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SMR1777/17

School on Nonlinear Differential Equations

(9 - 27 October 2006)

Cantor families of periodic solutions for completely resonant nonlinear wave equations

Massimiliano Berti
Università di Napoli "Federico II"
Dipartimento di Matematica e Applicazioni
Napoli, Italia

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Massimiliano Berti[*], Philippe Bolle[†]

Duke Mathematical Journal, 134, 2, 359-419, 2006

Abstract: We prove existence of small amplitude, $2\pi/\omega$-periodic in time solutions of completely resonant nonlinear wave equations with Dirichlet boundary conditions, for any frequency ω belonging to a Cantor-like set of asymptotically full measure and for a new set of nonlinearities. The proof relies on a suitable Lyapunov-Schmidt decomposition and a variant of the Nash-Moser Implicit Function Theorem. In spite of the complete resonance of the equation we show that we can still reduce the problem to a *finite* dimensional bifurcation equation. Moreover, a new simple approach for the inversion of the linearized operators required by the Nash-Moser scheme is developed. It allows to deal also with nonlinearities which are not odd and with finite spatial regularity.

Keywords: Nonlinear Wave Equation, Infinite Dimensional Hamiltonian Systems, Periodic Solutions, Variational Methods, Lyapunov-Schmidt reduction, small divisors, Nash-Moser Theorem.[1]

2000AMS subject classification: 35L05, 37K50, 58E05.

Contents

[*]Dipartimento di Matematica e Applicazioni R. Caccioppoli Universitá degli Studi Napoli Federico II, Via Cintia, Monte S. Angelo I-80126 Napoli, Italy, m.berti@unina.it.

[†]Département de mathématiques, Université d'Avignon, 33, rue Louis Pasteur, 84000 Avignon, France, philippe.bolle@univ-avignon.fr.

[1]Supported by MIUR Variational Methods and Nonlinear Differential Equations.

1 Introduction

We consider the *completely resonant* nonlinear wave equation

$$\begin{cases} u_{tt} - u_{xx} + f(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{1}$$

where the nonlinearity

$$f(x,u) = a_p(x)u^p + O(u^{p+1}), \qquad p \geq 2$$

is analytic in u but is only H^1 with respect to x.

We look for small amplitude, $2\pi/\omega$-periodic in time solutions of equation (1) for all frequencies ω in some Cantor set of *positive measure*, actually of full density at $\omega = 1$.

Equation (1) is an infinite dimensional Hamiltonian system possessing an elliptic equilibrium at $u = 0$. The frequencies of the linear oscillations at 0 are $\omega_j = j$, $\forall j = 1, 2, \ldots$, and therefore satisfy *infinitely many resonance* relations. Any solution $v = \sum_{j\geq 1} a_j \cos(jt + \theta_j)\sin(jx)$ of the linearized equation at $u = 0$,

$$\begin{cases} u_{tt} - u_{xx} = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{2}$$

is 2π-periodic in time. For this reason equation (1) is called a completely resonant Hamiltonian PDE.

Existence of periodic solutions close to a completely resonant elliptic equilibrium for finite dimensional Hamiltonian systems has been proved in the celebrated theorems of Weinstein [27], Moser [22] and Fadell-Rabinowitz [14]. The proofs are based on the classical Lyapunov-Schmidt decomposition which splits the problem into two equations: the *range equation*, solved through the standard Implicit Function Theorem, and the *bifurcation equation*, solved via variational arguments.

For proving existence of small amplitude periodic solutions of completely resonant Hamiltonian PDEs like (1) two main difficulties must be overcome:

(i) a "*small denominators*" problem which arises when solving the range equation;

(ii) the presence of an *infinite dimensional* bifurcation equation: which solutions v of the linearized equation (2) can be continued to solutions of the nonlinear equation (1)?

The "small denominators" problem (i) is easily explained: the eigenvalues of the operator $\partial_{tt} - \partial_{xx}$ in the spaces of functions $u(t,x)$, $2\pi/\omega$-periodic in time and such that, say, $u(t,.) \in H_0^1(0,\pi)$ for all t, are $-\omega^2 l^2 + j^2$, $l \in \mathbf{Z}$, $j \geq 1$. Therefore, for almost every $\omega \in \mathbf{R}$, the eigenvalues accumulate to 0. As a consequence, for most ω, the inverse operator of $\partial_{tt} - \partial_{xx}$ is unbounded and the standard Implicit Function Theorem is not applicable.

The appearence of small denominators is a common feature of Hamiltonian PDEs. This problem was first solved by Kuksin [19] and Wayne [26] using KAM theory (other existence results of quasi-periodic solutions with KAM theory were obtained e.g. in [21], [23], [10], see also [20] and references therein).

In [12] Craig-Wayne introduced for Hamiltonian PDEs the Lyapunov-Schmidt reduction method and solved the range equation via a Nash-Moser Implicit function technique. The major difficulty concerns the inversion of the *linearized operators* obtained at any step of the Nash-Moser iteration because the eigenvalues may be arbitrarily small (this is the small divisor problem (i)). The Craig-Wayne method to control such inverses is based on the Frölich-Spencer technique [15] and (in the wave equation with Dirichlet boundary conditions) works for nonlinearities $f(x,u)$ which can be extended to analytic, odd, periodic functions so that the Dirichlet problem on $[0,\pi]$ is equivalent to the 2π-periodic problem within the space of all odd functions. A key property exploited in this case is that the "off-diagonal" terms of the linearized operator (seen as an infinite dimensional matrix in Fourier basis) decay exponentially fast away from the diagonal. At the end of the Nash-Moser iteration, due to the "small divisor problem" (i), the range equation is solved only for a Cantor set of parameters.

We mention that the Craig-Wayne approach has been extended by Su [25] to some case where the nonlinearity has only low Sobolev regularity (for periodic conditions) and by Bourgain to find also quasi-periodic solutions [7]-[8].

The previous results apply for example to non-resonant or partially resonant Hamiltonian PDEs like $u_{tt}-u_{xx}+a_1(x)u = f(x,u)$ where the bifurcation equation is finite dimensional (2 dimensional in [12] and $2m$ dimensional in [13]). With a non-degeneracy assumption ("twist condition") the bifurcation equation is solved in [12]-[13] by the Implicit function Theorem finding a smooth path of solutions which intersects "transversally", for a positive measure set of frequencies, the Cantor set where also the range equation has been solved.

On the other hand, for completely resonant PDE like (1) where $a_1(x) \equiv 0$, both small divisor difuculties and infinite dimensional bifurcation phenomena occur. It was quoted in [11] as an important problem.

The first existence results for small amplitude periodic solutions of (1) have been obtained in[2] [18] for the nonlinearity $f(x,u) = u^3$, and in [2] for $f(x,u) = u^3 + O(u^5)$, imposing on the frequency ω the "strongly non-resonance" condition $|\omega l - j| \geq \gamma/l$, $\forall l \neq j$. For $0 < \gamma < 1/6$, the frequencies ω satisfying such condition accumulate to $\omega = 1$ but form a set $\mathcal{W}_\gamma$ of zero measure. For such ω's the spectrum of $\partial_{tt} - \partial_{xx}$ does not accumulate to 0 and so the small divisor problem (i) is by-passed. Next, problem (ii) is solved by means of the Implicit Function Theorem, observing that the $0th$-order bifurcation equation (which is an approximation of the exact bifurcation equation) possesses, for $f(x,u) = u^3$, non-degenerate periodic solutions, see [3].

In [5]-[6], for the same set $\mathcal{W}_\gamma$ of strongly non-resonant frequencies, existence and multiplicity of periodic solutions has been proved for any nonlinearity $f(u)$. The novelty of [5]-[6] was to solve the bifurcation equation via a variational principle at fixed frequency which, jointly with min-max arguments, enables to find solutions of (1) as critical points of the Lagrangian action functional. More precisely, the bifurcation equation is, for any fixed $\omega \in \mathcal{W}_\gamma$, the Euler Lagrange equation of a "reduced Lagrangian action functional" which possesses non trivial critical points of Mountain-pass type [1], see also remark 1.4.

Unlike [2]-[5]-[6], a new feature of the results of this paper is that the set of frequencies ω for which we prove existence of $2\pi/\omega$-periodic in time solutions of (1) has positive measure, actually has full density at $\omega = 1$.

The existence of periodic solutions for a set of frequencies of positive measure has been proved in [9] in the case of periodic boundary conditions in x and for the nonlinearity $f(x,u) = u^3 + \sum_{4 \leq j \leq d} a_j(x) u^j$ where the $a_j(x)$ are trigonometric cosine polynomials in x. The nonlinear equation $u_{tt} - u_{xx} + u^3 = 0$ possesses a continuum of small amplitude, analytic and non-degenerate periodic solutions in the form of traveling waves $u(t,x) = \delta p_0(\omega t + x)$ where $\omega^2 = 1 + \delta^2$ and p_0 is a non-trivial 2π-periodic solution of the ordinary differential equation $p_0'' = -p_0^3$. With these properties at hand the small divisors problem (i) is solved via a Nash-Moser Implicit function Theorem adapting the estimates of Craig-Wayne [12] for non-resonant PDEs.

Recently, the existence of periodic solutions of (1) for frequencies ω in a set of positive measure has been proved in [16] using the Lindstedt series method to solve the small divisor problem. [16] applies to odd analytic nonlinearities like $f(u) = au^3 + O(u^5)$ with $a \neq 0$ (the term u^3 guarantees a non-degeneracy property). The reason for which $f(u)$ is odd is that the solutions are obtained as analytic sine-series in x, see remark 1.1.

[2]Actually [18] deals with the case of periodic boundary conditions in x, i.e. $u(t, x + 2\pi) = u(t,x)$.

We also quote the recent paper [17] on the standing wave problem for a perfect fluid under gravity and with infinite depth which leads to a nonlinear and completely resonant second order equation.

In this paper we prove the existence of $2\pi/\omega$-periodic solutions of the completely resonant wave equation (1) with Dirichlet boundary conditions for a set of frequencies ω's with full density at $\omega = 1$ and for a new set of nonlinearities $f(x, u)$, including for example $f(x, u) = u^2$.

We do *not* require that $f(x, u)$ can be extended on $(-\pi, \pi) \times \mathbf{R}$ to a function $g(x, u)$, smooth w.r.t. u, satisfying the oddness assumption $g(-x, -u) = -g(x, u)$ and we assume only H^1-regularity in the spatial variable x, see assumption **(H)**.

To deal with these cases we develop a new approach for the inversion of the *linearized operators* which is different from the one of Craig-Wayne-Bourgain [12]-[7]-[8]. Our method -presented in section 4- is quite elementary especially requiring that the frequencies ω satisfy the Diophantine first order Melnikov non-resonance condition of Definition 3.3 with $1 < \tau < 2$, see comments regarding the (P)-equation in subsection 1.2.2.

To handle the presence of an infinite dimensional bifurcation equation (and the connected problems which arise in a direct application of the Craig-Wayne method, see subsection 1.2.2) we perform a further finite dimensional Lyapunov-Schmidt reduction. Under the condition that the $0th$-order bifurcation equation possesses a non-degenerate solution we find periodic solutions of (1) for asymptotically full measure sets of frequencies.

We postpone to subsection 1.2 a detailed description of our method of proof.

1.1 Main result

Normalizing the period to 2π, we look for solutions of

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + f(x, u) = 0 \\ u(t, 0) = u(t, \pi) = 0 \end{cases} \tag{3}$$

in the Hilbert space

$$X_{\sigma,s} := \Big\{ u(t, x) = \sum_{l \in \mathbf{Z}} \exp(\mathrm{i}lt)\ u_l(x) \ \Big| \ u_l \in H_0^1((0, \pi), \mathbf{R}), \quad u_l(x) = u_{-l}(x)\ \forall l \in \mathbf{Z},$$
$$\text{and} \quad \|u\|^2_{\sigma,s} := \sum_{l \in \mathbf{Z}} \exp(2\sigma|l|)(l^{2s} + 1)\|u_l\|^2_{H^1} < +\infty \Big\}.$$

For $\sigma > 0, s \geq 0$, the space $X_{\sigma,s}$ is the space of all even, 2π-periodic in time functions with values in $H_0^1((0, \pi), \mathbf{R})$, which have a bounded analytic extension in the complex strip $|\text{Im } t| < \sigma$ with trace function on $|\text{Im } t| = \sigma$ belonging to $H^s(\mathbf{T}, H_0^1((0, \pi), \mathbf{C}))$.

For $2s > 1$, $X_{\sigma,s}$ is a Banach algebra with respect to multiplication of functions, namely[3]

$$u_1, u_2 \in X_{\sigma,s} \implies u_1 u_2 \in X_{\sigma,s} \qquad \text{and} \qquad \|u_1 u_2\|_{\sigma,s} \leq C\|u_1\|_{\sigma,s}\|u_2\|_{\sigma,s}.$$

It is natural to look for solutions of (3) which are even in time because equation (1) is reversible.

A weak solution $u \in X_{\sigma,s}$ of (3) is a classical solution because the map $x \mapsto u_{xx}(t, x) = \omega^2 u_{tt}(t, x) - f(x, u(t, x))$ belongs to $H_0^1(0, \pi)$ for all $t \in \mathbf{T}$ and hence $u(t, \cdot) \in H^3(0, \pi) \subset C^2([0, \pi])$.

Remark 1.1 *Let us explain why we have chosen $H_0^1((0, \pi), \mathbf{R})$ as configuration space instead of $Y := \{u(x) = \sum_{j \geq 1} u_j \sin(jx) \mid \sum_j \exp(2aj) j^{2\rho} |u_j|^2 < +\infty\}$ as in [12]-[16] which is natural if the nonlinearity $f(x, u)$ can be extended to an analytic in both variables odd function. For non odd nonlinearities f (even analytic) it is not possible in general to find a non trivial, smooth solution of (1) with $u(t, \cdot) \in Y$ for all t. For example assume that $f(x, u) = u^2$. Deriving twice the equation w.r.t. x and using that $u(t, 0) = 0$, $u_{xx}(t, 0) = 0$, $u_{ttxx}(t, 0) = 0$, we deduce $-u_{xxxx}(t, 0) + 2u_x^2(t, 0) = 0$. Now $u_{xxxx}(t, 0) = 0$, $\forall t$, because all the even derivatives of any function in Y vanish at $x = 0$. Hence $u_x^2(t, 0) = 0$, $\forall t$. But this implies, using again the equation, that $\partial_x^k u(t, 0) = 0$, $\forall k$, $\forall t$. Hence, by the analyticity of $u(t, \cdot) \in Y$, $u \equiv 0$.*

[3] The proof is as in [24] recalling that $H_0^1((0, \pi), \mathbf{R})$ is a Banach algebra with respect to multiplication of functions.

The space of the solutions of the linear equation $v_{tt} - v_{xx} = 0$ that belong to $H^1_0(\mathbf{T} \times (0,\pi), \mathbf{R})$ and are even in time is

$$V := \Big\{ v(t,x) = \sum_{l \geq 1} 2\cos(lt) u_l \sin(lx) \ \Big| \ u_l \in \mathbf{R}\,,\ \sum_{l\geq 1} l^2 |u_l|^2 < +\infty \Big\}.$$

V can also be written as

$$V := \Big\{ v(t,x) = \eta(t+x) - \eta(t-x) \ \Big| \ \eta \in H^1(\mathbf{T}, \mathbf{R}) \text{ with } \eta \text{ odd} \Big\}.$$

We assume that the nonlinearity f satisfies

(H) $f(x,u) = \sum_{k\geq p} a_k(x) u^k$, $p \geq 2$, and $a_k(x) \in H^1((0,\pi),\mathbf{R})$ verify $\sum_{k \geq p} \|a_k\|_{H^1} \rho^k < +\infty$ for some $\rho > 0$.

Theorem 1.1 *Assume that $f(x,u)$ satisfies assumption* **(H)** *and*

$$f(x,u) = \begin{cases} a_2 u^2 + \sum_{k\geq 4} a_k(x) u^k & a_2 \neq 0 \\ or & \\ a_3(x) u^3 + \sum_{k \geq 4} a_k(x) u^k & \langle a_3 \rangle := (1/\pi) \int_0^\pi a_3(x) dx \neq 0. \end{cases}$$

Then, $s > 1/2$ being given, there exist $\delta_0 > 0$, $\overline{\sigma} > 0$ and a C^∞-curve $[0,\delta_0) \ni \delta \to u(\delta) \in X_{\overline{\sigma}/2,s}$ with the following properties:

- *(i)* $\big\| u(\delta) - \delta \overline{v} \big\|_{\overline{\sigma}/2,s} = O(\delta^2)$ *for some $\overline{v} \in V \cap X_{\overline{\sigma},s}$, $\overline{v} \neq \{0\}$;*
- *(ii) There exists a Cantor set $\mathcal{C} \subset [0,\delta_0)$ of asymptotically full measure, i.e. satisfying*

$$\lim_{\eta \to 0^+} \frac{\text{meas}(\mathcal{C} \cap (0,\eta))}{\eta} = 1\,, \qquad (4)$$

such that, $\forall\ \delta \in \mathcal{C}$, $u(\delta)$ is a 2π-periodic, even in time, classical solution of (3) with respectively

$$\omega = \omega(\delta) = \begin{cases} \sqrt{1 - 2\delta^2} \\ or \\ \sqrt{1 + 2\delta^2 \text{sign}\langle a_3 \rangle}. \end{cases}$$

As a consequence, $\forall \delta \in \mathcal{C}$, $\widetilde{u}(\delta)(t,x) := u(\delta)(\omega(\delta)t, x)$ is a $2\pi/\omega(\delta)$-periodic, even in time, classical solution of equation (1).

By (4) also the Cantor-like set $\{\omega(\delta) \mid \delta \in \mathcal{C}\}$ has asymptotically full measure at $\omega = 1$.

Remark 1.2 *The same conclusions of Theorem 1.1 hold true also for $f(x,u) = a_4 u^4 + O(u^8)$ with $\omega^2 = 1 - 2\delta^6$. This was recently proved in [4] as a further application of the techniques of the present paper, see remark 1.5.*

Theorem 1.1 is related to Theorem 1.2 stated in the next subsection.

Remark 1.3 *Under the hypotheses of Theorem 1.1 we could also get multiplicity of periodic solutions as a consequence of Theorem 1.2 and Lemmas 6.1 and 6.3. More precisely, there exist $n_0 \in \mathbf{N}$ and a Cantor-like set $\mathcal{C}$ of asymptotically full measure, such that $\forall \delta \in \mathcal{C}$, equation (1) has a $2\pi/(n\omega(\delta))$-periodic solution u_n for any $n_0 \leq n \leq N(\delta)$ with $\lim_{\delta \to 0} N(\delta) = \infty$ (u_n is in particular $2\pi/\omega(\delta)$-periodic). This can be seen as an analogue for (1) of the well known multiplicity results of Weinstein-Moser [27]-[22] and Fadell-Rabinowitz [14] which hold in finite dimension. Multiplicity of solutions of (1) was also obtained in [6], but only for the zero measure set of "strongly non-resonant" frequencies $\mathcal{W}_\gamma$.*

1.2 The Lyapunov-Schmidt reduction

Instead of looking for solutions of (3) in a shrinking neighborhood of 0 it is a convenient devise to perform the rescaling

$$u \to \delta u\,, \qquad \delta > 0$$

obtaining

$$\begin{cases} \omega^2 u_{tt} - u_{xx} + \delta^{p-1} g_\delta(x,u) = 0 \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{5}$$

where

$$g_\delta(x,u) := \frac{f(x,\delta u)}{\delta^p} = a_p(x)u^p + \delta a_{p+1}(x)u^{p+1} + \dots\,.$$

To find solutions of (5) we try to implement the Lyapunov-Schmidt reduction according to the orthogonal decomposition

$$X_{\sigma,s} = (V \cap X_{\sigma,s}) \oplus (W \cap X_{\sigma,s})$$

where

$$W := \Big\{ w = \sum_{l\in\mathbf{Z}} \exp(\mathrm{i}lt)\ w_l(x) \in X_{0,s} \mid w_{-l} = w_l \ \text{ and } \int_0^\pi w_l(x)\sin(lx)\,dx = 0,\ \forall l \in \mathbf{Z} \Big\} \tag{6}$$

(the lth time-Fourier coefficient $w_l(x)$ must be orthogonal to $\sin(lx)$).

Looking for solutions $u = v + w$ with $v \in V$, $w \in W$, we are led to solve the bifurcation equation (called the (Q)-equation) and the range equation (called the (P)-equation)

$$\begin{cases} -\dfrac{(\omega^2-1)}{2}\Delta v = \delta^{p-1}\Pi_V g_\delta(x, v+w) & (Q) \\ L_\omega w = \delta^{p-1}\Pi_W g_\delta(x, v+w) & (P) \end{cases} \tag{7}$$

where

$$\Delta v := v_{xx} + v_{tt}, \qquad\qquad L_\omega := -\omega^2\partial_{tt} + \partial_{xx}$$

and $\Pi_V : X_{\sigma,s} \to V$, $\Pi_W : X_{\sigma,s} \to W$ denote the projectors respectively on V and W.

1.2.1 The 0th order bifurcation equation

In order to find non-trivial solutions of (7) we impose a suitable relation between the frequency ω and the amplitude δ (ω must tend to 1 as $\delta \to 0$).

The simplest situation occurs when

$$\Pi_V(a_p(x)v^p) \not\equiv 0\,. \tag{8}$$

Assumption (8) amounts to require that

$$\exists v \in V \quad \text{such that} \quad \int_\Omega a_p(x)v^{p+1}(t,x)\,dtdx \neq 0, \qquad \Omega := \mathbf{T} \times (0,\pi) \tag{9}$$

which is verified iff $a_p(\pi - x) \not\equiv (-1)^p a_p(x)$, see Lemma 7.1 in the Appendix.

When condition (8) (equivalently (9)) holds, we set the "frequency-amplitude" relation

$$\frac{\omega^2-1}{2} = \varepsilon\,, \qquad |\varepsilon| := \delta^{p-1}\,,$$

so that system (7) becomes

$$\begin{cases} -\Delta v = \Pi_V g(\delta, x, v+w) & (Q) \\ L_\omega w = \varepsilon \Pi_W g(\delta, x, v+w) & (P) \end{cases} \tag{10}$$

where

$$g(\delta, x, u) := s^* g_\delta(x, u) = s^* \Big(a_p(x) u^p + \delta a_{p+1}(x) u^{p+1} + \dots \Big) \qquad \text{and} \qquad s^* := \text{sign}(\varepsilon) \, .$$

When $\delta = 0$ (and hence $\varepsilon = 0$), system (10) reduces to $w = 0$ and the "0th-order bifurcation equation"

$$-\Delta v = s^* \Pi_V (a_p(x) v^p) \tag{11}$$

which is the Euler-Lagrange equation of the functional $\Phi_0 : V \to \mathbf{R}$

$$\Phi_0(v) = \frac{\|v\|_{H^1}^2}{2} - s^* \int_\Omega a_p(x) \frac{v^{p+1}}{p+1} \, dx dt \tag{12}$$

where $\|v\|_{H^1}^2 := \int_\Omega v_t^2 + v_x^2 \, dx \, dt$.

By the Mountain-pass theorem [1], taking

$$s^* := \begin{cases} 1 & \text{i.e. } \varepsilon > 0, \, \omega > 1 \, , \quad \text{if } \exists \, v \in V \text{ such that } \int_\Omega a_p(x) v^{p+1} > 0 \\ -1 & \text{i.e. } \varepsilon < 0, \, \omega < 1 \, , \quad \text{if } \exists \, v \in V \text{ such that } \int_\Omega a_p(x) v^{p+1} < 0 \end{cases} \tag{13}$$

there exists at least one nontrivial critical point of Φ_0, i.e. a solution of (11).

We shall say that a solution $\overline{v} \in V$ of equation (11) is non degenerate if 0 is the only solution of the linearized equation at $\overline{v}$, i.e. $\ker \Phi_0''(\overline{v}) = \{0\}$.

If condition (8) is violated (as for $f(x, u) = a_2 u^2$) the right hand side of equation (11) vanishes. In this case the correct $0th$-order non-trivial bifurcation equation will involve higher order nonlinear terms and another "frequency-amplitude" relation is required, see subsection 1.2.3.

For the sake of clarity we shall develop all the details when the $0th$ order bifurcation equation is (11). In subsection 6.2 we shall describe the changes for dealing with other cases.

We can also look for $2\pi/n$-time-periodic solutions of the 0th order bifurcation equation (11) (they are particular 2π-periodic solutions). Let

$$\begin{aligned} V_n \; &:= \; \Big\{ v \in V \mid v \; \text{ is } 2\pi/n \text{ periodic in time} \Big\} \\ &= \; \Big\{ v(t, x) = \eta(nt + nx) - \eta(nt - nx) \mid \eta \in H^1(\mathbf{T}, \mathbf{R}) \; \text{ with } \; \eta \text{ odd} \Big\} \, . \end{aligned} \tag{14}$$

If $v \in V_n$ then $\Pi_V(a_p(x) v^p) \in V_n$, and the critical points of $\Phi_{0|V_n}$ are the solutions of equation (11) which are $2\pi/n$ periodic. Also $\Phi_{0|V_n}$ possesses a Mountain pass critical point for any n, see [6].

We shall say that a solution $\overline{v} \in V_n$ of (11) is non degenerate in V_n if 0 is the only solution in V_n of the linearized equation at $\overline{v}$, i.e. $\ker \Phi''_{0|V_n}(\overline{v}) = \{0\}$.

Theorem 1.2 *Let f satisfy (8) and* **(H)**. *Assume that $\overline{v} \in V_n$ is a non trivial solution of the $0th$ order bifurcation equation* (11) *which is non degenerate in V_n.*

Then the conclusions of Theorem 1.1 hold with $\omega = \omega(\delta) = \sqrt{1 + 2 s^* \delta^{p-1}}$.

