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ON THE ONE-STEP-BRACKET-GENERATING MOTION PLANNING PROBLEM

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Abstract. We consider the general motion planning problem for a sub-Riemannian metric with one-step bracket-generating distribution. Our results generalize earlier results in the corank-one case. Mostly, we completely solve the problem in generic situation for corank smaller or equal to 3. Our results are constructive: we explicitly construct the asymptotically optimal solutions.

1. Introduction. Statement of results

1.1. **Basic concepts, statement of the problems.** A "motion planning problem" on an n-dimensional manifold Ξ is the data $\Sigma = (\Delta, g, \Gamma)$ of a smooth curve $\Gamma : [0,1] \to \Xi$, well parametrized, i.e., $\dfrac{d\Gamma}{dt} \neq 0$ for all t, without double points, and a sub-Riemannian metric (Δ, g) on Ξ. Here, Δ is assumed to be a one-step-bracket-generating distribution over Ξ, and g is a Riemannian metric over Δ.

In fact, we need only a germ along Γ of such an SR metric (Δ, g). Then, we may assume that $\Xi \subset \mathbb{R}^n$ is an open connected subset, and we denote by $\mathcal{S}$ the set of such smooth (C^∞) couples of a curve Γ and an SR metric on Ξ, endowed with the C^∞ topology (there is no need here for the Whitney topology, since Γ is compact).

Along the paper, ε is a small parameter. Let d denote the SR distance function, then $\mathcal{T}_\varepsilon$ is the ε-sub-Riemannian tube $\mathcal{T}_\varepsilon = \{q \in \mathbb{R}^n \mid d(q, \Gamma) \leq \varepsilon\}$, and $\mathcal{C}_\varepsilon = \{q \in \mathbb{R}^n \mid d(q, \Gamma) = \varepsilon\}$ is the corresponding cylinder.

Two functions f_1 and f_2 in ε tending to $+\infty$ as ε tends to zero are said to be strongly equivalent, $f_1 \eqsim_s f_2$ (respectively, f_1 is "weakly equivalent" to f_2, $f_1 \eqsim_w f_2$) if $\lim\limits_{\varepsilon \to 0} \dfrac{f_1(\varepsilon)}{f_2(\varepsilon)} = 1$ (respectively, $k_1\ f_1(\varepsilon) \leq f_2(\varepsilon) \leq k_2 f_1(\varepsilon)$

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for sufficiently small ε and certain constants $k_1, k_2 > 0$). We also write $f_1 \geq_s f_2$ if $\liminf\limits_{\varepsilon\to 0} \dfrac{f_1(\varepsilon)}{f_2(\varepsilon)} \geq 1$. This notation is consistent.

Let $\gamma : [0, \theta_\gamma] \to \mathcal{T}_\varepsilon$ be a (smooth, piecewise-smooth, Lipschitz) arclength-parametrized admissible curve (i.e., almost everywhere tangent to Δ) connecting $\Gamma(0)$ and $\Gamma(1) : \gamma(0) = \Gamma(0)$, $\gamma(\theta_\gamma) = \Gamma(1)$. Then the SR length of γ is θ_γ. The question is to find the minimum of this length θ_γ among all the curves γ. This minimum is denoted by $\theta^*(\varepsilon)$.

Then the class of strong (respectively, weak) equivalence of the function $MC_\Sigma(\varepsilon) = \dfrac{1}{\varepsilon}\theta^*(\varepsilon)$ as ε tends to zero is called the strong (respectively, weak) metric complexity of the given motion planning problem. This notion has been introduced by F. Jean (see [12, 13, 14]). It is an important notion, coming from practitioners in robotics. For $T < 1$, we also denote by $MC_\Sigma(\varepsilon, T)$ the metric complexity of the piece of the curve $\Gamma : \{\Gamma(t) | t \leq T\}$.

A strong (respectively, weak) *asymptotic optimal synthesis* is an (ε-dependent) control strategy γ_ε that realizes a strong (respectively, weak) equivalence of the metric complexity as ε tends to zero, i.e., it is a family $\{\gamma_\varepsilon\}$ of admissible curves, $\gamma_\varepsilon([0, \theta_\varepsilon]) \subset \mathcal{T}_\varepsilon$, $\gamma_\varepsilon(0) = \Gamma(0)$, $\gamma_\varepsilon(\theta_\varepsilon) = \Gamma(1)$, such that $\dfrac{1}{\varepsilon}SR\text{-length}(\gamma_\varepsilon) \approx MC_\Sigma(\varepsilon)$. Here, we will construct an "*approximate asymptotic optimal synthesis*," i.e., a two-parameter family $\{\gamma_\varepsilon^\omega\}$ of curves such that $MC_\Sigma(\varepsilon) \geq_s \dfrac{1}{\varepsilon(1+\omega)} SR\text{-length}(\gamma_\varepsilon^\omega)$, for all $\omega > 0$.

The case of a corank-one distribution (eventually, not one-step-bracket-generating) has been addressed and solved in [7, 8]: explicit expressions for the strong metric complexity were exhibited (at least in generic cases) in terms of the basic invariants of the given motion planning problem Σ. Also, explicit asymptotic optimal syntheses were constructed.

Our aim in this paper is to generalize these results for one-step-bracket-generating distributions, i.e., $\Delta_x + [\Delta, \Delta]_x = T_x\mathbb{R}^n$ for all $x \in \Gamma([0, 1])$.

1.2. **Statement of the results and organization of the paper.** Along the curve $\Gamma : [0, 1] \to \Xi$, there is a well-defined function $\alpha : [0, 1] \to \mathbb{R}_+^*$, which is a fundamental invariant of the given motion planning problem. This invariant $\alpha(t)$ is defined in Sec. 2.7 below, and it is also denoted by $\mathrm{eig}(\tilde{\Omega}_t)$.

In the paper, we will prove the following results.

1 (arbitrary corank). For any one-step-bracket-generating problem Σ such that $\forall t \in [0, 1]$, $\dot{\Gamma}(t) \notin \Delta(\Gamma(t))$, the metric complexity satisfies the following relation:

$$MC_\Sigma(\varepsilon, T) \geq_s \frac{2}{\varepsilon^2} \int_0^T \frac{dt}{\alpha(t)}.$$

This is Theorem 3.1 proved in Sec. 3.

2 (corank 2 or 3). There is an open-dense subset $\mathcal{S}^*$ of the set $\mathcal{S}$ of motion planning problems (one-step-bracket-generating), such that the metric complexity satisfies the relation

$$MC_\Sigma(\varepsilon, T) \simeq_s \frac{2}{\varepsilon^2} \int_0^T \frac{dt}{\alpha(t)}.$$

This is Theorem 4.1. The purpose of Sec. 4 is to prove this theorem. Moreover, our proof is *constructive*: we exhibit an "approximate asymptotic optimal synthesis."

Of course, this expression of the metric complexity coincides with the results of [7, 8] in the corank-one case.

Section 2 gives a certain number of preliminary results, in particular, the definition of the set $\mathcal{S}^*$ and of the invariant α.

In the appendix (Sec. 5), we give very elementary technical results that we need in the paper.

