) \cdot f(b) < 0 \implies$

f has a zero in $]a,b[$



The shooting method

Dirichlet problem $\begin{cases} \ddot{u} = F(t,u) \\ u(0) = u(1) = 0 \end{cases}$

$F : [0,1] \times \mathbb{R} \to \mathbb{R}$ continuous /

- uniqueness for the initial value problem (i.v.p)
- every solution of the i.v.p is well defined in $[0,1]$

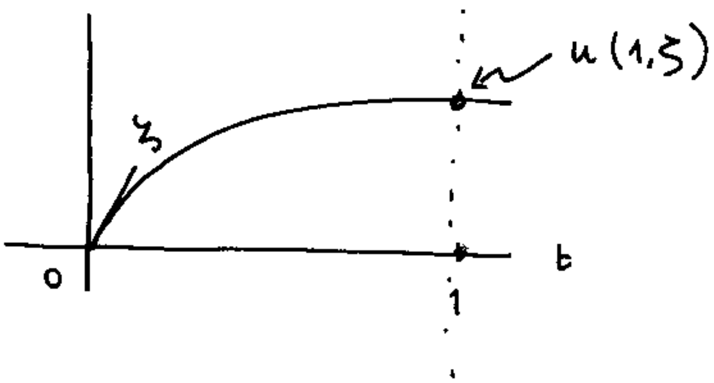
For each $\xi \in \mathbb{R}$, $u(t,\xi)$ is the solution with

$$u(0) = 0, \quad \dot{u}(0) = \xi$$

By continuous dependence with respect to initial conditions, $(t,\xi) \in [0,1] \times \mathbb{R} \mapsto u(t,\xi)$ is continuous.

Define $f : \mathbb{R} \to \mathbb{R}$, $f(\xi) = u(1,\xi)$

To solve the Dirichlet problem we look for zeros of f



We test the method proving the following result:

If F is bounded then the Dirichlet problem has a solution.

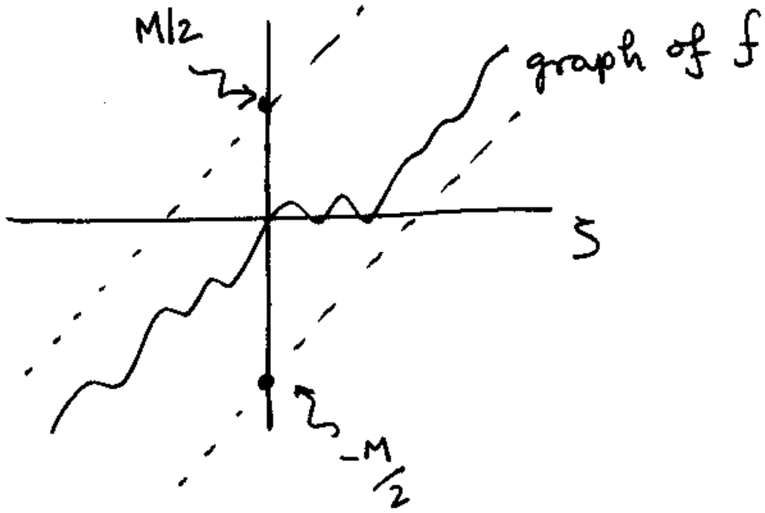
Proof $u(t,\xi)$ satisfies

$$u(t,\xi) = \xi t + \int_0^t (t-s)\, F(s, u(s,\xi))\, ds$$

If $|F(t,u)| \leq M \quad \forall (t,u)$,

$$|u(t,\xi) - \xi t| \leq M \int_0^t (t-s)\, ds \leq \frac{M}{2}$$

$$\Longrightarrow \; |f(\xi) - \xi| \leq \frac{M}{2} \quad \forall \xi \in \mathbb{R}$$

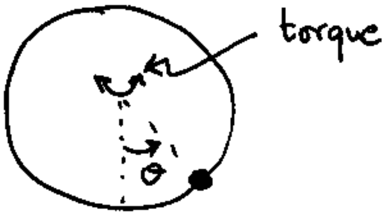


As $\lim_{\xi \to \pm\infty} f(\xi) = \pm\infty$, f has a zero.

Example The forced pendulum

$$\ddot{\Theta} + \sin\Theta = f(t), \qquad f : [0,1] \to \mathbb{R} \text{ continuous}$$

"torque"



Given any torque f, there is a motion starting and finishing $(t=0, t=1)$ at the lowest position

Next we present a result about multiplicity of solutions

Assume that F is bounded, of class C^1,

$$F(t,0)=0, \quad \pi^2 < -F_u(t,0) < 4\pi^2 \quad \forall t$$

Then the Dirichlet problem has at least 3 solutions (one of them is $u=0$)

Proof The theorem of differentiability w.r.t. initial conditions implies that $(t,\xi) \mapsto u(t,\xi)$ is C^1.

Moreover, $\frac{\partial u}{\partial \xi}(t,\xi)$ is the solution of

$$\ddot{v} = F_u(t, u(t,\xi))v, \quad v(0)=0, \ \dot{v}(0)=1.$$

We know that $u(t,0)\equiv 0$ and so $f(0)=0$. Let us compute $f'(0)$. It is given by

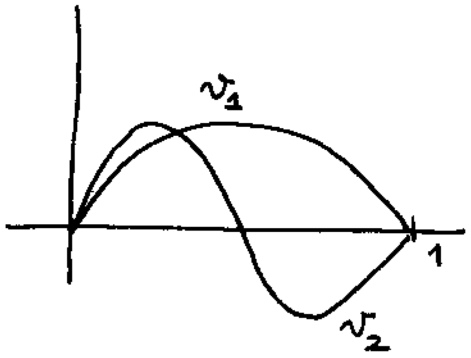
$$f'(0) = \frac{\partial u}{\partial \xi}(1,0) = v(1)$$

where $\quad \ddot{v} = F_u(t,0)v, \ v(0)=0, \ \dot{v}(0)=1.$

By Sturm comparison theory the zeros of $v(t)$

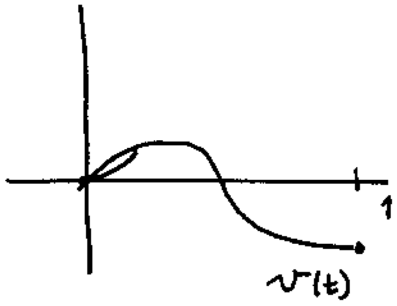
can be compared to those of $\ddot{v}_1 + \pi^2 v_1 = 0$ and $\ddot{v}_2 + 4\pi^2 v_2 = 0$,