1.2.2 About the proof of Theorem 1.2

Sections 2-5 are devoted to the proof of Theorem 1.2. Without genuine loss of generality, the proof is carried out for $n = 1$, and we shall explain why it works for $n > 1$ as well at the end of section 5.

The natural way to deal with (10) is to solve first the (P)-equation (for example through a Nash-Moser procedure) and then to insert the solution $w(\delta, v)$ in the (Q)-equation. However, since here V is infinite dimensional a serious difficulty arises: if $v \in V \cap X_{\sigma_0, s}$ then the solution $w(\delta, v)$ of the range equation, obtained with any Nash-Moser iteration scheme will have a lower regularity, e.g. $w(\delta, v) \in X_{\sigma_0/2, s}$. Therefore in solving next the bifurcation equation for $v \in V$, the best estimate we can obtain is $v \in V \cap X_{\sigma_0/2, s+2}$, which makes the scheme incoherent. Moreover we have to ensure that the $0th$-order bifurcation equation (11) has solutions $v \in V$ which are analytic, a necessary property to initiate an analytic Nash-Moser scheme (in [12]-[13] these problems do not arise since the bifurcation equation is finite dimensional).

We overcome these difficulties thanks to a reduction to a *finite dimensional* bifurcation equation on a subspace of V of dimension N independent of ω. This reduction can be implemented, in spite of the complete resonance of equation (1), thanks to the compactness of the operator $(-\Delta)^{-1}$.

We introduce the decomposition $V = V_1 \oplus V_2$ where

$$\begin{cases} V_1 := \Big\{ v \in V \mid v(t,x) = \sum_{l=1}^{N} 2\cos(lt) u_l \sin(lx)\,,\ u_l \in \mathbf{R} \Big\} \\ V_2 := \Big\{ v \in V \mid v(t,x) = \sum_{l \geq N+1} 2\cos(lt) u_l \sin(lx)\,,\ u_l \in \mathbf{R} \Big\}\,. \end{cases}$$

Setting $v := v_1 + v_2$, with $v_1 \in V_1, v_2 \in V_2$, system (10) is equivalent to

$$\begin{cases} -\Delta v_1 = \Pi_{V_1} g(\delta, x, v_1 + v_2 + w) & (Q1) \\ -\Delta v_2 = \Pi_{V_2} g(\delta, x, v_1 + v_2 + w) & (Q2) \\ L_\omega w = \varepsilon \Pi_W g(\delta, x, v_1 + v_2 + w) & (P) \end{cases} \tag{15}$$

where $\Pi_{V_i} : X_{\sigma,s} \to V_i$ $(i = 1, 2)$, denote the orthogonal projectors on V_i $(i = 1, 2)$.

Our strategy to find solutions of system (15) - and hence to prove Theorem 1.2- is the following.

Solution of the $(Q2)$ equation. We solve first the $(Q2)$-equation obtaining $v_2 = v_2(\delta, v_1, w) \in V_2 \cap X_{\sigma,s+2}$ when $w \in W \cap X_{\sigma,s}$, by the Contraction Mapping Theorem provided we have chosen N large enough and $0 < \sigma \leq \overline{\sigma}$ small enough, depending on the nonlinearity f but *independent of* δ, see section 2.

Solution of the (P) equation. Next we solve the (P)-equation, obtaining $w = w(\delta, v_1) \in W \cap X_{\overline{\sigma}/2,s}$ by means of a Nash-Moser type Implicit Function Theorem for (δ, v_1) belonging to some Cantor-like set B_∞ of parameters, see Theorem 3.1 in section 3.

Our approach for the inversion of the *linearized operators* at any step of the Nash-Moser iteration is different from the Craig-Wayne-Bourgain method. We develop $u(t,\cdot) \in H_0^1((0,\pi), \mathbf{R})$ in time-Fourier expansion only and we distinguish the "diagonal part" $D = \mathrm{diag}\{D_k\}_{k \in \mathbf{Z}}$ of the operator that we want to invert. Next, using Sturm-Liouville theory (see Lemma 4.1), we diagonalize each D_k in a suitable basis of $H_0^1((0,\pi), \mathbf{R})$ (close to, but different from $(\sin jx)_{j \geq 1}$). Assuming a "first order Melnikov non resonance condition" (Definition 3.3) we prove that its eigenvalues are polynomially bounded away from 0 and so we invert D with sufficiently good estimates (Corollary 4.2). The presence of the "off-diagonal" Toepliz operators requires to analyze the "small divisors": for our method it is sufficient to prove that the product of two "small divisors" is larger than a constant if the corresponding "singular sites" are close enough, see Lemma 4.5. This holds true if the Diophantine exponent $\tau \in (1,2)$ by the lower bound of Lemma 4.3. Moreover, for $\tau \in (1,2)$ the non-resonance Diophantine conditions are particularly simple, see Definition 3.3 and the Cantor set B_∞ in Theorem 3.1. This restriction for the values of the exponent τ simplifies also the proof of Lemma 4.9 where the loss of derivatives due to the small divisors is compensated by the regularizing property of the map v_2.

Solution of the $(Q1)$-equation. Finally, in section 5 we consider the finite dimensional $(Q1)$-equation.

We could define a smooth functional $\Psi : [0, \delta_0) \times V_1 \to \mathbf{R}$ such that any critical point $v_1 \in V_1$ of $\Psi(\delta, \cdot)$ with $(\delta, v_1) \in B_\infty$ ($\equiv$ the Cantor-like set of parameters for which the (P)-equation is solved exactly) gives rise to an exact solution of (3), see [5]. Moreover it would be possible to prove the existence of a critical point $v_1(\delta)$ of $\Psi(\delta, \cdot)$, $\forall \delta > 0$ small enough, using the Mountain pass theorem [1].

However, since the section $E_\delta := \{v_1 \mid (\delta, v_1) \in B_\infty\}$ has "gaps" (except for δ in a zero measure set, see remark 1.4), the difficulty is to prove that $(\delta, v_1(\delta)) \in B_\infty$ for a large set of δ's. Although B_∞ is in some sense a "large" set, this property is not obvious. In this paper, we prove that it holds at least if the path $(\delta \mapsto v_1(\delta))$ is C^1 (Proposition 3.2) and so intersects "transversally" the Cantor set B_∞.

This is why we require in Theorem 1.2 non-degenerate solutions of the 0th-order bifurcation equation (11). This condition enables to use the Implicit function theorem, yielding a smooth path $(\delta \to v_1(\delta))$ of solutions of the $(Q1)$-equation.

Remark 1.4 *The section E_δ has "no gaps" iff the frequency $\omega(\delta) = \sqrt{1 + 2s^* \delta^{p-1}}$ belongs to the uncountable zero-measure set $\mathcal{W}_\gamma := \{|\omega l - j| \geq \gamma/l,\ \forall j \neq l, l \geq 0, j \geq 1\}$ of [2]. This explains why in [5]-[6] we had been able to prove the existence of periodic solutions for* ANY *nonlinearity f, solving the bifurcation equation with variational methods.*

We lay the stress on the fact that the parts on the $(Q2)$ and (P)-equations do not use the non-degeneracy condition. We hope that we will be able to improve our results relaxing the non-degeneracy condition in a subsequent work, using the variational formulation of the $(Q1)$ equation and results on properties of critical sets for parameter depending functionals.

1.2.3 About the proof of Theorem 1.1

To deduce Theorem 1.1 when $f(x,u) = a_3(x)u^3 + O(u^4)$ and $\langle a_3 \rangle \neq 0$ we just have to prove that the 0th order bifurcation equation[4]

$$-\Delta v = s^* \Pi_V (a_3(x) v^3) \tag{16}$$

possesses, at least for n large, a non-degenerate solution in V_n. Choosing $s^* \in \{-1, 1\}$ so that $s^* \langle a_3 \rangle > 0$ this is proved in Lemma 6.1.

In the case $f(x,u) = a_2 u^2 + O(u^4)$, condition (8) is violated because $\Pi_V v^2 \equiv 0$, and we have to use a development in δ of higher order, as in [5]. Imposing in (7) the frequency-amplitude relation

$$\frac{\omega^2 - 1}{2} = -\delta^2 \tag{17}$$

the correct 0th-order bifurcation equation turns out to be (see subsection 6.2)

$$-\Delta v + 2a_2^2 \Pi_V (v L^{-1}(v^2)) = 0 \tag{18}$$

where $L^{-1} : W \to W$ is the inverse operator of $-\partial_{tt} + \partial_{xx}$. (18) is the Euler-Lagrange equation of

$$\Phi_0(v) = \frac{\|v\|_{H^1}^2}{2} + \frac{a_2^2}{2} \int_\Omega v^2 L^{-1} v^2 \tag{19}$$

which again possesses Mountain pass critical points because $\int_\Omega v^2 L^{-1} v^2 < 0$, $\forall v \in V$, see [5].

The existence of a *non-degenerate* critical point of $(\Phi_0)_{|V_n}$ for n large enough is proved in Lemma 6.3. This implies, as in Theorem 1.2, the conclusions of Theorem 1.1.

Remark 1.5 *Also when $f(x,u) = a_4 u^4 + O(u^8)$ condition (8) is violated because $\Pi_V v^4 \equiv 0$. Imposing the frequency-amplitude relation $\omega^2 - 1 = -2\delta^6$, the correct 0th-order bifurcation equation turns out to be*

$$-\Delta v + 4a_4^2 \Pi_V (v^3 L^{-1}(v^4)) = 0\,. \tag{20}$$

The existence of a solution of (20) which is non-degenerate in V_n for n large enough is proved in [4] . This implies the conclusions of Theorem 1.1.

Acknowledgments: Part of this paper was written when the second author was visiting SISSA in Trieste.

2 Solution of the $(Q2)$-equation

The main assumption of Theorem 1.2 says that at least one of the critical points of Φ_0 defined in (12) or of the restriction of Φ_0 to some V_n, called $\overline{v}$, is non-degenerate. For definiteness, we shall assume that $\overline{v}$ is non-degenerate in the whole space V.

[4]Note that $\langle a_3 \rangle \neq 0$ implies condition (9) because $a_3(\pi - x) \not\equiv -a_3(x)$ and so $\Pi_V(a_3(x)v^3) \not\equiv 0$.

By the regularizing property

$$(-\Delta)^{-1} : V \cap H^k(\Omega) \to V \cap H^{k+2}(\Omega)\,, \qquad \forall k \geq 0\,,$$

and a direct bootstrap argument, $\overline{v} \in H^k(\Omega)\ \forall k \geq 0$. Therefore[5] $\overline{v} \in V \cap C^\infty(\Omega)$.

In the sequel of the paper $s > 1/2$ is fixed once for all. We also fix some $R > 0$ such that

$$\|\overline{v}\|_{0,s} < R\,. \tag{21}$$

By the analyticity assumption (**H**) on the nonlinearity f and the Banach algebra property of $X_{\sigma,s}$, there is a constant $K_0 > 0$ such that

$$\begin{aligned} \Big\| g(\delta, x, u) \Big\|_{\sigma,s} &= \Big\| \sum_{k\geq p} a_k(x)\delta^{k-p}u^k \Big\|_{\sigma,s} \leq \sum_{k\geq p} \|a_k\|_{H^1}\delta^{k-p}K_0^{k-1}\|u\|^k_{\sigma,s} \\ &\leq C\|u\|^p_{\sigma,s} \sum_{k\geq p} \|a_k\|_{H^1}(\delta K_0\|u\|_{\sigma,s})^{k-p} \leq C'\|u\|^p_{\sigma,s} \end{aligned} \tag{22}$$

in the open domain $\mathcal{U}_\delta := \{u \in X_{\sigma,s} \mid \delta K_0\, \|u\|_{\sigma,s} < \rho\}$ because the power series $\sum_{k\geq p} \|a_k\|_{H^1}\rho^{k-p} < +\infty$ by (**H**). The Nemitsky operator

$$X_{\sigma,s} \ni u \to g(\delta, x, u) \in X_{\sigma,s} \quad \text{is in} \quad C^\infty(\mathcal{U}_\delta, X_{\sigma,s})\,.$$

We specify that all the norms $\|\ \|_{\sigma,s}$ are equivalent on V_1. In the sequel

$$B(\rho, V_1) := \{v_1 \in V_1 \mid \|v_1\|_{0,s} \leq \rho\}.$$

The fact that $\overline{v} \in V \cap X_{\sigma,s}$ for some $\sigma > 0$ is a consequence of the following Lemma.

Lemma 2.1 (Solution of the $(Q2)$-equation) *There exists $N \in \mathbf{N}_+$, $\overline{\sigma} := \ln 2/N > 0$, $\delta_0 > 0$ such that:*

a) *$\forall 0 \leq \sigma \leq \overline{\sigma}$, $\forall \|v_1\|_{0,s} \leq 2R$, $\forall \|w\|_{\sigma,s} \leq 1$, $\forall \delta \in [0, \delta_0)$, there exists a unique $v_2 = v_2(\delta, v_1, w) \in V_2 \cap X_{\sigma,s}$ with $\|v_2(\delta, v_1, w)\|_{\sigma,s} \leq 1$ which solves the $(Q2)$-equation.*

b) *$v_2(0, \Pi_{V_1}\overline{v}, 0) = \Pi_{V_2}\overline{v}$.*

c) *$v_2(\delta, v_1, w) \in X_{\sigma,s+2}$, the function[6] $v_2(\cdot,\cdot,\cdot) \in C^\infty\Big([0, \delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s}), V_2 \cap X_{\sigma,s+2}\Big)$ and $D^k v_2$ is bounded on $[0, \delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s})$ for any $k \in \mathbf{N}$.*

d) *If in addition $\|w\|_{\sigma,s'} < +\infty$ for some $s' \geq s$, then (provided δ_0 has been chosen small enough) $\|v_2(\delta, v_1, w)\|_{\sigma,s'+2} \leq K(s', \|w\|_{\sigma,s'})$.*

PROOF. Fixed points of the nonlinear operator $\mathcal{N}(\delta, v_1, w, \cdot) : V_2 \to V_2$ defined by

$$\mathcal{N}(\delta, v_1, w, v_2) := (-\Delta)^{-1}\Pi_{V_2} g(\delta, x, v_1 + w + v_2)$$

are solutions of equation $(Q2)$. For $w \in W \cap X_{\sigma,s}$, $v_2 \in V_2 \cap X_{\sigma,s}$ we have $\mathcal{N}(\delta, v_1, w, v_2) \in V_2 \cap X_{\sigma,s+2}$ since $g(\delta, x, v_1 + w + v_2) \in X_{\sigma,s}$ and because of the *regularizing property* of the operator $(-\Delta)^{-1}\Pi_{V_2} : X_{\sigma,s} \to V_2 \cap X_{\sigma,s+2}$.

Proof of a). Let $B := \{v_2 \in V_2 \cap X_{\sigma,s} \mid \|v_2\|_{\sigma,s} \leq 1\}$. We claim that there exists $N \in \mathbf{N}$, $\overline{\sigma} > 0$ and $\delta_0 > 0$, such that $\forall 0 \leq \sigma \leq \overline{\sigma}$, $\|v_1\|_{0,s} \leq 2R$, $\|w\|_{\sigma,s} \leq 1$, $\delta \in [0, \delta_0)$ the operator $v_2 \to \mathcal{N}(\delta, v_1, w, v_2)$ is a contraction in B, more precisely

- *(i)* $\|v_2\|_{\sigma,s} \leq 1 \Rightarrow \|\mathcal{N}(\delta, v_1, w, v_2)\|_{\sigma,s} \leq 1$;
- *(ii)* $v_2, \widetilde{v}_2 \in B \Rightarrow \|\mathcal{N}(\delta, v_1, w, v_2) - \mathcal{N}(\delta, v_1, w, \widetilde{v}_2)\|_{\sigma,s} \leq (1/2)\|v_2 - \widetilde{v}_2\|_{\sigma,s}$.

[5] Even if $a_p(x) \in H^1((0,\pi), \mathbf{R})$ only, because the projection Π_V has a regularizing effect in the variable x.

[6] "$l \in C^\infty(A, Y)$" means, if A is not open, that there is an open neighborhood U of A and an extension $\widetilde{l} \in C^\infty(U, Y)$ of l.

Let us prove (i). $\forall u \in X_{\sigma,s}$, $\|(-\Delta)^{-1}\Pi_{V_2} u\|_{\sigma,s} \le (C/(N+1)^2)\|u\|_{\sigma,s}$ and so, $\forall \|w\|_{\sigma,s} \le 1$, $\|v_1\|_{0,s} \le 2R$, $\delta \in [0,\delta_0)$, using (22),

$$\begin{aligned}\|\mathcal{N}(\delta,v_1,w,v_2)\|_{\sigma,s} &\le \frac{C}{(N+1)^2}\Big\|g(\delta,x,v_1+v_2+w)\Big\|_{\sigma,s} \le \frac{C'}{(N+1)^2}\Big(\|v_1\|^p_{\sigma,s}+\|v_2\|^p_{\sigma,s}+\|w\|^p_{\sigma,s}\Big)\\ &\le \frac{C'}{(N+1)^2}\Big(\exp(\sigma pN)\|v_1\|^p_{0,s}+\|v_2\|^p_{\sigma,s}+1\Big) \le \frac{C'}{(N+1)^2}\Big((4R)^p+\|v_2\|^p_{\sigma,s}+1\Big)\end{aligned}$$

for $\exp(\sigma N) \le 2$, where we have used that $\|v_1\|_{\sigma,s} \le \exp(\sigma N)\|v_1\|_{0,s} \le 4R$.

For N large enough (depending on R) we get

$$\|v_2\|_{\sigma,s} \le 1 \quad \Rightarrow \quad \Big\|\mathcal{N}(\delta,v_1,w,v_2)\Big\|_{\sigma,s} \le \frac{C'}{(N+1)^2}\Big((4R)^p+1+1\Big) \le 1$$

and (i) follows, taking $\overline{\sigma} := \ln 2/N$. Property (ii) can be proved similarly and the existence of a unique solution $v_2(\delta,v_1,w) \in B$ follows by the Contraction Mapping Theorem.

Proof of b). We may assume that N has been chosen so large that $\|\Pi_{V_2}\overline{v}\|_{0,s} \le 1/2$. Since $\overline{v}$ solves equation (11), $\Pi_{V_2}\overline{v}$ solves the $(Q2)$-equation associated with $(\delta,v_1,w) = (0,\Pi_{V_1}\overline{v},0)$. Since $\Pi_{V_2}\overline{v} = \mathcal{N}(0,\Pi_{V_1}\overline{v},0,\Pi_{V_2}\overline{v})$ and $\Pi_{V_2}\overline{v} \in B$, we deduce $\Pi_{V_2}\overline{v} = v_2(0,\Pi_{V_1}\overline{v},0)$.

Proof of c). As a consequence of (ii) the linear operator $I - D_{v_2}\mathcal{N}$ is invertible at the fixed point of $\mathcal{N}(\delta,v_1,w,\cdot)$. Since the map $(\delta,v_1,w,v_2) \mapsto \mathcal{N}(\delta,v_1,w,v_2)$ is C^∞, by the Implicit function Theorem $v_2 : \{(\delta,v_1,w) \mid \delta \in [0,\delta_0), \|v_1\|_{0,s} \le 2R, \|w\|_{\sigma,s} \le 1\} \to V_2 \cap X_{\sigma,s}$ is a C^∞ map. Hence, since $(-\Delta)^{-1}\Pi_{V_2}$ is a continuous linear operator from $X_{\sigma,s}$ to $V_2 \cap X_{\sigma,s+2}$ and

$$v_2(\delta,v_1,w) = (-\Delta)^{-1}\Pi_{V_2}\Big(g(\delta,x,v_1+w+v_2(\delta,v_1,w)\Big), \tag{23}$$

by the regularity of the Nemitsky operator induced by g, $v_2(\cdot,\cdot,\cdot) \in C^\infty([0,\delta_0) \times B(2R;V_1) \times B(1;W \cap X_{\sigma,s}), V_2 \cap X_{\sigma,s+2})$. The estimates for the derivatives can be obtained similarly.

Proof of d). Let us firstly prove the following : if $\delta\|u\|_{\sigma,s}$ is small enough then

$$u \in X_{\sigma,r} \quad \Rightarrow \quad g(\delta,x,u) \in X_{\sigma,r}, \quad \forall r \ge s\,. \tag{24}$$

We first observe that, since $r \ge s > 1/2$, for $u,v \in H^r(\mathbf{R}/2\pi\mathbf{Z})$, we have $\|uv\|_{H^r} \le C_r(\|u\|_\infty\|v\|_{H^r} + \|v\|_\infty\|u\|_{H^r})$. This is a consequence of the Gagliardo-Nirenberg inequalities. Hence there is a positive constant K_r such that

$$\|u^l\|_{H^r} \le K_r^{l-1}\|u\|^{l-1}_\infty\|u\|_{H^r} \le K_r^{l-1}\|u\|^{l-1}_{H^s}\|u\|_{H^r}\,, \qquad \forall u \in H^r(\mathbf{R}/2\pi\mathbf{Z})\,, \forall l \ge 1\,.$$

Considering the extension of a function $u \in X_{\sigma,r}$ to the complex strip of width σ and using that $H^1_0(0,\pi)$ is a Banach algebra, we can derive that $\forall r \ge s$, $\|u^l\|_{\sigma,r} \le K_r^{l-1}\|u\|^{l-1}_{\sigma,s}\|u\|_{\sigma,r}$. Therefore

$$\begin{aligned}\Big\|g(\delta,x,u)\Big\|_{\sigma,r} &= \Big\|\sum_{k\ge p} a_k(x)\delta^{k-p}u^k\Big\|_{\sigma,r} \le \|u\|^p_{\sigma,r}\sum_{k\ge p}\|a_k\|_{H^1}\Big\|(\delta u)^{k-p}\Big\|_{\sigma,r}\\ &\le \|u\|^p_{\sigma,r}\Big[\|a_p\|_{H^1} + \sum_{k>p}\|a_k\|_{H^1}C^{k-p}\Big(\delta\|u\|_{\sigma,s}\Big)^{k-p-1}(\delta\|u\|_{\sigma,r})\Big] < +\infty\end{aligned}$$

for $\delta\|u\|_{\sigma,s}$ small enough.

Now, assume that $\|w\|_{\sigma,s'} < +\infty$ for some $s' \ge s$. Since $v_2(\delta,v_1,w) \in X_{\sigma,s}$ solves equation (23), by a direct bootstrap argument using the regularizing properties of $(-\Delta)^{-1}\Pi_{V_2} : X_{\sigma,r} \to V_2 \cap X_{\sigma,r+2}$ and that $\|v_1\|_{\sigma,r} < +\infty$, $\forall r \ge s$, we derive that $v_2(\delta,v_1,w) \in X_{\sigma,s'+2}$ and $\|v_2(\delta,v_1,w)\|_{\sigma,s'+2} \le K(s',\|w\|_{\sigma,s'})$. ■

Remark 2.1 *Lemma 2.1 implies, in particular, that the solution $\overline{v}$ of the 0th-order bifurcation equation (11) is not only in $V \cap C^\infty(\Omega)$ but actually belongs to $V \cap X_{\overline{\sigma},s+2}$ and therefore is analytic in t and hence in x.*

We stress that we shall consider as *fixed* the constants N and $\overline{\sigma}$ obtained in Lemma 2.1, which depend only on the nonlinearity f and on $\overline{v}$. On the contrary, we shall allow δ_0 to decrease in the next sections.

3 Solution of the (P)-equation

By the previous section we are reduced to solve the (P)-equation with $v_2 = v_2(\delta, v_1, w)$, namely

$$L_\omega w = \varepsilon \Pi_W \Gamma(\delta, v_1, w) \tag{25}$$

where

$$\Gamma(\delta, v_1, w)(t,x) := g\Big(\delta, x, v_1(t,x) + w(t,x) + v_2(\delta, v_1, w)(t,x)\Big). \tag{26}$$

The solution $w = w(\delta, v_1)$ of the (P)-equation (25) will be obtained by means of a Nash-Moser Implicit Function Theorem for (δ, v_1) belonging to a Cantor-like set of parameters.

We consider the orthogonal splitting $W = W^{(n)} \oplus W^{(n)\perp}$ where

$$W^{(n)} = \Big\{ w \in W \ \Big| \ w = \sum_{|l| \le L_n} \exp(\mathrm{i}lt) \ w_l(x) \Big\}, \quad W^{(n)\perp} = \Big\{ w \in W \ \Big| \ w = \sum_{|l| > L_n} \exp(\mathrm{i}lt) \ w_l(x) \Big\} \tag{27}$$

and L_n are integer numbers (we will choose $L_n = L_0 2^n$ with $L_0 \in \mathbf{N}$ large enough). We denote by

$$P_n : W \to W^{(n)} \qquad \text{and} \qquad P_n^\perp : W \to W^{(n)\perp}$$

the orthogonal projectors onto $W^{(n)}$ and $W^{(n)\perp}$.

The convergence of the recursive scheme is based on properties $(P1)$-$(P2)$-$(P3)$ below.

- **(P1)** (**Regularity**) $\Gamma(\cdot,\cdot,\cdot) \in C^\infty\Big([0,\delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s}), X_{\sigma,s}\Big)$. Moreover, $D^k\Gamma$ is bounded on $[0,\delta_0) \times B(2R, V_1) \times B(1; W \cap X_{\sigma,s})$ for any $k \in \mathbf{N}$.

$(P1)$ is a consequence of the C^∞-regularity of the Nemitsky operator induced by $g(\delta, x, u)$ on $X_{\sigma,s}$ and of the C^∞-regularity of the map $v_2(\cdot,\cdot,\cdot)$ proved in Lemma 2.1.

