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Stable Invariant Manifold and Stabilization Problem for Semilinear Parabolic Equation

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STABLE INVARIANT MANIFOLD
AND STABILIZATION PROBLEM FOR SEMILINEAR
PARABOLIC EQUATION

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1. Stable Invariant Manifold

In these lectures we will study stable invariant manifolds for semilinear parabolic equation. For simplicity we consider one concret example of such similinear equation: Ginzburg-Landau equation. But for better understanding we firstly introduce the notion of invariant subspace in the simplest case of linear parabolic equation.

1.1. **Invariant subspaces for a linear parabolic equation.** Let $G \subset \mathbb{R}^n$, be a bounded domain with C^∞-boundary ∂G. We consider a linear parabolic equation

$$\partial_t v(t,x) - \nu\Delta v(t,x) - v(t,x) = 0, \quad x \in G, t > 0 \tag{1.1}$$

with boundary and initial conditions

$$v(t,\cdot)|_{x\in\partial G} = 0, \tag{1.2}$$

$$v(t,x)|_{t=0} = v_0(x), \quad x \in G, \tag{1.3}$$

where $\partial_t v = \partial v/\partial t$, $\nu > 0$ is a parameter, $-\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ is Laplace operator $v_0(x) \in H^2(G) \cap H_0^1(G)$ are given functions. Recall that $H^k(G)$ is the Sobolev space of functions belonging to $L_2(G)$ together with all their derivaties up to the order k. The norm of this space is defined as follows:

$$\|v\|^2_{H^2(G)} = \sum_{|\alpha|\le k} \int_G |D^\alpha v(x)|^2 dx, \tag{1.4}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j, j = 1, \dots, n$ are integer, nonegative numbers, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $D^\alpha v(x) = \partial^{|\alpha|} v(x)/\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$. Recall also that

$$H_0^1(G) = \{u(x) \in H^1(G) : u|_{\partial G} = 0\}$$

Let

$$\{e_k(x), \lambda_k\}, \qquad \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \to \infty \quad \text{as} k \to \infty \tag{1.5}$$

be the eigenfunctions and the eigenvalues of the spectral problem

$$-\nu\Delta e - e(x) = \lambda e(x),\ x \in G \quad e|\partial G = 0. \tag{1.6}$$

We assume the parameter ν is small enough and such that eigenvalues λ_k of the spectral problem (1.6) satisfy the condition:

$$\lambda_1 \leq \cdots \leq \lambda_N < 0 < \lambda_{N+1} \leq \cdots \leq \lambda_k \tag{1.7}$$

We introduce the subspaces

$$V_+ \equiv V_+(G) = [e_1, \dots, e_N], \quad V_- \equiv V_-(G) = [e_{N+1}, e_{N+2} \dots], \tag{1.8}$$

In other words subspace V_+ is generated of eigenfunctions possessing negative eigenvalues and V_- is generated of eigenfunctions with positive eigenvalues. Therefore the solutions $e^{-\lambda_k t}e_k(x)$ of the equation (1.1) tend to infinity as $t \to \infty$ for $k = 1, \dots, N$, and tend to zero as $t \to \infty$ for $k > N$. Besides, solving problem (1.1)- (1.3) by Fourier method one can see that if initial condition v_0 belongs to V_+ (or V_-) then solution $v(t,x)$ of (1.1)- (1.3) for each $t > 0$ belongs to V_+ (or, respectively, to V_-). That is why V_- is called stable invariant subspace for problem (1.1)- (1.3) and V_+ is called unsatable invariant subspace for the same problem.

1.2. **Ginzburg-Landau equation.** Let $G \subset \mathbb{R}^n, n = 1, 2, 3$ be a bounded domain with C^∞-boundary ∂G. We consider Ginzburg-Landau equation

$$\partial_t v(t,x) - \nu\Delta v(t,x) - v(t,x) + v^3(t,x) = f(x), \quad x \in G, t > 0 \tag{1.9}$$

with boundary and initial conditions

$$v(t,\cdot)|_{x\in\partial G} = 0, \tag{1.10}$$

$$v(t,x)|_{t=0} = v_0(x), \quad x \in G, \tag{1.11}$$

where $\nu > 0$, $f(x) \in L_2(G), v_0(x) \in H^2(G) \cap H_0^1(G)$ are given functions.

As a phase space of the dynamical system generated by (1.9), (1.10) we take the functional space

$$V \equiv V(G) = H^2(G) \cap H_0^1(G) \tag{1.12}$$

Let $\widehat{v}(x) \in V$ be a steady-state solution of (1.9), (1.10), i.e. a solution of the problem

$$-\nu\Delta\widehat{v}(t,x) - \widehat{v}(t,x) + \widehat{v}^3(t,x) = f(x), \quad x \in G, \quad \widehat{v}|_{\partial G} = 0 \tag{1.13}$$

To study the structure of the dynamical system (1.9), (1.10) in a neighborhood of $\widehat{v}(x)$ we make the change of unknown functions in (1.9), (1.10):

$$v(t,x) = \widehat{v}(x) + y(t,x) \tag{1.14}$$

After substitution (1.14) into (1.9)–(1.11) and taking into account (1.13) we get:

$$\partial_t y(t,x) - \nu\Delta y(t,x) - q(x)y(t,x) + B(x, y(t,x)) = 0, \quad x \in G, t > 0 \tag{1.15}$$

$$y(t,\cdot)|_{x\in\partial G} = 0, \tag{1.16}$$

$$y(t,x)|_{t=0} = y_0(x) = v_0(x) - \widehat{v}(x), \quad x \in G, \tag{1.17}$$

where

$$q(x) = 3\widehat{v}^2(x) - 1, \qquad B(x,y) = y^3 - 3\widehat{v}(x)y^2 \tag{1.18}$$

Similarly to previous subsection let

$$\{e_k(x), \lambda_k\}, \qquad \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \to \infty \quad \text{as} k \to \infty \tag{1.19}$$

be the eigenfunctions and the eigenvalues of the spectral problem

$$Ae \equiv -\nu\Delta e(x) + q(x)e(x) = \lambda e(x), \; x \in G \quad e|\partial G = 0. \tag{1.20}$$

We assume that eigenvalues λ_k of the spectral problem (1.20) satisfy the condition:

$$\lambda_1 \le \cdots \le \lambda_N < 0 < \lambda_{N+1} \le \cdots \le \lambda_k \tag{1.21}$$

Therefore the solutions $e^{-\lambda_k t} e_k(x)$ of the linear equation

$$\frac{\partial y}{\partial t} + Ay = 0 \tag{1.22}$$

tend to infinity as $t \to \infty$ for $k = 1, \ldots, N$, and tend to zero as $t \to \infty$ for $k > N$.