$v_1(t) = \sin \pi t$, $v_2(t) = \sin 2\pi t$



$\Longrightarrow$ $v(t)$ has one zero in $]0,1]$

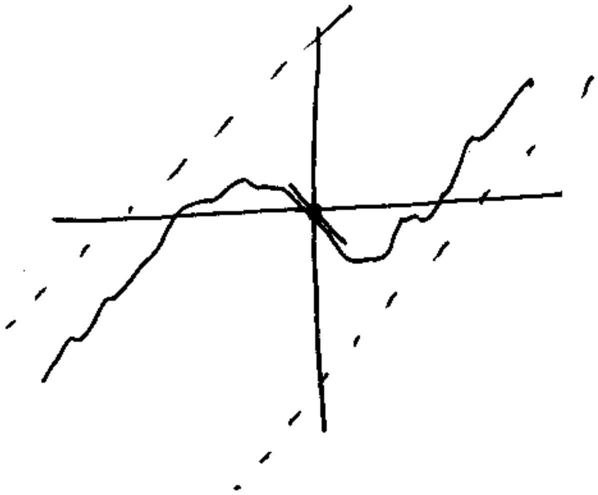
$\Longrightarrow$ $v(1) < 0$ (Notice that the zeros of $v(t)$ are simple)



Thus $f'(0) < 0$.

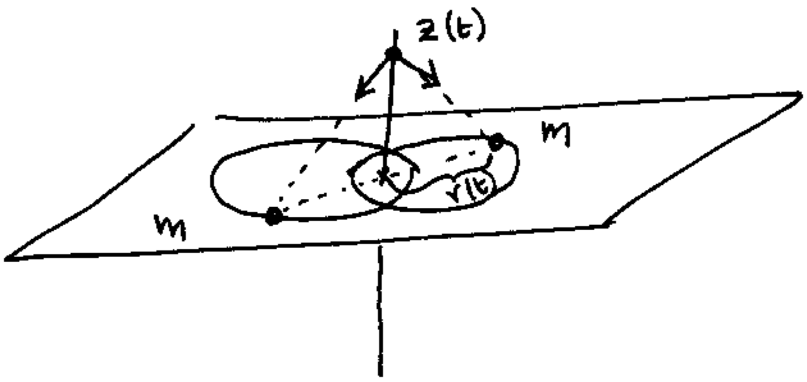
We know from the previous proof (F is bounded)*

that $\lim_{\zeta \to \pm\infty} f(\zeta) = \pm\infty$



f has at least one positive zero and one negative

Example The Sitnikov problem



$$\ddot{z} = \frac{-2z}{(z^2 + r(t)^2)^{3/2}}$$

$r(t)$ distance of the primaries to the origin

$z \equiv 0$ equilibrium

$$F_z(t,0) = \frac{-2}{r(t)^3}$$

If $\pi^2 < \frac{2}{r(t)^3} < 4\pi^2 \quad \forall t$

the asteroid can move in such a way that the initial and final position are at the center of mass.

Exercise The Neumann problem

$$\begin{cases} \ddot{u} = F(t,u) \\ \dot{u}(0) = \dot{u}(1) = 0 \end{cases}$$

$F : [0,1] \times \mathbb{R} \to \mathbb{R}$ continuous, bounded, uniqueness i. v. p.

$\exists \rho > 0 : \quad F(t,R) < 0 < F(t,-R) \quad$ if $R \geq \rho$

Then $\exists$ solution of the Neumann problem

Problems which cannot be treated with the same technique (Bolzano Th)

1 The Dirichlet problem for systems

$$\begin{cases} \ddot{u}_i = F_i(t, u_1, \dots u_N), \; i = 1, \dots, N \\ u_i(0) = u_i(1) = 0 \end{cases}$$

Now $f : \mathbb{R}^N \to \mathbb{R}^N, \quad f(\xi) = u(1,\xi)$

② The periodic problem

$$\begin{cases} \ddot{u} = F(t,u) \\ u(0)=u(1),\ \dot{u}(0)=\dot{u}(1) \end{cases}$$

$u(t,\xi,\eta)$ is the solution with $u(0)=\xi$, $\dot{u}(0)=\eta$

Under assumptions of uniqueness and extendability for the initial value problem, we look for zeros of

$$f:\mathbb{R}^2 \longrightarrow \mathbb{R}^2,\quad f(\xi,\eta)=\big(u(1,\xi,\eta)-\xi,\ \dot{u}(1,\xi,\eta)-\eta\big)$$

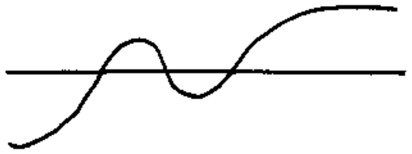
Question How to prove the existence of zeros for systems?

How to generalise Bolzano's Th. to higher dimensions?

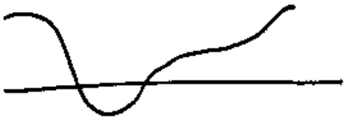
Degree in $\mathbb{R}$

$f:[a,b]\to\mathbb{R}$ continuous , $f(x)\neq 0$ if $x=a,b$

$$\deg(f,]a,b[) = \begin{cases} 1 & \text{if } f(a)<0<f(b) \\ -1 & \text{if } f(b)<0<f(a) \\ 0 & \text{otherwise} \end{cases}$$



$\deg = 1$



$\deg = 0$

Re-statement of Bolzanó s Theorem :

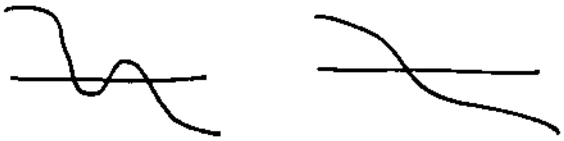
If $\deg(f,]a,b[) \neq 0 \Rightarrow f$ has a zero in $]a,b[$

Given any open and bounded set $\Omega \subset \mathbb{R}$,
it can be expressed as a disjoint and countable
union of open intervals , $\Omega = \bigcup_n I_n$

$f:\overline{\Omega}\to\mathbb{R}$ continuous, $f(x)\neq 0 \ \forall x\in\partial\Omega$

$$\deg(f,\Omega) = \sum_n \deg(f, I_n)$$

Example: $\Omega =]0,1[\cup]2,3[$



$\deg = -2$

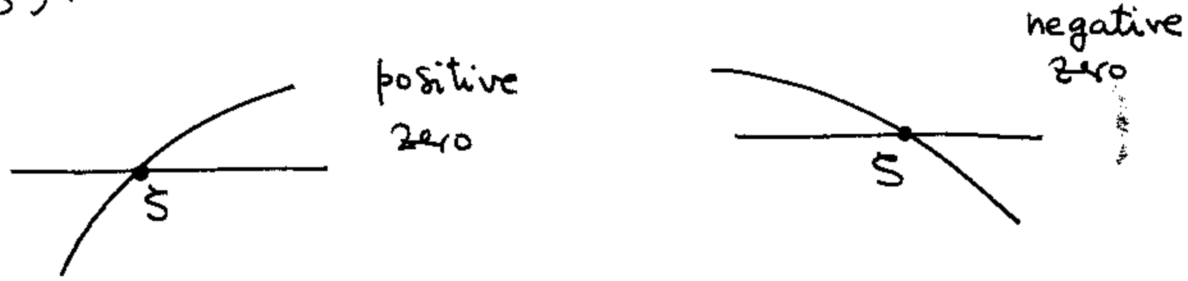
We observe that f can vanish only at a finite number of components I_n and so the sum is always meaningful.