- **(P2)** (**Smoothing estimate**) $\forall\, w \in W^{(n)\perp} \cap X_{\sigma,s}$ and $\forall\, 0 \le \sigma' \le \sigma$, $\|w\|_{\sigma',s} \le \exp^{(-L_n(\sigma - \sigma'))} \|w\|_{\sigma,s}$.

The standard property $(P2)$ follows from

$$\begin{aligned} \|w\|^2_{\sigma',s} &= \sum_{|l|>L_n} \exp(2\sigma'|l|)(l^{2s}+1)\|w_l\|^2_{H^1} = \sum_{|l|>L_n} \exp(-2(\sigma-\sigma')|l|)\exp(2\sigma|l|)(l^{2s}+1)\|w_l\|^2_{H^1} \\ &\le \exp(-2(\sigma-\sigma')L_n)\|w\|^2_{\sigma,s}\,. \end{aligned}$$

The next property $(P3)$ is an *invertibility property* of the linearized operator $\mathcal{L}_n(\delta, v_1, w) : D(\mathcal{L}_n) \subset W^{(n)} \to W^{(n)}$ defined by

$$\mathcal{L}_n(\delta, v_1, w)[h] := L_\omega h - \varepsilon P_n \Pi_W D_w \Gamma(\delta, v_1, w)[h]\,. \tag{28}$$

Throughout the proof, w will be the approximate solution obtained at a given step of the Nash-Moser iteration.

The invertibility of $\mathcal{L}_n(\delta, v_1, w)$ is obtained excising the set of parameters (δ, v_1) for which 0 is an eigenvalue of $\mathcal{L}_n(\delta, v_1, w)$. Moreover, in order to have bounds for the norm of the inverse operator $\mathcal{L}_n^{-1}(\delta, v_1, w)$ which are sufficiently good for the recursive scheme, we also excise the parameters (δ, v_1) for which the eigenvalues of $\mathcal{L}_n(\delta, v_1, w)$ are too small.

We prefix some definitions.

Definition 3.1 (Mean value) *For $\Omega := \mathbf{T} \times (0,\pi)$ we define*

$$M(\delta, v_1, w) := \frac{1}{|\Omega|}\int_\Omega \partial_u g\Big(\delta, x, v_1(t,x) + w(t,x) + v_2(\delta, v_1, w)(t,x)\Big)\, dxdt.$$

Note that $M(\cdot,\cdot,\cdot) : [0,\delta_0) \times B(2R; V_1) \times B(1; W \cap X_{\sigma,s}) \to \mathbf{R}$ is a C^∞-function.

Definition 3.2 *We define for* $1 < \tau < 2$

$$[w]_{\sigma,s} := \inf\Big\{ \sum_{i=0}^{q} \frac{\|h_i\|_{\sigma_i,s}}{(\sigma_i - \sigma)^{\frac{2(\tau-1)}{\beta}}} \ ; \ \forall q \geq 1, \ \overline{\sigma} \geq \sigma_i > \sigma, \quad h_i \in W^{(i)}, \ w = \sum_{i=0}^{q} h_i \Big\}$$

where $\beta := \frac{2-\tau}{\tau}$ *and we set* $[w]_{\sigma,s} := \infty$ *if the above set is empty.*

Definition 3.3 (First order Melnikov non-resonance condition) *Let* $0 < \gamma < 1$ *and* $1 < \tau < 2$. *We define (recall that* $\omega = \sqrt{1 + 2s^* \delta^{p-1}}$ *and* $\varepsilon = s^* \delta^{p-1}$*)*

$$\begin{aligned} \Delta_n^{\gamma,\tau}(v_1, w) \quad := \quad & \Big\{ \delta \in [0, \delta_0) \ \Big| \ |\omega l - j| \geq \frac{\gamma}{(l+j)^\tau}, \ \Big|\omega l - j - \varepsilon \frac{M(\delta, v_1, w)}{2j}\Big| \geq \frac{\gamma}{(l+j)^\tau} \\ & \forall l \in \mathbf{N}, \ j \geq 1, \ l \neq j, \ \frac{1}{3|\varepsilon|} < l, \ l \leq L_n, \ j \leq 2L_n \Big\}. \end{aligned}$$

The set $\Delta_n^{\gamma,\tau}(v_1, w)$ contains a whole interval $[0, \eta_n)$ for some $\eta_n > 0$ small enough (note that $\Delta_n^{\gamma,\tau}(v_1, w)$ is defined by a finite set of inequalities).

Remark 3.1 *The intersections of the sets* $\Delta_n^{\gamma,\tau}(v_1, w)$ *over all possible* (v_1, w) *in a neighborhood of* 0 *and over all* n *contains, for* $|\varepsilon|\gamma^{-1}$ *small, the zero measure, uncountable set* $\mathcal{W}_\gamma := \{\omega \in \mathbf{R} | \ |\omega l - j| \geq \gamma/l, \ \forall l \neq j, \ l \geq 0, \ j \geq 1\}$, $0 < \gamma < 1/6$ *introduced in [2]. See remark 1.4 for consequences on the existence of periodic solutions.*

We claim that:

- **(P3) (Invertibility of $\mathcal{L}_n$)** There exist positive constants μ, δ_0 such that, if $[w]_{\sigma,s} \leq \mu$, $\|v_1\|_{0,s} \leq 2R$ and $\delta \in \Delta_n^{\gamma,\tau}(v_1, w) \cap [0, \delta_0)$ for some $0 < \gamma < 1$, $1 < \tau < 2$, then $\mathcal{L}_n(\delta, v_1, w)$ is invertible and the inverse operator $\mathcal{L}_n^{-1}(\delta, v_1, w) : W^{(n)} \to W^{(n)}$ satisfies

$$\Big\| \mathcal{L}_n^{-1}(\delta, v_1, w)[h] \Big\|_{\sigma,s} \leq \frac{C}{\gamma} (L_n)^{\tau-1} \|h\|_{\sigma,s} \tag{29}$$

  for some positive constant $C > 0$.

Property $(P3)$ is the real core of the convergence proof and where the analysis of the small divisors enters into play. Property $(P3)$ is proved in section 4.

3.1 The Nash-Moser scheme

We are going to define recursively a sequence $\{w_n\}_{n\geq 0}$ with $w_n = w_n(\delta, v_1) \in W^{(n)}$, defined on smaller and smaller sets of "non-resonant" parameters (δ, v_1), $A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_1 \subseteq A_0 := \{(\delta, v_1) \mid \delta \in [0, \delta_0), \ \|v_1\|_{0,s} \leq 2R\}$. The sequence $(w_n(\delta, v_1))$ will converge to a solution $w(\delta, v_1)$ of the (P)-equation (25) for $(\delta, v_1) \in A_\infty := \cap_{n\geq 1} A_n$. The main goal of the construction is to show that, at the end of the recurrence, the set of parameters $A_\infty := \cap_{n\geq 1} A_n$ for which we have the solution $w(\delta, v_1)$ remains sufficiently large.

We define inductively the sequence $\{w_n\}_{n\geq 0}$. Define the "loss of analyticity" γ_n by

$$\gamma_n := \frac{\gamma_0}{n^2 + 1}, \qquad \sigma_0 = \overline{\sigma}, \qquad \sigma_{n+1} = \sigma_n - \gamma_n, \qquad \forall \ n \geq 0,$$

where we choose $\gamma_0 > 0$ small such that the "total loss of analyticity"

$$\sum_{n\geq 0} \gamma_n = \sum_{n\geq 0} \frac{\gamma_0}{(n^2 + 1)} \leq \frac{\overline{\sigma}}{2}, \qquad \text{i.e. } \sigma_n \geq \frac{\overline{\sigma}}{2} > 0, \ \forall n.$$

We also assume

$$L_n := L_0 2^n, \qquad \forall \ n \geq 0,$$

for some large integer L_0 specified in the next Proposition.

Proposition 3.1 (Induction) *Let* $A_0 := \{(\delta, v_1) \mid \delta \in [0, \delta_0),\ \|v_1\|_{0,s} \le 2R\}$. $\exists\ L_0 := L_0(\gamma, \tau) > 0$, $\varepsilon_0 := \varepsilon_0(\gamma, \tau) > 0$, *such that for* $\delta_0^{p-1}\gamma^{-1} < \varepsilon_0$, *there exists a sequence* $\{w_n\}_{n\ge 0}$, $w_n = w_n(\delta, v_1) \in W^{(n)}$, *of solutions of the equation*

$$(P_n) \qquad L_\omega w_n - \varepsilon P_n \Pi_W \Gamma(\delta, v_1, w_n) = 0\,,$$

defined inductively for $(\delta, v_1) \in A_n \subseteq A_{n-1} \subseteq \ldots \subseteq A_1 \subseteq A_0$ *where*

$$A_n := \Big\{ (\delta, v_1) \in A_{n-1} \mid \delta \in \Delta_n^{\gamma,\tau}(v_1, w_{n-1}) \Big\} \subseteq A_{n-1}\,, \tag{30}$$

$w_n(\delta, v_1) = \sum_{i=0}^n h_i(\delta, v_1)$, *and* $h_i = h_i(\delta, v_1) \in W^{(i)}$ *satisfy* $\|h_0\|_{\sigma_0,s} \le |\varepsilon| K_0$, $\|h_i\|_{\sigma_i,s} \le |\varepsilon|\gamma^{-1} \exp(-\chi^i)$ $\forall 1 \le i \le n$ *for some* $1 < \chi < 2$ *and some constant* $K_0 > 0$.

We define

$$A_\infty := \cap_{n\ge 0} A_n\,.$$

Remark 3.2 *For a given* (δ, v_1), *the sequence* (w_n) *may be finite because the iterative process stops after* w_{k-1} *if* $\delta \notin \Delta_k^{\gamma,\tau}(v_1, w_{k-1})$, *i.e. if* $(\delta, v_1) \notin A_k$. *However, from this possibly finite sequences, we shall define a* C^∞ *map* $\widetilde{w}(\delta, v_1)$ *on the whole set* A_0 *(Lemma 3.3) and Cantor-like set* B_∞, *such that* $B_\infty \subset A_\infty$ *and* $\forall (\delta, v_1) \in B_\infty$, $\widetilde{w}(\delta, v_1)$ *is an exact solution of the* (P)*-equation. It will be justified in Proposition 3.2 that* B_∞ *is a "large" set. As a consequence also* A_∞ *is "large".*

PROOF. The proof proceeds by induction.

First step: initialization. Let L_0 be given. If $|\omega - 1| L_0 \le 1/2$ then $L_{\omega|W^{(0)}}$ is invertible and $\|L_\omega^{-1} h\|_{\sigma_0,s} \le 2\|h\|_{\sigma_0,s}$, $\forall h \in W^{(0)}$. Indeed the eigenvalues of $L_{\omega|W^{(0)}}$ are $-\omega^2 l^2 + j^2$, $\forall\ 0 \le l \le L_0$, $j \ge 1$, $j \ne l$, and

$$|-\omega^2 l^2 + j^2| = |-\omega l + j|(\omega l + j) \ge \Big(|j - l| - |\omega - 1| L_0\Big)(\omega l + j) \ge \Big(1 - \frac{1}{2}\Big)\,.$$

By the Implicit function theorem, using Property $(P1)$, there exist $K_0 > 0$, $\varepsilon_1 := \varepsilon_1(\gamma, L_0) > 0$ such that if $|\varepsilon|\gamma^{-1} < \varepsilon_1$ and $\forall v_1 \in B(2R, V_1)$, equation (P_0) has a unique solution $w_0(\delta, v_1)$ satisfying

$$\|w_0(\delta, v_1)\|_{\sigma_0,s} \le K_0 |\varepsilon|\,.$$

Moreover, for $\delta_0^{p-1}\gamma^{-1} < \varepsilon_1$, the map $(\delta, v_1) \mapsto w_0(\delta, v_1)$ is in $C^\infty(A_0, W^{(0)})$ and $\|D^k w_0(\delta, v_1)\|_{\sigma_0,s} \le C(k)$.

Second step: iteration. Fix some $\chi \in (1, 2)$. Let $\varepsilon_2 := \varepsilon_2(L_0, \gamma, \tau) \in (0, \varepsilon_1(\gamma, L_0))$ be small enough such that

$$\varepsilon_2 \max(1, e K_0 \gamma) \sum_{i\ge 0} \exp(-\chi^i) \Big(\frac{1 + i^2}{\gamma_0}\Big)^{\frac{2(\tau-1)}{\beta}} < \mu \tag{31}$$

where μ is defined in property $(P3)$ and $\beta := (2 - \tau)/\tau$.

Suppose we have already defined a solution $w_n = w_n(\delta, v_1) \in W^{(n)}$ of equation (P_n) satisfying the properties stated in the Proposition. We want to define

$$w_{n+1} = w_{n+1}(\delta, v_1) := w_n(\delta, v_1) + h_{n+1}(\delta, v_1), \qquad h_{n+1}(\delta, v_1) \in W^{(n+1)}$$

as an *exact* solution of the equation

$$(P_{n+1}) \qquad L_\omega w_{n+1} - \varepsilon P_{n+1} \Pi_W \Gamma(\delta, v_1, w_{n+1}) = 0\,.$$

In order to find a solution $w_{n+1} = w_n + h_{n+1}$ of equation (P_{n+1}) we write, for $h \in W^{(n+1)}$,

$$\begin{aligned} L_\omega(w_n + h) - \varepsilon P_{n+1} \Pi_W \Gamma(\delta, v_1, w_n + h) &= L_\omega w_n - \varepsilon P_{n+1} \Pi_W \Gamma(\delta, v_1, w_n) \\ &+ L_\omega h - \varepsilon P_{n+1} \Pi_W D_w \Gamma(\delta, v_1, w_n)[h] + R(h) \\ &= r_n + \mathcal{L}_{n+1}(\delta, v_1, w_n)[h] + R(h) \end{aligned} \tag{32}$$

where, since w_n solves equation (P_n),

$$\begin{cases} r_n := L_\omega w_n - \varepsilon P_{n+1}\Pi_W \Gamma(\delta, v_1, w_n) = -\varepsilon P_n^\perp P_{n+1}\Pi_W \Gamma(\delta, v_1, w_n) \in W^{(n+1)} \\ R(h) := \ -\varepsilon P_{n+1}\Pi_W \Big(\Gamma(\delta, v_1, w_n + h) - \Gamma(\delta, v_1, w_n) - D_w\Gamma(\delta, v_1, w_n)[h]\Big). \end{cases}$$

The term r_n is "super-exponentially" small because, using properties $(P2)$ and $(P1)$,

$$\begin{aligned} \|r_n\|_{\sigma_{n+1},s} \ &\leq \ |\varepsilon| \ C \exp\left(-L_n\gamma_n\right)\Big\|P_{n+1}\Pi_W\Gamma(\delta, v_1, w_n)\Big\|_{\sigma_n,s} \leq |\varepsilon| \ C' \exp\left(-L_n\gamma_n\right)\Big\|\Gamma(\delta, v_1, w_n)\Big\|_{\sigma_n,s} \\ &\leq \ |\varepsilon| \ C'' \exp\left(-L_n\gamma_n\right) \end{aligned} \tag{33}$$

being $\|w_n\|_{\sigma_n,s}$ bounded independently of n since, by the induction hypothesis,

$$\|w_n\|_{\sigma_n,s} \leq \sum_{i=0}^{n} \|h_i\|_{\sigma_i,s} \leq \max(1, eK_0\gamma)|\varepsilon|\gamma^{-1} \sum_{i=0}^{\infty} \exp\left(-\chi^i\right), \tag{34}$$

with $h_0 := w_0$. The term $R(h)$ is "quadratic" in h, since, by property $(P1)$ and (34),

$$\begin{cases} \|R(h)\|_{\sigma_{n+1},s} \leq C|\varepsilon| \ \|h\|^2_{\sigma_{n+1},s} \\ \|R(h) - R(h')\|_{\sigma_{n+1},s} \leq C|\varepsilon| \ (\|h\|_{\sigma_{n+1},s} + \|h'\|_{\sigma_{n+1},s}) \ \|h - h'\|_{\sigma_{n+1},s} \end{cases} \tag{35}$$

for all $h, h' \in W^{(n+1)}$ with $\|h\|_{\sigma_{n+1},s}$, $\|h'\|_{\sigma_{n+1},s}$ small enough.

Since $w_n = \sum_{i=0}^n h_i$ with $\|h_i\|_{\sigma_i,s} \leq \max(1, eK_0\gamma)|\varepsilon|\gamma^{-1} \ \exp\left(-\chi^i\right)$, and $\sigma_i - \sigma_{n+1} \geq \gamma_i := \gamma_0/(1+i^2)$, $\forall i = 0, \ldots, n$

$$[w_n]_{\sigma_{n+1},s} \leq \sum_{i=0}^{n} \frac{\|h_i\|_{\sigma_i,s}}{(\sigma_i - \sigma_{n+1})^{\frac{2(\tau-1)}{\beta}}} \leq \max(1, eK_0\gamma)\frac{|\varepsilon|}{\gamma} \ \sum_{i\geq 0} \exp\left(-\chi^i\right)\Big(\frac{1+i^2}{\gamma_0}\Big)^{\frac{2(\tau-1)}{\beta}} < \mu$$

for $|\varepsilon|\gamma^{-1} \leq \varepsilon_2$ and by (31).

Hence, by property $(P3)$, the linear operator $\mathcal{L}_{n+1}(\delta, v_1, w_n) : D(\mathcal{L}_{n+1}) \subset W^{(n+1)} \to W^{(n+1)}$ is invertible for (δ, v_1) restricted to the set of parameters

$$A_{n+1} := \Big\{(\delta, v_1) \in A_n \ | \ \delta \in \Delta^{\gamma,\tau}_{n+1}(v_1, w_n)\Big\} \subseteq A_n\,, \tag{36}$$

and the inverse operator satisfies

$$\Big\|\mathcal{L}_{n+1}(\delta, v_1, w_n)^{-1}\Big\|_{\sigma_{n+1},s} \leq \frac{C}{\gamma}(L_{n+1})^{\tau-1}, \qquad \forall(\delta, v_1) \in A_{n+1}\,. \tag{37}$$

By (32), equation (P_{n+1}) for $w_{n+1} = w_n + h$ is equivalent to find $h \in W^{(n+1)}$ solving

$$h = -\mathcal{L}_{n+1}(\delta, v_1, w_n)^{-1}\Big(r_n + R(h)\Big)\,,$$

namely to look for a fixed point

$$h = \mathcal{G}(\delta, v_1, w_n, h)\,, \qquad h \in W^{(n+1)}\,, \tag{38}$$

of the nonlinear operator

$$\mathcal{G}(\delta, v_1, w_n, \cdot) : W^{(n+1)} \to W^{(n+1)}\,, \qquad \mathcal{G}(\delta, v_1, w_n, h) := -\mathcal{L}_{n+1}(\delta, v_1, w_n)^{-1}\Big(r_n + R(h)\Big).$$

To complete the proof of the Proposition we need the following Lemma.

Lemma 3.1 (Contraction) *There exist $L_0(\gamma, \tau) > 0$, $\varepsilon_0(L_0, \gamma, \tau)$, such that, $\forall|\varepsilon|\gamma^{-1} < \varepsilon_0$, the operator $\mathcal{G}(\delta, v_1, w_n, \cdot)$ is, for any $n \geq 0$, a contraction in the ball*

$$B(\rho_{n+1}; W^{(n+1)}) := \Big\{h \in W^{(n+1)} \ \ | \ \ \|h\|_{\sigma_{n+1},s} \leq \rho_{n+1} := \frac{|\varepsilon|}{\gamma} \ \exp\left(-\chi^{n+1}\right)\Big\}\,.$$

PROOF. We first prove that $\mathcal{G}(\delta, v_1, w_n, \cdot)$ maps the ball $B(\rho_{n+1}; W^{(n+1)})$ into itself.

By (37), (33) and (35),

$$\begin{aligned}\Big\|\mathcal{G}(\delta, v_1, w_n, h)\Big\|_{\sigma_{n+1},s} &= \Big\|\mathcal{L}_{n+1}(\delta, v_1, w_n)^{-1}\Big(r_n + R(h)\Big)\Big\|_{\sigma_{n+1},s} \\ &\leq \frac{C}{\gamma}(L_{n+1})^{\tau-1}\Big(\|r_n\|_{\sigma_{n+1},s} + \|R(h)\|_{\sigma_{n+1},s}\Big) \\ &\leq \frac{C'}{\gamma}(L_{n+1})^{\tau-1}\Big(|\varepsilon| \ \exp(-L_n\gamma_n) + |\varepsilon| \ \|h\|^2_{\sigma_{n+1},s}\Big). \end{aligned} \tag{39}$$

By (39), if $\|h\|_{\sigma_{n+1},s} \leq \rho_{n+1}$ then

$$\Big\|\mathcal{G}(\delta, v_1, w_n, h)\Big\|_{\sigma_{n+1},s} \leq \frac{C'}{\gamma}(L_{n+1})^{\tau-1}|\varepsilon|\Big(\exp(-L_n\gamma_n) + \rho^2_{n+1}\Big) \leq \rho_{n+1}$$

provided that

$$C'\frac{|\varepsilon|}{\gamma}(L_{n+1})^{\tau-1} \ \exp(-L_n\gamma_n) \leq \frac{\rho_{n+1}}{2} \qquad \text{and} \qquad C'\frac{|\varepsilon|}{\gamma}(L_{n+1})^{\tau-1}\rho_{n+1} \leq \frac{1}{2}\,. \tag{40}$$

The first inequality in (40) becomes, for $\rho_{n+1} := |\varepsilon|\gamma^{-1} \ \exp(-\chi^{n+1})$,

$$C'(L_{n+1})^{\tau-1} \ \exp(-L_n\gamma_n) \leq \frac{1}{2} \ \exp(-\chi^{n+1})$$

which, for $L_n := L_0 2^n$, $\gamma_n := \gamma_0/(1+n^2)$ and $L_0 := L_0(\gamma, \tau) > 0$ large enough, is satisfied $\forall n \geq 0$.

Next, the second inequality in (40) becomes

$$C'\frac{|\varepsilon|^2}{\gamma^2}\Big(L_0(\gamma, \tau)2^{n+1}\Big)^{\tau-1} \ \exp(-\chi^{n+1}) \leq \frac{1}{2}$$

which is satisfied for $|\varepsilon|\gamma^{-1} \leq \varepsilon_0(L_0, \gamma, \tau)$ $(\leq \varepsilon_2)$ small enough, $\forall n \geq 0$.

With similar estimates, using (35), we can prove that $\forall h, h' \in B(\rho_{n+1}; W^{(n+1)})$,

$$\Big\|\mathcal{G}(\delta, v_1, w_n, h') - \mathcal{G}(\delta, v_1, w_n, h)\Big\|_{\sigma_{n+1},s} \leq \frac{1}{2}\|h - h'\|_{\sigma_{n+1},s}$$

again for L_0 large enough and $|\varepsilon|\gamma^{-1} \leq \varepsilon_0(L_0, \gamma, \tau)$ small enough, uniformly in n, and we conclude that $\mathcal{G}(\delta, v_1, w_n, \cdot)$ is a contraction on $B(\rho_{n+1}; W^{(n+1)})$. ■

By the standard Contraction Mapping Theorem we deduce the existence, for $L_0(\gamma, \tau)$ large enough and $|\varepsilon|\gamma^{-1} < \varepsilon_0(L_0, \gamma, \tau)$, of a unique $h_{n+1} \in W^{(n+1)}$ solving (38) and satisfying

$$\Big\|h_{n+1}\Big\|_{\sigma_{n+1},s} \leq \rho_{n+1} = \frac{|\varepsilon|}{\gamma} \ \exp(-\chi^{n+1})\,.$$

Summarizing, $w_{n+1}(\delta, v_1) = w_n(\delta, v_1) + h_{n+1}(\delta, v_1)$ is a solution in $W^{(n+1)}$ of equation (P_{n+1}), defined for $(\delta, v_1) \in A_{n+1} \subseteq A_n \subseteq \ldots \subseteq A_1 \subseteq A_0$, and $w_{n+1}(\delta, v_1) = \sum_{i=0}^{n+1} h_i(\delta, v_1)$ where $h_i = h_i(\delta, v_1) \in W^{(i)}$ satisfy $\|h_i\|_{\sigma_i,s} \leq |\varepsilon|\gamma^{-1} \ \exp(-\chi^i)$ for some $\chi \in (1,2)$, $\forall i = 1, \ldots, n+1$, $\|h_0\|_{\sigma_0,s} \leq K_0|\varepsilon|$. ■

Remark 3.3 *A difference with respect to the usual "quadratic" Nash-Moser scheme, is that $h_n(\delta, v_1)$ is found as an exact solution of equation (P_n), and not just a solution of the linearized equation $r_n + \mathcal{L}_{n+1}(\delta, v_1, w_n)[h] = 0$. It appears to be more convenient to prove the regularity of $h_n(\delta, v_1)$ with respect to the parameters (δ, v_1), see Lemma 3.2.*

Corollary 3.1 (Solution of the (P)-equation) *For $(\delta, v_1) \in A_\infty := \cap_{n\geq 0} A_n$, $\sum_{i\geq 0} h_i(\delta, v_1)$ converges in $X_{\overline{\sigma}/2,s}$ to a solution $w(\delta, v_1) \in W \cap X_{\overline{\sigma}/2,s}$ of the (P)-equation (25) and $\|w(\delta, v_1)\|_{\overline{\sigma}/2,s} \leq C|\varepsilon|\gamma^{-1}$. The convergence is uniform in A_∞.*

PROOF. By Proposition 3.1, for $(\delta, v_1) \in A_\infty := \cap_{n\geq 0} A_n$,

$$\sum_{i=0}^{\infty} \|h_i(\delta, v_1)\|_{\overline{\sigma}/2,s} \leq \sum_{i=0}^{\infty} \|h_i(\delta, v_1)\|_{\sigma_i,s} \leq \max(1, eK_0\gamma) \sum_{i=0}^{\infty} \frac{|\varepsilon|}{\gamma} \exp\left(-\chi^i\right) < +\infty\,. \tag{41}$$

Hence the series of functions $w = \sum_{i\geq 0} h_i : A_\infty \to W \cap X_{\overline{\sigma}/2,s}$ converges normally and, by (41), $\|w(\delta, v_1)\|_{\overline{\sigma}/2,s} \leq C|\varepsilon|\gamma^{-1}$ with $C := \max(1, eK_0\gamma) \sum_{i=0}^{\infty} \exp\left(-\chi^i\right)$.