Similarly to (1.8)we introduce the subspaces

$$V_+ \equiv V_+(G) = [e_1, \ldots, e_N], \quad V_- \equiv V_-(G) = [e_{N+1}, e_{N+2} \ldots], \tag{1.23}$$

of unstable and stable modes for equation (1.22). The following relation is true:

$$V_+(G) + V_-(G) = V(G) \tag{1.24}$$

1.3. **Stable invariant manifold.** It is well-known, that for each $y_0 \in V$ there exists a unique solution $y(t,x) \in C(0,T;V(G))$ of problem (1.15)-(1.18), where $T > 0$ is arbitrary fixed number. We denote by $S(t, y_0)$ the solution operator of the boundary value problem (1.15)-(1.18):

$$S(t, y_0) = y(t, \cdot) \tag{1.25}$$

where $y(t,x)$ is the solution of (1.15)-(1.18).

Recall now some commonly used definitions of stable invariant manifold (see, for instance, Chapter V in [1]) adopted for our case. The origin of the phase space V, i.e. the function $y(x) \equiv 0$, is, evidently, a steady-state solution of problem (1.10)-(1.12).

Definition 1.1. *The set $M_- \subset H$ defined in a neighborhood of the origin is called the stable invariant manifold if for each $y_0 \in M_-$ the solution $S(t, y_0)$ is well-defined and belongs to M_- for each $t > 0$, and*

$$\|S(t, y_0)\|_V \le ce^{-rt} \quad \textit{as} \quad t \to \infty \tag{1.26}$$

where $0 < r < \lambda_{N+1}$.

The stable invariant manifold can be defined as a graph in the phase space $V = V_+ \oplus V_-$ by the formula

$$M_- = \{y_- + F(y_-),\ y_- \in \mathcal{O}(V_-)\} \tag{1.27}$$

where $\mathcal{O}(V_-)$ is a neighborhood of the origin in the subspace V_-, and

$$F : \mathcal{O}(V_-) \to V_+ \tag{1.28}$$

is a certain map satisfying

$$\|F(y_-)\|_{V_+} / \|y_-\|_{V_-} \to 0 \quad \text{as} \quad \|y_-\|_{V_-} \to 0. \tag{1.29}$$

So, in order to construct the invariant manifold M_- we have to calculate the map (1.28), (1.29).

Remark 1.1. *Compareing Definition 1.1 with definition of stable invariant subspace from Subsection 1.1 we see that when we add nonlinear term to equation, the stable invariant subspace transformes in an neighborhood of origin to stable invariant manifold.*

Analogously it is possible to define a instable invariant manifold that is analog of unstable invariant subspace from Subsection 1.1. We will not do it.

2. Stabilization problem

Stable invariant manifolds play important role in the theory of stabilization of quasilinear parabolic equations and Navier-Stokes system. That is why they are so interesting for us. Below we explain how they are applied in stabilization theory.

2.1. **Formulation of stabilization problem.** Let $\Omega \subset \mathbb{R}^n, n = 1, 2, 3$ be a bounded domainwith C^∞ boundary $\partial\Omega$ that consists of two closed disjoint components Γ_0, Γ:

$$\partial\Omega = \Gamma_0 \cup \Gamma, \qquad \Gamma_0 \cap \Gamma = \emptyset \tag{2.1}$$

We consider the following problem for Ginzburg-Landau equation:

$$\partial_t w(t,x) - \nu\Delta w(t,x) - w(t,x) + w^3(t,x) = g(x), \quad x \in \Omega, t > 0 \tag{2.2}$$

$$w(t,\cdot)|_{x\in\Gamma_0} = 0, \quad w(t,\cdot)|_{x'\in\Gamma} = u(t,x') \tag{2.3}$$

$$w(t,x)|_{t=0} = w_0(x), \quad x \in \Omega, \tag{2.4}$$

where $\nu > 0$ is a parameter, $g(x) \in L_2(G), w_0(x) \in H^2(G), w_0|_{\Gamma_0} = 0$ are given functions, w, u are unknown functions, and u is a control.

Suppose that a steady-state solution $\widehat{w}(x) \in V$ of problem (2.2)–(2.4) is given:

$$-\nu\Delta\widehat{w}(x) - \widehat{w}(x) + \widehat{w}^3(x) = g(x), \quad x \in \Omega, \quad \widehat{v}|_{\Gamma_0} = 0, \tag{2.5}$$

and

$$\|\widehat{w}(x) - w_0(x)\|_{H^2(\Omega)} \le \varepsilon \tag{2.6}$$

where $\varepsilon > 0$ is sufficiently small. We suppose also that $\widehat{w}$ is unstable steady-state solution.

The stabilization problem for (2.2)–(2.4) is formulated as follows: find a control $u(t,x'), x' \in \Gamma, t > 0$ such that the solution $w(t,x)$ of boundary value problem (2.2)-(2.4) with chosen $u(t,x')$ satisfies the inequality:

$$\|w(t,\cdot) - \hat{w}\|_{H^2(\Omega)} \le ce^{-\sigma t} \quad \text{as } t > 0, \tag{2.7}$$

for certain $\sigma > 0$.

2.2. **Method of stabilization.** We extend the domain Ω through the part of the boundary Γ upto a bounded domain $G: \ \Omega \subset G$ with C^∞boundary ∂G:

$$G = \Omega \cup \Gamma \cup \omega, \quad \overline{\Omega} \cap \overline{\omega} = \Gamma, \quad \omega \text{ is a bounded domain} \tag{2.8}$$

Then we extend steady-state solution $\widehat{w}(x)$ into $\widehat{v}(x), \ x \in G$ with $\widehat{v}(x) \in H^2(G) \cap H^1_0(G)$. Substitute $\widehat{v}(x)$ into left side of (1.13) and get the extension $f(x), x \in G$ of g from (1.13). Now we take equations (1.9),(1.10) as extension of (2.2), (2.3) without condition $w|_\Gamma = u$ from $\mathbb{R}_+ \times \Omega$ in $\mathbb{R}_+ \times G$. To finish extension of (2.2), (2.3) (without condition $w|_\Gamma = u$) from $\mathbb{R}_+ \times \Omega$ in $\mathbb{R}_+ \times G$ we have to extend initial condition $w_0(x), x \in \Omega$ from (2.4) in G. This extension is the key point of the stabilization method described here. The following theorem holds:

Theorem 2.1. *Let $\varepsilon > 0$ from (2.6) is sufficiently small. Then there exists an operator E defined on the set $B_\varepsilon(\Omega) = \{w_0 \in H^2(\Omega), w_0|_{\Gamma_0} = 0, \|w_0 - \widehat{w}\|_{H^2(\Omega)} < \varepsilon\}$ and acting to the stable invariant manifold $\widehat{v} + M_-$ defined near steady-state solution $\widehat{v}(x), x \in G$ where M_- is the set (1.27):*

$$E : B_\varepsilon(\Omega) \longrightarrow \widehat{v} + M_- \tag{2.9}$$

This theorem had been proved in [2].