[Proof. Assume by contradiction that $f(x_n)=0$ with $x_n \in I_{\sigma(n)}$ all of them different. After extracting a subsequence, $x_n \to x_*$. By continuity $f(x_*)=0$ and so $x_* \notin \partial\Omega$. Thus $x_* \in I_m$ for some m and so $x_n \in I_m$ for n large. This is a contradiction]

An important formula

Assume now that $f \in C^1[a,b]$, $f(x) \neq 0$ if $x=a,b$. Given a zero $\xi \in]a,b[$ we say that it is simple if

$$f'(\xi) \neq 0$$



Assume now that all the zeros of f are simple, then there is a finite number of them and they can be labeled as

$$a < \xi_1 < \xi_2 < \dots < \xi_n < b.$$

We observe that positive and negative zeros alternate, leading to

$$\boxed{\deg(f,]a,b[) = \sum_{i=1}^{n} \operatorname{sign} f'(\xi_i)}$$

We are using the convention

$$\sum_{\emptyset} = 0.$$

Degree in $\mathbb{R}^2$

Lemma 1 Assume that $\alpha : [a,b] \to \mathbb{R}^2 - \{0\}$ is continuous. Then there exists a continuous function $\theta : \cancel{\mathbb{R}}\, [a,b] \to \mathbb{R}$ such that

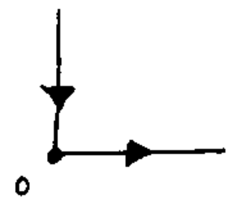
$$\alpha(t) = r(t)\,\big(\cos\theta(t), \sin\theta(t)\big), \quad t \in [a,b]$$

with $r(t) = \|\alpha(t)\|$ (euclidean norm).

Moreover $\theta(t)$ is unique up to an additive constant 2π

Lemma 2 Assume that $\alpha_n : [a,b] \to \mathbb{R}^2 - \{0\}$ is a sequence of continuous functions converging uniformly to $\alpha : [a,b] \to \mathbb{R}^2 - \{0\}$. If $\theta_n(a) \to \theta(a)$ then $\theta_n \to \theta$ uniformly.

Remark Notice that the condition $\alpha([a,b]) \subset \mathbb{R}^2 - \{0\}$ is essential to find a continuous argument. This is shown by the curve

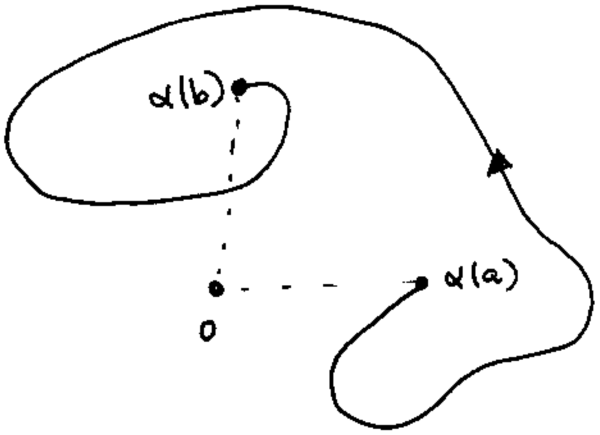


which is continuous but has a discontinuous argument.

Given $\alpha : [a,b] \to \mathbb{R}^2 - \{0\}$ continuous we observe that the number

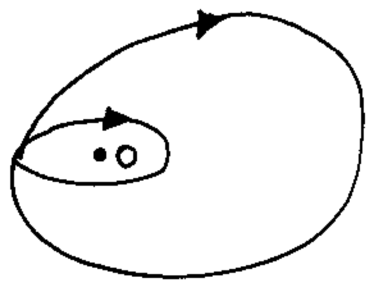
$$\frac{1}{2\pi}\left[\theta(b) - \theta(a)\right]$$

is independent of the chosen argument. We call it the winding number of the curve $\alpha(t)$ around the origin.



winding number $= \frac{1}{4}$

If the curve is closed $(\alpha(a)=\alpha(b))$ the winding number is an integer. Intuitively speaking $+n$ means n revolutions around the origin counterclockwise, $-n$ if it is clockwise.



-2

Assume now that Ω is a Jordan domain in $\mathbb{R}^2$. This means that Ω is the bounded component of $\mathbb{R}^2 - \Gamma$, where Γ is a Jordan curve. We assume

that $\alpha : [0,1] \to \mathbb{R}^2$ is a positive parameterization of Γ,

$\alpha(0) = \alpha(1)$

$\alpha|_{[0,1[}$ one-to-one

$\alpha([0,1]) = \Gamma$

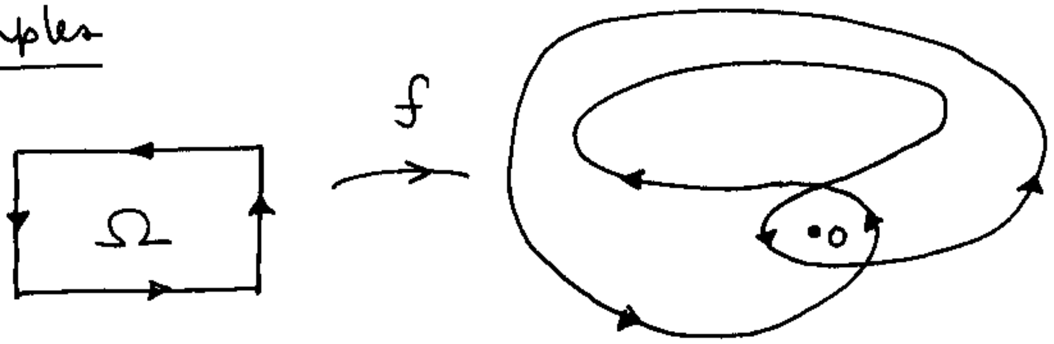
Given $f : \overline{\Omega} \to \mathbb{R}^2$ continuous,

$$f(x) \neq 0 \quad \forall x \in \partial\Omega = \Gamma$$

We define

$$\deg(f,\Omega) = \text{winding number of } f\circ\alpha \text{ around the origin}$$