Let us justify that $L_\omega w = \varepsilon \Pi_W \Gamma(\delta, v_1, w)$. Since w_n solves equation (P_n) ,

$$L_\omega w_n = \varepsilon P_n \Pi_W \Gamma(\delta, v_1, w_n) = \varepsilon \Pi_W \Gamma(\delta, v_1, w_n) - \varepsilon P_n^\perp \Pi_W \Gamma(\delta, v_1, w_n)\,. \tag{42}$$

We have, by $(P2)$, $(P1)$ and since $\sigma_n - (\overline{\sigma}/2) \geq \gamma_n := \gamma_0/(n^2+1)$,

$$\Big\| P_n^\perp \Pi_W \Gamma(\delta, v_1, w_n) \Big\|_{\overline{\sigma}/2,s} \leq C \exp\Big(- L_n(\sigma_n - (\overline{\sigma}/2)) \Big) \leq C \exp\Big(- \gamma_0 \frac{L_0 2^n}{(n^2+1)} \Big).$$

Hence, by $(P1)$, the right hand side in (42) converges in $X_{\overline{\sigma}/2,s}$ to $\Gamma(\delta, v_1, w)$. Moreover, since $(w_n) \to w$ in $X_{\overline{\sigma}/2,s}$, $(L_\omega w_n) \to L_\omega w$ in the sense of distributions. Hence $L_\omega w = \varepsilon \Pi_W \Gamma(\delta, v_1, w)$. ■

3.2 C^∞ extension

Before proving the key property $(P3)$ on the linearized operator we prove a "Whitney-differentiability" property for $w(\delta, v_1)$ extending $w(\cdot, \cdot)$ in a C^∞-way on the whole A_0.

For this, some bound on the derivatives of $h_n = w_n - w_{n-1}$ is required.

Lemma 3.2 (Estimates for the derivatives of h_n and w_n) *For $\varepsilon_0\gamma^{-1} = \delta_0^{p-1}\gamma^{-1}$ small enough, the function $(\delta, v_1) \to h_n(\delta, v_1)$ is in $C^\infty(A_n, W^{(n)})$, $\forall n \geq 0$, and the k^{th}-derivative $D^k h_n(\delta, v_1)$ satisfies*

$$\Big\| D^k h_n(\delta, v_1) \Big\|_{\sigma_n,s} \leq K_1(k, \overline{\chi})^n \exp(-\overline{\chi}^n) \tag{43}$$

for $\overline{\chi} \in (1, \chi)$ and a suitable positive constant $K_1(k, \overline{\chi})$, $\forall n \geq 0$.

As a consequence, the function $(\delta, v_1) \to w_n(\delta, v_1) = \sum_{i=0}^n h_i(\delta, v_1)$ is in $C^\infty(A_n, W^{(n)})$ and the k^{th}-derivative $D^k w_n(\delta, v_1)$ satisfies

$$\Big\| D^k w_n(\delta, v_1) \Big\|_{\sigma_n,s} \leq K_2(k) \tag{44}$$

for a suitable positive constant $K_2(k)$.

PROOF. By the first step in the proof of Proposition 3.1, $h_0 = w_0$ depends smoothly on (δ, v_1), and $\|D^k w_0(\delta, v_1)\|_{\sigma_0,s} \leq C(k)$.

Next, assume by induction that $h_n = h_n(\delta, v_1)$ is a C^∞ map defined in A_n. We shall prove that $h_{n+1} = h_{n+1}(\delta, v_1)$ is C^∞ too.

First recall that $h_{n+1} = h_{n+1}(\delta, v_1)$ is defined, in Proposition 3.1, for $(\delta, v_1) \in A_{n+1}$ as a solution in $W^{(n+1)}$ of equation (P_{n+1}), namely

$$(P_{n+1}) \qquad\qquad U_{n+1}\Big(\delta, v_1, h_{n+1}(\delta, v_1)\Big) = 0$$

where

$$U_{n+1}(\delta, v_1, h) := L_\omega(w_n + h) - \varepsilon P_{n+1} \Pi_W \Gamma(\delta, v_1, w_n + h)\,.$$

We claim that $D_h U_{n+1}(\delta, v_1, h_{n+1}) = \mathcal{L}_{n+1}(\delta, v_1, w_{n+1})$ is invertible and

$$\Big\| \Big(D_h U_{n+1}(\delta, v_1, h_{n+1}) \Big)^{-1} \Big\|_{\sigma_{n+1},s} = \Big\| \mathcal{L}_{n+1}(\delta, v_1, w_{n+1})^{-1} \Big\|_{\sigma_{n+1},s} \leq \frac{C'}{\gamma} (L_{n+1})^{\tau-1}\,. \tag{45}$$

Now Equation (P_{n+1}) can be written as $h + q_{n+1}(\delta, v_1, h) = 0$, where

$$q_{n+1}(\delta, v_1, h) = (\Delta)^{-1}[L_\omega w_n - (\omega^2+1)h_{tt} - \varepsilon P_{n+1}\Pi_W \Gamma(\delta, v_1, w_n + h)].$$

The map $q_{n+1} : [0, \delta_0) \times V_1 \times W^{(n+1)} \to W^{(n+1)}$ is C^∞ and the invertibility of $\mathcal{L}_{n+1}(\delta, v_1, w_{n+1})$ implies the injectivity and hence (noting that $D_h q_{n+1}(\delta, v_1, h_{n+1})$ is compact) the invertibility of $I + D_h q_{n+1}(\delta, v_1, h_{n+1})$. As a consequence, by the Implicit Function Theorem, the map $(\delta, v_1) \mapsto h_{n+1}(\delta, v_1)$ is in $C^\infty(A_{n+1}, W^{(n+1)})$.

Let us prove (45). Using $(P1)$ and $\|w_{n+1} - w_n\|_{\sigma_{n+1},s} = \|h_{n+1}\|_{\sigma_{n+1},s} \le |\varepsilon|\gamma^{-1}\exp(-\chi^{n+1})$, we get

$$\begin{aligned}\Big\|\mathcal{L}_{n+1}(\delta, v_1, w_{n+1}) - \mathcal{L}_{n+1}(\delta, v_1, w_n)\Big\|_{\sigma_{n+1},s} &= \Big\|\varepsilon P_{n+1}\Pi_W\Big(D_w\Gamma(\delta, v_1, w_{n+1}) - D_w\Gamma(\delta, v_1, w_n)\Big)\Big\|_{\sigma_{n+1},s} \\ &\le C|\varepsilon|\ \|h_{n+1}\|_{\sigma_{n+1},s} \le C\frac{\varepsilon^2}{\gamma}\exp(-\chi^{n+1})\,. \qquad (46)\end{aligned}$$

Therefore

$$\mathcal{L}_{n+1}(\delta, v_1, w_{n+1}) = \mathcal{L}_{n+1}(\delta, v_1, w_n)\Big[\mathrm{Id} + \mathcal{L}_{n+1}(\delta, v_1, w_n)^{-1}\Big(\mathcal{L}_{n+1}(\delta, v_1, w_{n+1}) - \mathcal{L}_{n+1}(\delta, v_1, w_n)\Big)\Big] \quad (47)$$

is invertible whenever (recall (37) and (46))

$$\begin{aligned}\Big\|\mathcal{L}_{n+1}(\delta, v_1, w_n)^{-1}\Big(\mathcal{L}_{n+1}(\delta, v_1, w_{n+1}) - \mathcal{L}_{n+1}(\delta, v_1, w_n)\Big)\Big\|_{\sigma_{n+1},s} &\le \frac{C}{\gamma}(L_{n+1})^{\tau-1}\frac{\varepsilon^2}{\gamma}\exp(-\chi^{n+1}) \\ &< \frac{1}{2} \qquad (48)\end{aligned}$$

which is true, provided that $|\varepsilon|\gamma^{-1}$ is small enough, for all $n \ge 0$ (note that $(L_{n+1})^{\tau-1} = (L_0 2^{n+1})^{\tau-1} << \exp(\chi^{n+1})$ for n large). Furthermore, by (47), (37), (48), estimate (45) holds.

We now prove in detail estimate (43) for $k = 1$. Differentiating equation (P_{n+1}) with respect to some coordinate λ of $(\delta, v_1) \in A_{n+1}$, we obtain

$$(P'_{n+1}) \qquad \mathcal{L}_{n+1}(\delta, v_1, w_{n+1})\Big[\partial_\lambda h_{n+1}(\delta, v_1)\Big] = -(\partial_\lambda U_{n+1})\Big(\delta, v_1, h_{n+1}(\delta, v_1)\Big)$$

and therefore, by (45),

$$\Big\|\partial_\lambda h_{n+1}\Big\|_{\sigma_{n+1},s} \le \frac{C}{\gamma}(L_{n+1})^{\tau-1}\Big\|(\partial_\lambda U_{n+1})(\delta, v_1, h_{n+1})\Big\|_{\sigma_{n+1},s}. \qquad (49)$$

To estimate the right hand side of (49), first notice that since $w_n = w_n(\delta, v_1)$ solves

$$L_\omega w_n = \varepsilon P_n \Pi_W \Gamma(\delta, v_1, w_n)\,, \qquad \forall (\delta, v_1) \in A_n\,,$$

we have

$$U_{n+1}(\delta, v_1, h) = L_\omega h + \varepsilon(P_n\Pi_W\Gamma(\delta, v_1, w_n) - P_{n+1}\Pi_W\Gamma(\delta, v_1, w_n + h)).$$

Let us write

$$(\partial_\lambda U_{n+1})(\delta, v_1, h) = (\partial_\lambda U_{n+1})(\delta, v_1, 0) + r(\delta, v_1, h) \qquad (50)$$

where

$$\begin{aligned}(\partial_\lambda U_{n+1})(\delta, v_1, 0) &= (P_n - P_{n+1})\Pi_W\partial_\lambda[\varepsilon(\delta)\Gamma(\delta, v_1, w_n(\delta, v_1))] \\ &= -\varepsilon P_n^\perp P_{n+1}\Pi_W\Big[(\partial_\lambda\Gamma)(\delta, v_1, w_n) + (\partial_w\Gamma)(\delta, v_1, w_n)[\partial_\lambda w_n]\Big] \\ &\quad -\partial_\lambda(\varepsilon(\delta))P_n^\perp P_{n+1}\Pi_W\Gamma(\delta, v_1, w_n) \qquad (51)\end{aligned}$$

and

$$
\begin{aligned}
r(\delta, v_1, h) \quad := \quad & -\varepsilon P_{n+1}\Pi_W\Big[(\partial_\lambda\Gamma)(\delta, v_1, w_n + h) - (\partial_\lambda\Gamma)(\delta, v_1, w_n)\Big] \\
& -\varepsilon P_{n+1}\Pi_W\Big[(\partial_w\Gamma)(\delta, v_1, w_n + h) - (\partial_w\Gamma)(\delta, v_1, w_n)\Big][\partial_\lambda w_n] \\
& +\partial_\lambda(L_{\omega(\delta)}h) + \partial_\lambda(\varepsilon(\delta))P_{n+1}\Pi_W(\Gamma(\delta, v_1, w_n) - \Gamma(\delta, v_1, w_n + h))\,, \qquad (52)
\end{aligned}
$$

with $\partial_\lambda(L_{\omega(\delta)}h) = 0$, $\partial_\lambda(\varepsilon(\delta)) = 0$ if $\lambda \neq \delta$ and

$$
\partial_\delta(L_{\omega(\delta)}h) = -2(p-1)\delta^{p-2}h_{tt}, \quad \partial_\delta(\varepsilon(\delta)) = (p-1)\delta^{p-2}. \qquad (53)
$$

By $(P1)$, (34), (52) and (53), for $h \in W^{(n+1)}$,

$$
\Big\|r(\delta, v_1, h)\Big\|_{\sigma_{n+1},s} \leq C|\varepsilon|\ \|h\|_{\sigma_{n+1},s}\Big(1 + \|\partial_\lambda w_n\|_{\sigma_{n+1},s}\Big) + CL_{n+1}^2\|h\|_{\sigma_{n+1},s}. \qquad (54)
$$

We now estimate $(\partial_\lambda U_{n+1})(\delta, v_1, 0)$. By (51), properties $(P2)$, $(P1)$

$$
\begin{aligned}
\Big\|(\partial_\lambda U_{n+1})(\delta, v_1, 0)\Big\|_{\sigma_{n+1},s} \quad \leq \quad & \exp(-L_n\gamma_n)\Big[|\varepsilon|\Big\|(\partial_\lambda\Gamma)(\delta, v_1, w_n) + (\partial_w\Gamma)(\delta, v_1, w_n)[\partial_\lambda w_n]\Big\|_{\sigma_n,s} \\
& +\Big\|\Gamma(\delta, v_1, w_n)\Big\|_{\sigma_n,s}\Big] \\
\leq \quad & C\exp(-L_n\gamma_n)\Big(1 + \|\partial_\lambda w_n\|_{\sigma_n,s}\Big)\,. \qquad (55)
\end{aligned}
$$

Combining (49), (50), (54), (55) and the bound $\|h_{n+1}\|_{\sigma_{n+1},s} \leq |\varepsilon|\gamma^{-1}\exp(-\chi^{n+1})$, we get

$$
\begin{aligned}
\Big\|\partial_\lambda h_{n+1}\Big\|_{\sigma_{n+1},s} \quad \leq \quad & \frac{C}{\gamma}(L_{n+1})^{\tau+1}\Big(\frac{|\varepsilon|}{\gamma}\exp(-\chi^{n+1}) + \exp(-L_n\gamma_n)\Big)\Big(1 + \|\partial_\lambda w_n\|_{\sigma_n,s}\Big) \\
\leq \quad & C(\overline{\chi})\ \exp(-\overline{\chi}^{n+1})\Big(1 + \sum_{i=0}^{n}\|\partial_\lambda h_i\|_{\sigma_i,s}\Big) \qquad (56)
\end{aligned}
$$

for any $\overline{\chi} \in (1, \chi)$. By (56), the sequence $a_n := \|\partial_\lambda h_n\|_{\sigma_n,s}$ satisfies

$$
a_0 \leq C \qquad \text{and} \qquad a_{n+1} \leq C(\overline{\chi})\ \exp(-\overline{\chi}^{n+1})\Big(1 + a_0 + \ldots + a_n\Big)
$$

which implies (by induction)

$$
\Big\|\partial_\lambda h_n\Big\|_{\sigma_n,s} = a_n \leq K(\overline{\chi})^n\ \exp(-\overline{\chi}^n)\,, \qquad \forall n \geq 0,
$$

provided that $K(\overline{\chi})$ has been chosen large enough. We can prove in the same way the general estimate (43) from which (44) follows. ■

Since, by (43), $h_n(\delta, v_1) = O(\varepsilon\gamma^{-1}\exp(-\overline{\chi}^n))$ and the "non-resonant" set A_n is obtained at each step deleting strips of size $O(\gamma/L_n^\tau)$, we can define (by interpolation, say) a C^∞-extension $\widetilde{w}(\delta, v_1)$ of $w(\delta, v_1)$ for all $(\delta, v_1) \in A_0$.

Let

$$
\widetilde{A}_n := \Big\{(\delta, v_1) \in A_n \mid \text{dist}((\delta, v_1), \partial A_n) \geq \frac{2\nu}{L_n^3}\Big\} \subset A_n
$$

where $\nu\gamma^{-1} > 0$ is some small constant to be specified later, see Lemma 3.4.

Lemma 3.3 (Whitney C^∞ Extension $\widetilde{w}$ of w on A_0) *There exists a function $\widetilde{w}(\delta, v_1) \in C^\infty(A_0, W \cap X_{\overline{\sigma}/2,s})$ satisfying*

$$
\Big\|\widetilde{w}(\delta, v_1)\Big\|_{\overline{\sigma}/2,s} \leq \frac{|\varepsilon|}{\gamma}C, \quad \Big\|D^k\widetilde{w}(\delta, v_1)\Big\|_{\overline{\sigma}/2,s} \leq \frac{C(k)}{\nu^k}, \qquad \forall(\delta, v_1) \in A_0\,,\ \forall k \geq 1\,, \qquad (57)
$$

for some $C(k) > 0$, *such that,*

$$\forall\,(\delta, v_1) \in \widetilde{A}_\infty := \cap_{n\geq 0}\widetilde{A}_n\,, \quad \widetilde{w}(\delta, v_1) \text{ solves the } (P) - \text{equation } (25)\,.$$

Moreover there exists a sequence $\widetilde{w}_n \in C^\infty(A_0,\, W^{(n)})$ *such that*

$$\widetilde{w}_n(\delta, v_1) = w_n(\delta, v_1)\,, \qquad \forall(\delta, v_1) \in \widetilde{A}_n \tag{58}$$

and

$$\Big\|\widetilde{w}(\delta, v_1) - \widetilde{w}_n(\delta, v_1)\Big\|_{\overline{\sigma}/2,s} \quad \leq \quad \frac{|\varepsilon|C}{\gamma}\ \exp(-\widetilde{\chi}^n), \tag{59}$$

$$\Big\|D^k\widetilde{w}(\delta, v_1) - D^k\widetilde{w}_n(\delta, v_1)\Big\|_{\overline{\sigma}/2,s} \quad \leq \quad \frac{C(k)}{\nu^k}\ \exp(-\widetilde{\chi}^n)\,, \quad \forall(\delta, v_1) \in A_0\,, \tag{60}$$

for some $\widetilde{\chi} \in (1, \overline{\chi})$.

PROOF. Let $\varphi : \mathbf{R}\times V_1 \to \mathbf{R}_+$ be a C^∞ function supported in the open ball $B(0,1)$ of center 0 and radius 1 with $\int_{\mathbf{R}\times V_1} \varphi\ d\mu = 1$. Here μ is the Borelian positive measure of $\mathbf{R}\times V_1$ defined by $\mu(E) = m(L^{-1}(E))$ where L is some automorphism from $\mathbf{R}^{N+1}$ to $\mathbf{R}\times V_1$ and m is the Lebesgue measure in $\mathbf{R}^{N+1}$.

Let $\varphi_n : \mathbf{R}\times V_1 \to \mathbf{R}_+$ be the "mollifier"

$$\varphi_n(\lambda) := \Big(\frac{L_n^3}{\nu}\Big)^{N+1} \varphi\Big(\frac{L_n^3}{\nu}\lambda\Big)$$

(here $\lambda := (\delta, v_1)$) which is a C^∞ function satisfying

$$\mathrm{supp}\,\varphi_n \subset B\Big(0, \frac{\nu}{L_n^3}\Big) \qquad \text{and} \qquad \int_{\mathbf{R}\times V_1} \varphi_n\ d\mu = 1\,. \tag{61}$$

Next we define $\psi_n : \mathbf{R}\times V_1 \to \mathbf{R}_+$ as

$$\psi_n(\lambda) := \Big(\varphi_n * \chi_{A_n^*}\Big)(\lambda) = \int_{\mathbf{R}\times V_1} \varphi_n(\lambda - \eta)\, \chi_{A_n^*}(\eta)\ d\mu(\eta)$$

where $\chi_{A_n^*}$ is the characteristic function of the set

$$A_n^* := \Big\{\lambda = (\delta, v_1) \in A_n \mid \mathrm{dist}(\lambda, \partial A_n) \geq \frac{\nu}{L_n^3}\Big\} \subset A_n\,,$$

namely $\chi_{A_n^*}(\lambda) := 1$ if $\lambda \in A_n^*$, and $\chi_{A_n^*}(\lambda) := 0$ if $\lambda \notin A_n^*$.

The function ψ_n is C^∞ and, $\forall k \in \mathbf{N}$, $\forall \lambda \in \mathbf{R}\times V_1$,

$$\begin{aligned}
|D^k\psi_n(\lambda)| &= \Big|\int_{\mathbf{R}\times V_1} D^k\varphi_n(\lambda - \eta)\, \chi_{A_n^*}(\eta)\ d\mu(\eta)\Big| \\
&\leq \int_{\mathbf{R}\times V_1} \Big|\Big(\frac{L_n^3}{\nu}\Big)^k \Big(\frac{L_n^3}{\nu}\Big)^{N+1} (D^k\varphi)\Big(\frac{L_n^3}{\nu}(\lambda - \eta)\Big)\Big|\ d\mu(\eta) \\
&= \Big(\frac{L_n^3}{\nu}\Big)^k \int_{\mathbf{R}\times V_1} |D^k\varphi|\ d\mu = \Big(\frac{L_n^3}{\nu}\Big)^k C(k)
\end{aligned} \tag{62}$$

where $C(k) := \int_{\mathbf{R}\times V_1} |D^k\varphi|\ d\mu$. Furthermore, by (61) and the definition of A_n^* and $\widetilde{A}_n$,

$$0 \leq \psi_n(\lambda) \leq 1\,, \quad \mathrm{supp}\,\psi_n \subset \mathrm{int}A_n \quad \text{and} \quad \psi_n(\lambda) = 1 \ \text{ if } \ \lambda \in \widetilde{A}_n\,.$$

Finally we can define $\widetilde{w}_n : A_0 \to W^{(n)}$ by

$$\widetilde{w}_0(\lambda) := w_0(\lambda)\,, \qquad \widetilde{w}_{n+1}(\lambda) := \widetilde{w}_n(\lambda) + \widetilde{h}_{n+1}(\lambda) \in W^{(n+1)}\,,$$

where

$$\widetilde{h}_{n+1}(\lambda) := \begin{cases} \psi_{n+1}(\lambda)h_{n+1}(\lambda) & \text{if} \quad \lambda \in A_{n+1} \\ 0 & \text{if} \quad \lambda \notin A_{n+1} \end{cases}$$

is in $C^\infty(A_0, W^{(n+1)})$ because supp $\psi_{n+1} \subset$ int A_{n+1} and, by Lemma 3.2, $h_{n+1} \in C^\infty(A_{n+1}, W^{(n+1)})$. Therefore we have

$$\widetilde{w}_n(\lambda) = \sum_{i=0}^{n} \widetilde{h}_i(\lambda), \qquad \widetilde{w}_n \in C^\infty(A_0, W^{(n)})$$

and (58) holds.

By the bounds (62) and (43) we obtain, $\forall k \in \mathbf{N}$, $\forall \lambda \in A_0$, $\forall n \geq 0$,

$$\left\|\widetilde{h}_{n+1}(\lambda)\right\|_{\sigma_{n+1},s} \leq \frac{|\varepsilon|K}{\gamma} \exp(-\overline{\chi}^n), \ \left\|D^k\widetilde{h}_{n+1}(\lambda)\right\|_{\sigma_{n+1},s} \leq C(k,\overline{\chi})^n \Big(\frac{L_{n+1}^3}{\nu}\Big)^k \exp(-\overline{\chi}^n) \leq \frac{K(k)}{\nu^k} \exp(-\widetilde{\chi}^n)$$

for some $1 < \widetilde{\chi} < \overline{\chi}$ and some positive constant $K(k)$ large enough. As a consequence, the sequence $(\widetilde{w}_n)$ (and all its derivatives) converges uniformly in A_0 for the norm $\| \ \|_{\overline{\sigma}/2,s}$ on W, to some function $\widetilde{w}(\delta, v_1) \in C^\infty(A_0, W \cap X_{\overline{\sigma}/2,s})$ which satisfies (57), (59) and (60).

Finally, note that if $\lambda \notin A_\infty := \cap_{n\geq 0} A_n$ then the series $\widetilde{w}(\lambda) = \sum_{n\geq 1} \widetilde{h}_n(\lambda)$ is a finite sum. On the other hand, if $\lambda \in \widetilde{A}_\infty := \cap_{n\geq 0}\widetilde{A}_n$ then $\widetilde{w}(\lambda) = w(\lambda)$ solves the (P)-equation (25). ■

Remark 3.4 *If $(\delta, v_1) \notin \widetilde{A}_\infty$ we claim that $\widetilde{w}(\delta, v_1)$ solves the (P)-equation up to exponentially small remainders: there exists $\alpha > 0$, $\delta_0(\gamma, \tau) > 0$ such that, $\forall 0 < \delta \leq \delta_0(\gamma, \tau)$*

$$\left\|L_\omega \widetilde{w}(\delta, v_1) - \varepsilon \Pi_W \Gamma\Big(\delta, v_1, \widetilde{w}(\delta, v_1)\Big)\right\|_{\overline{\sigma}/4,s} \leq \frac{|\varepsilon|}{\gamma} \exp\Big(-\frac{1}{\delta^\alpha}\Big).$$

Since we shall not use this property in the present paper we do not give here the proof.

3.3 Measure estimate

We now replace the set $\widetilde{A}_\infty$ with a smaller Cantor-like set B_∞ which has the advantage of being independent of the iteration steps. This is more convenient for the measure estimates required in section 5 (this issue is discussed differently in [12]).

Define

$$B_n := \Big\{(\delta, v_1) \in \widetilde{A}_0 \mid \delta \in \Delta_n^{2\gamma,\tau}(v_1, \widetilde{w}(\delta, v_1))\Big\} \tag{63}$$

where we have replaced γ with 2γ in the definition of $\Delta_n^{\gamma,\tau}$, see Definition 3.3. Note that B_n does not depend on the approximate solution w_n but only on the fixed function $\widetilde{w}$.