Now we extend initial condition w_0 from (2.4) by the formula

$$v_0 = E(w_0), \tag{2.10}$$

and take this v_0 as initial condition in (1.12). Since $v_0 \in \widehat{v} + M_-$, in virtue of Definition 1.1 of M_- solution $v(t,x)$ of the problem (1.9)–(1.11) satisfies the inequality

$$\|v(t,\cdot) - \hat{v}\|_{H^2(G)} \leq ce^{-\sigma t} \quad \text{as } t > 0, \tag{2.11}$$

with some $\sigma > 0$. Define now the solution $(w(t,x), u(t,x'))$ of stabilization problem (2.2)–(2.5) by the relation:

$$(w(t,\cdot), u(t\cdot)) = (v(t,\cdot)|_\Omega, v(t,\cdot)|_\Gamma), \tag{2.12}$$

where v is the solution of (1.9)–(1.11) with initial condition v_0 constructed in (2.10), $|_\Omega, |_\Gamma$ are restriction operators on Ω and Γ correspondingly.

Definition (2.12) implies that the pair $(w(t,x), u(t,x))$ satisfies (2.2)–(2.4) and inequality (2.7) folows from (2.11). So we have proved that the pair (w,u) defined in (2.12) is the solution of stabilization problem (2.2)–(2.5).

2.3. **Stabilization of linear parabolic equation.** Let us consider now stabilization of linear parabolic equation. In this more simple case it will be easier to explain the the main steps in the proof of Theorem 2.1.

We consider linear analog of problem (2.1)-(2.4)on the domain Ω with the boundary $\partial\Omega = \Gamma_0 \cup \Gamma, \qquad \Gamma_0 \cap \Gamma = \emptyset$:

$$\partial_t w(t,x) - \nu\Delta w(t,x) - w(t,x) = 0, \quad x \in \Omega, \ t > 0 \tag{2.13}$$

$$w(t,\cdot)|_{x\in\Gamma_0} = 0, \quad w(t,\cdot)|_{x'\in\Gamma} = u(t,x') \tag{2.14}$$

$$w(t,x)|_{t=0} = w_0(x), \quad x \in \Omega, \tag{2.15}$$

where $w_0(x) \in H^2(G), w_0|_{\Gamma_0} = 0$, and u is a control.

The stabilization problem for (2.13)–(2.15) is formulated as above: find a control $u(t,x'), x' \in \Gamma, t > 0$ such that the solution $w(t,x)$ of (2.13)-(2.15) with chosen $u(t,x')$ satisfies the inequality:

$$\|w(t,\cdot)\|_{H^2(\Omega)} \leq ce^{-\sigma t} \quad \text{as } t > 0, \tag{2.16}$$

for certain $\sigma > 0$.

As in the case of Ginzburg-Landau equation we extend the domain Ω through the part of the boundary Γ upto a bounded domain G with C^∞boundary ∂G. Now we take equations (1.1),(1.2) as extension of (2.13), (2.14) in G. To finish our extension we have to extend initial condition $w_0(x), x \in \Omega$ from (2.15) in initial condition v_0 on G by such a way that solution $v(t,x)$ of boundary value problem (1.1)–(1.3) tends to zero as $t \to \infty$ exponentially.

To do this note that using Fourier method for solution of (1.1)–(1.3) we get:

$$v(t,x) = \sum_{k=1}^{\infty} v_{0k} e^{-kt} e_k(x) \tag{2.17}$$

where $\{e_k(x)\}$ is basis from eigenfunctions (1.5) and $v_{0,k}$ is Fourier coefficients of v_0 in this basis. Since righ side of (2.17) exponentially tends to zero, it belongs to V_- for every $t > 0$, and therefore the relations hold:

$$v_{0,k} = 0 \quad \text{for} \quad r = 1, \ldots, N \tag{2.18}$$

This means that extension of w_0 in G should belong to $V_-(G)$

Lemma 2.1. *There exists extension operator*

$$E : L_2(\Omega) \longrightarrow V_-(G), \quad ((Ew)|_\Omega(x) \equiv w(x), \tag{2.19}$$

such that $v_0(x) = (Ew)(x)$ *satisfies (2.18).*

Proof. We define operator (2.19) by the formula

$$Ew_0(x) = \begin{cases} w_0(x), & x \in \Omega \\ z(x), & x \in G \setminus \Omega \end{cases} \tag{2.20}$$

where the function $z(x)$ should be defined. In virtue of (2.18) $z(x)$ should satify the system of equations:

$$\int_{G\setminus\Omega} e_k(x)z(x)dx = -\int_\Omega e_k(x)w_0(x)dx, \quad k = 1,\dots,N$$

We look for $z(x)$ in a form

$$z(x) = \sum_{j=1}^N \hat{z}(j)e_j(x)$$

Substitution this equation into previous one gives the system of equations that determines $\hat{z}(j)$:

$$\sum_{j=1}^N a_{kj}\hat{z}(j) = -\int_\Omega e_k(x)w_0(x)dx, \quad k = 1,\dots,N,$$

where coefficients a_{jk} are determined by relations:

$$a_{kj} = \int_{G\setminus\Omega} e_j(x)e_k(x)dx$$

The matrix $A =\| a_{kj} \|$ is positive definite. Indeed, if

$$\Phi = \{\hat{\varphi}_k, k = 1,\dots,N\} \quad \text{and} \quad \varphi(x) = \sum_k \hat{\varphi}_k e_k(x),$$

then

$$(A\Phi,\Phi) = \sum_{k,j} a_{j,k}\hat{\varphi}_j\hat{\varphi}_k = \int\limits_{G\setminus\Omega} |\varphi(x)|^2 dx \geq 0.$$