2. Preliminaries

2.1. **Notation and terminology.**

1. Relevant motion planning problems. The distribution Δ has corank $p \geq 2$.

Definition 2.1. A motion planning problem $\Sigma = (\Delta, g, \Gamma)$ is said to be *relevant* if for all $t \in [0, 1]$, $\dot{\Gamma}(t) \notin \Delta(\Gamma(t))$.

Clearly (by standard transversality arguments), this condition defines an open-dense subset of $\mathcal{S}$ that we still denote by $\mathcal{S}$. Note that in the corank-1 case, this set is not dense.

2. Frames. A motion planning problem can be specified by a couple (Γ, F), where $F = (F_1, \ldots, F_{n-p})$ is a frame of vector fields that generate Δ and that are orthonormal for g. Hence, we will also write $\Sigma = (\Gamma, F)$. If a global coordinate system (x, y, w) is given on Ξ, where $x \in \mathbb{R}^{n-p}$, $y \in \mathbb{R}^{p-1}$, and $w \in \mathbb{R}$, then we write

$$F_j = \sum_{i=1}^{n-p} \mathcal{Q}_{i,j}(x, y, w) \frac{\partial}{\partial x_i} + \sum_{i=1}^{p-1} \mathcal{L}_{i,j}(x, y, w) \frac{\partial}{\partial y_i} + \mathcal{M}_j(x, y, w) \frac{\partial}{\partial w},$$
$$j = 1, \ldots, n-p.$$

Then the SR metric is specified by the triple $(\mathcal{Q}, \mathcal{L}, \mathcal{M})$ of smooth x, y, w-dependent matrices, and we write also $\Sigma = (\Gamma, \mathcal{Q}, \mathcal{L}, \mathcal{M})$. It will often happen that Γ, in coordinates, will be the curve: $\Gamma(t) = (0, 0, t)$. In this case, we will write $\Sigma = (\mathcal{Q}, \mathcal{L}, \mathcal{M})$.

2.2. **Normal coordinates.** Consider $\Sigma \in \mathcal{S}$ and fix a (well-) parametrized surface S in Ξ, $(y, w) \to S(y, w)$, with the following properties: y and w are coordinates on S that are global in a neighborhood of Γ (and we restrict S to this neighborhood). Also, $y \in \mathbb{R}^{p-1}$, $w \in \mathbb{R}$, and $S(0, w) = \Gamma(w)$ for all $w \in [0, 1]$. Moreover, we require that S be transversal to Δ. This is always possible, since $\dot{\Gamma} \notin \Delta$ (Σ is relevant). Let us define $\mathcal{T}_\varepsilon^S = \{q \in \mathbb{R}^n \mid d(q, S) \leq \varepsilon\}$ and $\mathcal{C}_\varepsilon^S = \{q \in \mathbb{R}^n \mid d(q, S) = \varepsilon\}$, the sub-Riemannian S-tube and cylinder.

Lemma 2.1 (normal coordinates with respect to S)**.** *There are mappings $x : \Xi \to \mathbb{R}^{n-p}$, $y : \Xi \to \mathbb{R}^{p-1}$, and $w : \Xi \to \mathbb{R}$ such that $\xi = (x, y, w)$ is a coordinate system on Ξ (possibly, restricting Ξ to some neighborhood of S) such that:*

0. $S(y, w) = (0, y, w)$;
1. $\Delta|_S = \ker dw \cap \bigcap_{i=1,\dots,p-1} \ker dy_i$, $g|_S = \sum_{i=1}^{n-p} (dx_i)^2$;
2. $\mathcal{C}_\varepsilon^S = \left\{\xi \;\Big|\; \sum_{i=1}^{n-p} {x_i}^2 = \varepsilon^2\right\}$;
3. *geodesics (from the Pontryagin maximum principle [15]) satisfying the transversality conditions with respect to S are the straight lines through S, contained in the planes $P_{y_0,w_0} = \{\xi \mid (y, w) = (y_0, w_0)\}$ (hence, they are orthogonal to S).*

These normal coordinates are unique up to changes of coordinates of the form

$$\tilde{x} = T(y, w)x, (\tilde{y}, \tilde{w}) = (y, w), \tag{2.1}$$

where $T(y, w) \in O(n - p)$, the $(n - p)$ orthogonal group.

The proof of this lemma is similar to proofs in [1, 2, 4, 5], where these coordinates were introduced for a curve transversal to the distribution (not for a surface). In [1], this is done in the three-dimensional case. In [2], in the general contact case. In [4], this is done in quasi-contact case for even dimension. The elder paper [5] introduces these coordinates in a “formal” way. In fact, the proof is always the same, and works as soon as the curve (surface) is transversal to the distribution.

2.3. **Normal form.** Consider $\Sigma \in \mathcal{S}$ and fix a surface S just as in Sec. 2.2. Fix a normal coordinate system $\xi = (x, y, w)$ given by Lemma 2.1.

Theorem 2.1 (normal form)**.** *There is a unique orthonormal frame $F = (\mathcal{Q}, \mathcal{L}, \mathcal{M})$ for (Δ, g) with the following properties:*

1. $\mathcal{Q}(x, y, w)$ *is symmetric,* $\mathcal{Q}(0, y, w) = \mathrm{Id}$ *(the identity matrix);*
2. $\mathcal{Q}(x, y, w)x = x$;
3. $\mathcal{L}(x, y, w)x = 0$, $\mathcal{M}(x, y, w)x = 0$.

4. *Conversely, if* $\xi = (x, y, w)$ *is a coordinate system such that conditions* 1–3 *hold, then* ξ *is a normal coordinate system for the SR metric defined by the orthonormal frame* F*, with respect to the parametrized surface* $\{(0, y, w)\}$.

Clearly, this normal form is invariant with respect to the changes of normal coordinates (2.1).

Let us write

$$\begin{aligned} \mathcal{Q}(x,y,w) &= \mathrm{Id} + Q_1(x,y,w) + Q_2(x,y,w) + \dots, \\ \mathcal{L}(x,y,w) &= 0 + L_1(x,y,w) + L_2(x,y,w) + \dots, \\ \mathcal{M}(x,y,w) &= 0 + M_1(x,y,w) + M_2(x,y,w) + \dots, \end{aligned}$$

where Q_i, L_i, and M_i are matrices depending on ξ, the coefficients of which have order i with respect to x (i.e., they are in the ith power of the ideal of $C^\infty(x, y, w)$ generated by x_r, $r = 1, \dots, n-p$). Then, in particular, Q_1 is linear in x, Q_2 is quadratic. We set $u = (u_1, \dots, u_{n-p}) \in \mathbb{R}^{n-p}$. Then

$$\sum_{j=1}^{p-1} L_{1_j}(x,y,w)u_j = L_{1,y,w}(x,u)$$

is quadratic in (x, u) and $\mathbb{R}^{p-1}$-valued. Its ith component is the quadratic expression denoted by $L_{1,i,y,w}(x, u)$. Similarly,

$$\sum_{j=1}^{p-1} M_{1_j}(x,y,w)u_j = M_{1,y,w}(x,u)$$

is a quadratic expression in (x, u). The corresponding matrices are denoted by $L_{1,i,y,w}$, $i = 1, \dots, p-1$, and $M_{1,y,w}$.