Lemma 3.4 *If $\nu\gamma^{-1} > 0$ and $|\varepsilon|\gamma^{-1}$ are small enough, then*

$$B_n \subset \widetilde{A}_n, \qquad \forall n \geq 0.$$

Hence $B_\infty := \cap_{n\geq 1} B_n \subset \widetilde{A}_\infty \subset A_\infty$ and so, if $(\delta, v_1) \in B_\infty$ then $\widetilde{w}(\delta, v_1)$ solves the (P)-equation (25).

PROOF. We shall prove the Lemma by induction. First $B_0 \subset \widetilde{A}_0$. Suppose next that $B_n \subset \widetilde{A}_n$ holds. In order to prove that $B_{n+1} \subset \widetilde{A}_{n+1}$, take any $(\delta, v_1) \in B_{n+1}$. We have to justify that the ball $B((\delta, v_1), 2\nu/L_{n+1}^3) \subset A_{n+1}$.

First, since $B_{n+1} \subset B_n \subset \widetilde{A}_n$, $(\delta, v_1) \in \widetilde{A}_n$. Hence, since $L_{n+1} > L_n$, $B((\delta, v_1), 2\nu/L_{n+1}^3) \subset A_n$.

Let $(\delta', v_1') \in B((\delta, v_1), 2\nu/L_{n+1}^3)$. Since $(\delta, v_1) \in \widetilde{A}_n$ we have $\widetilde{w}_n(\delta, v_1) = w_n(\delta, v_1)$. Moreover by (44) $\|Dw_n\|_{\overline{\sigma}/2,s} \leq C$. By (59) we can derive

$$\begin{aligned} \left\|w_n(\delta', v_1') - \widetilde{w}(\delta, v_1)\right\|_{\overline{\sigma}/2,s} &\leq \left\|w_n(\delta', v_1') - w_n(\delta, v_1)\right\|_{\overline{\sigma}/2,s} + \left\|w_n(\delta, v_1) - \widetilde{w}(\delta, v_1)\right\|_{\overline{\sigma}/2,s} \\ &\leq \frac{2\nu C}{L_{n+1}^3} + \frac{C|\varepsilon|}{\gamma} \exp(-\widetilde{\chi}^n). \end{aligned}$$

Hence, by (63), setting $\omega' := \sqrt{1+2(\delta')^{p-1}}$ and $\varepsilon' := (\delta')^{p-1}$ (for simplicity of notation suppose $s^*=1$)

$$\begin{aligned}\Big|\omega' l - j - \varepsilon' \frac{M(\delta', v_1', w_n(\delta', v_1'))}{2j}\Big| &\geq \Big|\omega l - j - \varepsilon \frac{M(\delta, v_1, \widetilde{w}(\delta, v_1))}{2j}\Big| - l\frac{C\nu}{L_{n+1}^3} - C\frac{|\varepsilon|\nu}{L_{n+1}^3} - C\frac{|\varepsilon|^2}{\gamma}\exp(-\widetilde{\chi}^n) \\ &\geq \frac{2\gamma}{(l+j)^\tau} - \frac{C\nu}{L_{n+1}^2} - C\frac{|\varepsilon|^2}{\gamma}\exp(-\widetilde{\chi}^n) \geq \frac{\gamma}{(l+j)^\tau}\end{aligned}$$

for all $1/3|\varepsilon| < l < L_{n+1}$, $l \neq j$, $j \leq 2L_{n+1}$, whenever

$$\frac{\gamma}{(3L_{n+1})^\tau} \geq C\Big(\frac{\nu}{L_{n+1}^2} + \frac{|\varepsilon|^2}{\gamma}\exp(-\widetilde{\chi}^n)\Big). \tag{64}$$

(64) holds true, for $|\varepsilon|\gamma^{-1}$ and $\nu\gamma^{-1}$ small, for all $n \geq 0$, because $\tau < 2$ and $\lim_{n\to\infty} L_{n+1}^\tau \exp(-\widetilde{\chi}^n) = 0$. It results that $B((\delta, v_1), 2\nu/L_{n+1}^3)) \subset A_{n+1}$. ■

Up to now, we have not justified that

$$B_\infty \subset \widetilde{A}_\infty \subset A_\infty \tag{65}$$

are not reduced to $\{\delta = 0\} \times B(2R, V_1)$. It is a consequence of the following result which shall be applied in section 5.

Proposition 3.2 (Measure estimate of B_∞) *Let $\mathcal{V}_1 : [0, \delta_0) \to V_1$ be a C^1 function. Then*

$$\lim_{\eta\to 0^+} \frac{\text{meas}\{\delta \in [0,\eta) \mid (\delta, \mathcal{V}_1(\delta)) \in B_\infty\}}{\eta} = 1\,. \tag{66}$$

Proof. Let $0 < \eta < \delta_0$. Define

$$\mathcal{C}_{\mathcal{V}_1,\eta} := \Big\{\delta \in (0,\eta) \mid (\delta, \mathcal{V}_1(\delta)) \in B_\infty\Big\} \qquad \text{and} \qquad \mathcal{D}_{\mathcal{V}_1,\eta} := (0,\eta)\backslash\mathcal{C}_{\mathcal{V}_1,\eta}\,.$$

By the definition $B_\infty := \cap_{n\geq 1} B_n$ (see also the expression of B_∞ in the statement of Theorem 3.1 where for simplicity of notation we suppose $s^* = 1$)

$$\begin{aligned}\mathcal{D}_{\mathcal{V}_1,\eta} = \Big\{\delta \in (0,\eta) \;\Big|\; &\Big|\omega(\delta)l - j - \frac{\delta^{p-1}m(\delta)}{2j}\Big| < \frac{2\gamma}{(l+j)^\tau} \quad \textit{or} \quad \Big|\omega(\delta)l - j\Big| < \frac{2\gamma}{(l+j)^\tau} \\ &\text{for some } l > \frac{1}{3\delta^{p-1}},\ l \neq j\Big\}\end{aligned}$$

where $m(\delta) := M(\delta, \mathcal{V}_1(\delta), \widetilde{w}(\delta, \mathcal{V}_1(\delta)))$ is a function in $C^1([0,\delta_0), \mathbf{R})$ since $\widetilde{w}(\cdot,\cdot)$ is in $C^\infty(A_0, W\cap X_{\overline{\sigma}/2,s}))$ and $\mathcal{V}_1$ is C^1. This implies, in particular,

$$|m(\delta)| + |m'(\delta)| \leq C, \qquad \forall \delta \in [0, \delta_0/2] \tag{67}$$

for some positive constant C.

We claim that, for any interval $[\delta_1/2, \delta_1] \subset [0,\eta] \subset [0, \delta_0/2]$ the following measure estimate holds:

$$\text{meas}\Big(\mathcal{D}_{\mathcal{V}_1,\eta} \cap \Big[\frac{\delta_1}{2}, \delta_1\Big]\Big) \leq K_1(\tau)\gamma\eta^{(p-1)(\tau-1)}\text{meas}\Big(\Big[\frac{\delta_1}{2}, \delta_1\Big]\Big) \tag{68}$$

for some constant $K_1(\tau) > 0$.

Before proving (68) we show how to conclude the proof of the Lemma. Writing $(0,\eta] = \cup_{n\geq 0}\,[\eta/2^{n+1}, \eta/2^n]$ and applying the measure estimate (68) to any interval $[\delta_1/2, \delta_1] = [\eta/2^{n+1}, \eta/2^n]$, we get

$$\text{meas}(\mathcal{D}_{\mathcal{V}_1,\eta} \cap [0,\eta]) \leq K_1(\tau)\gamma\eta^{(p-1)(\tau-1)}\eta\,,$$

whence $\lim_{\eta\to 0^+}$ meas $(\mathcal{C}_{\mathcal{V}_1,\eta} \cap (0,\eta))/\eta = 1$, proving Proposition 3.2.

We now prove (68). We have

$$\mathcal{D}_{\mathcal{V}_1,\eta} \bigcap \Big[\frac{\delta_1}{2}, \delta_1\Big] \subset \bigcup_{(l,j)\in \mathcal{I}_R} \mathcal{R}_{l,j}(\delta_1) \tag{69}$$

where

$$\mathcal{R}_{l,j}(\delta_1) := \Big\{\delta \in \Big[\frac{\delta_1}{2}, \delta_1\Big] \mid \Big|\omega(\delta)l - j - \frac{\delta^{p-1}m(\delta)}{2j}\Big| < \frac{2\gamma}{(l+j)^\tau} \quad \text{or} \quad \Big|\omega(\delta)l - j\Big| < \frac{2\gamma}{(l+j)^\tau}\Big\}$$

and

$$I_R := \Big\{(l,j) \mid l > \frac{1}{3\delta_1^{p-1}},\ l \neq j,\ \frac{j}{l} \in [1 - c_0\delta_1^{p-1}, 1 + c_0\delta_1^{p-1}]\Big\}$$

(note indeed that $\mathcal{R}_{j,l}(\delta_1) = \emptyset$ unless $j/l \in [1-c_0\delta_1^{p-1}, 1+c_0\delta_1^{p-1}]$ for some constant $c_0 > 0$ large enough).

Next, let us prove that

$$\text{meas}(\mathcal{R}_{lj}(\delta_1)) = O\Big(\frac{\gamma}{l^{\tau+1}\delta_1^{p-2}}\Big). \tag{70}$$

Define $f_{lj}(\delta) := \omega(\delta)l - j - (\delta^{p-1}m(\delta)/2j)$ and $\mathcal{S}_{j,l}(\delta_1) := \{\delta \in [\delta_1/2, \delta_1] \ : \ |f_{l,j}(\delta)| < 2\gamma/(l+j)^\tau\}$. Provided δ_0 has been chosen small enough (recall that $j, l \geq 1/3\delta_0^{p-1}$),

$$\begin{aligned} |\partial_\delta f_{lj}(\delta)| &= \Big|\frac{l(p-1)\delta^{p-2}}{\sqrt{1+2\delta^{p-1}}} - \frac{(p-1)\delta^{p-2}m(\delta)}{2j} - \frac{\delta^{p-1}m'(\delta)}{2j}\Big| \\ &\geq \frac{(p-1)\delta^{p-2}}{2}\Big(l - \frac{C}{j}\Big) \geq \frac{(p-1)\delta^{p-2}l}{4} \end{aligned}$$

and therefore $|\partial_\delta f_{lj}(\delta)| \geq (p-1)\delta_1^{p-2}l/2^p$ for any $\delta \in [\delta_1/2, \delta_1]$. This implies

$$\text{meas}(\mathcal{S}_{lj}(\delta_1)) \leq \frac{4\gamma}{(l+j)^\tau} \times \Big(\min_{\delta\in[\delta_1/2,\delta_1]} |\partial_\delta f_{lj}(\delta)|\Big)^{-1} \leq \frac{4\gamma}{(l+j)^\tau} \times \frac{2^p}{(p-1)l\delta_1^{p-2}} = O\Big(\frac{\gamma}{l^{\tau+1}\delta_1^{p-2}}\Big).$$

Similarly we can prove

$$\text{meas}\Big(\Big\{\delta \in \Big[\frac{\delta_1}{2}, \delta_1\Big] \ : \ |\omega(\delta)l - j| < \frac{2\gamma}{(l+j)^\tau}\Big\}\Big) = O\Big(\frac{\gamma}{l^{\tau+1}\delta_1^{p-2}}\Big)$$

and the measure estimate (70) follows.

Now, by (69), (70) and since, for a given l, the number of j for which $(l,j) \in I_R$ is $O(\delta_1^{p-1}l)$,

$$\text{meas}\Big(\mathcal{D}_{\mathcal{V}_1,\eta} \cap \Big[\frac{\delta_1}{2}, \delta_1\Big]\Big) \leq \sum_{(l,j)\in I_R} \text{meas}(\mathcal{R}_{j,l}(\delta_1)) \leq C \sum_{l \geq 1/3\delta_1^{p-1}} \delta_1^{p-1} l \times \frac{\gamma}{l^{\tau+1}\delta_1^{p-2}} \leq K_2(\tau)\gamma\delta_1^{1+(p-1)(\tau-1)}$$

whence we obtain (68) since $0 < \delta_1 < \eta$. ■

We summarize the main result of this section as follows:

Theorem 3.1 (Solution of the (P)-equation) *For $\delta_0 := \delta_0(\gamma, \tau) > 0$ small enough, there exist a C^∞-function $\widetilde{w} : A_0 := \{(\delta, v_1) \mid \ \delta \in [0, \delta_0),\ \|v_1\|_{0,s} \leq 2R\} \to W \cap X_{\overline{\sigma}/2,s}$ satisfying (57), and a "large" -see (66)- Cantor set*

$$\begin{aligned} B_\infty \ := \ \Big\{(\delta, v_1) \in A_0 \ : \ &\Big|\omega(\delta)l - j - s^*\delta^{p-1}\frac{M(\delta, v_1, \widetilde{w}(\delta, v_1))}{2j}\Big| \geq \frac{2\gamma}{(l+j)^\tau}, \\ &\Big|\omega(\delta)l - j\Big| \geq \frac{2\gamma}{(l+j)^\tau},\ \forall l \geq \frac{1}{3\delta^{p-1}},\ l \neq j\ \Big\} \subset A_0\,, \end{aligned}$$

where $\omega(\delta) = \sqrt{1 + 2s^*\delta^{p-1}}$ and $M(\delta, v_1, w)$ is defined in Definition 3.1, such that

$$\forall (\delta, v_1) \in B_\infty\,, \quad \widetilde{w}(\delta, v_1) \text{ solves the } (P) - \text{equation (25)}\,.$$

4 Analysis of the linearized problem: proof of (P3)

We prove in this section the key property $(P3)$ on the inversion of the linear operator $\mathcal{L}_n(\delta, v_1, w)$ defined in (28).

Through this section we shall use the notations

$$F_k := \Big\{ f \in H_0^1((0,\pi); \mathbf{R}) \ \mid \ \int_0^\pi f(x) \sin(kx)\, dx = 0 \Big\} = \langle \sin(kx) \rangle^\perp$$

whence the space W, defined in (6), writes

$$W = \Big\{ h = \sum_{k \in \mathbf{Z}} \exp{(\mathrm{i}kt)} h_k \in X_{0,s} \ \mid h_k = h_{-k}, \ h_k \in F_k\,, \ \forall k \in \mathbf{Z} \Big\}$$

and the corresponding projector $\Pi_W : X_{\sigma,s} \to W$ is

$$(\Pi_W h)(t,x) = \sum_{k \in \mathbf{Z}} \exp{(\mathrm{i}kt)} (\pi_k h_k)(x) \tag{71}$$

where $\pi_k : H_0^1((0,\pi); \mathbf{R}) \to F_k := \langle \sin(kx) \rangle^\perp$ is the L^2-orthogonal projector onto F_k

$$(\pi_k f)(x) := f(x) - \Big(\frac{2}{\pi} \int_0^\pi f(x) \sin(kx) dx \Big) \sin(kx)\,.$$

Note that $\pi_{-k} = \pi_k$. Hence, since $h_k = h_{-k}$, $\pi_k h_k = \pi_{-k} h_{-k}$.

4.1 Decomposition of $\mathcal{L}_n(\delta, v_1, w)$

Recalling (26), the operator $\mathcal{L}_n(\delta, v_1, w) : D(\mathcal{L}_n) \subset W^{(n)} \to W^{(n)}$ writes

$$\begin{aligned} \mathcal{L}_n(\delta, v_1, w)[h] &:= L_\omega h - \varepsilon P_n \Pi_W D_w \Gamma(\delta, v_1, w)[h] \\ &= L_\omega h - \varepsilon P_n \Pi_W \Big(\partial_u g(\delta, x, v_1 + w + v_2(\delta, v_1, w)) \Big(h + \partial_w v_2(\delta, v_1, w)[h] \Big) \Big) \\ &= L_\omega h - \varepsilon P_n \Pi_W \Big(a(t,x)\ h \Big) - \varepsilon P_n \Pi_W \Big(a(t,x)\ \partial_w v_2(\delta, v_1, w)[h] \Big) \end{aligned} \tag{72}$$

where, for brevity, we have set

$$a(t,x) := \partial_u g \Big(\delta, x, v_1(t,x) + w(t,x) + v_2(\delta, v_1, w)(t,x) \Big)\,. \tag{73}$$

In order to invert $\mathcal{L}_n$ it is convenient to perform a Fourier expansion and represent the operator $\mathcal{L}_n$ as a matrix, distinguishing a "diagonal" matrix D and a "off-diagonal Toepliz" matrix. The main difference with respect to the analogue procedure of Craig-Wayne-Bourgain [12]-[8] is that we shall develop $\mathcal{L}_n$ only in *time*-Fourier basis and not also in the spatial fixed basis formed by the eigenvectors $\sin(jx)$ of the linear operator $-\partial_{xx}$. The reason is that this is more convenient to deal with nonlinearities $f(x,u)$ with finite regularity in x and without oddness assumptions. Each diagonal element D_k is a differential operator acting on functions of x. Next, using Sturm-Liouville theory, we shall diagonalize each D_k in a suitable basis of eigenfunctions close, but different, from $\sin jx$, see Lemma 4.1 and Corollary 4.1.

Performing a time-Fourier expansion, the operator $L_\omega := -\omega^2 \partial_{tt} + \partial_{xx}$ is diagonal since

$$L_\omega \Big(\sum_{|k| \le L_n} \exp(\mathrm{i}kt) h_k \Big) = \sum_{|k| \le L_n} \exp(\mathrm{i}kt) (\omega^2 k^2 + \partial_{xx}) h_k\,. \tag{74}$$

The operator $h \to P_n \Pi_W (a(t,x)\ h)$ is the composition of the multiplication operator for the function $a(t,x) = \sum_{l \in \mathbf{Z}} \exp(\mathrm{i}lt) a_l(x)$ with the projectors Π_W and P_n. As usual, in Fourier expansion, the multiplication operator is described by a "Toepliz matrix"

$$a(t,x)\ h(t,x) = \sum_{|k| \le L_n, l \in \mathbf{Z}} \exp(\mathrm{i}lt) a_{l-k}(x) h_k(x)$$

and, recalling (71) and (27),

$$P_n\Pi_W(a(t,x)\ h) = \sum_{|k|,|l|\le L_n} \exp(\mathrm{i}lt)\pi_l(a_{l-k}(x)h_k)$$
$$= \sum_{|k|\le L_n} \exp(\mathrm{i}kt)\pi_k(a_0(x)h_k) + \sum_{|k|,|l|\le L_n, k\ne l} \exp(\mathrm{i}lt)\pi_l(a_{l-k}h_k) \tag{75}$$

where we have distinguished the "diagonal" term

$$\sum_{|k|\le L_n} \exp(\mathrm{i}kt)\pi_k(a_0(x)h_k) = P_n\Pi_W(a_0(x)\ h) \tag{76}$$

with $a_0(x) := \frac{1}{2\pi}\int_0^{2\pi} a(t,x)\,dt$, from the "off-diagonal Toepliz" term

$$\sum_{|k|,|l|\le L_n, k\ne l} \exp(\mathrm{i}lt)\pi_l(a_{l-k}h_k) = P_n\Pi_W(\overline{a}(t,x)\ h) \tag{77}$$

where

$$\overline{a}(t,x) := a(t,x) - a_0(x)$$

has zero time-average.

By (72), (75), (76) and (77), we can decompose

$$\mathcal{L}_n(\delta, v_1, w) = D - \mathcal{M}_1 - \mathcal{M}_2$$

where D, $\mathcal{M}_1$, $\mathcal{M}_2$ are the linear operators

$$\begin{cases} Dh := L_\omega h - \varepsilon P_n\Pi_W(a_0(x)\ h) \\ \mathcal{M}_1 h := \varepsilon P_n\Pi_W(\overline{a}(t,x)\ h) \\ \mathcal{M}_2 h := \varepsilon P_n\Pi_W(a(t,x)\ \partial_w v_2[h])\,. \end{cases} \tag{78}$$

To invert $\mathcal{L}_n$ we first (Step 1) prove that, assuming the "first order Melnikov non-resonance condition $\delta \in \Delta_n^{\gamma,\tau}(v_1, w)$ (see Definition 3.3) the diagonal (in time) linear operator D is invertible, see Corollary 4.2. Next (Step 2) we prove that the "off-diagonal Toepliz" operator $\mathcal{M}_1$ (Lemma 4.8) and $\mathcal{M}_2$ (Lemma 4.9) are small enough with respect to D, yielding the invertibility of the whole $\mathcal{L}_n$ (note that we do not decompose the term $\mathcal{M}_2$ in a diagonal and "off-diagonal term"). More precisely, the crucial bounds of Lemma 4.5 enable us to prove via Lemma 4.6 that the operator $|D|^{-1/2}\mathcal{M}_1|D|^{-1/2}$ has small norm, whereas the norm of $|D|^{-1/2}\mathcal{M}_2|D|^{-1/2}$ is controlled thanks to the regularizing properties of the map v_2.

4.2 Step 1: Inversion of D

The first aim is to diagonalize (both in time and space) the linear operator D, see Corollary 4.1.
By (74) and (76), the operator D is yet diagonal in time-Fourier basis, and, $\forall\ h \in W^{(n)}$, the k^{th} time Fourier coefficient of Dh is

$$(Dh)_k = (\omega^2 k^2 + \partial_{xx})h_k - \varepsilon\pi_k(a_0(x)h_k) \equiv D_k h_k$$

where $D_k : \mathcal{D}(D_k) \subset F_k \to F_k$ is the operator

$$D_k u = \omega^2 k^2 u - S_k u \qquad \text{and} \qquad S_k u := -\partial_{xx} u + \varepsilon\pi_k(a_0(x)\ u)\,.$$

Note that $S_k = S_{-k}$.

We now have to diagonalize (in space) each Sturm-Liouville type operator S_k and to study its spectral properties.

In the next Lemma 4.1 we shall find a basis of eigenfunctions $v_{k,j}$ of $S_k : \mathcal{D}(S_k) \subset F_k \to F_k$ which are orthonormal for the scalar product of F_k,

$$\langle u, v \rangle_\varepsilon := \int_0^\pi u_x v_x + \varepsilon a_0(x) uv \, dx \, .$$

For $|\varepsilon||a_0|_\infty < 1$, $\langle \ , \ \rangle_\varepsilon$ actually defines a scalar product on $F_k \subset H^1_0((0,\pi); \mathbf{R})$ and its associated norm is equivalent to the H^1-norm defined by $\|u\|^2_{H^1} := \int_0^\pi u_x^2(x) \, dx$, since

$$\|u\|^2_{H^1}\Big(1 - |\varepsilon| \, |a_0|_\infty\Big) \leq \|u\|^2_\varepsilon \leq \|u\|^2_{H^1}\Big(1 + |\varepsilon| \, |a_0|_\infty\Big) \qquad \forall u \in F_k \, . \tag{79}$$

(79) follows from[7] $\int_0^\pi u(x)^2 \, dx \leq \int_0^\pi u_x(x)^2 \, dx$, $\forall u \in H^1_0(0,\pi)$, and

$$\Big| \int_0^\pi \varepsilon a_0(x) u^2 \, dx \Big| \leq |\varepsilon| \, |a_0|_\infty \int_0^\pi u^2 \, dx \, .$$

Lemma 4.1 (Sturm-Liouville) *The operator $S_k : \mathcal{D}(S_k) \subset F_k \to F_k$ possesses a $\langle \ , \ \rangle_\varepsilon$-orthonormal basis $(v_{k,j})_{j \geq 1, j \neq |k|}$ of eigenvectors with positive, simple eigenvalues*

$$0 < \lambda_{k,1} < \ldots < \lambda_{k,|k|-1} < \lambda_{k,|k|+1} < \ldots < \lambda_{k,j} < \ldots \qquad \text{with} \qquad \lim_{j \to \infty} \lambda_{k,j} = +\infty$$

and $\lambda_{k,j} = \lambda_{-k,j}$, $v_{-k,j} = v_{k,j}$.
Moreover, $(v_{k,j})_{j \geq 1, j \neq |k|}$ is an orthogonal basis also for the L^2-scalar product in F_k.

The asymptotic expansion as $j \to +\infty$ of the eigenfunctions $\varphi_{k,j} := v_{k,j}/\|v_{k,j}\|_{L^2}$ of S_k and its eigenvalues $\lambda_{k,j}$ is

$$\Big| \varphi_{k,j} - \sqrt{\frac{2}{\pi}} \sin(jx) \Big|_{L^2} = O\Big(\frac{\varepsilon |a_0|_\infty}{j}\Big)$$

and

$$\lambda_{k,j} = \lambda_{k,j}(\delta, v_1, w) = j^2 + \varepsilon M(\delta, v_1, w) + O\Big(\frac{\varepsilon \|a_0\|_{H^1}}{j}\Big) \tag{80}$$

where $M(\delta, v_1, w)$, introduced in Definition 3.1, is the mean value of $a_0(x)$ on $(0,\pi)$.

Proof. In the Appendix. We remark that we do not directly apply some known result for Sturm-Liouville operators because of the projection π_k. ■

By Lemma 4.1, each linear operator $D_k : \mathcal{D}(D_k) \subset F_k \to F_k$ possesses a $\langle \ , \ \rangle_\varepsilon$-orthonormal basis $(v_{k,j})_{j \geq 1, j \neq |k|}$ of real eigenvectors with real eigenvalues $(\omega^2 k^2 - \lambda_{k,j})_{j \geq 1, j \neq |k|}$.

As a consequence

Corollary 4.1 (Diagonalization of D) *The operator D (acting in $W^{(n)}$) is the diagonal operator* $\mathrm{diag}\{\omega^2 k^2 - \lambda_{k,j}\}$ *in the basis $\{\cos(kt)\varphi_{k,j} \ ; \ k \geq 0, j \geq 1, j \neq k\}$ of $W^{(n)}$.*

By Lemma 4.1,

$$\min_{|k| \leq L_n} |\omega^2 k^2 - \lambda_{k,j}| \to +\infty \qquad \text{as} \qquad j \to +\infty \, ,$$

and so, by Corollary 4.1, the linear operator D is invertible iff all its eigenvalues $\{\omega^2 k^2 - \lambda_{k,j}\}_{|k| \leq L_n, j \geq 1, j \neq |k|}$ are different from zero.