If for some Φ we have equality here then

$$\varphi(x) = \sum \hat{\varphi}_k e_k(x) \equiv 0, \ x \in G \setminus \Omega \quad \text{and therefore } \varphi_k = 0 \quad \forall k.$$

The last assertion is rather deep and is proved in [2]. Thus, $\det \| a_{kj} \| \neq 0$ and therefore corresponding system define uniquely operator (2.19),(2.20) that satisfies all conditions of Lemma. □

2.4. **Some motivations.** We see in the Lemma 2.1 proof that since stable invariant subspace V_- is defined by N linear equations, construction of extension operator (2.19) is reduced to solution of $N \times N$-system of linear equaitons. The main idea used for construction of extension operator (2.9)in the case of Ginzburg-Landau equation is as follows. In virtue of (1.29) stable invariant manifold M_- in small neighborhood of steady-state solution $\widehat{w}$ is close to stable invariant subspace V_- that corresponds to linearization of Ginzburg-Landau equation at $\widehat{w}$. That is why a proper small perturbation of extension operator (2.19),(2.20) gives extension operator(2.9). The construction of such perturbation is the main content of Theorem 2.1. Note that construction of extension operator from Theorem 2.1 can not be applied directly in numerical calculations because description of stable invariant manifold used in Theorem 2.1is too absract.

We have to emphasize that the main reason to develop stabilization theory is to provide reliable stable algorithms for numerical stabilization [1] To construct such algorithms it is very desirable to have a simple description for infinite-dimensional invariant manifold M_- allowing to calculate it easily in arbitrary point. Just such description gives functional-analytic decomposition of M_-. In these lectures we investigate the possibility of such description for M_- in the case of Ginzburg-Landau equation.

3. Preliminaries

To get functional-analytic decompozition of the map F that defines stable invariant manifold, we have to derive differential equation for F

3.1. **Equation for F.**

First of all we recall derivation of well-known equation for map (1.28) that determines invariant manifold M_-. After that we recall definitions of certain notions that we use later.

First of all we introduce several notations. We rewrite equations (1.15),(1.18) using definition (1.20) of operator A as follows:

$$\partial_t y(t) + Ay(t) + B(\cdot, y(t)) = 0 \tag{3.1}$$

Define the orthoprojectors

$$P_+ : V \to V_+, \quad P_- : V \to V_- \tag{3.2}$$

and introduce notations

$$P_+ y = y_+, \quad P_- y = y_-, \quad P_+ S(t, y_0) = S_+(t, y_0), \quad P_- S(t, y_0) = S_-(t, y_0) \tag{3.3}$$

Taking into account that the spaces V_+, V_- are invariant with respect to acting of operator A and using notation (3.3) we can rewrite (3.1) as follows:

$$\begin{aligned} &\partial_t y_+(t) + Ay_+(t) + P_+ B(\cdot, y_+(t) + y_-(t)) = 0 \\ &\partial_t y_-(t) + Ay_-(t) + P_- B(\cdot, y_+(t) + y_-(t)) = 0 \end{aligned} \tag{3.4}$$

Let $y_0 \in M_-$. Then by (1.27) it has the form $y_0 = y_- + F(y_-)$. By definition of an invariant manifold for each $t \in \mathbb{R}_+$ $S(t, y_0) \in M_-$ or, what is equivalent

$$S_+(t, y_- + F(y_-)) = F(S_-(t, y_- + F(y_-)))$$

We differentiate this equation with respect to t and express t-derivatives with help of equations (3.4). As a result we get:

$$\begin{aligned} &AS_+(t, y_- + F(y_-)) + P_+ B(\cdot, S(t, y_- + F(y_-))) \\ =&\langle F'(S_-(t, y_- + F(y_-))), AS_-(t, y_- + F(y_-)) \\ &+ P_- B(\cdot, S_+(t, y_- + F(y_-)) + S_-(t, y_- + F(y_-)))\rangle \end{aligned} \tag{3.5}$$

[1] Indeed, the existence theorems for exact controllability problem (see, for instance, [3]) are stronger than existence rezultes for corresponding stabilization problem. The other point is that exact controllability problems are ill-posed in the case of parabolic equations and therefore they can not be solved numerically by adequate way.

where by $\langle F'(z), h\rangle$ we denote the value of derivative $F'(z)$ on vector h. Passing to limit in (1.27) as $t \to 0$ we get the desired equation for F:

$$AF(y_-) + P_+B(\cdot, y_- + F(y_-)) = \langle F'(y_-), Ay_- + P_-B(\cdot, y_- + F(y_-))\rangle \qquad (3.6)$$

3.2. **Analytic maps.** Let H_i be Hilbert spaces with the scalar products $(\cdot,\cdot)_i$ and the norms $\|\cdot\|_i$ where $i = 1, 2$. We denote by $(H_1)^k = H_1 \times \cdots \times H_1$ (k times) the direct product of k copies of H_1 and define by $F_k : (H_1)^k \to H_2$ a k-linear operator $F_k(h_1, \dots, h_k)$, i.e. the operator that is linear with respect to each variable $h_i, i = 1, \dots, k$. Then

$$\|F_k\| = \sup_{\|h_i\|_1=1, i=1,\dots,k} \|F_k(h_1, \dots, h_k)\|_2 \qquad (3.7)$$

Restriction of k-linear operator $F_k(h_1, \dots, h_k)$ to diagonal $h_1 = \cdots = h_k = h$ is called power operator of order k:

$$F_k(h) = F_k(h, \dots, h) \qquad (3.8)$$

Using derivatives one can restore k-linear operator $F_k(h_1, \dots, h_k)$ by power operator $F_k(h)$.

Denote by $\mathcal{O}(H_1)$ a neighbourhood of origin in the space H_1. The map

$$F : \mathcal{O}(H_1) \to H_2 \qquad (3.9)$$

is called analytic if it can be decomposed in the series

$$F(h) = F_0 + \sum_{k=1}^{\infty} F_k(h) \qquad (3.10)$$

where $F_0 \in H_2$ and $F_k(h)$ are power operators of order k. Series (2.4) converges if the numerical series $\|F_0\|_2 + \sum_{k=1}^{\infty} \|F_k(h)\|_2$ converges.