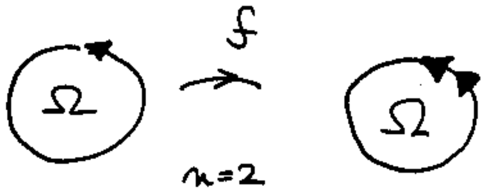
Examples

i) 

$$\deg(f,\Omega) = 2$$

ii) $\mathbb{C} \equiv \mathbb{R}$, $f(z) = z^n$, $n = 1, 2, \ldots$

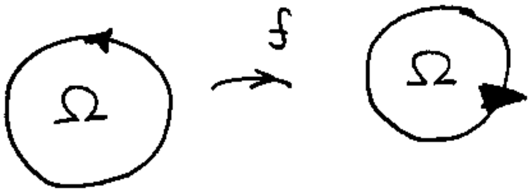
Ω = unit disk, $\alpha(t) = e^{2\pi i t}$



$f(\alpha(t)) = e^{2\pi i n t}$

$$\deg(f,\Omega) = n$$

iii) $f(z) = \bar{z}$, Ω unit disk $\qquad f(\alpha(t)) = e^{-2\pi i t}$



$$\deg(f,\Omega) = -1$$

Lemma 2 implies the continuity of the degree. If

$f_n, f : \overline{\Omega} \to \mathbb{R}^2$ continuous $f_n(x) \neq 0$, $f(x) \neq 0$ $\forall x \in \partial\Omega$

$f_n \to f$ uniformly on $\partial\Omega$. Then $\deg(f_n, \Omega) \to \deg(f, \Omega)$.

As $\deg(f_n, \Omega)$ is an integer it means that it must be eventually constant. From here we deduce the important

<u>Homotopy property</u> $H : \overline{\Omega} \times [0,1] \to \mathbb{R}^2$, $H = H(x, \lambda)$ continuous

$$H(x,\lambda) \neq 0 \quad \text{if } x \in \partial\Omega,\ \lambda \in [0,1].$$

Then $\deg(H(\cdot,\lambda), \Omega)$ is independent of λ.

<u>Proof</u>. $\lambda \in [0,1] \longmapsto \deg(H(\cdot,\lambda), \Omega)$ is continuous and takes integer values.

<u>The existence property</u> $f : \overline{\Omega} \to \mathbb{R}^2$ continuous

$$f(x) \neq 0 \quad \forall x \in \partial\Omega$$

$\deg(f, \Omega) \neq 0 \Rightarrow f$ has a zero in Ω

<u>Proof</u>. Ω unit ball $H(x,\lambda) = f(\lambda x)$.

By contradiction, if f has no zeros H is a homotopy implying that

$$\deg(H(\cdot,0), \Omega) = \deg(H(\cdot,1), \Omega) = \deg(f, \Omega) \neq 0$$

But $H(x,0) = f(0)$ and a contant function has degree zero (Notice that $f \circ \alpha$ has constant argument).

If Ω is any Jordan domain, $H(x,\lambda) = f(\Phi(x,\lambda))$ where $\Phi : \overline{\Omega} \times [0,1] \to \overline{\Omega}$ continuous,

$$\Phi(\cdot, 1) = \text{identity}, \quad \Phi(\cdot, 0) = \text{constant}$$

This is always possible since $\overline{\Omega}$ is homeomorphic to the closed unit ball.

Appendix: proofs of the lemmas

Lemma 1 First we prove the uniqueness. If $\theta_1, \theta_2 : [a,b] \to \mathbb{R}$ are continuous and $(\cos\theta_1(t), \sin\theta_1(t)) = (\cos\theta_2(t), \sin\theta_2(t))$ then $\theta_2(t) = \theta_1(t) + 2\pi k(t)$ for each t, with $k(t) \in \mathbb{Z}$.
As $t \mapsto k(t)$ is continuous and integer-valued, k is constant.

Existence for $\alpha \in C^1([a,b], \mathbb{R}^2)$

Before the proof assume that the result is valid and let us try to find a formula for $\dot\theta$

$\alpha = r(\cos\theta, \sin\theta) \Longrightarrow \dot\alpha = \dot r(\cos\theta, \sin\theta) + r\dot\theta(-\sin\theta, \cos\theta)$

$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ rotation of 90° ↺

$\langle \dot\alpha, J\alpha \rangle = r^2 \dot\theta$.

Now we can start the proof: define

$$\theta(t) = \theta_0 + \int_a^t \frac{-\alpha_2(s)\dot\alpha_1(s) + \dot\alpha_2(s)\alpha_1(s)}{\alpha_1(s)^2 + \alpha_2(s)^2}\, ds,$$

with θ_0 chosen so that $\alpha(a) = \|\alpha(a)\| (\cos\theta_0, \sin\theta_0)$.

The function θ is C^1 and

$x_1 = \cos\theta$, $x_2 = \sin\theta$ is a solution of the system of differential equations

$$\dot x_1 = -\dot\theta(t) x_2, \quad \dot x_2 = \dot\theta(t) x_1$$

Next we prove that $y_1 = \frac{\alpha_1}{r}$, $y_2 = \frac{\alpha_2}{r}$ is also a solution of this system.

$r^2 = \alpha_1^2 + \alpha_2^2 \Longrightarrow r\dot r = \alpha_1\dot\alpha_1 + \alpha_2\dot\alpha_2$

$$\dot y_1 = \frac{\dot\alpha_1 r - \alpha_1 \dot r}{r^2} = \frac{\dot\alpha_1 r^2 - \alpha_1 r\dot r}{r^3} = \frac{\dot\alpha_1 r^2 - \alpha_1^2\dot\alpha_1 - \alpha_1\alpha_2\dot\alpha_2}{r^3}$$

$$= \frac{\alpha_2^2\dot\alpha_1 - \alpha_1\alpha_2\dot\alpha_2}{r^3} = \frac{\alpha_2\dot\alpha_1 - \alpha_1\dot\alpha_2}{r^2}\,\frac{\alpha_2}{r} = -\dot\theta(t) y_2$$

With a similar computation, $\dot{y}_2 = \dot{\theta}(t) y_1$.
The solutions x_1, x_2 and y_1, y_2 satisfy the same initial condition at $t = a$. By uniqueness of the Cauchy problem,

$$x_1 = y_1, \quad x_2 = y_2.$$

An estimate $\quad |\theta_1 - \theta_2| \leq \frac{\pi}{4} \Rightarrow |\theta_1 - \theta_2| \leq \sqrt{2}\, |e^{i\theta_1} - e^{i\theta_2}|$