In this case, we can define D^{-1} as well as $|D|^{-1/2} : W^{(n)} \to W^{(n)}$ by

$$|D|^{-1/2} h := \sum_{|k| \leq L_n} \exp(\mathrm{i}kt) |D_k|^{-1/2} h_k \, , \qquad \forall h = \sum_{|k| \leq L_n} \exp(\mathrm{i}kt) h_k$$

[7] Because the least eigenvalue of $-\partial_{xx}$ with Dirichlet B.C. on $(0,\pi)$ is 1.

where $|D_k|^{-1/2} : F_k \to F_k$ is the diagonal operator defined by

$$|D_k|^{-1/2} v_{k,j} := \frac{v_{k,j}}{\sqrt{|\omega^2 k^2 - \lambda_{k,j}|}}, \qquad \forall j \geq 1, \ j \neq |k| . \tag{81}$$

The "small divisor problem" (i) is that some of the eigenvalues of D, $\omega^2 k^2 - \lambda_{k,j}$, can become arbitrarily small for $(k,j) \in \mathbf{Z}^2$ sufficiently large and therefore the norm of $|D|^{-1/2}$ can become arbitrarily large as $L_n \to \infty$.

In order to quantify this phenomenon we define for all $|k| \leq L_n$

$$\alpha_k := \min_{j \neq |k|} |\omega^2 k^2 - \lambda_{k,j}|. \tag{82}$$

Note that $\alpha_{-k} = \alpha_k$.

Lemma 4.2 *Suppose $\alpha_k \neq 0$. Then D_k is invertible and, for ε small enough,*

$$\Big\| |D_k|^{-1/2} u \Big\|_{H^1} \leq \frac{2}{\sqrt{\alpha_k}} \|u\|_{H^1} . \tag{83}$$

PROOF. For any $u = \sum_{j \neq |k|} u_j v_{k,j} \in F_k$, by (81), and using that $(v_{k,j})_{j \neq |k|}$ is an orthonormal basis for the $\langle \ , \ \rangle_\varepsilon$ scalar product on F_k,

$$\Big\| \, |D_k|^{-1/2} u \, \Big\|_\varepsilon^2 = \Big\| \sum_{j \neq |k|} \frac{u_j \, v_{k,j}}{\sqrt{|\omega^2 k^2 - \lambda_{k,j}|}} \Big\|_\varepsilon^2 = \sum_{j \neq |k|} \frac{|u_j|^2}{|\omega^2 k^2 - \lambda_{k,j}|} \leq \frac{1}{\alpha_k} \sum_{j \neq |k|} |u_j|^2 = \frac{\|u\|_\varepsilon^2}{\alpha_k} .$$

Hence, since, by (79), the norms $\| \cdot \|_\varepsilon$ and $\| \cdot \|_{H^1}$ are equivalent, (83) follows (for ε small enough). ■

The condition "$\alpha_k \neq 0$, $\forall |k| \leq L_n$" depends very sensitively on the parameters (δ, v_1). Assuming the "first order Melnikov non-resonance condition" $\delta \in \Delta_n^{\gamma,\tau}(v_1, w)$ (see Definition 3.3) with $\tau \in (1,2)$, we obtain, in Lemma 4.3, a lower bound of the form $c\gamma / |k|^{\tau - 1}$ for the moduli of the eigenvalues of D_k (namely $\alpha_k \geq c\gamma/|k|^{\tau-1}$) and, therefore, in Corollary 4.2, sufficiently good estimates for the inverse of D.

Lemma 4.3 (Lower bound for the eigenvalues of D) *There is $c > 0$ such that if $\delta \in \Delta_n^{\gamma,\tau}(v_1, w) \cap [0, \delta_0)$ and δ_0 is small enough (depending on γ), then*

$$\alpha_k := \min_{j \geq 1, j \neq |k|} |\omega^2 k^2 - \lambda_{k,j}| \geq \frac{c\,\gamma}{|k|^{\tau - 1}} > 0, \qquad \forall \ 0 < |k| \leq L_n . \tag{84}$$

Moreover $\alpha_0 \geq 1/2$.

PROOF. Since $\alpha_{-k} = \alpha_k$ it is sufficient to consider $k \geq 0$. By the asymptotic expansion (80) for the eigenvalues $\lambda_{k,j}$, using that $\|a_0\|_{H^1}$, $|M(\delta, v_1, w)| \leq C$,

$$\begin{aligned}
|\omega^2 k^2 - \lambda_{k,j}| &= \Big| \omega^2 k^2 - j^2 - \varepsilon M(\delta, v_1, w) + O\Big(\frac{\varepsilon \|a_0\|_{H^1}}{j} \Big) \Big| \\
&= \Big| \Big(\omega k - \sqrt{j^2 + \varepsilon M(\delta, v_1, w)} \Big) \Big(\omega k + \sqrt{j^2 + \varepsilon M(\delta, v_1, w)} \Big) + O\Big(\frac{|\varepsilon|}{j} \Big) \Big| \\
&\geq \Big| \omega k - j - \varepsilon \frac{M(\delta, v_1, w)}{2j} + O\Big(\frac{\varepsilon^2}{j^3} \Big) \Big| \omega k - C \frac{|\varepsilon|}{j} \\
&\geq \Big| \omega k - j - \varepsilon \frac{M(\delta, v_1, w)}{2j} \Big| \, \omega k - C' \Big(\frac{\varepsilon^2 k}{j^3} + \frac{|\varepsilon|}{j} \Big) \geq \frac{\gamma \omega k}{(k+j)^\tau} - C \Big(\frac{\varepsilon^2 k}{j^3} + \frac{|\varepsilon|}{j} \Big),
\end{aligned} \tag{85}$$

since $\delta \in \Delta_n^{\gamma,\tau}(v_1, w)$. If $\alpha_k := \min_{j \geq 1, j \neq k} |\omega^2 k^2 - \lambda_{k,j}|$ is attained at $j = j(k)$, i.e. $\alpha_k = |\omega^2 k^2 - \lambda_{k,j}|$, then $|\omega k - j| \leq 1$ (provided $|\varepsilon|$ is small enough). Therefore, using that $1 < \tau < 2$ and $|\omega - 1| \leq 2|\varepsilon|$, we can derive (84) from (85), for $|\varepsilon|$ small enough. ■

Corollary 4.2 (Estimate of $|D|^{-1/2}$) *If* $\delta \in \Delta_n^{\gamma,\tau}(v_1,w) \cap [0,\delta_0)$ *and* δ_0 *is small enough, then* $D : \mathcal{D}(D) \subset W^{(n)} \to W^{(n)}$ *is invertible and* $\forall s' \geq 0$

$$\Big\| |D|^{-1/2} h \Big\|_{\sigma,s'} \leq \frac{C}{\sqrt{\gamma}} \|h\|_{\sigma,s'+\frac{\tau-1}{2}} \qquad \forall h \in W^{(n)}. \tag{86}$$

PROOF. Since $|D|^{-1/2}h := \sum_{|k|\leq L_n} \exp(\mathrm{i}kt)|D_k|^{-1/2}h_k$, we get, using (83) and (84),

$$\begin{aligned}
\Big\| |D|^{-1/2} h \Big\|^2_{\sigma,s'} &= \sum_{|k|\leq L_n} \exp(2\sigma|k|)(1+k^{2s'}) \Big\| |D_k|^{-1/2} h_k \Big\|^2_{H^1} \\
&\leq \sum_{|k|\leq L_n} \exp(2\sigma|k|)(1+k^{2s'}) \frac{4}{\alpha_k} \|h_k\|^2_{H^1} \\
&\leq 8\|h_0\|^2_{H^1} + C \sum_{0<|k|\leq L_n} \exp(2\sigma|k|)(1+k^{2s'}) \frac{|k|^{(\tau-1)}}{\gamma} \|h_k\|^2_{H^1} \\
&\leq \frac{C'}{\gamma} \|h\|^2_{\sigma,s'+\frac{\tau-1}{2}}
\end{aligned}$$

proving (86). ■

4.3 Step 2: Inversion of $\mathcal{L}_n$

To show the invertibility of $\mathcal{L}_n : D(\mathcal{L}_n) \subset W^{(n)} \to W^{(n)}$ it is a convenient devise to write

$$\mathcal{L}_n = D - \mathcal{M}_1 - \mathcal{M}_2 = |D|^{1/2}\Big(U - \mathcal{R}_1 - \mathcal{R}_2\Big)|D|^{1/2}$$

where

$$U := |D|^{-1/2} D |D|^{-1/2} = |D|^{-1} D \qquad \text{and} \qquad \mathcal{R}_i := |D|^{-1/2} \mathcal{M}_i |D|^{-1/2}, \quad i = 1,2\,.$$

We shall prove the invertibility of $U - \mathcal{R}_1 - \mathcal{R}_2$ showing that, for ε small enough, $\mathcal{R}_1$ and $\mathcal{R}_2$ are small perturbations of U.

Lemma 4.4 (Estimate of $\|U^{-1}\|$) $U : W^{(n)} \to W^{(n)}$ *is an invertible operator and its inverse* U^{-1} *satisfies,* $\forall s' \geq 0$,

$$\Big\| U^{-1} h \Big\|_{\sigma,s'} = \|h\|_{\sigma,s'} \Big(1 + O(\varepsilon \|a_0\|_{H^1})\Big) \qquad \forall\ h \in W^{(n)}. \tag{87}$$

PROOF. $U_k := |D_k|^{-1} D_k : F_k \to F_k$ being orthogonal for the $\langle\ ,\ \rangle_\varepsilon$ scalar product, it is invertible and $\forall u \in F_k, \|U_k^{-1}u\|_\varepsilon = \|u\|_\varepsilon$. Hence, by (79),

$$\forall u \in F_k, \quad \|U_k^{-1}u\|_{H^1} = \|u\|_\varepsilon (1 + O(\varepsilon\|a_0\|_{H^1}))\,.$$

Therefore, $U = |D|^{-1}D$, being defined by $(Uh)_k = U_k h_k$, $\forall\ |k| \leq L_n$, U is invertible, $(U^{-1}h)_k = U_k^{-1} h_k$ and (87) holds. ■

The estimate of the "off-diagonal" operator $\mathcal{R}_1 : W^{(n)} \to W^{(n)}$ requires a careful analysis of the "*small divisors*" and the use of the "first order Melnikov non-resonance condition" $\delta \in \Delta_n^{\gamma,\tau}(v_1,w)$, see Definition 3.3. For clarity, we enounce such property separately.

Lemma 4.5 (Analysis of the Small Divisors) *Let* $\delta \in \Delta_n^{\gamma,\tau}(v_1,w) \cap [0,\delta_0)$, *with* δ_0 *small. There exists* $C > 0$ *such that,* $\forall l \neq k$,

$$\frac{1}{\alpha_k \alpha_l} \leq C \frac{|k-l|^{2\frac{\tau-1}{\beta}}}{\gamma^2 |\varepsilon|^{\tau-1}} \qquad \text{where} \qquad \beta := \frac{2-\tau}{\tau}. \tag{88}$$

PROOF. To obtain (88) we distinguish different cases.

- FIRST CASE: $|k-l| \geq (1/2)[\max(|k|,|l|)]^\beta$. Then $(\alpha_k \alpha_l)^{-1} \leq C|k-l|^{2\frac{\tau-1}{\beta}}/\gamma^2$.

Indeed we can estimate both α_k, α_l with the lower bound (84), $\alpha_k \geq c\gamma/|k|^{\tau-1}$, $\alpha_l \geq c\gamma/|l|^{\tau-1}$. Using that $0 < \beta < 1$, we obtain

$$\frac{1}{\alpha_k \alpha_l} \leq C \frac{|k|^{\tau-1}|l|^{\tau-1}}{\gamma^2} \leq C \frac{[\max(|k|,|l|)]^{2(\tau-1)}}{\gamma^2} \leq C' \frac{|k-l|^{2\frac{\tau-1}{\beta}}}{\gamma^2}.$$

In the other cases we have $0 < |k-l| < (1/2)[\max(|k|,|l|)]^\beta$. We observe that in this situation, $\mathrm{sign}(l) = \mathrm{sign}(k)$ and, to fix the ideas, we assume in the sequel that $l, k \geq 0$ (the estimate for $k, l < 0$ is the same, since $\alpha_k \alpha_l = \alpha_{-k}\alpha_{-l}$). Moreover, since $\beta \leq 1$, we have $\max(k,l) = k$ or $l - k \leq (1/2)l^\beta \leq (1/2)l$. Hence $l \leq 2k$; similarly $k \leq 2l$.

- SECOND CASE: $0 < |k-l| < (1/2)[\max(|k|,|l|)]^\beta$ and ($|k| \leq 1/3|\varepsilon|$ or $|l| \leq 1/3|\varepsilon|$). Then $(\alpha_k\alpha_l)^{-1} \leq C/\gamma$.

Suppose, for example, that $0 \leq k \leq 1/3|\varepsilon|$. We claim that if ε is small enough, then $\alpha_k \geq (k+1)/8$. Indeed, $\forall j \neq k$,

$$|\omega k - j| = \Big|\omega k - k + k - j\Big| \geq |k-j| - |\omega - 1|\ |k| \geq 1 - 2|\varepsilon|\ k \geq \frac{1}{3}.$$

Therefore $\forall 1 \leq k < 1/3|\varepsilon|$, $\forall j \neq k$, $j \geq 1$, $|\omega^2 k^2 - j^2| = |\omega k - j|\ |\omega k + j| \geq (\omega k + 1)/3 \geq (k+1)/6$ and so

$$\alpha_k := \min_{j \geq 1, k \neq j} \Big|\omega^2 k^2 - \lambda_{k,j}\Big| = \min_{j\geq 1, k\neq j} \Big|\omega^2 k^2 - j^2 - \varepsilon M(\delta, v_1, w) + O\Big(\frac{\varepsilon \|a_0\|_{H^1}}{j}\Big)\Big| \geq \frac{k+1}{6} - |\varepsilon|\ C \geq \frac{k+1}{8}.$$

Next, we estimate α_l. If $0 \leq l \leq 1/3|\varepsilon|$ then $\alpha_l \geq 1/8$ and therefore $(\alpha_k\alpha_l)^{-1} \leq 64$. If $l > 1/3|\varepsilon|$, we estimate α_l with the lower bound (84) and so, since $l \leq 2k$ and $1 < \tau < 2$

$$\frac{1}{\alpha_k\alpha_l} \leq C \frac{l^{\tau-1}}{k\gamma} \leq \frac{C'}{k^{2-\tau}\gamma} \leq \frac{C'}{\gamma}.$$

In the remaining cases we consider $|k-l| < (1/2)[\max(|k|,|l|)]^\beta$ and both $|k|, |l| > 1/3|\varepsilon|$. We have to distinguish two sub-cases. For this, $\forall k \in \mathbf{Z}$, let $j = j(k) \geq 1$ be an integer such that $\alpha_k := \min_{n \neq |k|} |\omega^2 k^2 - \lambda_{k,n}| = |\omega^2 k^2 - \lambda_{k,j}|$. Analogously let $i = i(k) \geq 1$ be an integer such that $\alpha_l = |\omega^2 l^2 - \lambda_{l,i}|$.

- THIRD CASE: $0 < |k-l| < (1/2)[\max(|k|,|l|)]^\beta$, $|k|, |l| > 1/3|\varepsilon|$ and $k - l = j - i$. Then $(\alpha_k\alpha_l)^{-1} \leq C/\gamma|\varepsilon|^{\tau-1}$.

Indeed $|(\omega k - j) - (\omega l - i)| = |\omega(k-l) - (j-i)| = |\omega - 1||k-l| \geq |\varepsilon|/2$ and therefore $|\omega k - j| \geq |\varepsilon|/4$ or $|\omega l - i| \geq |\varepsilon|/4$. Assume for instance that $|\omega k - j| \geq |\varepsilon|/4$. Then $|\omega^2 k^2 - j^2| = |\omega k - j|\ |\omega k + j| \geq |\varepsilon|\omega k/2 \geq |\varepsilon|(1 - 2|\varepsilon|)k/2$ and so, for ε small enough, $|\alpha_k| \geq |\varepsilon|k/4$. Hence, since $l \leq 2k$ and $k > 1/3|\varepsilon|$,

$$\frac{1}{\alpha_k\alpha_l} \leq C \frac{l^{\tau-1}}{\gamma|\varepsilon|k} \leq \frac{C}{\gamma k^{2-\tau}|\varepsilon|} \leq \frac{C}{\gamma|\varepsilon|^{\tau-1}}.$$

- FOURTH CASE: $0 < |k-l| < (1/2)[\max(k,l)]^\beta$, $k, l > 1/3|\varepsilon|$ and $k - l \neq j - i$. Then $(\alpha_k\alpha_l)^{-1} \leq C/\gamma^2$.

Using that ω is γ-τ-Diophantine, we get

$$\Big|(\omega k - j) - (\omega l - i)\Big| = \Big|\omega(k-l) - (j-i)\Big| \geq \frac{\gamma}{|k-l|^\tau} \geq \frac{C\gamma}{[\max(k,l)]^{\beta\tau}} \geq \frac{C}{2}\Big(\frac{\gamma}{k^{\beta\tau}} + \frac{\gamma}{l^{\beta\tau}}\Big)$$

so that $|\omega k - j| \geq C\gamma/2k^{\beta\tau}$ or $|\omega l - i| \geq C\gamma/2l^{\beta\tau}$. Therefore $|\omega^2 k^2 - j^2| \geq C'\gamma k^{1-\beta\tau} = C'\gamma k^{\tau-1}$, since $\beta := (2-\tau)/\tau$. Hence, for ε small enough, $\alpha_k \geq C'\gamma k^{\tau-1}/2$. We estimate α_l with the worst possible lower bound and so, using also $l \leq 2k$, we obtain

$$\frac{1}{\alpha_k \alpha_l} \leq \frac{Cl^{\tau-1}}{\gamma^2 k^{\tau-1}} \leq \frac{C}{\gamma^2}.$$

Collecting the estimates of all the previous cases, (88) follows. ■

Remark 4.1 *The analysis of the small divisors in the Cases II)-III)-IV) of the previous Lemma corresponds, in the language of [12], to the property of "separation of the singular sites".*

Lemma 4.6 (Bound of an off-diagonal operator) *Assume that $\delta \in \Delta_n^{\gamma,\tau}(v_1, w) \cap [0, \delta_0)$ and let, for some $s' \geq s$, $b(t,x) \in X_{\sigma, s'+\frac{\tau-1}{\beta}}$ satisfy $b_0(x) = 0$, i.e. $\int_0^{2\pi} b(t,x)\, dt \equiv 0$, $\forall x \in (0,\pi)$. Define the operator $T_n : W^{(n)} \to W^{(n)}$ by*

$$T_n h := |D|^{-1/2} P_n \Pi_W \Big(b(t,x)\ |D|^{-1/2} h \Big).$$

There is a constant $\widetilde{C}$, independent of $b(t,x)$ and of n, such that

$$\Big\| T_n h \Big\|_{\sigma,s'} \leq \frac{\widetilde{C}}{|\varepsilon|^{\frac{\tau-1}{2}}\gamma} \|b\|_{\sigma, s'+\frac{\tau-1}{\beta}} \|h\|_{\sigma,s'} \qquad \forall h \in W^{(n)}.$$

Proof. For $h \in W^{(n)}$, we have $(T_n h)(t,x) = \sum_{|k| \leq L_n} (T_n h)_k(x) \exp(\mathrm{i}kt)$, with

$$(T_n h)_k = |D_k|^{-1/2} \pi_k \Big(b\ |D|^{-1/2} h \Big)_k = |D_k|^{-1/2} \pi_k \Big[\sum_{|l| \leq L_n} b_{k-l} |D_l|^{-1/2} h_l \Big]. \tag{89}$$

Set $B_m := \|b_m(x)\|_{H^1}$. From (89) and (83), using that $B_0 := \|b_0(x)\|_{H^1} = 0$,

$$\Big\| (T_n h)_k \Big\|_{H^1} \leq C \sum_{|l| \leq L_n, l \neq k} \frac{B_{k-l}}{\sqrt{\alpha_k}\sqrt{\alpha_l}} \|h_l\|_{H^1}. \tag{90}$$

Hence, by (88)

$$\Big\| (T_n h)_k \Big\|_{H^1} \leq \frac{C}{\gamma |\varepsilon|^{\frac{\tau-1}{2}}} s_k \qquad \text{where} \qquad s_k := \sum_{|l| \leq L_n} B_{k-l} |k-l|^{\frac{\tau-1}{\beta}} \|h_l\|_{H^1}. \tag{91}$$

By (91), setting $\widetilde{s}(t) := \sum_{|k| \leq L_n} s_k \exp(\mathrm{i}kt)$ (with $s_{-k} = s_k$),

$$\begin{aligned} \Big\| T_n h \Big\|^2_{\sigma,s'} &= \sum_{|k| \leq L_n} \exp(2\sigma|k|)(k^{2s'}+1) \Big\| (T_n h)_k \Big\|^2_{H^1} \\ &\leq \frac{C^2}{\gamma^2 |\varepsilon|^{\tau-1}} \sum_{|k| \leq L_n} \exp(2\sigma|k|)(k^{2s'}+1) s_k^2 = \frac{C^2}{\gamma^2 |\varepsilon|^{\tau-1}} \|\widetilde{s}\|^2_{\sigma,s'}. \end{aligned} \tag{92}$$

It turns out that $\widetilde{s} = P_n(\widetilde{b}\widetilde{c})$ where $\widetilde{b}(t) := \sum_{l \in \mathbf{Z}} |l|^{\frac{\tau-1}{\beta}} B_l \exp(\mathrm{i}lt)$ and $\widetilde{c}(t) := \sum_{|l| \leq L_n} \|h_l\|_{H^1} \exp(\mathrm{i}lt)$. Therefore, by (92) and since $s' > 1/2$,

$$\Big\| T_n h \Big\|_{\sigma,s'} \leq \frac{C}{\gamma |\varepsilon|^{\frac{\tau-1}{2}}} \|\widetilde{b}\widetilde{c}\|_{\sigma,s'} \leq \frac{C}{\gamma |\varepsilon|^{\frac{\tau-1}{2}}} \|\widetilde{b}\|_{\sigma,s'} \|\widetilde{c}\|_{\sigma,s'} \leq \frac{C}{\gamma |\varepsilon|^{\frac{\tau-1}{2}}} \|b\|_{\sigma, s'+\frac{\tau-1}{\beta}} \|h\|_{\sigma,s'}$$

since $\|\widetilde{b}\|_{\sigma,s'} \leq \|b\|_{\sigma,s'+\frac{\tau-1}{\beta}}$ and $\|\widetilde{c}\|_{\sigma,s'} = \|h\|_{\sigma,s'}$. ■

Before proving the smallness of the "off-diagonal" operator $\mathcal{R}_1$ and of $\mathcal{R}_2$ we need the following preliminary Lemma which gives a suitable estimate of the multiplicative function $a(t,x)$.

Lemma 4.7 *There are* $\mu > 0$, $\delta_0 > 0$ *and* $C > 0$ *with the following property : if* $\|v_1\|_{0,s} \le 2R$, $[w]_{\sigma,s} \le \mu$ *and* $\delta \in [0, \delta_0)$, *then* $\|a\|_{\sigma, s + \frac{2(\tau-1)}{\beta}} \le C$.