Proposition 3.1. *Let norms (3.7) of power operator $F_k(h)$ from (3.10) satisfy*

$$\|F_k\| \le \gamma\rho^{-k} \qquad (3.11)$$

where $\gamma > 0, \rho > 0$ do not depend on k. [2] *Then series (3.10) converges for each h belonging to the ball $B_\rho(H_1) = \{h \in H_1 : \|h\|_1 < \rho\}$.*

Proof. There exists $\varepsilon > 0$ such that $\|h\|_1 \le \rho - \varepsilon$. Then using (3.7), (3.11) we get

$$\|F(h)\|_2 \le \|F_0\|_2 + \sum_{k=1}^{\infty} \|F_k\| \|h\|_1^k \le \gamma \sum_{k=1}^{\infty} (\frac{\rho - \varepsilon}{\rho})^k < \infty$$

□

3.3. **Operators from equation for F.** We consider here operators from equation (3.5).

[2]For briefness we use for power operator $F_k(h)$ the norm (3.7) although an alternative definition is possible (see detailes in Chapter 1 of [4])

3.3.1. *Operator A and phase space V.* Phase space is defined in (1.12), and operator A is defined by the following relations (see (1.20),(1.18):

$$A : L_2(G) \longrightarrow L_2(G),\ D(A) = V,\ Av(x) = -\nu\Delta v(x) + q(x)v(x),\ \forall v \in V \quad (3.12)$$

Since operator A is symmetric in $L_2(G)$, the set (1.19) of its eigenfunctions $\{e_k\}$ forms orthogonal basis in $L_2(G)$. We can assume (have done normalization) that $\{e_k\}$ is orthonormal basis in $L_2(G)$. It is well-known that usual Sobolev H^2-norm in $V = H^2(G) \cap H_0^1(G)$ is equivalent to the norm

$$\|v\|_V^2 = \sum_{j=1}^{\infty} \lambda_j^2 |v_j|^2, \text{ where } v_j = \int_G v(x)e_j(x)dx, \text{ and} v(x) = \sum_{j=1}^{\infty} v_j e_j(x) \quad (3.13)$$

Evidently, $\{e_j\}$ forms orthogonal basis in V with respect to scalar product defined by norm (3.12). Below we suppose that the phase space V is supplied with the norm (3.12).

3.3.2. *Subspapces $V_\pm$ and projectors $P_\pm$.* Subcpasec V_+, V_- of V are defined in (1.23), and projectors$P_\pm$ are defined in (3.2). Orthogonality of decomposition (1.23) as well as orthogonality of projectors (3.2) take place with respect to the scalar product defined by (3.13). It is easy to see that

$$\|P_+\| \leq 1, \qquad \|P_-\| \leq 1 \quad (3.14)$$

Kernels $\widehat{P}_\pm$ of operators $P_\pm$, i.e. distributions on $G \times G$ such that

$$(P_\pm v)(x) = \int_G \widehat{P}_\pm(x,\xi)v(\xi)d\xi \quad (3.15)$$

are defined as follows:

$$\widehat{P}_+(x,\xi) = \sum_{k=1}^{N} e_k(x)e_k(\xi), \quad \widehat{P}_-(x,\xi) = \delta(x-\xi) - \sum_{k=1}^{N} e_k(x)e_k(\xi), \quad (3.16)$$

where $\delta(x-\xi)$ is Dirac δ-function. Note that integral in (3.16) in the case $\widehat{P}_-(x,\xi)$ is understood (at each fixed x) as value of distribution $\delta(x-\xi) - \sum_{k=1}^N e_k(x)e_k(\xi)$ on the test function $v(x)$. Such notation for values of distributions will be often used below without additional expleinations.

3.3.3. *Analyticity of the map $B(\cdot, y)$.* We intend to decompose operator $B(\cdot, y)$ defined in (1.18) in series (3.7). For this we use that the phase space V is the algebra, i.e. in this space the operation of multiplcation of functions is well-defined.

Define the operator of multiplcation Γ_k as follows:

$$\Gamma_k : V^k \longrightarrow V, \qquad \Gamma_k(v_1, \ldots, v_k)(x) = v_1(x) \cdot \cdots \cdot v_k(x) \quad (3.17)$$

where $V^k = V \times \cdots \times V$ (k times),

Lemma 3.1. *Let $V = H^2(G) \cap H_0^1(G), G \subset \mathbb{R}^n, n = 1,2,3$. Then operator Γ_k defined in (3.17) is k-linear bounded operator. Moreover, there exists a constatnt $\gamma > 0$ such that for each k*

$$\|\Gamma_k(v_1, \ldots, v_k)\|_V \leq \gamma^{k-1} \|v_1\|_V \cdot \cdots \cdot \|v_k\|_V \quad (3.18)$$

Proof. Since norm (3.12) is equivalent to the norm of Sobolev space $H^2(G)$, we can use H^2-norm. Taking into account that embeddings $H^2(G) \subset C(\overline{G})$ and $H^2(G) \subset W_4^1(G)$ are continious we get:

$$\|v_1 \cdot v_2\|_{H^2(G)} = (\sum_{\|\alpha| \le 2} \int |D^\alpha(v_1(x)v_2(x))|^2 dx)^{1/2}$$

$$\le \|v_1\|_{H^2}\|v_2\|_C + \|v_1\|_C\|v_2\|_{H^2} + 2\|v_1\|_{W_4^1}\|v_2\|_{W_4^1} \le \gamma\|v_1\|_{H^2}\|v_2\|_{H^2}$$

Using this inequality we obtain (3.17) by induction in k □

It follows from Lemma 3.1 and (1.18) that for $y \in V$

$$B(x, y(x)) = \Gamma_3(y, y, y)(x) - 3\widehat{v}(x)\Gamma_2(y, y)(x) \tag{3.19}$$

Therefore operator B is analytic, and relation (3.19) is its decomposition in series (3.10). The kernels of operators from (3.19) are as follows:

$$\widehat{\Gamma}_3(x; \xi_1, \xi_2, \xi_3) = \delta(x - \xi_1)\delta(x - \xi_2)\delta(x - \xi_3) \tag{3.20}$$

$$3\widehat{v}(x)\widehat{\Gamma}_2(x; \xi_1, \xi_2) = 3\widehat{v}(x)\delta(x - \xi_1)\delta(x - \xi_2) \tag{3.21}$$

3.4. **Series for operator F.** Let us consider the special case when $H_1 = V_-, H_2 = V_+$ with Hilbert spaces V_-, V_+ defined in(1.23). In this case analytic map (3.9), (3.10) can be rewritten as follows:

$$F : \mathcal{O}(V_-) \to V_+, \quad F(y_-) = \sum_{k=2}^{\infty} F_k(y_-) \tag{3.22}$$

It is convinient for us to bound ourselves with the case when $F_0 = 0,\ F_1 = 0$ because by (1.29) the map F defining stable invariant manifold M_- has just this form.