We have the following proposition.

Proposition 2.1. 1. $Q_1 = 0$.
2. $L_{1,i,y,w}$*,* $i = 1, \dots, p-1$*, and* $M_{1,y,w}$ *are skew-symmetric matrices.*

Item 1 is, in fact, an SR-analog of a Bianchi identity in Riemannian geometry.

The proof of Theorem 2.1 and Proposition 2.1 are given in [1, 2, 4, 5], in different cases for corank-one distributions: contact for [2], quasi-contact for [4], etc.

In fact, again, the proofs are easily (obviously) generalized to our case. Again, it is not even necessary that the distribution be bracket-generating. The only important point is still that the surface S is transversal to Δ.

2.4. **Cylinder-box theorem in normal coordinates.** The following theorem is a result that can be easily obtained from the normal form and the ball-box Theorem of SR geometry (see, e.g., [10]).

Let $\xi = (x, y, w)$ be a normal coordinate system and $F = (\mathcal{Q}, \mathcal{L}, \mathcal{M})$ be the associated normal form. Assume that Δ is one-step-bracket-generating.

Theorem 2.2 (normal cylinder-box theorem)**.**

1. *Let* $\xi = (x, y, w) \in \mathcal{T}_\varepsilon$. *Then*

$$\|x\|_2 \le \varepsilon, \quad \|y\|_2 \le k_2\varepsilon^2$$

for some $k_2 > 0$.

2. *We take* $0 < \omega < 1$ *and set*

$$K_\varepsilon^{k_1,\omega} = \Big\{\xi = (x, y, w) \;\Big|\; \|x\|_2 \le \omega\varepsilon,\ \|y\|_2 \le k_1\varepsilon^2\Big\}.$$

Then for sufficiently small k_1, $K_\varepsilon^{k_1,\omega} \subset \mathcal{T}_\varepsilon$.

Item 1 of this theorem, together with Theorem 2.1 and Proposition 2.1, immediately implies the following lemma.

Lemma 2.2. *In normal coordinates restricted to the tubes* $\mathcal{T}_\varepsilon$, *the normal form of linear combinations of* $F = (F_1, \dots, F_{n-p})$ *is as follows, with* $u = (u_1, \dots, u_{n-p}) \in \mathbb{R}^{n-p}$:

$$\sum_{j=1}^{n-p} F_j u_j = \sum_{j=1}^{n-p} u_j \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{i=1}^{p-1} L_w^i(x, u)\frac{\partial}{\partial y_i} + \frac{1}{2}M_w(x, u)\frac{\partial}{\partial w} + O^2(\varepsilon), \quad (2.2)$$

where $L_w^i(x, u)$ *and* $M_w(x, u)$ *are skew-symmetric bilinear forms depending smoothly on* w *and* $O^2(\varepsilon)$ *is a smooth vector field, the components of which are bounded by* $K\varepsilon^2$ *for some* $K > 0$, *independent of* ε.

In the matrix form, we will write

$$\begin{aligned} L_w^i(x, u) &= x' L_w^i u, \qquad i = 1, \dots, p-1, \\ M_w(x, u) &= x' M_w u, \end{aligned} \quad (2.3)$$

where the prime denotes transposition and L_w^i and M_w are smooth one-parameter families of skew-symmetric matrices.

In this notation, w is not an index, and one should not confuse the notation M_w and M_i. The matrix M_w is equal to $M_{1,0,w}$ in the notation of Sec. 2.3.

In the following lemma and corollary, we give useful rough estimates that are also consequences of the standard SR ball-box theorem.

Lemma 2.3. *In the normal coordinates* $\xi = (x, y, w)$ *on a compact neighborhood* N *of the curve* Γ: *for all* $0 < \omega < 1$, *there are positive constants* $k_1(\omega)$ *and* k_2 *such that the balls* B_{ε,w_0} *of radius* ε *centered at points* $(0, 0, w_0)$ *of* Γ *satisfy the following conditions*:

a. $B_{\varepsilon,w_0} \subset \Big\{\xi \;\Big|\; \|x\|_2 \le \varepsilon;\ |y_i| \le k_2\varepsilon^2,\ i = 1, \dots, p-1;\ |w - w_0| \le k_2\varepsilon^2\Big\}$;

b. $\left\{\xi \,\middle|\, \|x\|_2 \leq \omega\varepsilon;\ |y_i| \leq k_1(\omega)\varepsilon^2,\ i=1,\dots,p-1;\ |w-w_0| \leq k_1(\omega)\varepsilon^2\right\} \subset B_{\varepsilon,w_0}$.

Lemma 2.3 implies the following assertion.

Corollary 2.1. *In the normal coordinates* $\xi = (x, y, w)$ *on a compact neighborhood* N *of the curve* Γ*, there are constants* $h_1, h_2 > 0$ *such that if* ξ_1*,* $\xi_2 \in \mathcal{T}_\varepsilon$*, then the time* (*or the arclength*) *to go from* ξ_1 *to* ξ_2*, remaining inside* $\mathcal{T}_\varepsilon$*, is less than* $(w_2 - w_1)\frac{h_1}{\varepsilon} + h_2\varepsilon$.

2.5. **The affine space of fundamental 2-forms.** Given $\Sigma = (\Gamma, \Delta, g) \in \mathcal{S}$, we consider the 1-forms α defined on Ξ such that

$$\alpha(\Delta) = 0,\ \alpha(\dot{\Gamma}) = 1.$$

This space of one-forms is invariant with respect to multiplication by a function which is 1 on Γ. Now let us consider the space Ω_t of 2-forms obtained by taking the exterior derivative of α, and by restricting to $\Delta(\Gamma(t))$ for all $t \in [0, 1]$.

Then Ω_t is the affine space of 2-forms on $\Delta(\Gamma(t))$, and each $\omega_t \in \Omega_t$ can be written as follows:

$$\omega_t = \omega_t^0 + \sum_{i=1}^{p-1} \lambda_i \omega_t^i, \quad \lambda_i \in \mathbb{R},$$

and we will see below (Lemma 2.5) that, owing to the bracket-generating assumption, ω_t^0 and ω_t^i *are independent*.

We can also define an affine space $\bar{\Omega}_t$, $t \in [0, 1]$, of skew-symmetric endomorphisms $\bar{\omega}_t$ of $\Delta(\Gamma(t))$, as follows:

$$\bar{\Omega}_t = \left\{\bar{\omega}_t \,\middle|\, \langle\bar{\omega}_t(X), Y\rangle_g = \omega_t(X, Y)\ \forall X, Y \in \Delta(\Gamma(t)),\ \omega_t \in \Omega_t\right\}.$$

If, moreover, an orthonormal frame is specified on each $\Delta(\Gamma(t))$ (for example, this is the case where a normal coordinate system is fixed, with respect to an arbitrary surface S), we call $\tilde{\Omega}_t$ the affine space of matrices obtained by taking the matrices of $\bar{\omega}_t \in \bar{\Omega}_t$ with respect to this frame. Then, $\tilde{\Omega} = \{\tilde{\Omega}_t,\ t \in [0, 1]\}$ is a field along Γ of p-dimensional affine spaces of skew-symmetric matrices A_t.