PROOF. By the Definition 3.2 of $[w]_{\sigma,s}$ there are $h_i \in W^{(i)}$, $0 \le i \le q$, and a sequence $(\sigma_i)_{0 \le i \le q}$ with $\sigma_i > \sigma$, such that $w = h_0 + h_1 + \ldots + h_q$ and

$$\sum_{i=0}^{q} \frac{\|h_i\|_{\sigma_i,s}}{(\sigma_i - \sigma)^{\frac{2(\tau-1)}{\beta}}} \le 2[w]_{\sigma,s} \le 2\mu\,. \tag{93}$$

An elementary calculus, using that $\max_{k \ge 1} k^\alpha \exp\{-(\sigma_i - \sigma)k\} \le C(\alpha)/(\sigma_i - \sigma)^\alpha$, gives

$$\|h_i\|_{\sigma, s + \frac{2(\tau-1)}{\beta}} \le C(\tau) \frac{\|h_i\|_{\sigma_i,s}}{(\sigma_i - \sigma)^{\frac{2(\tau-1)}{\beta}}}\,. \tag{94}$$

Hence, by (93)-(94)

$$\|w\|_{\sigma, s + \frac{2(\tau-1)}{\beta}} \le \sum_{i=0}^{q} \|h_i\|_{\sigma, s + \frac{2(\tau-1)}{\beta}} \le \sum_{i=0}^{q} C(\tau) \frac{\|h_i\|_{\sigma_i,s}}{(\sigma_i - \sigma)^{\frac{2(\tau-1)}{\beta}}} \le C(\tau) 2\mu\,.$$

By Lemma 2.1-d), provided δ_0 is small enough, also $\|v_2(\delta, v_1, w)\|_{\sigma, s + \frac{2(\tau-1)}{\beta}} \le C'$ and therefore

$$\|a\|_{\sigma, s + \frac{2(\tau-1)}{\beta}} = \Big\| (\partial_u g)\Big(\delta, x, v_1 + w + v_2(\delta, v_1, w)\Big) \Big\|_{\sigma, s + \frac{2(\tau-1)}{\beta}} \le C.$$

This bound is a consequence of the analyticity assumption (**H**) on the nonlinearity f, the Banach algebra property of $X_{\sigma, s + 2(\tau-1)/\beta}$, and can be obtained as in (22). ■

Lemma 4.8 (Estimate of $\mathcal{R}_1$) *Under the hypotheses of (P3), there exists a constant* $C > 0$ *depending on* μ *such that*

$$\Big\| \mathcal{R}_1 h \Big\|_{\sigma, s + \frac{\tau-1}{2}} \le |\varepsilon|^{\frac{3-\tau}{2}} \frac{C}{\gamma} \|h\|_{\sigma, s + \frac{\tau-1}{2}} \qquad \forall h \in W^{(n)}.$$

PROOF. Recalling the definition of $\mathcal{R}_1 := |D|^{-1/2} \mathcal{M}_1 |D|^{-1/2}$ and $\mathcal{M}_1$, and using Lemma 4.6 since $\overline{a}(t,x)$ has zero time-average,

$$\begin{aligned}
\Big\| \mathcal{R}_1 h \Big\|_{\sigma, s + \frac{\tau-1}{2}} &= \Big\| |D|^{-1/2} \mathcal{M}_1 |D|^{-1/2} h \Big\|_{\sigma, s + \frac{\tau-1}{2}} = |\varepsilon| \, \Big\| |D|^{-1/2} P_n \Pi_W \Big(\overline{a}\; |D|^{-1/2} h \Big) \Big\|_{\sigma, s + \frac{\tau-1}{2}} \\
&\le |\varepsilon| \, \frac{\widetilde{C}}{|\varepsilon|^{\frac{\tau-1}{2}} \gamma} \|\overline{a}\|_{\sigma, s + \frac{\tau-1}{2} + \frac{\tau-1}{\beta}} \|h\|_{\sigma, s + \frac{\tau-1}{2}} \le |\varepsilon|^{\frac{3-\tau}{2}} \, \frac{\widetilde{C}}{\gamma} \|\overline{a}\|_{\sigma, s + \frac{2(\tau-1)}{\beta}} \|h\|_{\sigma, s + \frac{\tau-1}{2}} \\
&\le |\varepsilon|^{\frac{3-\tau}{2}} \frac{C}{\gamma} \|h\|_{\sigma, s + \frac{\tau-1}{2}}
\end{aligned}$$

since $0 < \beta < 1$ and, by Lemma 4.7, $\|\overline{a}\|_{\sigma, s + \frac{2(\tau-1)}{\beta}} \le \|a\|_{\sigma, s + \frac{2(\tau-1)}{\beta}} \le C$. ■

The "smallness" of $\mathcal{R}_2 := |D|^{-1/2} \mathcal{M}_2 |D|^{-1/2}$ w.r.t. U, is just a consequence of Lemma 4.7 and of the regularizing property of $\partial_w v_2 : X_{\sigma,s} \to X_{\sigma, s+2}$ proved in Lemma 2.1: by (86) the "loss of $\tau - 1$ derivatives" due to $|D|^{-1/2}$ applied twice, is compensated by the gain of 2 derivatives due to $\partial_w v_2 : X_{\sigma,s} \to X_{\sigma,s+2}$.

Lemma 4.9 (Estimate of $\mathcal{R}_2$) *Under the hypotheses of (P3), there exists a constant* $C > 0$ *depending on* μ *such that*

$$\Big\| \mathcal{R}_2 h \Big\|_{\sigma, s + \frac{\tau-1}{2}} \le C \frac{|\varepsilon|}{\gamma} \; \|h\|_{\sigma, s + \frac{\tau-1}{2}} \qquad \forall h \in W^{(n)}.$$

PROOF. By (86) and the regularizing estimates $\|\partial_w v_2[u]\|_{\sigma,s+2} \le C\|u\|_{\sigma,s}$ of Lemma 2.1 we get

$$\begin{aligned}
\Big\|\mathcal{R}_2 h\Big\|_{\sigma,s+\frac{\tau-1}{2}} &\le \frac{C}{\sqrt{\gamma}}\Big\|\mathcal{M}_2|D|^{-1/2}h\Big\|_{\sigma,s+\tau-1} = C\frac{|\varepsilon|}{\sqrt{\gamma}}\ \Big\|P_n\Pi_W\Big(a\ \partial_w v_2\Big[|D|^{-1/2}h\Big]\Big)\Big\|_{\sigma,s+\tau-1}\\
&\le C\frac{|\varepsilon|}{\sqrt{\gamma}}\ \|a\|_{\sigma,s+\tau-1}\Big\|\partial_w v_2\Big[|D|^{-1/2}h\Big]\Big\|_{\sigma,s+\tau-1}\\
&\le C'\frac{|\varepsilon|}{\sqrt{\gamma}}\ \|a\|_{\sigma,s+\tau-1}\Big\|\partial_w v_2\Big[|D|^{-1/2}h\Big]\Big\|_{\sigma,s+2}\\
&\le C\frac{|\varepsilon|}{\sqrt{\gamma}}\ \|a\|_{\sigma,s+\tau-1}\Big\||D|^{-1/2}h\Big\|_{\sigma,s} \le C'\frac{|\varepsilon|}{\gamma}\ \|h\|_{\sigma,s+\frac{\tau-1}{2}}
\end{aligned}$$

since $\tau < 3$ and by Lemma 4.7, $\|a\|_{\sigma,s+\tau-1} \le \|a\|_{\sigma,s+\frac{2(\tau-1)}{\beta}} \le C$. ∎

PROOF OF PROPERTY $(P3)$ COMPLETED. Under the hypothesis of $(P3)$, the linear operator U is invertible by Lemma 4.4 and, by Lemmas 4.9 and 4.8, provided δ is small enough,

$$\Big\|U^{-1}\mathcal{R}_1\Big\|_{\sigma,s+\frac{\tau-1}{2}}\,,\ \Big\|U^{-1}\mathcal{R}_2\Big\|_{\sigma,s+\frac{\tau-1}{2}} < \frac{1}{4}\,.$$

Therefore also the linear operator $U - \mathcal{R}_1 - \mathcal{R}_2$ is invertible and its inverse satisfies

$$\Big\|(U-\mathcal{R}_1-\mathcal{R}_2)^{-1}h\Big\|_{\sigma,s+\frac{\tau-1}{2}} = \Big\|(I-U^{-1}\mathcal{R}_1-U^{-1}\mathcal{R}_2)^{-1}U^{-1}h\Big\|_{\sigma,s+\frac{\tau-1}{2}} \tag{95}$$

$$\le 2\|U^{-1}h\|_{\sigma,s+\frac{\tau-1}{2}} \le C\|h\|_{\sigma,s+\frac{\tau-1}{2}} \qquad \forall h \in W^{(n)}. \tag{96}$$

Hence $\mathcal{L}_n$ is invertible, $\mathcal{L}_n^{-1} = |D|^{-1/2}(U-\mathcal{R}_1-\mathcal{R}_2)^{-1}|D|^{-1/2} : W^{(n)} \to W^{(n)}$, and by (86), (95),

$$\begin{aligned}
\|\mathcal{L}_n^{-1}h\|_{\sigma,s} &= \Big\||D|^{-1/2}(U-\mathcal{R}_1-\mathcal{R}_2)^{-1}|D|^{-1/2}h\Big\|_{\sigma,s} \le \frac{C}{\sqrt{\gamma}}\Big\|(U-\mathcal{R}_1-\mathcal{R}_2)^{-1}|D|^{-1/2}h\Big\|_{\sigma,s+\frac{\tau-1}{2}}\\
&\le \frac{C'}{\sqrt{\gamma}}\Big\||D|^{-1/2}h\Big\|_{\sigma,s+\frac{\tau-1}{2}} \le \frac{C''}{\gamma}\|h\|_{\sigma,s+\tau-1} \le \frac{C''}{\gamma}(L_n)^{\tau-1}\|h\|_{\sigma,s}
\end{aligned}$$

because $h \in W^{(n)}$. This completes the proof of property $(P3)$.

5 Solution of the $(Q1)$-equation

Once the $(Q2)$ and (P)-equations are solved (with "gaps" for the latter), the last step is to find solutions of the finite dimensional $(Q1)$-equation

$$-\Delta v_1 = \Pi_{V_1}\mathcal{G}(\delta, v_1) \tag{97}$$

where

$$\mathcal{G}(\delta,v_1)(t,x) := g\Big(\delta, x, v_1(t,x) + \widetilde{w}(\delta,v_1)(t,x) + v_2(\delta,v_1,\widetilde{w}(\delta,v_1))(t,x)\Big)\,.$$

We are interested in solutions (δ, v_1) which belong to the Cantor set B_∞.

5.1 The $(Q1)$-equation for $\delta = 0$

For $\delta = 0$ the $(Q1)$-equation (97) reduces to

$$-\Delta v_1 = \Pi_{V_1}\mathcal{G}(0,v_1) = s^*\Pi_{V_1}\Big(a_p(x)(v_1 + v_2(0,v_1,0))^p\Big) \tag{98}$$

which is the Euler-Lagrange equation of $\Psi_0 : B(2R, V_1) \to \mathbf{R}$,

$$\Psi_0(v_1) := \Phi_0(v_1 + v_2(0,v_1,0))\,, \tag{99}$$

where $\Phi_0 : V \to \mathbf{R}$ is defined in (12).

In fact, since $v_2(0, v_1, 0)$ solves the $(Q2)$-equation (for $\delta = 0$, $w = 0$), $d\Phi_0(v_1 + v_2(0, v_1, 0))[k] = 0$, $\forall k \in V_2$. Moreover, since $\forall h \in V_1$, $D_{v_1} v_2(0, v_1, 0)[h] \in V_2$,

$$\begin{aligned} d\Psi_0(v_1)[h] &= d\Phi_0(v_1 + v_2(0, v_1, 0))\Big[h + D_{v_1} v_2(0, v_1, 0)[h]\Big] = d\Phi_0(v_1 + v_2(0, v_1, 0))[h] \\ &= \int_\Omega \Big[- \Delta v_1 - s^* \Pi_{V_1} \Big(a_p(x)(v_1 + v_2(0, v_1, 0))\Big)^p \Big] h \,. \end{aligned} \tag{100}$$

Hence $\overline{v}_1$ is a critical point of Ψ_0 iff it is a solution of equation (98).

Lemma 5.1 *Let $\overline{v}$ be the non-degenerate solution of equation* (11) *introduced in Theorem 1.2. Then $\overline{v}_1 := \Pi_{V_1} \overline{v} \in B(R; V_1)$ is a non-degenerate solution of (98).*

Proof. By Lemma 2.1-**b)**, $\Pi_{V_2} \overline{v} = v_2(0, \overline{v}_1, 0)$. Hence, since $\overline{v}$ solves (11), $\overline{v}_1$ solves (98). Now assume that $h_1 \in V_1$ is a solution of the linearized equation at $\overline{v}_1$ of (98). This means

$$-\Delta h_1 = s^* \Pi_{V_1} \Big(p a_p(x)(\overline{v}_1 + v_2(0, \overline{v}_1, 0))^{p-1}(h_1 + h_2)\Big) \tag{101}$$

where $h_2 := D_{v_1} v_2(0, \overline{v}_1, 0)[h_1] \in V_2$. Now, by the definition of the map v_2, we have

$$-\Delta v_2(0, v_1, 0) = s^* \Pi_{V_2} \Big(a_p(x)(v_1 + v_2(0, v_1, 0))^p \Big) , \qquad \forall v_1 \in B(2R, V_1) \ ,$$

from which we derive, taking the differential at $\overline{v}_1$,

$$-\Delta h_2 = s^* \Pi_{V_2} \Big(p a_p(x)(\overline{v}_1 + v_2(0, \overline{v}_1, 0))^{p-1}(h_1 + h_2)\Big) \,. \tag{102}$$

Summing (101) and (102), we obtain that $h = h_1 + h_2$ is a solution of the linearized form at $\overline{v}$ of equation (11). Since $\overline{v}$ is a non-degenerate solution of (11), $h = 0$, hence $h_1 = 0$. As a result, $\overline{v}_1 = \Pi_{V_1} \overline{v}$ is a non-degenerate solution of (98). ■

5.2 Proof of Theorem 1.2

By assumption, $\overline{v}$ is a non-degenerate solution of equation (11). Hence, by Lemma 5.1, $\overline{v}_1 = \Pi_{V_1} \overline{v} \in B(R, V_1)$ is a non degenerate solution of (98).

Since the map $(\delta, v_1) \to -\Delta v_1 - \Pi_{V_1} \mathcal{G}(\delta, v_1)$ is in $C^\infty([0, \delta_0) \times V_1; V_1)$, by the Implicit Function Theorem, there is a C^∞ path

$$\delta \mapsto v_1(\delta) \in B(2R, V_1)$$

such that $v_1(\delta)$ is a solution of (97) and $v_1(0) = \overline{v}_1$.

By Theorem 3.1, the function

$$\widetilde{u}(\delta) := \delta \Big[v_1(\delta) + v_2\Big(\delta, v_1(\delta), \widetilde{w}(\delta, v_1(\delta))\Big) + \widetilde{w}(\delta, v_1(\delta))\Big] \in X_{\overline{\sigma}/2, s} \tag{103}$$

is a solution of equation (3) if δ belongs to the Cantor-like set

$$\mathcal{C} := \Big\{ \delta \in [0, \delta_0) \mid (\delta, v_1(\delta)) \in B_\infty \Big\} \,.$$

By Proposition 3.2, the smoothness of $v_1(\cdot)$ implies that the Cantor set $\mathcal{C}$ has full density at the origin, i.e. satisfies the measure estimate (4).

Finally, by (103), since $\overline{v} = \overline{v}_1 + v_2(0, \overline{v}_1, 0)$,

$$\begin{aligned} \Big\| \widetilde{u}(\delta) - \delta \overline{v} \Big\|_{\overline{\sigma}/2, s} &= \delta \Big\| (v_1(\delta) - \overline{v}_1) + \Big(v_2(\delta, v_1(\delta), \widetilde{w}(\delta, v_1(\delta))) - v_2(0, \overline{v}_1, 0)\Big) + \widetilde{w}(\delta, v_1(\delta))\Big\|_{\overline{\sigma}/2, s} \\ &\le \delta \Big(\| v_1(\delta) - \overline{v}_1 \|_{\overline{\sigma}/2, s} + \Big\| v_2(\delta, v_1(\delta), \widetilde{w}(\delta, v_1(\delta))) - v_2(0, \overline{v}_1, 0)\Big\|_{\overline{\sigma}/2, s} \\ &+ \ \| \widetilde{w}(\delta, v_1(\delta)) \|_{\overline{\sigma}/2, s} \Big) = O(\delta^2) \end{aligned}$$

by (57).

This proves Theorem 1.2 in the case when $\overline{v}$ is non-degenerate in the whole space V.

Now, we can look for $2\pi/n$ time-periodic solutions of (3) as well (they are particular 2π periodic solutions). Let

$$X_{\sigma,s,n} := \Big\{ u \in X_{\sigma,s} \mid u \text{ is } \frac{2\pi}{n} \text{ time} - \text{periodic} \Big\} = V_n \oplus W_n$$

where V_n (defined in (14)) and W_n are the subspaces of V and W formed by the functions $2\pi/n$-periodic in t.

Introducing an appropriate finite dimensional subspace $V_{1,n} \subset V_n$ we split $V_n = V_{1,2} \oplus V_{2,n}$ and we obtain associated $(Q1)$,$(Q2)$,(P)-equations as in (15).

With the arguments of sections 2 and 3 we can solve the $(Q2)$ and (P)-equations exactly as in the case $n = 1$.

The $0th$-order bifurcation equation is again equation (11), but in V_n, and the corresponding functional is just the restriction of Φ_0 to V_n.

The main assumption of Theorem 1.2 -that at least one of the critical points of $(\Phi_0)_{|V_n}$, called $\overline{v}$, is non-degenerate- allows to find a C^∞-path $\delta \mapsto v_1(\delta) \in V_{1,n}$ of solutions of equation (97).

As above this implies the conclusions of Theorem 1.2.

6 Proof of Theorem 1.1

For this section we define the linear map $\mathcal{H}_n : V \to V$ by:

$$\text{for } \; v(t,x) = \eta(t+x) - \eta(t-x) \in V \,, \quad (\mathcal{H}_n v)(t,x) := \eta(n(t+x)) - \eta(n(t-x))$$

so that $V_n = \mathcal{H}_n V$.

6.1 Case $f(x,u) = a_3(x)u^3 + O(u^4)$

Lemma 6.1 *Let $\langle a_3 \rangle := (1/\pi) \int_0^\pi a_3(x) \neq 0$. Taking $s^* = \text{sign}\langle a_3 \rangle$, $\exists\, n_0 \in \mathbf{N}$ such that $\forall n \geq n_0$ the $0th$ order bifurcation equation (16) has a solution $\overline{v} \in V_n$ which is non degenerate in V_n.*

Proof. Equation (16) is the Euler-Lagrange equation of

$$\Phi_0(v) = \frac{\|v\|_{H^1}^2}{2} - s^* \int_\Omega a_3(x) \frac{v^4}{4} \,. \tag{104}$$

The functional $\Phi_n(v) := \Phi_0(\mathcal{H}_n v)$ has the following development: for $v(t,x) = \eta(t+x) - \eta(t-x) \in V$ we obtain, using that $\int_\Omega v^4 = \int_\Omega (\mathcal{H}_n v)^4$,

$$\Phi_n(v) = 2\pi n^2 \int_{\mathbf{T}} \dot{\eta}^2(t)dt - s^* \langle a_3 \rangle \int_\Omega \frac{v^4}{4} \; - s^* \int_\Omega \Big(a_3(x) - \langle a_3 \rangle \Big) \frac{(\mathcal{H}_n v)^4}{4}.$$

We choose $s^* = \text{sign}\langle a_3 \rangle$ so that $s^* \langle a_3 \rangle > 0$. To simplify notations take $\langle a_3 \rangle > 0$ so $s^* = 1$.

$$\begin{aligned} \Phi_n\Big(\frac{\sqrt{2}n}{\sqrt{\langle a_3 \rangle}} v \Big) &= \frac{8\pi n^4}{\langle a_3 \rangle} \Big[\frac{1}{2} \int_{\mathbf{T}} \dot{\eta}^2(s)ds - \frac{1}{8\pi} \int_\Omega v^4 \; + \frac{1}{8\pi} \int_\Omega \Big(\frac{a_3(x)}{\langle a_3 \rangle} - 1 \Big) (\mathcal{H}_n v)^4 \, dt\, dx \Big] \\ &= \frac{8\pi n^4}{\langle a_3 \rangle} [\Psi(\eta) + \mathcal{R}_n(v)] \end{aligned}$$

where

$$\Psi(\eta) := \frac{1}{2} \int_{\mathbf{T}} \dot{\eta}^2(s)ds - \frac{1}{4} \int_{\mathbf{T}} \eta^4(s)ds - \frac{3}{8\pi} \Big(\int_{\mathbf{T}} \eta^2(s)ds \Big)^2,$$

$$\mathcal{R}_n(v) := \frac{1}{8\pi} \int_\Omega b(x)(\mathcal{H}_n v)^4 \, dt\, dx, \quad b(x) := \frac{a_3(x)}{\langle a_3 \rangle} - 1.$$

Let $E := \{\eta \in H^1(\mathbf{T}) \mid \eta \text{ is odd}\}$. It is enough to prove that $\Psi : E \to \mathbf{R}$ has a non-degenerate critical point $\overline{\eta}$ and that $\mathcal{R}_n$ is small for large n, see Lemma 6.2. Indeed the operator $\Psi''(\overline{\eta})$ has the form Id+Compact so that if its kernel is 0 then $\Psi''(\overline{\eta})$ is invertible. Hence, by the implicit function theorem, for n large enough, Φ_n too (hence $\Phi_{0|V_n}$) has a non-degenerate critical point.

The critical points of Ψ in E are the 2π-periodic odd solutions of

$$\ddot{\eta} + \eta^3 + 3\langle \eta^2 \rangle \eta = 0 \,. \tag{105}$$

By [2] it is known that there exists a solution of (105) which is a non-degenerate critical point of Ψ in E. It remains to prove Lemma 6.2.■

Lemma 6.2 *There holds*

$$\|D\mathcal{R}_n(v)\|,\ \|D^2\mathcal{R}_n(v)\| \to 0 \qquad \text{as} \qquad n \to +\infty \tag{106}$$

uniformly for v in bounded sets of E.

PROOF. We shall prove the estimate only for $D^2\mathcal{R}_n$. We have

$$D^2\mathcal{R}_n(v)[h,k] = \frac{3}{2\pi}\int_\Omega b(x)(\mathcal{H}_n v)^2(\mathcal{H}_n h)\ (\mathcal{H}_n k) = \frac{3}{2\pi}\int_0^\pi b(x) g(nx)\,dx$$

where $g(y)$ is the π-periodic function defined by

$$g(y) := \int_{\mathbf{T}} (\eta(t+y) - \eta(t-y))^2 (\beta(t+y) - \beta(t-y))(\gamma(t+y) - \gamma(t-y))dt\ ,$$

β and γ being associated with h and k as η is with v. Developing in Fourier series $g(y) = \sum_{l\in\mathbf{Z}} g_l \exp(\mathrm{i}2ly)$ we have $g(nx) = \sum_{l\in\mathbf{Z}} g_l \exp(\mathrm{i}2lnx)$. Extending $b(x)$ to a π-periodic function, we also write $b(x) = \sum_{l\in\mathbf{Z}} b_l \exp(\mathrm{i}2lx)$, with $b_0 = \langle b\rangle = 0$. Therefore

$$\begin{aligned} |D^2\mathcal{R}_n(v_n)[h,k]| &= \frac{3}{2}\Big|\sum_{l\neq 0} g_l b_{-ln}\Big| \leq \frac{3}{2}\Big(\sum_{l\neq 0} g_l^2\Big)^{1/2}\Big(\sum_{l\neq 0} b_{ln}^2\Big)^{1/2} \leq \frac{3}{2}\|g\|_{L^2(0,\pi)}\Big(\sum_{l\neq 0} b_{ln}^2\Big)^{1/2} \\ &\leq C\|\eta\|_\infty^2\|\beta\|_\infty\|\gamma\|_\infty\ \Big(\sum_{l\neq 0} b_{ln}^2\Big)^{1/2} \leq C\|v_0\|_{H^1}^2\|h\|_{H^1}\|k\|_{H^1}\ \Big(\sum_{l\neq 0} b_{ln}^2\Big)^{1/2}. \end{aligned}$$

Since $(\sum_{l\neq 0} b_{ln}^2)^{1/2} \to 0$ as $n \to \infty$ it proves (106). With a similar calculus we can prove that $D\mathcal{R}_n(v) \to 0$ as $n \to +\infty$. ■

6.2 Case $f(x,u) = a_2 u^2 + O(u^4)$

With the frequency-amplitude relation (17) system (7) with $p = 2$ becomes

$$\begin{cases} -\Delta v = -\delta^{-1}\Pi_V g_\delta(x, v+w) & (Q) \\ L_\omega w = \delta \Pi_W g_\delta(x, v+w) & (P) \end{cases} \tag{107}$$

where

$$g_\delta(x,u) = \frac{f(x,\delta u)}{\delta^2} = a_2 u^2 + \delta^2 a_4(x) u^4 + \ldots. \tag{108}$$

With the further rescaling

$$w \to \delta w$$

and since $v^2 \in W$, system (107) is equivalent to

$$\begin{cases} -\Delta v = \Pi_V\Big(- 2a_2 vw - a_2\delta w^2 - \delta r(\delta, x, v+\delta w)\Big) & (Q) \\ L_\omega w = a_2 v^2 + \delta \Pi_W \Big(2a_2 vw + \delta a_2 w^2 + \delta r(\delta, x, v+\delta w)\Big) & (P) \end{cases} \tag{109}$$

where $r(\delta, x, u) = \delta^{-4}(f(x, \delta u) - a_2\delta^2 u^2) = a_4(x)u^4 + ...$

For $\delta = 0$ system (109) reduces to

$$\begin{cases} -\Delta v = -2a_2\Pi_V(vw) \\ Lw = a_2 v^2 \end{cases} \tag{110}$$

where $L := -\partial_{tt} + \partial_{xx}$, and it is equivalent to $w = a_2 L^{-1}v^2$, $-\Delta v = -2a_2^2\Pi_V(vL^{-1}v^2)$, namely to the 0th order bifurcation equation (18).

Lemma 6.3 *If $a_2 \neq 0$, $\exists\, n_0 \in \mathbf{N}$ such that $\forall n \geq n_0$ the 0th order bifurcation equation (18) has a solution $\overline{v} \in V_n$ which is non-degenerate in V_n.*

Proof. We have to prove that $\Phi_n(v) := \Phi_0(\mathcal{H}_n v)$, where Φ_0 is defined in (19), possesses non-degenerate critical points at least for n large.

Φ_n admits the following development (Lemmas 3.7 and 3.8 in [6]): for $v(t,x) = \eta(t+x) - \eta(t-x)$

$$\Phi_n(v) = 2\pi n^2 \int_{\mathbf{T}} \dot\eta^2(t)dt - \frac{\pi^2 a_2^2}{12}\Big(\int_{\mathbf{T}} \eta^2(t)\,dt\Big)^2 + \frac{a_2^2}{2n^2}\Big(\int_\Omega v^2 L^{-1}v^2 + \frac{\pi^2}{6}\Big(\int_{\mathbf{T}} \eta^2(t)\,dt\Big)^2\Big).$$

Hence we can write

$$\Phi_n\Big(\frac{\sqrt{12}n}{\sqrt{\pi}a_2}v\Big) = \frac{48n^4}{a_2^2}\Big[\frac{1}{2}\int_{\mathbf{T}} \dot\eta^2(s)\;ds - \frac{1}{4}\Big(\int_{\mathbf{T}} \eta^2(s)\;ds\Big)^2 + \frac{1}{n^2}\mathcal{R}(\eta)\Big] = \frac{48n^4}{a_2^2}\Big[\Psi(\eta) + \frac{1}{n^2}\mathcal{R}(\eta)\Big] \tag{111}$$

where

$$\Psi(\eta) = \frac{1}{2}\int_{\mathbf{T}} \dot\eta^2(s)\;ds - \frac{1}{4}\Big(\int_{\mathbf{T}} \eta^2(s)\;ds\Big)^2$$

and $\mathcal{R} : E \to \mathbf{R}$ is a smooth functional defined on $E := \{\eta \in H^1(\mathbf{T}) \mid \eta \text{ odd}\}$. By (111), in order to prove that Φ_n has a non-degenerate critical point for n large enough, it is enough to prove the following Lemma. ■

Lemma 6.4 $\Psi : E \to \mathbf{R}$ *possesses a non degenerate critical point.*

Proof. The critical points of Ψ in E are the 2π-periodic odd solutions of the equation

$$\ddot\eta + \Big(\int_{\mathbf{T}} \eta^2(t)\,dt\Big)\eta = 0\,. \tag{112}$$

Equation (112) has a 2π-periodic solution of the form $\bar\eta(t) = (1/\sqrt{\pi})\sin t$.