Now we define kernels $\widehat{F}_k(x; \xi_1, \ldots, \xi_k)$ of k-linear operator $F_k(\cdot; y_1, \ldots, y_k), y_j \in V_-, j = 1, \ldots, k$. Let

$$V \subset L_2(G) \subset V' \tag{3.23}$$

where V' is the space dual to V with respect to duality generated by scalar product in $L_2(G)$. Define

$$V'_- = \{u(x) \in V' : \int u(x)\varphi(x)dx = 0 \quad \forall \varphi \in V_+\} = V_+^\perp \tag{3.24}$$

Below we use the following notation:

$$\overline{\xi^k} = (\xi_1, \ldots, \xi_k), \quad \text{where } \xi_j \in G,\ j = 1, \ldots, k \tag{3.25}$$

The kernel $\widehat{F}_k(x; \overline{\xi^k}), x \in G, \overline{\xi^k} \in G^k \equiv G \times \cdots \times G$ (k times) is an element of the space $V_+ \otimes (\overset{k}{\otimes} V'_-)$ where $\overset{k}{\otimes} V'_- = V' \otimes \cdots \otimes V'$ (k times). In other words $\widehat{F}_k(x; \overline{\xi^k})$ is a distribution on G^k with values in V_+, such that for each $y_j \in V, j = 1, \ldots, k$ the value

$$F_k(x, y_1, \ldots, y_k) = \int \widehat{F}_k(x; \overline{\xi^k}) y_1(\xi_1) \cdot \cdots \cdot y_k(\xi_k) d\overline{\xi^k} \tag{3.26}$$

($d\overline{\xi^k} = d\xi_1 \ldots d\xi_k$) of distribution $\widehat{F}_k(x; \overline{\xi^k})$ on test function $y(\xi_1) \cdot \cdots \cdot y(\xi_k)$ is well-defined. Moreover, if $y_j \in V_+$ at least for one $j \in \{1, \ldots, k\}$ then right hand side of equality (3.26) equals zero. Moreover, since $F_k(\cdot, y_1, \ldots, y_k)$ is symmetric with respect to $(y_1, \dots, y_k)$, i.e. $F_k(\cdot, y_1, \dots, y_k) = F_k(\cdot, y_{j_1}, \dots, y_{j_k})$ for each permutation $(j_1, \dots, j_k)$ of $(1, \dots, k)$, the distribution $\widehat{F}_k(x; \overline{\xi^k})$ is symmetric with respect to $(\xi_1, \dots, \xi_k)$

Now using relation (3.26) and notation

$$y_-(\overline{\xi^k}) = y_-(\xi_1) \cdot \dots \cdot y_-(\xi_k) \tag{3.27}$$

we can rewrite the series from (3.22) in the form:

$$F(x, y_-) = \sum_{k=2}^{\infty} \int \widehat{F}_k(x; \overline{\xi^k}) y_(\overline{\xi^k}) d\overline{\xi^k} \tag{3.28}$$

For each function or distribution $K(\eta_1, \dots, \eta_r)$ defined on G^r we determine the function $\sigma_{\overline{\eta^r}} K(\eta_1, \dots, \eta_r)$ which is simmetric with respect to arbitrary permutation $(\eta_{j_1}, \dots, \eta_{j_r})$ of variables $(\eta_1, \dots, \eta_r)$ by the formula:

$$\sigma_{\overline{\eta^r}} K(\eta_1, \dots \eta_r) = \frac{1}{r!} \sum_{(j_1, \dots, j_r)} K(\eta_{j_1}, \dots, \eta_{j_r}) \tag{3.29}$$

where the sum in the r.h.s. of (2.12) performs over all permutations $(j_1, \dots, j_r)$ of the set $(1, \dots, r)$.

Lemma 3.2. *Let $K(\eta_1, \dots, \eta_r)$ be defined on G^r. Then*

(a) *The following equality is true:*

$$\sum_{\overline{\eta^r}} K(\eta_1, \dots, \eta_r) h(\eta_1) \dots h(\eta_r) = \sigma_{\overline{\eta^r}} \sum_{\overline{\eta^r}} K(\eta_1, \dots, \eta_r) h(\eta_1) \dots h(\eta_r)$$

for any $h(\eta_r)$ such that the serie in the l.h.s. converges,

(b) *For any function $G(\eta_1, \dots, \eta_r)$ simmetric in its arguments*

$$G(\overline{\eta^r}) \sigma_{\overline{\eta^r}} K(\overline{\eta^r}) = \sigma_{\overline{\eta^r}} [G(\overline{\eta^r}) K(\overline{\eta^r})] \tag{3.30}$$

Furthmore

$$\sup_{\overline{\eta^r}} |\sigma_{\overline{\eta^r}} K(\overline{\eta^r})| \leq \sup_{\overline{\eta^r}} |K(\overline{\eta^r})| \tag{3.31}$$

(c) *If all distributions $F_k(x; \overline{\eta^k})$ from (3.28) are symmetric in their arguments $\overline{\eta^k}$ then these distributions are defined uniquely by values of analytic functions $F(y_-)$ from (3.28).*

The proof of this Lemma is evident.

4. Formal construction of the map F

We look for the map defining stable invariant manifold in the form of a series (3.28).In this section we find recurrence relations for kernals $\widehat{F}_k(x; \overline{\xi^k})$.