This field is well defined and unique if an orthonormal frame (or a normal coordinate system) is chosen along Γ. It is easy to see that a change of normal coordinates (2.1) acts on A_t by the transformation $T(0, t)A_tT(0, t)'$.

This field Ω (respectively, $\bar{\Omega}$, $\tilde{\Omega}$) of affine spaces of 2-forms (respectively, endomorphisms of $\Delta(\Gamma(t))$, skew-symmetric matrices), is called the field of *fundamental* 2-forms (respectively, fundamental endomorphisms, fundamental matrices).

A simple computation (we omit it) proves the following assertion.

Lemma 2.4 (fundamental matrices in normal coordinates)**.** *Let $\xi = (x, y, w)$ be a normal coordinate system. The field of fundamental matrices is*

$$\tilde{\Omega}_t = \Big\{ M_t + \sum_{i=1}^{p-1} \lambda_i L_t^i,\ \lambda_i \in \mathbb{R} \Big\}, \tag{2.4}$$

where M_t, L_t^i, $i = 1, \dots, p-1$, are the matrices appearing in the normal form (2.2) *and* (2.3)*.*

Remark 2.1. Note that the matrices M_t and L_t^i themselves depend on the surface S, and even on its parametrization. They depend also on the normal coordinates. However, the associated field $\tilde{\Omega}_t$ of affine spaces of skew-symmetric (with respect to g) endomorphisms of $\Delta(\Gamma(t))$ does not. It depends only on the given $\Sigma \in \mathcal{S}$.

2.6. **Brackets.** For a 1-form α and vector fields X and $\tilde{X}$, the standard formula

$$d\alpha(X, \tilde{X}) = \alpha([X, \tilde{X}]) - L_X \alpha(\tilde{X}) + L_{\tilde{X}} \alpha(X)$$

shows that the mapping $\Delta_q \times \Delta_q \to T_q\Xi/\Delta_q$, $(X, \tilde{X}) \to [X, \tilde{X}]$ is well defined.

Fix $\Sigma \in \mathcal{S}$, together with a normal coordinate system $\xi = (x, y, w)$. Then at a point $\xi_0 = (0, 0, w_0) \in \Gamma$, the tangent plane $T_{\xi_0} S \approx S$ with coordinates (y, w) is identified with $T_{\xi_0}\Xi/\Delta_{\xi_0}$ and Δ_{ξ_0} is identified with the horizontal plane $P_{0,w_0} = \{\xi | y = 0, w = w_0\}$, and finally, the mapping $[\cdot, \cdot]$ is just the mapping

$$(x, \tilde{x}) \to (x' L_w^1 \tilde{x}, \dots, x' L_w^{p-1} \tilde{x}, x' M_w \tilde{x}), \tag{2.5}$$

where M_w, L_w^i, $i = 1, \dots, p-1$ are the skew-symmetric matrices appearing in formulas (2.2) and (2.3).

Recall that all the normal forms are valid not only for bracket-generating distributions (we need only that Δ be transversal to Γ). In fact, formula (2.5) implies the following lemma.

Lemma 2.5. *The distribution Δ is one-step-bracket-generating* (*in a neighborhood of* Γ) *if and only if the skew-symmetric matrices M_w and L_w^i, $i = 1, \dots, p-1$, are independent for all w.*

2.7. **Genericity.** Consider the mapping Λ: $\mathcal{S} \times [0, 1] \to \mathcal{A}(p, \mathfrak{so}(n-p))$, $(\Sigma = (F, \Gamma), t) \to \tilde{\Omega}_t$, where $\tilde{\Omega}_t$ is the affine space of skew-symmetric matrices corresponding to the choice of $F(\Gamma(t))$ for an orthonormal frame in $\Delta(\Gamma(t))$. Here, $\mathfrak{so}(n-p)$ denotes as usual the set of skew-symmetric matrices of size $(n-p)$, and $\mathcal{A}(p, \mathfrak{so}(n-p))$ denotes the set of affine spaces (of dimension $p-1$) of skew-symmetric matrices of size $(n-p)$.

Proposition 2.2. *The mapping Λ is a surjective submersion.*

The proof of this proposition is similar to the proof given in [7], in the case where $p = 1$, i.e., the affine spaces are reduced to points (dimension zero).

Denote by $\mathcal{V}$ the subset of $\mathfrak{so}(n-p)$ formed by the matrices with distinct nonzero eigenvalues, and with kernel of dimension at most 2. $\mathcal{V}$ is semi-algebraic, and its complement has codimension 3 at least for $n-p \geq 3$ (see, e.g., [7]). Let $\mathcal{E}_1$ be the subset of $\mathcal{A}(p, \mathfrak{so}(n-p))$ formed by affine subspaces whose elements (matrices) belong to $\mathcal{V}$.

Proposition 2.3. *$\mathcal{E}_1$ is semi-algebraic, and its complement has codimension at least $4-p$ for $n-p \geq 3$.*

Let $\tilde{\omega} \in \mathcal{A}(p, \mathfrak{so}(n-p))$ and let $\operatorname{eig}(\tilde{\omega})$ denote the minimum over the affine space $\tilde{\omega}$ of the maximum moduli of eigenvalues of the skew-symmetric matrices $A \in \tilde{\omega}$ (or of the norm $\|A\|_2$ of such matrices).

Lemma 2.6. *The function* $\operatorname{eig} : \mathcal{A}(p, \mathfrak{so}(n-p)) \to \mathbb{R}_+$ *is semi-algebraic and continuous.*

Proof. First,

$$\mathcal{A}(p, \mathfrak{so}(n-p)) = \left\{ \left\{ A + \sum_{i=1}^{p-1} \lambda_i B_i \;\middle|\; \lambda_i \in \mathbb{R} \right\} \;\middle|\; B_i \text{ are independent} \right\}.$$

In the abbreviated notation, we write $A + \sum_{i=1}^{p} \lambda_i B_i = A + \lambda B$ and we identify $\mathcal{A}(p, \mathfrak{so}(n-p))$ with the set of p-tuples of skew-symmetric matrices $(A, B) = (A, B_1, \ldots, B_{p-1})$. We set

$$\alpha(A, B, \lambda) = \max\{\text{modulus of eigenvalues of } A + \lambda B\} = \|A + \lambda B\|_2.$$

The function α is continuous, proper in the restriction to any compact set of the (A, B) space (if λ is not bounded, $\alpha(A + \lambda B)$ is not bounded, since the B_i's are independent). By Lemma 5.1 (see Appendix 5), the function eig is continuous.

The fact that eig is semi-algebraic follows from the Tarski–Seidenberg theorem. □

Let $\mathcal{A}^+(p, \mathfrak{so}(n-p))$ denote the set of affine spaces that are not vector subspaces. The following lemma is obvious.