It is enough to prove

$$0 \leq \theta \leq \frac{\pi}{4} \Longrightarrow \theta \leq \sqrt{2}\, |e^{i\theta} - 1| \qquad (\theta = \theta_1 - \theta_2)$$

$$|e^{i\theta} - 1| \geq |\cos\theta - 1| + |\sin\theta| \geq |\sin\theta|$$

$$\geq |\cos\zeta|\, \theta, \quad \text{for some } \zeta \in [0, \tfrac{\pi}{4}]$$

$$\Longrightarrow |e^{i\theta} - 1| \geq \frac{1}{\sqrt{2}}\, \theta$$

Existence for α continuous

We construct a sequence $\alpha_n \in C^1([a,b], \mathbb{R}^2)$ such that $\alpha_n \longrightarrow \alpha$ uniformly. For n large enough $\alpha_n(t) \neq 0 \ \forall t \in [a,b]$. We construct the argument function θ_n according to the above step. We find N such that

$$\left\| \frac{\alpha_n(t)}{\|\alpha_n(t)\|} - \frac{\alpha_m(t)}{\|\alpha_m(t)\|} \right\| < \frac{\pi}{8\sqrt{2}}, \quad n, m \geq N, \ t \in [a,b]$$

We can also assume that $\theta_n(a)$ is convergent to θ_0, so that $|\theta_n(a) - \theta_m(a)| < \frac{\pi}{4}$. From the estimate,

$$|\theta_n(t) - \theta_m(t)| < \frac{\pi}{8}$$

as soon as $|\theta_n(t) - \theta_m(t)| < \frac{\pi}{4}$. This implies that $\quad |\theta_n(t) - \theta_m(t)| < \frac{\pi}{4} \quad \forall t,\ n, m \geq N.$

Going back to the estimate,

$$|\theta_n(t) - \theta_m(t)| \leq \sqrt{2}\ \left\| \frac{\alpha_n(t)}{\|\alpha_n(t)\|} - \frac{\alpha_m(t)}{\|\alpha_m(t)\|} \right\|$$

and so $\{\theta_n(t)\}$ is uniformly a Cauchy sequence.(*)

Thus $\theta_n \to \theta$ uniformly and, by a passage to the limit, θ is the searched argument.

Lemma 2 The proof is a repetition of the last arguments.

(*) Notice that $\|\alpha_n(t)\| \geq \delta_0$, $n \geq N$, $\forall t$

Applications of the degree in the plane

The fundamental theorem of Algebra

Let $p \in \mathbb{C}[z]$, $p(z) = z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$ be a polynomial with $n \geq 1$. Then p has a root.

Proof First we recall the estimate for any possible root ζ,

$$|\zeta| \leq \max\{1, |a_0| + \dots + |a_{n-1}|\}.$$

Define

$$H(z,\lambda) = z^n + \lambda a_{n-1} z^{n-1} + \dots + \lambda a_1 z + \lambda a_0$$

and let Ω be a ball centered at the origin with radius $R > \max\{1, |a_0| + \dots + |a_{n-1}|\}$.

Then $H(z,\lambda) \neq 0 \quad \forall z \in \partial\Omega, \lambda \in [0,1]$

and so

$$\deg(p,\Omega) = \deg(H(\cdot,1),\Omega)$$
$$= \deg(H(\cdot,0),\Omega) = n > 0.$$

Then existence property implies that p has a zero

Brouwer's fixed point theorem

$$B = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$$

$\Phi : B \to B$ continuous. Then Φ has a fixed point

<u>Proof</u> Assume that $\Phi(x) \neq x \ \forall x \in \partial B$, for otherwise the Theorem is already proven. Define

$$H(x,\lambda) = x - \lambda \Phi(x)$$

and observe that if $\|x\| = 1$, $\lambda \in [0,1[$,

$$\|H(x,\lambda)\| \geq \|x\| - \lambda \|\Phi(x)\| \geq 1 - \lambda > 0.$$

Thus $H(x,\lambda) \neq 0$ if $x \in \partial B$, $\lambda \in [0,1]$ and so

$$\deg(\mathrm{id} - \Phi, \mathring{B}) = \deg(H(\cdot,1), \mathring{B})$$
$$= \deg(H(\cdot,0), \mathring{B}) = \deg(\mathrm{id}, \mathring{B}) = 1$$

Thus $\mathrm{id} - \Phi$ has a zero and this is a fixed point of Φ.

<u>Exercise</u> Extend Brouwer's Theorem to any space which is homeomorphic to B.

A difference between 1 and 2 dimensions

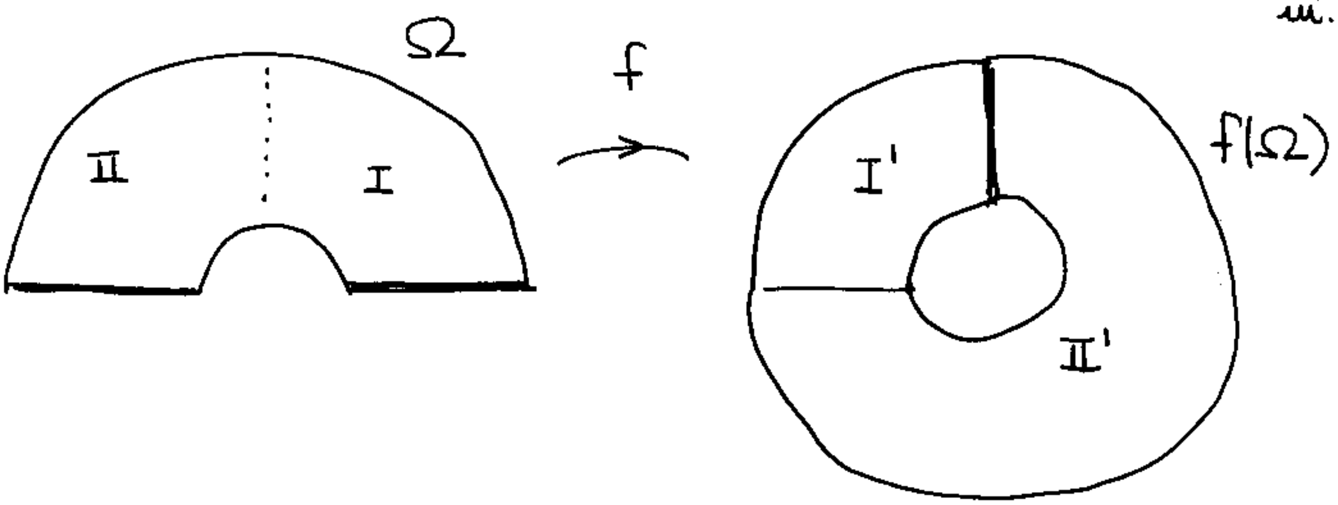
$\Phi : [a,b] \to \mathbb{R}$ continuous, $[a,b] \subset \Phi([a,b])$

$\Longrightarrow$ Φ has a fixed point

[Use Bolzano]