4.1. **Calculation of $\widehat{F}_2(x;\overline{\xi^2})$.** Below using k-linear operators $F_k(x;y) = F_k(x;y,\dots,y)$ (k times) we omitt sometimes variable x writing $F_k(y)$. After substitution (3.28) into (3.6) we get taking into account (1.18) that for each $y = y_- \in V_-$

$$\sum_{q=2}^{\infty} AF_q(y) + P_+\left(\sum_{j=0}^{3}\binom{3}{j}F^j(y)y^{3-j} - 3\widehat{v}\sum_{j=0}^{2}\binom{2}{j}F^j(y)y^{2-j}\right)$$
$$+\sum_{q=2}^{\infty} qF_q\left(y,\dots,y,Ay + P_-\left(\sum_{j=0}^{3}\binom{3}{j}F^j(y)y^{3-j} - 3\widehat{v}\sum_{j=0}^{2}\binom{2}{j}F^j(y)y^{2-j}\right)\right) \tag{4.1}$$

Let us equate the terms from (4.1) of the second order with respect to y:

$$AF_2(y,y) - 3P_+(\widehat{v}y^2) = 2F_2(y,Ay)$$

Using kernels of bilinear operator $F_2(y,y)$ we can rewrite this relation as follows:

$$\int [\widehat{F}_2(x;\overline{\xi^2})(A_{\xi_1} + A_{\xi_2})y(\overline{\xi^2})) - A_x\widehat{F}_2(x;\overline{\xi^2})y(\overline{\xi^2})]d\overline{\xi^2} = -3P_+(\widehat{v}y^2)(x) \tag{4.2}$$

where subscript of operator A indicates independent variable of a function to that this operator A is applied. We will use notation:

$$A_{\overline{\xi^k}} = \sum_{j=1}^{k} A_{\xi_j} \tag{4.3}$$

Carrying operator $A_{\overline{\xi^2}}$ from $y(\overline{\xi^2})$ to $\widehat{F}_2(x;\overline{\xi^2})$ and using operator (3.17) in right side of (4.2) we get:

$$\int (A_{\overline{\xi^k}} - A_x)\widehat{F}_2(x;\overline{\xi^2})y(\overline{\xi^2})d\overline{\xi^2} = -3\int \widehat{P}_+(x,\eta)\widehat{v}(\eta)\widehat{\Gamma}_2(\eta;\overline{\xi^2})y(\overline{\xi^2})d\eta d\overline{\xi^2} \tag{4.4}$$

Since $y \in V_-$ and subspaces V_+, V_-' are invariant with respect of operator A, we obtain from (4.4) the relation determining $\widehat{F}_2$:

$$\widehat{F}_2(x;\overline{\xi^2}) = -3(A_{\overline{\xi^k}} - A_x)^{-1}\int \widehat{P}_+(x,\eta)\widehat{v}(\eta)\widehat{\Gamma}_2(\eta;\overline{\zeta^2})\widehat{P}_-(\overline{\zeta^2};\overline{\xi^2})d\overline{\zeta^2} \tag{4.5}$$

Note that operator $(A_{\overline{\xi^k}} - A_x)^{-1}$ is well-defined. Moreover, the following assertion hold:

Lemma 4.1. *Operator*

$$(A_{\overline{\xi^k}} - A_x)^{-1} : V_+ \otimes (\overset{k}{\otimes} V_-') \longrightarrow V_+ \otimes (\overset{k}{\otimes} V_-') \tag{4.6}$$

is well-defined and bounded, and for its norm the following estimate holds:

$$\|(A_{\overline{\xi^k}} - A_x)^{-1}\| \le q_0(k+1) \tag{4.7}$$

with certain constant $q_0 > 0$

At last we write down the recurrence relation for the kernel $\widehat{F}_(x;\overline{\xi^3})$ that can be obtained cemilarly to the formula (4.5)

$$\widehat{F}_3(x;\overline{\xi^3}) = (A_{\overline{\xi^k}} - A_x)^{-1} S_{\overline{\xi^3}} \Big[\int \widehat{P}_+(x,\eta)\widehat{\Gamma}_3(\eta;\overline{\zeta^3})\widehat{P}_-(\overline{\zeta^3};\overline{\xi^3}) d\eta d\overline{\zeta^3}$$

$$-6\int \widehat{P}_+(x,\eta)\widehat{v}(\eta)\widehat{F}_2(\eta;\overline{\zeta^2})\delta(\eta-\zeta_3)\widehat{P}_-(\overline{\zeta^3};\overline{\xi^3}) d\eta d\overline{\zeta^3} \tag{4.8}$$

$$+2\int \widehat{F}_2(x;\overline{\eta^2})\widehat{P}_-(\eta_1,\xi_3)\widehat{P}_-(\eta_2,s)\widehat{v}(s)\widehat{\Gamma}_2(s;\overline{\zeta^2})\widehat{P}_-(\overline{\zeta^2};\overline{\xi^2}) d\overline{\eta^2} ds d\overline{\zeta^2} \Big]$$

4.2. **Calculation of $\widehat{F}_q(x;\overline{\xi^q})$.** Let us equate the terms from (4.1) of the order q with respect to y:

$$qAF_q(y,\dots,y,Ay) - AF_q(y) = P_+ \big[3y^2F_{q-2}(y) - 6\widehat{v}yF_{q-1}(y)$$

$$+3\,y\sum_{m_1+m_2=q-1} F_{m_1}F_{m_2} - 3\widehat{v}\sum_{m_1+m_2=q} F_{m_1}F_{m_2} + \sum_{m_1+m_2+m_3=q} F_{m_1}F_{m_2}F_{m_3}\Big]$$

$$+(q-1)F_{q-1}(y,\dots,y,P_-(3\widehat{v}y^2)) - (q-2)F_{q-2}(y,\dots,y,P_-(y^3))$$

$$+\sum_{l+m=q} mF_m(y,\dots,y,P_-(6\widehat{v}yF_l(y))) - \sum_{l+m=q-1} mF_m(y,\dots,y,P_-(3y^2F_l(y)))$$

$$+\sum_{l_1+l_2+m=q+1} mF_m(y,\dots,y,P_-(3\widehat{v}F_{l_1}F_{l_2})) - \sum_{l_1+l_2+m=q} mF_m(y,\dots,y,P_-(3yF_{l_1}F_{l_2}))$$

$$-\sum_{l_1+l_2+l_3+m=q+1} mF_m(y,\dots,y,P_-(F_{l_1}F_{l_2}F_{l_3})) \tag{4.9}$$

Remark 4.1. *Note that this formula is true only for $q \geq 7$. To get formula for $q = 6$ we have to omit the last (12-th) term in righ side of (4.9),in formula for $q = 5$ we have to omit in addition 5-th, 7-th, and 11-th terms, for $q = 4$ we have to omit in addition 3-rd, 9-th, and 10-th terms, for $q = 3$ we omit also 1-th,4-th,6-th, and 8-th terms. At last in formula for $q = 2$ we resrve in right side of (4.9) the second term only.*