Lemma 2.7. *The function* eig *is bounded from below by a strictly positive number a_Ω, in restriction to any compact subset of $\mathcal{A}^+(p, \mathfrak{so}(n-p))$.*

For $\omega \in \mathcal{A}(p, \mathfrak{so}(n-p))$, denote by $\Lambda(\omega)$ the set of $A \in \omega$ reaching the minimum $\operatorname{eig}(\omega)$. Let L be the set of all couples (ω, A), $A \in \Lambda(\omega)$, and let Π be the projection on the first component, $\Pi : L \to \mathcal{A}(p, \mathfrak{so}(n-p))$. It is easy to see that L is semi-algebraic, closed, and that L is bounded vertically (Π is proper). Then, by the theorems on stratification of mappings (see [11]), there exists an analytic section $s : U \subset \mathcal{A}(p, \mathfrak{so}(n-p)) \to L$, where U is open and dense. Let $\mathcal{E}_2$ denote the complement of U in $\mathcal{A}(p, \mathfrak{so}(n-p))$. It is subanalytic, closed, of codimension at least one. As a consequence, we have the following lemma.

Lemma 2.8. *A smooth curve $\gamma : [0,1] \to \mathcal{A}(p, \mathfrak{so}(n-p))$ transversal to $\mathcal{E}_2$ has the following property $(\mathcal{P}_1)$, by construction: there is a (bounded) section s over γ, $\Pi \circ s = \mathrm{Id}_\gamma$, $s(\gamma(t)) \in \Lambda(\gamma(t))$, which is smooth except for a finite number of points.*

It follows from Propositions 2.2 and 2.3, Lemma 2.8, and transversality theorems for closed Whitney-stratified subsets (see [9]) that there is an open-dense subset $\mathcal{S}^* \subset \mathcal{S}$ of $\Sigma = (\Gamma, F)$ such that the mapping $t \to \tilde{\Omega}_t$ is transversal to the closure of the subsets $\mathcal{E}_2$ and Complement$(\mathcal{E}_1)$.

Hence, we have the following lemma.

Lemma 2.9. *For $\Sigma \in \mathcal{S}^*$, we have:*

1. *For $p = 2$ $(n \geq 5)$:*
   (a) *all the matrices in $\tilde{\Omega}_t$ have simple nonzero eigenvalues, and the kernel of dimension at most* 2;
   (b) *the mapping $t \to \mathrm{eig}(\tilde{\Omega}_t)$ is continuous, piecewise-smooth (it is not smooth at most on a finite set);*[1]
   (c) *there is a section Λ of the field $\tilde{\Omega}$ $(\Lambda(t) \in \tilde{\Omega}_t)$ such that $\|\Lambda(t)\|_2 = \mathrm{eig}(\tilde{\Omega}_t)$ and Λ is bounded and smooth except possibly for a finite set.*
2. *For $p = 3$ $(n \geq 6)$, properties* (b) *and* (c) *still hold and property* (a) *holds except for a finite set.*
3. *For arbitrary p, properties* (b) *and* (c) *still hold.*

3. Minorant of the metric complexity

In this section, we will use the properties of the one-step-bracket-generating motion planning problems described above: for $\Sigma = (\Gamma, F) \in \mathcal{S}$, the function: $[0,1] \to \mathbb{R}_+$, $t \to \mathrm{eig}(\tilde{\Omega}_t)$ is continuous (Lemma 2.6).

Also, the skew-symmetric matrices M_w, L^i_w, $i = 1, \dots, p-1$ are independent (Lemma 2.5). By Lemma 2.4, they generate $\tilde{\Omega}_t$ and, therefore, $\tilde{\Omega}_t \in \mathcal{A}^+(p, \mathfrak{so}(n-p))$ for all $t \in [0,1]$). Then $\mathrm{eig}(\tilde{\Omega}_t)$ is strictly positive by Lemma 2.7.

We prove the following theorem.

[1] It can be verified that, for $n = 6$ and $p = 2$, this set of isolated points may actually be nonempty, for an open set of systems.

Theorem 3.1. *For $\Sigma = (\Gamma, F) \in \mathcal{S}$, the metric complexity $MC_\Sigma(\varepsilon, T)$ satisfies the inequality*

$$MC_\Sigma(\varepsilon, T) \geq_s \frac{2}{\varepsilon^2} \int_0^T \frac{dt}{\operatorname{eig}(\tilde{\Omega}_t)}. \tag{3.1}$$

Proof. To prove this, we will use the normal form on the tubes $\mathcal{T}_\varepsilon$ (see (2.2) and (2.3)).

Let us consider the control system

$$\frac{d\xi}{dt} = \sum_{j=1}^{n-p} F_j u_j,$$

where $\sum_{j=1}^{n-p} (u_j)^2 = 1$ and the time is the arclength.

Reading the expression of the fields F_j in (2.2), we obtain that, along a trajectory $\xi(t) = (x(t), y(t), w(t))$ entirely contained in $\mathcal{T}_\varepsilon$, with $w(0) = w_1 = 0$ and $w(T) = w_2$, we have

$$\frac{dw}{dt} = \frac{1}{2} x' M_w u + O^2(\varepsilon) \leq \frac{\varepsilon}{2} ||M_w||_2 + +O^2(\varepsilon). \tag{3.2}$$

Let us make the following change of coordinates: $\tilde{w} = w + \sum_{i=1}^{p-1} \lambda_i(w) y_i$, where $\lambda_i(w)$ are arbitrary smooth functions. Such changes of variables are changes of coordinates in the surface S that leave the w axis invariant, i.e., leave invariant the curve Γ together with its parametrization. Reading again the normal form (2.2), we obtain:

$$\frac{d\tilde{w}}{dt} = \frac{1}{2} x' \left(M_w + \sum_{i=1}^{p-1} \lambda_i(w) L_w^i \right) u + \sum_{i=1}^{p-1} y_i \frac{d\lambda_i(w)}{dw} \dot{w} + O^2(\varepsilon).$$

By the cylinder-box theorem, the y_i's have order ε^2. Also, $\dot{w}$ has order ε. Since Γ is compact ($0 \leq w \leq 1$), $\frac{d\lambda_i(w)}{dw}$ are bounded (smooth). Therefore, we obtain

$$\frac{d\tilde{w}}{dt} = x' \left(M_w + \sum_{i=1}^{p-1} \lambda_i(w) L_w^i \right) u + O^2(\varepsilon).$$

We replace w by $\tilde{w} - \sum_{i=1}^{p-1} \lambda_i(w) y_i$, expand the result, and take again into account the following facts:

(a) y has order $O(\varepsilon^2)$, x has order $O(\varepsilon)$,
(b) $\|u\|_2 = 1$.

Then we obtain

$$\frac{d\tilde{w}}{dt} \leq \frac{\varepsilon}{2}\left\| M_{\tilde{w}} + \sum_{i=1}^{p-1} \lambda_i(\tilde{w}) L^i_{\tilde{w}} \right\|_2 + \varepsilon^2 \tilde{K}(\lambda).$$

This shows that

$$\frac{2}{\varepsilon} \int\limits_{w_1+\lambda(w_1)y_1}^{w_2+\lambda(w_2)y_2} \frac{d\tilde{w}}{\left\| M_{\tilde{w}} + \sum\limits_{i=1}^{p-1} \lambda_i(\tilde{w}) L^i_{\tilde{w}} \right\|_2} \leq (1+\varepsilon K(\lambda))T,$$

where $K(\lambda)$ is positive, depending on the choice of the arbitrary functions $\lambda_i(w)$.

As we have already said at the beginning of the section, the denominator above is bounded from below by a strictly positive constant.