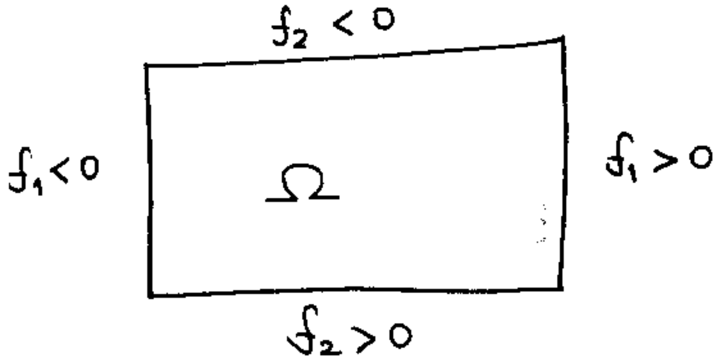
$\Phi : \overline{\Omega} \to \mathbb{R}^2$ continuous, Ω Jordan domain

$\overline{\Omega} \subset \Phi(\overline{\Omega}) \not\Longrightarrow \Phi$ has a fixed point



A Generalization of Bolzanó's Th. (Poincaré- Miranda)

$\Omega =]-a, a[\times]-b, b[$



$f_1(a,y) > 0 > f_1(-a,y)$

$f_2(x,b) < 0 < f_2(x,-b)$

$f = (f_1, f_2) : \overline{\Omega} \to \mathbb{R}^2$ continuous.

Then f has a zero in Ω.

Proof We first observe that the map $f^*(x,y) = (x,-y)$ is in the assumptions of the theorem. The degree is easily computed

$\deg(f^*, \Omega) = -1$

Define

$$H(x,y,\lambda) = (\lambda f_1(x,y) + (1-\lambda)x,\ \lambda f_2(x,y) - (1-\lambda)y)$$

We observe that for each $\lambda \in [0,1]$, $H(\cdot,\cdot,\lambda)$ is in the conditions of the Theorem and so

$$H(x,y,\lambda) \neq 0 \quad \text{if } x = \pm a \text{ or } y = \pm b$$
$$\Leftrightarrow (x,y) \in \partial\Omega$$

Thus,

$$\deg(f,\Omega) = \deg(H(\cdot,1),\Omega) = \deg(H(\cdot,0),\Omega)$$
$$= \deg(f^*,\Omega) = -1.$$

Exercise Construct variants of the above theorem

Periodic solutions of a system

$$\dot{u}_1 = F_1(t,u_2), \quad \dot{u}_2 = F_2(t,u_1)$$

$F_1, F_2 : [0,1] \times \mathbb{R} \to \mathbb{R}$ continuous, uniqueness for i.v.p.

F_2 is bounded,

$$\begin{aligned} F_1(t,\rho) &< 0 < F_1(t,-\rho) \\ F_2(t,\rho) &> 0 > F_2(t,-\rho) \end{aligned} \quad \text{if } t \in [0,1],\ \rho \geq R.$$

Then there is a 1-periodic solution.

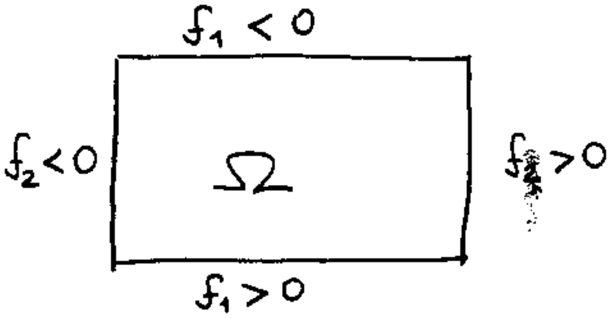
Proof First we observe that the solutions of the initial value problem are defined in $[0,1]$. Indeed, as F_2 is bounded, $u_2(t)$ is bounded and so it cannot blow up. From there we know that $F_1(t, u_2(t))$ remains bounded and so $u_1(t)$ is also bounded.

We can consider

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(\xi,\eta) = (u_1(1,\xi,\eta) - \xi,\ u_2(1,\xi,\eta) - \eta)$$

and we are going to prove that it has a zero using a variant of P-M th.

$$\Omega =]-a,a[\times]-b,b[,$$



Let $M > 0$: $|F_2| \le M$ everywhere and $M^* > 0$:

$$|F_1| \le M^* \quad \text{if} \quad |u_2| \le R + 2M.$$

Define $a = R + M^*$, $b = R + M$. From the second equation

$$|u_2(t,\xi,\eta) - \eta| \le M.$$

If $(\xi,\eta) \in \overline{\Omega}$, $|u_2(t,\xi,\eta)| \le R + 2M \Longrightarrow$

$$|F_1(t, u_2(t,\xi,\eta))| \le M^*. \text{ From here,}$$

$$|u_1(t,\xi,\eta) - \xi| \le M^*.$$

If $(\xi,\eta) \in \overline{\Omega}$ with $\eta = b = R + M$, $u_2(t,\xi,\eta) \ge \eta - M$
$= R$

$$\Longrightarrow F_1(t, u_2(t,\xi,b)) < 0 \Longrightarrow$$

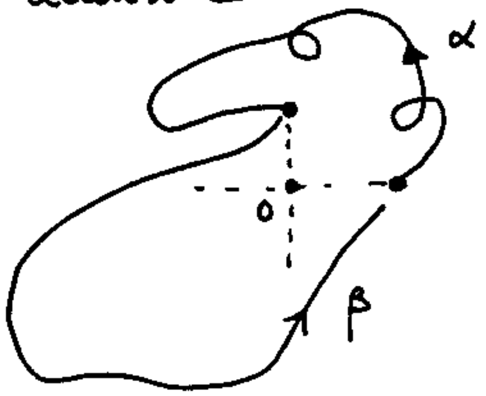
$$f_1(\xi,\underset{b}{\overset{\shortparallel}{\eta}}) = u_1(1,\xi,\underset{b}{\overset{\shortparallel}{\eta}}) - \xi = \int_0^1 F_1(t, u_2(t,\xi,b))\,dt < 0$$

In a similar way, $f_1(\xi, -b) > 0$ and

$f_2(a, \eta) > 0 > f_2(-a, \eta)$.