Using kernels of operators $F_q(y)$ similarly to bilinear case, we can derive from (4.9)recurrence relation for kernals $\widehat{F}_q(x;\overline{\xi^q})$:

$$\widehat{F}_q(x;\overline{\xi^q}) = (A_{\overline{\xi^q}} - A_x)^{-1} S_{\overline{\xi^q}}(\widehat{I}_1 + \dots + \widehat{I}_9)(x;\overline{\xi^q}) \tag{4.10}$$

where

$$\widehat{I}_1(x;\overline{\xi^q}) = 3\int \widehat{P}_+(x,\eta)\widehat{F}_{q-2}(\eta;\overline{\zeta^{q-2}})\widehat{\Gamma}_2(\eta;\zeta_{q-1},\zeta_q)P_-(\overline{\zeta^q};\overline{\xi^q}) d\eta d\overline{\zeta^q}, \tag{4.11}$$

$$\widehat{I}_2(x;\overline{\xi^q}) = -6\int \widehat{P}_+(x,\eta)\widehat{F}_{q-1}(\eta;\overline{\zeta^{q-1}})\widehat{v}(\eta)\delta(\eta-\zeta_q)P_-(\overline{\zeta^q};\overline{\xi^q}) d\eta d\overline{\zeta^q}, \tag{4.12}$$

$$\widehat{I}_3 = 3\sum_{m_1+m_2=q-1}\int \widehat{P}_+(x,\eta)(\widehat{F}_{m_1}(\eta;\cdot)\widehat{F}_{m_2}(\eta;\cdot))(\overline{\zeta^{q-1}})\delta(\eta-\zeta_q)P_-(\overline{\zeta^q};\overline{\xi^q}) d\eta d\overline{\zeta^q}, \tag{4.13}$$

$$\widehat{I}_4(x;\overline{\xi^q}) = -3\sum_{m_1+m_2=q}\int \widehat{P}_+(x,\eta)\widehat{v}(\eta)(\widehat{F}_{m_1}(\eta;\cdot)\widehat{F}_{m_2}(\eta;\cdot))(\overline{\zeta^q})P_-(\overline{\zeta^q};\overline{\xi^q}) d\eta d\overline{\zeta^q}, \tag{4.14}$$

$$\widehat{I}_5 = \sum_{m_1+m_2+m_3=q-1} \int \widehat{P}_+(x,\eta)(\widehat{F}_{m_1}(\eta;\cdot)\widehat{F}_{m_2}(\eta;\cdot)\widehat{F}_{m_3}(\eta;\cdot))(\overline{\zeta^q})P_-(\overline{\zeta^q};\overline{\xi^q})d\eta d\overline{\zeta^q}. \tag{4.15}$$

$$\widehat{I}_6(x;\overline{\xi^q}) = 3(q-1)\int \widehat{F_{q-1}P_-}(x;\overline{\zeta^{q-2}},s)\widehat{v}(s)\widehat{\Gamma}_2(s;\zeta_{q-1},\zeta_q)P_-(\overline{\zeta^q};\overline{\xi^q})dsd\overline{\zeta^q}, \tag{4.16}$$

$$\widehat{I}_7(x;\overline{\xi^q}) = (q-2)\int \widehat{F_{q-2}P_-}(x;\overline{\zeta^{q-3}},s)\widehat{\Gamma}_3(s;\zeta_{q-2},\zeta_{q-1},\zeta_q)P_-(\overline{\zeta^q};\overline{\xi^q})dsd\overline{\zeta^q}, \tag{4.17}$$

$$\widehat{I}_8(x;\overline{\xi^q}) = \sum_{l+m=q} 6m\int \widehat{v}(s)(\widehat{F_mP_-}(x;s,\cdot)\widehat{F}_l(s;\cdot))(\overline{\zeta^{q-1}})\delta(s-\zeta_q)P_-(\overline{\zeta^q};\overline{\xi^q})dsd\overline{\zeta^q}, \tag{4.18}$$

$$\widehat{I}_9(x;\overline{\xi^q}) = -\sum_{l+m=q-1} 3m\int (\widehat{F_mP_-}(x;s,\cdot)\widehat{F}_l(s;\cdot))(\overline{\zeta^{q-2}})\widehat{\Gamma}_2(s;\zeta_{q-1},\zeta_q)P_-(\overline{\zeta^q};\overline{\xi^q})dsd\overline{\zeta^q}, \tag{4.19}$$

$$\widehat{I}_{10} = \sum_{l_1+l_2+m=q+1} 3m\int (\widehat{F_mP_-}(x;s,\cdot)\widehat{F}_{l_1}(s;\cdot)\widehat{F}_{l_2}(s;\cdot))(\overline{\zeta^q})\widehat{v}(s)P_-(\overline{\zeta^q};\overline{\xi^q})dsd\overline{\zeta^q}, \tag{4.20}$$

$$\frac{-\widehat{I}_{11}}{3m} = \sum_{l_1+l_2+m=q} \int (\widehat{F_mP_-}(x;s,\cdot)\widehat{F}_{l_1}(s;\cdot)\widehat{F}_{l_2}(s;\cdot))(\overline{\zeta^{q-1}})\delta(s-\zeta_q)P_-(\overline{\zeta^q};\overline{\xi^q})dsd\overline{\zeta^q}, \tag{4.21}$$

$$\frac{-\widehat{I}_{12}}{m} = \sum_{l_1+l_2+l_3+m=q+1} \int (\widehat{F_mP_-}(x;s,\cdot)(\widehat{F}_{l_1}(s;\cdot)\widehat{F}_{l_2}(s;\cdot)\widehat{F}_{l_3}(s;\cdot))(\overline{\zeta^q})P_-(\overline{\zeta^q};\overline{\xi^q})dsd\overline{\zeta^q}. \tag{4.22}$$

Note that these recurrence relations are true for $q \geq 7$. To get recurrence relations for $\widehat{F}_q(x;\overline{\xi^q})$ with $q = 4,5,6$ one shoud take into account Remark 4.1

Using recurrence relations derived above one can prove that there exists a constants a, b such that for each $q \geq 2$ operators $F_q(y)$ from decomposition (3.22) of the map $F(y)$ satisfy the bounds.

$$\|F_q\| \leq ab^q$$

These estimates imply imidiately, that the serie (3.22) converges for $\|y\|_V < 1/b$ This result in the case of semilinear parabolic equation has been obtained in [5]. The proof in the case of Ginzburg-Landau equation of spatial dimension 2 and 3 will be pulished in some other place.

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