We claim that $\bar\eta$ is non-degenerate. The linearized equation of (112) at $\bar\eta$ is

$$\ddot h + h + \frac{2}{\pi}\Big(\int_{\mathbf{T}} \sin t\; h(t)\,dt\Big)\sin t = 0. \tag{113}$$

Developing in time-Fourier series $h(t) = \sum_{k\geq 1} a_k \sin kt$ we find out that any solution of the linearized equation (113) satisfies

$$-k^2 a_k + a_k = 0, \quad \forall k \geq 2, \qquad a_1 = 0$$

and therefore $h = 0$. ■

As in Theorem 1.2, the existence of a solution $\overline{v}$ of the 0th order bifurcation equation which is non degenerate in some V_n entails the conclusions of Theorem 1.2. To avoid cumbersome notations, we still give the main arguments assuming that $n = 1$.

Since for $\delta = 0$ the solution of the (P)-equation in (110) is $w = a_2 L^{-1}v^2$, it is convenient to perform the change of variable

$$w = a_2 L^{-1}v^2 + y\,, \quad y \in W\,. \tag{114}$$

System (109) is then written

$$\left\{\begin{array}{rcll} -\Delta v & = & -2a_2^2\Pi_V(vL^{-1}v^2)+\Pi_V\Big(-2a_2vy-a_2\delta w^2-\delta r(\delta,x,v+\delta w)\Big) & (Q') \\ L_\omega y & = & 2a_2\delta^2\mathcal{R}(v^2)+\delta\Pi_W\Big(2a_2vw+\delta a_2w^2+\delta r(\delta,x,v+\delta w)\Big) & (P') \end{array}\right. \tag{115}$$

where w is a function of v and y through (114), and the linear operator in W

$$\mathcal{R} := (1-\omega^2)^{-1}(I-L_\omega L^{-1}) = (2\delta^2)^{-1}(I-L_\omega L^{-1})$$

does not depend on ω and can be expressed as

$$\mathcal{R}\Big(\sum_{l\neq j} w_{l,j}\cos(lt)\sin(jx)\Big) = \sum_{l\neq j}\frac{l^2}{l^2-j^2}w_{l,j}\cos(lt)\sin(jx)\,.$$

Since $l^2|l^2-j^2|^{-1} = l^2|l+j|^{-1}|l-j|^{-1}\leq |l|$, the operator $\mathcal{R}$ satisfies the estimate

$$\forall w\in W,\quad \|\mathcal{R}w\|_{\sigma,s}\leq\|w\|_{\sigma,s+1}\,. \tag{116}$$

Splitting $V=V_1\oplus V_2$, the (Q')-equation is divided in two parts: the $(Q'1)$ and the $(Q'2)$-equations.

Setting

$$R := \|\overline{v}\|_{0,s}$$

the analogue of Lemma 2.1 is:

Lemma 6.5 *There exist $N\in\mathbf{N}_+$, $\overline{\sigma}=\ln 2/N>0$, $\delta_0>0$, such that, $\forall\, 0\leq\sigma\leq\overline{\sigma}$, $\forall\,\|v_1\|_{0,s}\leq 2R$, $\forall\,\|y\|_{\sigma,s}\leq 1$, $\forall\,\delta\in[0,\delta_0)$, there exists a unique solution $v_2(\delta,v_1,y)\in V_2\cap X_{\sigma,s}$ of the $(Q'2)$-equation with $\|v_2(\delta,v_1,y)\|_{\sigma,s}\leq 1$. Moreover $v_2(0,\Pi_{V_1}\overline{v},0)=\Pi_{V_2}\overline{v}$, $v_2(\delta,v_1,y)\in X_{\sigma,s+2}$ and the regularizing property*

$$\Big\|D_w v_2(\delta,v_1,y)[h]\Big\|_{\sigma,s+2}\leq C\|h\|_{\sigma,s} \tag{117}$$

holds, where C is some positive constant.

Substituting $v_2=v_2(\delta,v_1,y)$ into the (P')-equation yields

$$L_\omega y = \delta\Gamma(\delta,v_1,y) := \delta\widetilde{\Gamma}(\delta,v_1+v_2(\delta,v_1,y),y) \tag{118}$$

where

$$\begin{aligned}\widetilde{\Gamma}(\delta,v,y) \;:=\; 2\delta a_2\mathcal{R}(v^2) \;+\; &\Pi_W\Big(2a_2v(a_2L^{-1}(v^2)+y)+\delta a_2(a_2L^{-1}(v^2)+y)^2 \\ &+\delta r(\delta,x,v+\delta(a_2L^{-1}(v^2)+y))\Big)\,.\end{aligned}$$

The (P')-equation (118) can be solved as in sections 3-4 with slight changes that we specify.

Theorem 6.1 (Solution of the (P')-equation) *For $\delta_0>0$ small enough, there exists a C^∞-function $\widetilde{y}:[0,\delta_0)\times B(2R,V_1)\to W\cap X_{\overline{\sigma}/2,s}$ satisfying $\widetilde{y}(0,v_1)=0$, $\|\widetilde{y}\|_{\overline{\sigma}/2,s}=O(\delta)$, $\|D^k\widetilde{y}\|_{\overline{\sigma}/2,s}=O(1)$, and verifying the following property : let*

$$\begin{aligned}B_\infty \;:=\; &\Big\{(\delta,v_1)\in[0,\delta_0)\times B(2R,V_1)\;:\;\Big|\omega(\delta)l-j-\delta\frac{M(\delta,v_1,\widetilde{y}(\delta,v_1))}{2j}\Big|\geq\frac{2\gamma}{(l+j)^\tau}, \\ &\Big|\omega(\delta)l-j\Big|\geq\frac{2\gamma}{(l+j)^\tau},\ \forall l\geq\frac{1}{3\delta^2},\ l\neq j\Big\},\end{aligned}$$

where $\omega(\delta)=\sqrt{1-2\delta^2}$ and $M(\delta,v_1,y)$ is defined in (119). Then $\forall(\delta,v_1)\in B_\infty$, $\widetilde{y}(\delta,v_1)$ solves the (P')-equation (118).

PROOF. As before, the key point is the inversion, at each step of the iterative process, of a linear operator

$$\mathcal{L}_n(\delta, v_1, y)[h] = L_\omega h - \delta P_n \Pi_W D_y \Gamma(\delta, v_1, y)[h], \qquad h \in W^{(n)}.$$

We have

$$D_y\Gamma(\delta, v_1, y)[h] = D_y\widetilde{\Gamma}(\delta, v_1 + v_2(\delta, v_1, y), y)[h] + D_v\widetilde{\Gamma}(\delta, v_1 + v_2(\delta, v_1, y), y) D_y v_2(\delta, v_1, y)[h]$$

and, as it can be directly verified,

$$D_y\widetilde{\Gamma}(\delta, v, y)[h] = \Pi_W\Big((\partial_u g_\delta)(x, v + \delta w)h\Big)$$

where g_δ is defined in (108) and w is given by (114). As in section 4, setting $a(t,x) := (\partial_u g_\delta)(x, v(t,x) + \delta w(t,x))$, we can decompose $\mathcal{L}_n(\delta, v_1, y) = D - \mathcal{M}_1 - \mathcal{M}_2$ where (with the notations of section 4)

$$\begin{cases} Dh &:= L_\omega h - \delta P_n \Pi_W(a_0(x)h) \\ \mathcal{M}_1 h &:= \delta P_n \Pi_W(\overline{a}(t,x)h) \\ \mathcal{M}_2 h &:= \delta P_n \Pi_W D_v \widetilde{\Gamma}(\delta, v_1 + v_2(\delta, v_1, y), y) D_y v_2(\delta, v_1, y)[h]. \end{cases}$$

As in Lemma 4.1, the eigenvalues of the similarly defined operator S_k satisfy $\lambda_{k,j} = j^2 + \delta M(\delta, v_1, y) + O(\delta/j)$, where

$$M(\delta, v_1, y) := \frac{1}{|\Omega|}\int_\Omega (\partial_u g_\delta)\Big(x, v_1 + v_2(\delta, v_1, y) + \delta w(t,x)\Big)\, dx dt, \qquad w = a_2 L^{-1}(v^2) + y. \tag{119}$$

The bounds for the operator D (Lemma 4.3, Corollary 4.2) still hold assuming an analogous non resonance condition, and we can define in the same way the operators $\mathcal{U}$, $\mathcal{R}_1$, $\mathcal{R}_2$, with $\|\mathcal{U}^{-1}h\|_{\sigma,s'} = (1 + O(\delta))\|h\|_{\sigma,s'}$. With the same arguments we obtain for $\mathcal{R}_1$ the bound

$$\Big\|\mathcal{R}_1 h\Big\|_{\sigma, s+\frac{\tau-1}{2}} \le \delta^{2-\tau}\frac{C}{\gamma}\|h\|_{\sigma, s+\frac{\tau-1}{2}}$$

which is enough since $\tau < 2$.

For the estimate of $\mathcal{R}_2$ the most delicate term to deal with is $\delta^2 |D|^{-1/2} D_y F |D|^{-1/2}$, where

$$F(\delta, v_1, y) := \mathcal{R}\Big((v_1 + v_2(\delta, v_1, y))^2\Big),$$

because the operator $\mathcal{R}$ induces a loss of regularity, see (116). However, again the regularizing property (117) of the map v_2 enables to obtain the bound

$$\Big\|\mathcal{R}_2 h\Big\|_{\sigma, s+\frac{\tau-1}{2}} \le C\frac{\delta}{\gamma}\ \|h\|_{\sigma, s+\frac{\tau-1}{2}}. \tag{120}$$

The key point is that the loss of $(\tau - 1)$ derivatives due to $|D|^{-1/2}$ applied twice, added to the loss of 1 derivative due to $\mathcal{R}$ in (116) is compensated by the gain of 2 derivatives with v_2, whenever $\tau < 2$. Let us enter briefly into details.

$$\begin{aligned} \Big\|D_y F(\delta, v_1, y)[h]\Big\|_{\sigma, s+1} &= \Big\|2\mathcal{R}\Big((v_1 + v_2) D_y v_2(\delta, v_1, y)[h]\Big)\Big\|_{\sigma, s+1} \\ &\le 2\Big\|(v_1 + v_2) D_y v_2(\delta, v_1, y)[h]\Big\|_{\sigma, s+2} \\ &\le C\|(v_1 + v_2)\|_{\sigma, s+2}\Big\|D_y v_2(\delta, v_1, y)[h]\Big\|_{\sigma, s+2} \\ &\le K(N, R, \|y\|_{\sigma,s})\|h\|_{\sigma,s} \end{aligned}$$

by the regularizing property (117) of v_2. We can then derive (120) as in the proof of Lemma 4.9, using that $\tau < 2$. ■

Finally, inserting $\widetilde{y}(\delta, v_1)$ in the $(Q1')$-equation, we get

$$-\Delta v_1 = \mathcal{G}(\delta, v_1) \tag{121}$$

where

$$\mathcal{G}(0, v_1) := -\Pi_{V_1}\Big(2a_2(v_1 + v_2(0, v_1, 0))L^{-1}(v_1 + v_2(0, v_1, 0))^2\Big).$$

As in subsection 5.2, since $\Phi_0 : V \to \mathbf{R}$ possesses a non-degenerate critical point $\overline{v}$, the equation $-\Delta v_1 = \mathcal{G}(0, v_1)$ has the non-degenerate solution $\overline{v}_1 := \Pi_{V_1}\overline{v} \in B(R, V_1)$ and, by the Implicit function theorem, there exists a smooth path $\delta \mapsto v_1(\delta) \in B(2R, V_1)$ of solutions of (121) with $v_1(0) = \overline{v}$. As in Proposition 3.2 this implies that the set $\mathcal{C} = \{\delta \in (0, \delta_0) \mid (\delta, v_1(\delta)) \in B_\infty\}$ has asymptotically full measure at 0.

7 Appendix

Lemma 7.1 *If q is an even integer, then*

$$\int_\Omega a(x)v^q(t,x)\,dt\,dx = 0, \ \forall v \in V \iff \Big\{a(\pi - x) = -a(x), \ \forall x \in [0,\pi]\Big\}.$$

If $q \geq 3$ is an odd integer, then

$$\int_\Omega a(x)v^q(t,x)\,dt\,dx = 0, \ \forall v \in V \iff \Big\{a(\pi - x) = a(x), \ \forall x \in [0,\pi]\Big\}.$$

Proof. We first assume that $q = 2s$ is even. If $a(\pi - x) = -a(x)$ $\forall x \in (0,\pi)$, then, for all $v \in V$,

$$\begin{aligned}\int_\Omega a(x)v^{2s}(t,x)\,dt\,dx &= \int_\Omega a(\pi - x)v^{2s}(t, \pi - x)\,dt\,dx = \int_\Omega -a(x)(-v(t+\pi, x))^{2s}\,dt\,dx \\ &= -\int_\Omega a(x)v^{2s}(t,x)\,dt\,dx\end{aligned}$$

and so $\int_\Omega a(x)v^{2s}(t,x)\,dt\,dx = 0$.
Now assume that $\Sigma(v) := \int_\Omega a(x)v^{2s}(t,x)\,dt\,dx = 0$ $\forall v \in V$. Writing that $D^{2s}\Sigma = 0$, we get

$$\int_\Omega a(x)v_1(t,x)\ldots v_{2s}(t,x)\,dt\,dx = 0, \qquad \forall (v_1, \ldots, v_{2s}) \in V^{2s}.$$

Choosing $v_{2s}(t,x) = v_{2s-1}(t,x) = \cos(lt)\sin(lx)$, we obtain

$$\frac{1}{4}\int_\Omega a(x)v_1(t,x)\ldots v_{2(s-1)}(t,x)(\cos(2lt) + 1)(1 - \cos(2lx))\,dt\,dx = 0$$

Taking limits as $l \to \infty$, there results $\int_\Omega a(x)v_1(t,x)\ldots v_{2(s-1)}(t,x)\,dt\,dx = 0$ $\forall (v_1, \ldots, v_{2(s-1)}) \in V^{2(s-1)}$. Iterating this operation, we finally get

$$\forall (v_1, v_2) \in V^2 \quad \int_\Omega a(x)v_1(t,x)v_2(t,x)\,dt\,dx = 0, \quad \text{and} \quad \int_0^\pi a(x)\,dx = 0.$$

Choosing $v_1(t,x) = v_2(t,x) = \cos(lt)\sin(lx)$ in the first equality, we derive that $\int_0^\pi a(x)\sin^2(lx)\,dx = 0$. Hence $\forall l \in \mathbf{N}$ $\int_0^\pi a(x)\cos(2lx)\,dx = 0$. This implies that a is orthogonal in $L^2(0,\pi)$ to $F = \{b \in L^2(0,\pi) \mid b(\pi - x) = b(x)$ a.e.$\}$. Hence $a(\pi - x) = -a(x)$ a.e., and, since a is continuous, the identity holds everywhere.
We next assume that $q = 2s + 1$ is odd, $q \geq 3$. The first implication is derived in a similar way. Now assume that $\int_\Omega a(x)v^q(t,x)\,dt\,dx = 0$ $\forall v \in V$. We can prove exactly as in the first part that

$$\forall (v_1, v_2, v_3) \in V^3 \quad \int_\Omega a(x)v_1(t,x)v_2(t,x)v_3(t,x)\,dt\,dx = 0.$$

Choosing $v_1(t,x) = \cos(l_1 t)\sin(l_1 x)$, $v_2(t,x) = \cos(l_2 t)\sin(l_2 x)$, $v_3(t,x) = \cos((l_1+l_2)t)\sin((l_1+l_2)x)$ and using the fact that $\int_0^{2\pi} \cos(l_1 t)\cos(l_2 t)\cos((l_1+l_2)t)\,dt \neq 0$, we obtain

$$\begin{aligned}\int_0^\pi a(x)\Big[\sin^2(l_1x)\sin(l_2x)\cos(l_2x) + \sin^2(l_2x)\sin(l_1x)\cos(l_1x)\Big]\,dx = \\ \int_0^\pi a(x)\sin(l_1x)\sin(l_2x)\sin\Big((l_1+l_2)x\Big)\,dx = 0.\end{aligned} \tag{122}$$

Letting l_2 go to infinity and taking limits, (122) yields $\int_0^\pi (1/2)a(x)\sin(l_1x)\cos(l_1x)\ dx = 0$. Hence $\int_0^\pi a(x)\sin(2lx) = 0$, $\forall l > 0$. This implies that, in $L^2(0,\pi)$, a is orthogonal to $G = \{b \in L^2(0,\pi) \mid b(\pi - x) = -b(x) \text{ a.e.}\}$. Hence $a(\pi - x) = a(x)\ \forall x \in (0,\pi)$. ∎

Proof of Lemma 4.1. Let $K_k(\varepsilon) = S_k^{-1}(\varepsilon)$ be the self-adjoint compact operator of F_k defined by

$$\langle K_k(\varepsilon)u, v\rangle_\varepsilon = (u,v)_{L^2}, \qquad \forall u,v \in F_k$$

(in other words $K_k(\varepsilon)u$ is the unique weak solution $z \in F_k$ of $S_k z := u$).

Note that $K_k(\varepsilon)$ is a positive operator, i.e. $\langle K_k(\varepsilon)u,u\rangle_\varepsilon > 0$, $\forall u \neq 0$, and that $K_k(\varepsilon)$ is also self-adjoint for the L^2-scalar product.

By the spectral theory of compact self-adjoint operators in Hilbert spaces, there is a $\langle\ ,\ \rangle_\varepsilon$-orthonormal basis $(v_{k,j})_{j\geq 1, j\neq k}$ of F_k, such that $v_{k,j}$ is an eigenvector of $K_k(\varepsilon)$ associated to a positive eigenvalue $\nu_{k,j}(\varepsilon)$, the sequence $(\nu_{k,j}(\varepsilon))_j$ is non-increasing and tends to 0 as $j \to +\infty$. Each $v_{k,j}(\varepsilon)$ belongs to $D(S_k)$ and is an eigenvector of S_k with associated eigenvalue $\lambda_{k,j}(\varepsilon) = 1/\nu_{k,j}(\varepsilon)$, with $(\lambda_{k,j}(\varepsilon))_{j\geq 1} \to +\infty$ as $j \to +\infty$.

The map $\varepsilon \mapsto K_k(\varepsilon) \in \mathcal{L}(F_k,F_k)$ is differentiable and $K_k'(\varepsilon) = -K_k(\varepsilon)MK_k(\varepsilon)$, where $Mu := \pi_k(a_0u)$.

For $u = \sum_{j\neq k}\alpha_j v_{k,j}(\varepsilon) \in F_k$,

$$\langle u,u\rangle_\varepsilon = \sum_{j\neq k}|\alpha_j|^2 \quad \text{and} \quad (u,u)_{L^2} = \sum_{j\neq k}\frac{|\alpha_j|^2}{\lambda_{k,j}(\varepsilon)}.$$

As a consequence,

$$\lambda_{k,j}(\varepsilon) = \min\Big\{\max_{u\in F, \|u\|_{L^2}=1}\langle u,u\rangle_\varepsilon\ ;\ F \text{ subspace of } F_k \text{ of dimension } j \text{ (if } j<k)\ ,\ j-1 \text{ (if } j>k)\Big\}. \tag{123}$$

It is clear by inspection that $\lambda_{k,j}(0) = j^2$ and that we can choose $v_{k,j}(0) = \sqrt{2/\pi}\sin(jx)/j$. Hence, by (123), $|\lambda_{k,j}(\varepsilon) - j^2| \leq |\varepsilon|\ \|a_0\|_\infty < 1$, from which we derive

$$\forall l \neq j\ \ |\lambda_{k,l}(\varepsilon) - \lambda_{k,j}(\varepsilon)| \geq (l+j) - 2 \geq 2\min(l,j) - 1\ \ (\geq 1). \tag{124}$$

In particular, the eigenvalues $\lambda_{k,j}(\varepsilon)$ ($\nu_{k,j}(\varepsilon)$) are simple. By the variational characterization (123) we also see that $\lambda_{k,j}(\varepsilon)$ depends continuously on ε, and we can assume without loss of generality that $\varepsilon \mapsto v_{k,j}(\varepsilon)$ is a continuous map to F_k.

Let $\varphi_{k,j}(\varepsilon) := \sqrt{\lambda_{k,j}(\varepsilon)}v_{k,j}(\varepsilon)$. $(\varphi_{k,j}(\varepsilon))_{j\neq k}$ is a L^2-orthogonal family in F_k and

$$\forall\varepsilon \quad \begin{cases} K_k(\varepsilon)\varphi_{k,j}(\varepsilon) = \nu_{k,j}(\varepsilon)\varphi_{k,j}(\varepsilon) \\ (\varphi_{k,j}(\varepsilon),\varphi_{k,j}(\varepsilon))_{L^2} = 1 \end{cases}$$

We observe that the L^2-orthogonality w.r.t. $\varphi_{k,j}(\varepsilon)$ is equivalent to the $\langle\ ,\ \rangle_\varepsilon$-orthogonality w.r.t. $\varphi_{k,j}(\varepsilon)$, and that $E_{k,j}(\varepsilon) := [\varphi_{k,j}(\varepsilon)]^\perp$ is invariant under $K_k(\varepsilon)$. Using that $L_{k,j} := (K_k(\varepsilon) - \nu_{k,j}(\varepsilon)I)_{|E_{k,j}(\varepsilon)}$ is invertible, it is easy to derive from the Implicit Function Theorem that the maps $(\varepsilon \mapsto \nu_{k,j}(\varepsilon))$ and $(\varepsilon \mapsto \varphi_{k,j}(\varepsilon))$ are differentiable.

Denoting by P the orthogonal projector onto $E_{k,j}(\varepsilon)$, we have

$$\varphi_{k,j}'(\varepsilon) = L^{-1}(-PK_k'(\varepsilon)\varphi_{k,j}(\varepsilon)) = L^{-1}(PK_kMK_k\varphi_{k,j}(\varepsilon)) = \nu_{k,j}(\varepsilon)L^{-1}K_kPM\varphi_{k,j}(\varepsilon),$$

$$
\begin{aligned}
\nu'_{k,j}(\varepsilon) &= \Big(K'_k(\varepsilon)\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon)\Big)_{L^2} = -\Big(K_k M K_k \varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon)\Big)_{L^2} \\
&= -\Big(M K_k \varphi_{k,j}(\varepsilon), K_k \varphi_{k,j}(\varepsilon)\Big)_{L^2} = -\nu^2_{k,j}(\varepsilon)\Big(M \varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon)\Big)_{L^2}.
\end{aligned} \tag{125}
$$

We have

$$
\nu_{k,j} L^{-1} K_k \Big(\sum_{l \neq j} \alpha_l v_{k,l} \Big) = \sum_{l \neq j} \frac{\nu_{k,j}\nu_{k,l}}{\nu_{k,l} - \nu_{k,j}} \alpha_l v_{k,l} = \sum_{l \neq j} \frac{\alpha_l}{\lambda_{k,j} - \lambda_{k,l}} v_{k,l}.
$$

Hence, by (124), $|\nu_{k,j} L^{-1} K_k P u|_{L^2} \le |u|_{L^2}/j$. We obtain $|\varphi'_{k,j}(\varepsilon)|_{L_2} = O(|a_0|_\infty / j)$. Hence

$$
\Big| \varphi_{k,j}(\varepsilon) - \sqrt{\frac{2}{\pi}} \sin(jx) \Big|_{L^2} = O\Big(\frac{\varepsilon |a_0|_\infty}{j} \Big).
$$

Hence, by (125),

$$
\begin{aligned}
\lambda'_{k,j}(\varepsilon) &= \Big(M\varphi_{k,j}(\varepsilon), \varphi_{k,j}(\varepsilon)\Big)_{L^2} = \int_0^\pi a_0(x) (\varphi_{k,j})^2 \, dx \\
&= \frac{2}{\pi} \int_0^\pi a_0(x) (\sin(jx))^2 \, dx + O\Big(\frac{\varepsilon |a_0|^2_\infty}{j} \Big)
\end{aligned}
$$

Writing $\sin^2(jx) = (1 - \cos(2jx))/2$, and since $\int_0^\pi a_0(x) \cos(2jx) \, dx = -\int_0^\pi (a_0)_x(x) \sin(2jx)/2j \, dx$, we get

$$
\lambda'_{k,j}(\varepsilon) = \frac{1}{\pi} \int_0^\pi a_0(x) \, dx + O\Big(\frac{\|a_0\|_{H^1}}{j} \Big) = M(\delta, v_1, w) + O\Big(\frac{\|a_0\|_{H^1}}{j} \Big).
$$

Hence $\lambda_{k,j}(\varepsilon) = j^2 + \varepsilon M(\delta, v_1, w) + O(\varepsilon \|a_0\|_{H^1}/j)$, which is the first estimate in (80).

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