Then, using once more the fact that y has order $O^2(\varepsilon)$, we obtain

$$\frac{2}{\varepsilon} \int\limits_{w_1}^{w_2} \frac{d\tilde{w}}{\left\| M_{\tilde{w}} + \sum\limits_{i=1}^{p-1} \lambda_i(\tilde{w}) L^i_{\tilde{w}} \right\|_2} \leq (1+\varepsilon K(\lambda))T + \varepsilon S(\lambda)$$

or

$$T \geq \frac{2}{\varepsilon(1+\varepsilon K(\lambda))} \int\limits_{w_1}^{w_2} \frac{d\tilde{w}}{\left\| M_{\tilde{w}} + \sum\limits_{i=1}^{p-1} \lambda_i(\tilde{w}) L^i_{\tilde{w}} \right\|_2} - \varepsilon \bar{S}(\lambda),$$

for sufficiently small ε (depending on λ) and for some positive values $\bar{S}(\lambda)$ and $K(\lambda)$.

Dividing this inequality by the quantity

$$\Psi = \frac{2}{\varepsilon} \int\limits_{w_1}^{w_2} \frac{dw}{\operatorname{eig}(\tilde{\Omega}_w)} = \frac{2}{\varepsilon}\tilde{\Psi}$$

(which is also positive, as we have said at the beginning of the section), we have

$$\frac{T(\xi(\cdot))}{\Psi} \geq \frac{2}{\varepsilon(1+\varepsilon K(\lambda))\Psi} \int\limits_{w_1}^{w_2} \frac{d\tilde{w}}{\left\| M_{\tilde{w}} + \sum\limits_{i=1}^{p-1} \lambda_i(\tilde{w}) L^i_{\tilde{w}} \right\|_2} - \frac{\varepsilon}{\Psi}\bar{S}(\lambda).$$

The right-hand side of this formula is independent of the trajectory $\xi(\cdot)$ contained in $\mathcal{T}_\varepsilon$. Hence,

$$\Phi(\varepsilon) = \frac{\inf\limits_{\xi(\cdot)\subset\mathcal{T}_\varepsilon}(T(\xi(\cdot)))}{\Psi} \geq \frac{1}{(1+\varepsilon K(\lambda))\tilde{\Psi}} \times \int_{w_1}^{w_2} \frac{d\tilde{w}}{\left\| M_{\tilde{w}} + \sum\limits_{i=1}^{p-1} \lambda_i(\tilde{w}) L^i_{\tilde{w}} \right\|_2} - \frac{\varepsilon}{\Psi}\bar{S}(\lambda).$$

Therefore,

$$\liminf_{\varepsilon\to 0} \frac{\varepsilon MC_\Sigma(\varepsilon, w_2)}{\Psi(\varepsilon)} = \liminf_{\varepsilon\to 0} \Phi(\varepsilon) \geq \frac{1}{\tilde{\Psi}} \int_{w_1}^{w_2} \frac{dw}{\left\| M_w + \sum\limits_{i=1}^{p-1} \lambda_i(w) L^i_w \right\|_2}.$$

Assume that for all $\delta > 0$, we can find $\lambda(w)$ such that

$$\int_{w_1}^{w_2} \frac{dw}{\left\| M_w + \sum\limits_{i=1}^{p-1} \lambda_i(w) L^i_w \right\|_2} \geq (1-\delta)\tilde{\Psi}; \tag{3.3}$$

then we see that

$$MC_\Sigma(\varepsilon, w_2) \geq_s \frac{\Psi}{\varepsilon} = \frac{2}{\varepsilon^2} \int_{w_1}^{w_2} \frac{dw}{\operatorname{eig}(\tilde{\Omega}_w)},$$

as required.

Actually, it is not difficult to show that assumption (3.3) holds: by Lemma 2.6, the set

$$E = \left\{ (\lambda, w) \;\middle|\; \left\| M_w + \sum_{i=1}^{p-1} \lambda_i L^i_w \right\|_2 = \operatorname{eig}(\tilde{\Omega}_w) \right\}$$

is closed (since the function $\operatorname{eig}(\tilde{\Omega}_w)$ is continuous). It is also bounded, and for each w, there exists λ such that $(\lambda, w) \in E$ (recall that boundedness comes from the bracket-generating assumption, and the last claim follows from this boundedness). There exists a measurable and bounded section $w \to \lambda^*(w)$ such that $(w, \lambda^*(w)) \in E$ (see, e.g., [6]). But by the Lusin theorem, λ^* differs from a continuous (bounded by the same bounds) function on a set of arbitrarily small measure ε. Then we can find continuous λ such that $\tilde{\Psi}$ is arbitrarily close to

$$\int_{w_1}^{w_2} \frac{dw}{\left\| M_w + \sum\limits_{i=1}^{p-1} \lambda_i(w) L^i_w \right\|_2}.$$

Finally, we approximate this λ uniformly by an appropriate smooth section.

□

4. Majorant of the metric complexity for $p = 2$ and $p = 3$

First, let us fix $\Sigma \in \mathcal{S}^*$ (defined at the end of Sec. 2) and normal coordinates. Let us also fix $0 < \omega < 1$. As above, $\Gamma = \{(0,0,w) \mid 0 = w_1 \leq w \leq w_2\}$.

By Lemma 2.9 at the end of Sec. 2, there exists a finite number of special isolated points $\xi_1, \ldots, \xi_r \in \Gamma$ such that $\mathcal{T}_\varepsilon$ is the union of r small tubes of height $1-\omega$ around ξ_i and of $r+1$ tubes $\mathcal{T}_\varepsilon^i$, each of them of height h_i, such that $w_2 = \sum\limits_{i=1}^{r+1} h_i + r(1-\omega)$. By Corollary 2.1, the time to cross the r small tubes is $T_1 = (1-\omega)\dfrac{k_1}{\varepsilon} + k_2\varepsilon$ for some $k_1, k_2 > 0$.

Consider one of the closed tubes $\mathcal{T}_\varepsilon^i$, say $\mathcal{T}_\varepsilon^1$, containing $\tilde{\Gamma} = \{(0,0,w) \mid 0 \leq w \leq h_1\}$.

Now take the smooth function $\lambda(w)$ given by Lemma 2.9(c). We can find a smooth couple of vector fields $(X_{w,}^1\ X_w^2) \in \Delta_w \times \Delta_w$, which are orthogonal with respect to the SR metric and which generate along $\tilde{\Gamma}$ the maximum real eigenspace of $\left(M_w + \sum\limits_{i=1}^{p-1} \lambda_i(w) L^i w\right)$. By a change of normal coordinates according to Lemma 2.1, we form a new normal coordinate system such that

$$(X_w^1, X_w^2) = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right).$$

Also, we change the parametrization of the surface S, leaving $\tilde{\Gamma}$ invariant, setting:

$$\tilde{w} = w + \sum_{i=1}^{p-1} \lambda_i(w) y_i.$$

From now on, the new coordinate $\tilde{w}$ will be denoted simply by w. An easy computation shows that the normal form (2.2), (2.3) has the following property: the skew symmetric matrix M_w is block-diagonal: $M_w = \mathrm{blockdiag}(M_w^1, M_w^2)$,

$$M_w^1 = \begin{pmatrix} 0 & -\frac{1}{2}\alpha_1(w) \\ \frac{1}{2}\alpha_1(w) & 0 \end{pmatrix}, \tag{4.1}$$

where

$$\alpha_1(w) = \mathrm{eig}(\tilde{\Omega}_w) = \inf_\lambda \left(\left\| M_w + \sum_{i=1}^{p-1} \lambda_i L_w^i \right|_2 \right).$$

Then we come to the crucial point: Proposition 5.1 (see Appendix 5) shows that

$$L_{w1,2}^i = L_{w2,1}^i = 0 \quad \forall i = 1, \ldots, p-1. \tag{4.2}$$

To summarize, we are at the following point:

($\mathcal{P}$) we have a normal coordinate system, with the normal form (2.2), (2.3), which satisfies also relations (4.1) and (4.2).