Some properties of the degree

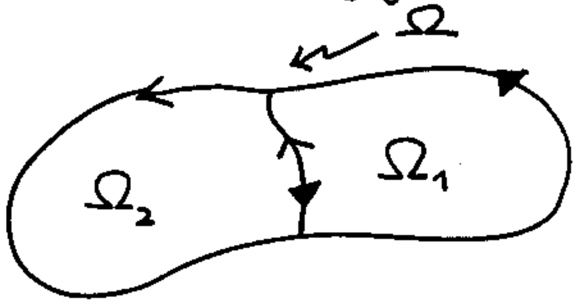
Given two continuous functions $\alpha_1 : [a,b] \to \mathbb{R}^2 - \{0\}$, $\alpha_2 : [b,c] \to \mathbb{R}^2 - \{0\}$, we can juxtapose and obtain the new function $\alpha_1 * \alpha_2 : [a,c] \to \mathbb{R}^2 - \{0\}$. It is clear from the definition that the winding number is additive



winding number

$$\left.\begin{array}{l} \alpha \longrightarrow \frac{1}{4} \\ \beta \longrightarrow \frac{3}{4} \end{array}\right] \Longrightarrow \alpha * \beta \longrightarrow 1$$

Assume now that we have three Jordan domains as indicated in the figure



and $f : \overline{\Omega} \to \mathbb{R}^2$, $f(x) \neq 0 \quad \forall x \in \partial\Omega_1 \cup \partial\Omega_2$, f continuous. Then

$$\deg(f, \Omega) = \deg(f, \Omega_1) + \deg(f, \Omega_2)$$

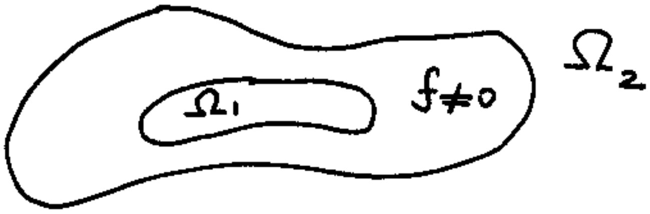
[Notice that $\partial\Omega \subset \partial\Omega_1 \cup \partial\Omega_2$]

With a similar argument we can deduce the excision property:

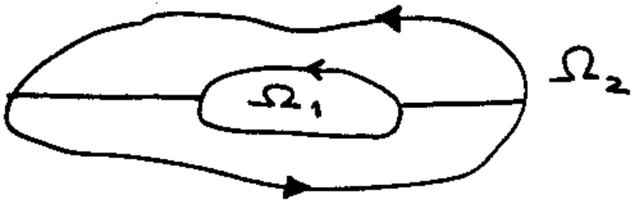
Ω_1, Ω_2 Jordan domains with $\overline{\Omega}_1 \subset \Omega_2$,

$f : \overline{\Omega}_2 \to \mathbb{R}^2$ continuous, $f(x) \neq 0 \quad \forall x \in \overline{\Omega}_2 - \Omega_1$

$$\Longrightarrow \quad \deg(f, \Omega_1) = \deg(f, \Omega_2)$$



Proof



Use additivity with 3 domains and the existence property

The degree of a linear map

Assume that L is a 2×2 matrix with $\det L \neq 0$ and let Ω be the unit ball centered at the origin. We observe that $Lx \neq 0 \ \forall x \in \partial\Omega$ and we want to compute $\deg(L, \Omega)$. We recall that the group

$$Gl(\mathbb{R}^2) = \{ L \in M_2(\mathbb{R}) : \det L \neq 0 \}$$

has two connected components: matrices with positive [resp negative] determinant.

Assume now that $\det L > 0$. Then we can define a continuous map $\lambda \in [0,1] \mapsto L_\lambda \in Gl_+(\mathbb{R}^2)$ with $L_0 = L$, $L_1 = I$. The map

$$H(x, \lambda) = L_\lambda x$$

is a homotopy since $L_\lambda x = 0 \Rightarrow x = 0$ and so

$$\deg(L, \Omega) = \deg(Id, \Omega) = 1$$

If $\det L < 0$ we make a homotopy with $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and we know that this map has degree -1 $(z \mapsto \bar{z})$.

Summing up,

$$\deg(L, \Omega) = \left\{ \begin{array}{ll} 1 & \text{if } \det L > 0 \\ & \\ -1 & \text{if } \det L < 0 \end{array} \right\} = \operatorname{sign}(\det L).$$

Linearization and degree

Assume now that Ω is an arbitrary Jordan domain and $f : \overline{\Omega} \to \mathbb{R}^2$ is C^1. Moreover f has a unique zero at $x_* \in \Omega$ and this zero is simple (this means that $\det(f'(x_*)) \neq 0$). Then

$$\deg(f, \Omega) = \deg(f'(x_*), \Omega) = \operatorname{sign}(\det f'(x_*)).$$

Proof The inverse function theorem implies that there exists a small open ball centered at x_*, B, such that f is a diffeomorphism from $\overline{B}$ onto its image. By excision,

$$\deg(f, \Omega) = \deg(f, B).$$

For simplicity we assume $x_* = 0$ and define the homotopy on $\overline{B}$,

$$H(x,\lambda) = \begin{cases} \frac{1}{\lambda} f(\lambda x) & \text{if } \lambda \in]0,1] \\ f'(0)x & \text{if } \lambda = 0. \end{cases}$$

It is easy to check that H is continuous and has no zeros on ∂B (in fact the only zero is $x=0$). Thus, $\deg(H(\cdot,1), B) = \deg(H(\cdot,0), B)$ and this completes the proof.

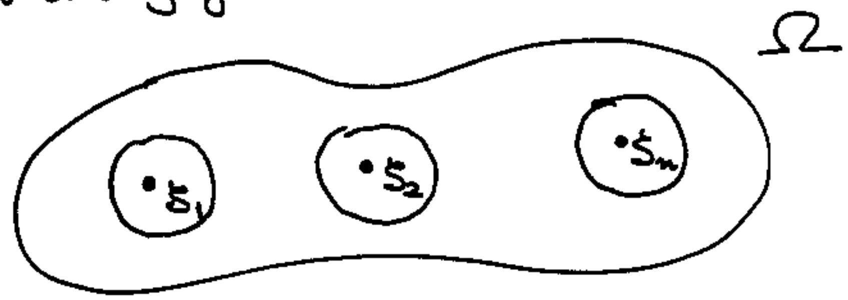
An important formula

Assume that Ω is a Jordan domain and $f : \overline{\Omega} \to \mathbb{R}^2$ is C^1 with $f(x) \neq 0$ $\forall x \in \partial\Omega$. In addition all its zeros are simple. We label them as $\xi_1, \dots, \xi_n \in \Omega$, then

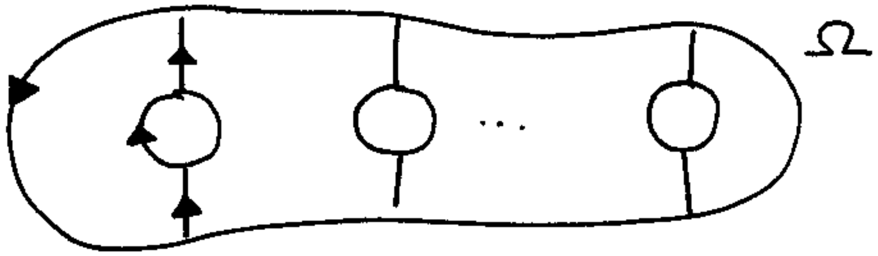
$$\deg(f, \Omega) = \sum_{i=1}^{n} \operatorname{sign}(\det f'(\xi_i)).$$

Remark Notice that a simple zero is isolated ($\Leftarrow$ inverse function th.) Since $\overline{\Omega}$ is compact $f^{-1}(0)$ must be finite.