The following "approximate asymptotic optimal synthesis" provides the majoration. The synthesis is described in several recurrent steps.

Step 1. Start from $(x, y, w) = (0, 0, 0)$, and go to $(x = x_1, 0, 0)$, where $x_1 = (\omega\varepsilon, 0, \dots, 0)$. This step costs a length (or a time) less than ε.

Step 2. For $\varepsilon > 0$, consider the $(p+1)$-dimensional cylinder:

$$\mathcal{C}^{\varepsilon,\omega} = \left\{ \xi \;\middle|\; \sqrt{(x_1)^2 + (x_2)^2} = \varepsilon\omega,\ (x_3)^2 + \cdots + (x_{n-p})^2 = 0 \right\}. \tag{4.3}$$

For sufficiently small ε, $\mathcal{C}^{\varepsilon,\omega}$ is transversal to Δ. The intersection of Δ with the tangent space to $\mathcal{C}^{\varepsilon,\omega}$ defines a field of tangent lines to $\mathcal{C}^{\varepsilon,\omega}$. Then it defines two opposite unitary vector fields. It is easy to calculate that the w-component of these vector fields is

$$\pm\frac{\omega\varepsilon}{2}\alpha_1(w)\frac{\partial}{\partial w} + O^2(\varepsilon)$$

(it never vanishes since $\pm\alpha_1(w)$ is also equal to $\pm\,\mathrm{eig}(\tilde{\Omega}_w)$, which does not vanish by Lemma 2.7). Again, $O^2(\varepsilon)$ is a function bounded by $\varepsilon^2 K$ for some $K > 0$. Denote by X the vector field along which w increases.

Step 2a. Now, follow the flow of X, till getting out of $\mathcal{T}_\varepsilon$. It costs a certain length (or time) T_1 that we will majorize at a moment to reach a certain height $w = W_1$ that we will bound from below later.

Step 2b. Follow a piece of radial admissible curve and return back to $\tilde{\Gamma}$. This can be done in time (or length) ε (since we are still inside $\mathcal{T}_\varepsilon$), and this makes eventually w decrease by $h\varepsilon^2$, for a certain $h > 0$, as we read on normal form (2.2). This $h\varepsilon^2$ can be compensated in time $h'\varepsilon > 0$, by Corollary 2.1. Finally, Step 2b costs $h''\varepsilon$ for some $h'' > 0$.

Repeat Steps 1 and 2 until the end of the curve $\tilde{\Gamma}$, $w = h_1$.

Now we estimate the time T_1 and height W_1.

Lemma 4.1.

$$T_1 \leq \frac{2}{\omega\varepsilon}\int_0^{W_1}\frac{dw}{\alpha_1(w)}(1 + \varepsilon H)$$

for some constant $H > 0$, and the time $T_1 > 0$ to stay inside $\mathcal{T}_\varepsilon$ can be taken independent of ε.

Proof. Normal form (2.2) and property (4.2) imply

$$X = \frac{\omega\varepsilon}{2}\alpha_1(w)\frac{\partial}{\partial w} + \frac{x_2}{\|x\|_2}\frac{\partial}{\partial x_1} - \frac{x_1}{\|x\|_2}\frac{\partial}{\partial x_2} + O^2(\varepsilon) \tag{4.4}$$

(this is a straightforward computation). In particular, $|\dot{y}_i| \leq \varepsilon^2 K$ for some $K > 0$. By item 2 of the cylinder-box theorem (Theorem 2.2), $|y_i|$ can grow from 0 up to $k\varepsilon^2$, for some $k > 0$, when the trajectory stays in $\mathcal{T}_\varepsilon$. Hence there is another constant $K' > 0$ such that $T_1 \geq K'$.

Now (4.4) shows that

$$\frac{2}{\omega\varepsilon}\frac{dw}{\alpha_1(w)} = dt(1 + O(\varepsilon)),$$

or, for a certain constant K'',

$$\frac{2}{\varepsilon\omega}\int_0^{W_1}\frac{dw}{\alpha_1(w)} \geq T_1(1 - \varepsilon K'').$$

Recall that in these computations $\alpha_1(w)$ is bounded from below by a strictly positive constant. □

Then, if we repeat Step 2 k times, where $k = A/\varepsilon$, we obtain a time T satisfying, for some $A' > 0$:

$$T \leq \frac{2}{\omega\varepsilon}\int_0^{h_1}\frac{dw}{\alpha_1(w)} + A'.$$

The constant A' depends on the ω fixed at the beginning of the section.

Finally, we see that the time T_f (arclength) to reach $w = w_2$ (the end of the curve Γ) satisfies

$$T_f \leq \frac{2}{\omega\varepsilon}\int_0^{w_2}\frac{dw}{\alpha_1(w)} + A'(\omega) + (1-\omega)\frac{k_1}{\varepsilon} + k_2\varepsilon.$$

Then

$$\inf_{\substack{\gamma\subset\mathcal{T}_\varepsilon\\ \gamma(0)=\Gamma(0)\\ \gamma(T_\gamma)=\Gamma(w_2)}}(T(\gamma)) \leq \frac{2}{\omega\varepsilon}\int_0^{w_2}\frac{dw}{\alpha_1(w)} + A'(\omega) + (1-\omega)\frac{k_1}{\varepsilon} + k_2\varepsilon$$

and

$$\frac{\varepsilon MC_\Sigma(\varepsilon, w_2)}{\frac{2}{\varepsilon}\int_0^{w_2}\frac{dw}{\alpha_1(w)}} \leq \frac{1}{\omega} + \varepsilon K_1(\omega) + (1-\omega)K_2,$$

for some $K_1(\omega), K_2 > 0$. Hence,

$$\limsup_{\varepsilon\to 0}\frac{\varepsilon MC_\Sigma(\varepsilon, w_2)}{\frac{2}{\varepsilon}\int_0^{w_2}\frac{dw}{\alpha_1(w)}} \leq \frac{1}{\omega} + (1-\omega)K_2.$$

Since this is true for all $0 < \omega < 1$, we have

$$MC_{\Sigma}(\varepsilon, w_2) \leq_s \frac{2}{\varepsilon^2} \int_0^{w_2} \frac{dw}{\alpha_1(w)}.$$

Combining this with Theorem 3.1, we obtain the following assertion.