Proof We find small disks around $\xi_1, \dots \xi_n$ as in the figure



Then we apply additivity and the linearization principle



<u>Exercise</u> Compute the degree of $f(x,y) = (e^{x+y}-1, e^{x-y}-1)$ in a neighborhood of the origin.

Degree in $\mathbb{R}^d$ (The Nagumo approach)

Given $\Omega \subset \mathbb{R}^d$ non-empty, bounded and open and $f : \overline{\Omega} \to \mathbb{R}^d$ continuous with

$$f(x) \neq 0 \quad \forall x \in \partial\Omega$$

we want to define $\deg(f, \Omega)$ so that the existence, homotopy and excision properties hold.

The idea will be to use the above formula as the definition of degree. We do the definition in three steps.

Step 1 f is C^1 and all its zeros are simple

$$\deg(f, \Omega) = \sum_{i=1}^{n} \operatorname{sign}(\det f'(\xi_i))$$

with $f^{-1}(0) = \{\xi_1, \dots \xi_n\}$. Convention $\sum_{\phi} = 0$.

Example $d = 2$, $\mathbb{R}^2 \equiv \mathbb{C}$, $f(z) = z^2 - \varepsilon$, $\varepsilon > 0$

Ω unit ball

$$f^{-1}(0) = \{\sqrt{\varepsilon}, -\sqrt{\varepsilon}\}, \qquad f'(x,y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

$$\deg(f, \Omega) = 2$$

Step 2 f is C^1 but its zeros are not necessarily simple

We apply Sard's Theorem which says that almost all vectors $v \in \mathbb{R}^d$ are regular values for f. This means that if $f(\xi)=v$ then $\det f'(\xi)\neq 0$. May be the zero vector $v=0$ is not a regular value but certainly we can find a sequence of vectors $v_n \in \mathbb{R}^d$ such that they are regular values and converge to 0. We can apply Step 1 to the map $f_n(x)=f(x)-v_n$ and define (*)

$$\deg(f,\Omega)=\lim_{n\to\infty}\deg(f_n,\Omega).$$

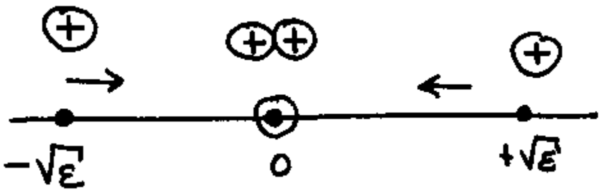
For this definition to make sense one must prove that the sequence $\deg(f_n,\Omega)$ becomes eventually constant and also that the limit is independent of the choice of v_n.

<u>Exercise</u> Prove the previous assertions for $d=2$ and Ω a Jordan domain.

<u>Example</u> $d=2$, $\mathbb{R}^2 \equiv \mathbb{C}$, $f(z)=z^2$, Ω unit ball

$$\deg(f,\Omega)=\lim_{\varepsilon\downarrow 0}\deg(f_\varepsilon,\Omega)=2$$

$f_\varepsilon(z)=z^2-\varepsilon$



(*) <u>Remark</u> If $f(x)\neq 0$ $\forall x\in\partial\Omega \implies$

$f_n(x)\neq 0$ $\forall x\in\partial\Omega$ and n large

<u>Step 3</u> (General Case) f is continuous

We can approximate f by a sequence of functions f_n of class C^1. This means that $f_n \to f$ uniformly in $\overline{\Omega}$. The maps f_n satisfy $f_n(x) \neq 0 \ \forall x \in \partial\Omega$ if n is large. and so $\deg(f_n, \Omega)$ is defined according to Step 2.

By definition,

$$\deg(f, \Omega) = \lim_{n\to\infty} \deg(f_n, \Omega).$$

It must be proven that this limit exists and is independent of the choice of f_n. Again we know how to prove this for $d = 2$.

<u>Example</u> $f(x_1, \dots, x_d) = (|x_1|, x_2, \dots x_d)$

Ω = unit ball in $\mathbb{R}^d$

$$f_\varepsilon(x_1, \dots x_d) = \left(\sqrt{x_1^2 + \varepsilon}, x_2, \dots, x_d\right)$$

According to Step 1,

$$\deg(f_\varepsilon, \Omega) = 0 \quad \text{since it has no zeros}$$

$$\Longrightarrow \deg(f, \Omega) = 0$$

Now we can generalize the result of the first lecture on the Dirichlet problem.

The Dirichlet problem for systems

$$\begin{cases} \ddot{u}_1 = F_1(t, u_1, \dots u_N), & u_1(0) = u_1(1) = 0 \\ \dots & \\ \ddot{u}_N = F_N(t, u_1, \dots u_N), & u_N(0) = u_N(1) = 0 \end{cases}$$

$F_i : [0,1] \times \mathbb{R}^N \to \mathbb{R}$ continuous and bounded

Uniqueness of the initial value problem

$\Longrightarrow$ There is a solution of the Dirichlet problem.

The proof is an easy consequence of

Lemma $f : \mathbb{R}^d \to \mathbb{R}^d$ continuous, $\exists M > 0$:

$$\|f(x) - x\| \leq M \quad \forall x \in \mathbb{R}^d.$$

Then f has a zero.

Proof $f(x) = x + \Phi(x)$ with $\|\Phi\| \leq M$

Define the homotopy

$$H(x, \lambda) = \lambda f(x) + (1-\lambda)x, \quad \lambda \in [0,1]$$

$$H(x,\lambda) = 0 \Longleftrightarrow x + \lambda \Phi(x) = 0 \Longrightarrow \|x\| \leq$$

$\|\Phi(x)\| \leq M$.

Let Ω be a ball centered at the origin

with radius $> M$.

The homotopy property implies that

$$\deg(f, \Omega) = \deg(H(\cdot, 1), \Omega)$$
$$= \deg(H(\cdot, 0), \Omega) = \deg(id, \Omega) \underset{\text{Step 1}}{=} 1$$

For applications to p.d.e.'s it is important to define the degree in spaces of infinite dimensions. This is not inmediate and cannot be done for <u>all</u> continuous mappings. We finish with an example of a map $f: H \to H$ wich is continuous, H is a Hilbert space and $f(B) \subset B$, $B = \{x \in H : \|x\| \leq 1\}$ and such that f has no fixed points. This shows that Brouwer's Fixed Point Theorem cannot be extended to infinite dimensions and the map $id - f$ is homotopic to id but has no zeras.

<u>Example</u> $H = \ell^2$, $\|\{x_n\}\| = \sqrt{\sum x_n^2}$

$$f(\{x_0, x_1, \ldots, x_n, \ldots\}) = \{ [illegible] \}$$
$$= \{\sqrt{1 - \|x\|^2}, x_0, x_1 \ldots\}$$

$f(B) \subset \partial B \subset B$, f has no fixed points