Theorem 4.1. *For $\Sigma \in \mathcal{S}^*$ (which is an open dense condition), for $p = 2$ or $p = 3$:*

$$MC_{\Sigma}(\varepsilon, w_2) \simeq_s \frac{2}{\varepsilon^2} \int_0^{w_2} \frac{dw}{\alpha_1(w)} = \frac{2}{\varepsilon^2} \int_0^{w_2} \frac{dw}{\operatorname{eig}(\tilde{\Omega}_w)}.$$

5. Appendix

5.1. **Continuity lemma.** For completeness, we prove here a very elementary lemma (but, curiously, we were not able to find a reference). Hence, we provide a simple proof.

Lemma 5.1. *Let X be a manifold and $f : \mathbb{R}^p \times X \to \mathbb{R}_+$ be a continuous function such that, for each compact set $K \subset X$, the restriction $f_K = f|_{\mathbb{R}^p \times K}$ is proper. Then $\varphi(x) = \inf_{t \in \mathbb{R}^p} f(t, x)$ is a well-defined continuous function on X.*

Proof. Fix $x^* \in X$. Consider t^* such that $f(t^*, x^*) = \inf_{t \in \mathbb{R}^p} f(t, x^*)$. Since $f(t, x^*)$ is proper, the domain $V = \{t \mid f(t, x^*) \leq f(0, x^*)\}$ is compact, and t^* does exist, realizing the minimum over V.

Consider a sequence x_n converging to x^*. It is entirely contained in a compact set $K \subset X$, where K is a neighborhood of x^*. Also, a sequence t_n such that $f(t_n, x_n) = \inf_{t \in \mathbb{R}^p} f(t, x_n)$ does exist for the same reason.

The sequence t_n is bounded: if it is not so, by the properness of f over $\mathbb{R}^p \times K$, $f(t_n, x_n)$ is also not bounded (if it is bounded, since f_K is proper, t_n is also bounded). Then, there exists a subsequence, also denoted by (t_n, x_n), such that $f(t_n, x_n) > f(t^*, x^*) + 1$. But, for sufficiently large n, by the continuity of f,

$$f(t^*, x_n) < f(t^*, x^*) + 1 < f(t_n, x_n).$$

This contradicts the fact that $f(t_n, x_n) = \inf_{t \in \mathbb{R}^p} f(t, x_n)$.

Then, since t_n is bounded, we extract again a sequence, still denoted by (t_n, x_n), such that $(t_n, x_n) \to (\bar{t}, x^*)$. If $f(\bar{t}, x^*) > f(t^*, x^*)$, then by the convergence and continuity, for sufficiently large n,

$$f(t_n, x_n) > \frac{1}{2} \left(f(t^*, x^*) + f(\bar{t}, x^*) \right).$$

On the other hand, by the continuity of f, we have

$$f(t^*, x_n) < \frac{1}{2}\left(f(t^*, x^*) + f(\bar{t}, x^*)\right) < f(t_n, x_n),$$

a contradiction since $f(t_n, x_n) = \inf_{t \in \mathbb{R}^p} f(t, x_n)$. Then it is impossible that $f(\bar{t}, x^*) > f(t^*, x^*)$.

Also, $f(\bar{t}, x^*) < f(t^*, x^*)$ is impossible, since

$$f(t^*, x^*) = \inf_{t \in \mathbb{R}^p} f(t, x^*).$$

Hence $f(\bar{t}, x^*) = f(t^*, x^*)$ and

$$\lim \varphi(x_n) = \lim f(t_n, x_n) = f(\bar{t}, x^*) = f(t^*, x^*) = \varphi(x^*).$$

Therefore, the function φ is continuous. □

5.2. **Some properties of pencils of skew-symmetric matrices.** Assume that A and B_i, $i = 1, \ldots, k$, are *independent*, skew-symmetric $(N \times N)$-matrices such that the affine pencil $P(\cdot)$, $P(\lambda) = A + \sum_{i=1}^{k} \lambda_i B_i$, $\lambda_i \in \mathbb{R}$, has no multiple nonzero eigenvalue and the kernel of dimension at most 2 for all $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$. For simplicity, we abuse the notation and write $P(\lambda) = A + \lambda B$.

Lemma 5.2. *There are p analytic functions of λ, $\alpha_1(\lambda), \ldots, \alpha_p(\lambda)$, $p = [N/2]$ or $[N/2] + 1$ (depending on the parity of N) such that:*

1. *$\alpha_1(\lambda), \ldots, \alpha_{p-1}(\lambda) > 0$ for all λ;*
2. *$\alpha_1(\lambda) > \cdots > \alpha_p(\lambda)$ and $\pm i\alpha_1(\lambda), \ldots, \pm i\alpha_p(\lambda)$ are the eigenvalues of $P(\lambda)$ for all λ.*

In particular,

$$\alpha_1(\lambda) = \|P(\lambda)\|_2 = \sup_{\|x\|_2 = \|y\|_2 = 1} y' P(\lambda) x. \tag{5.1}$$

This shows that $\alpha_1(\lambda)$ is a convex function (the supremum of affine functions). Moreover, it is *strictly convex*: assume that the minimum of $\alpha_1(\lambda)$ is reached on a nontrivial segment $I = \{\omega\lambda_0 + (1 - \omega)\lambda_1 \mid \omega \in [0, 1]\}$ (the minimum is attained on a convex set, compact since the B_i's are independent), then it is reached on the whole line containing I by analyticity, which is impossible.

Also, it is easy to see that we can find an analytic family $\{X(\lambda), Y(\lambda)\}$ such that for all λ,

$$\|X(\lambda)\|_2 = \|Y(\lambda)\|_2 = 1, \quad \langle X(\lambda), Y(\lambda)\rangle = 0,$$
$$\alpha_1(\lambda) = \sup_{\|x\|_2 = \|y\|_2 = 1} y' P(\lambda) x = Y(\lambda)' P(\lambda) X(\lambda).$$

Let λ^* be the unique λ such that $\inf_{\lambda} \alpha_1(\lambda) = \alpha_1(\lambda^*)$. The following proposition is crucial in Sec. 4.

Proposition 5.1. *For all* $i = 1, \dots, k$, $Y(\lambda^*)' B_i X(\lambda^*) = 0$.

Proof. Let us set

$$\begin{aligned} \alpha_1(\lambda) &= \alpha_1(\lambda^*) + O(\lambda - \lambda^*), \\ X(\lambda) &= X(\lambda^*) + \sum_{i=1}^{k} (\lambda_i - \lambda_i^*) \tilde{X}_i(\lambda^*) + O^2(\lambda - \lambda^*), \\ Y(\lambda) &= Y(\lambda^*) + \sum_{i=1}^{k} (\lambda_i - \lambda_i^*) \tilde{Y}_i(\lambda^*) + O^2(\lambda - \lambda^*), \end{aligned} \tag{5.2}$$

where $O^i(\lambda - \lambda^*)$ has components in the ith power of the ideal of $C^\omega(\mathbb{R}^k)$ generated by the functions $(\lambda_r - \lambda_r^*)$, $r = 1, \dots, k$.

Let us express $\|X(\lambda)\|_2 = \|Y(\lambda)\|_2 = 1$. This means

$$1 = \langle X(\lambda^*), X(\lambda^*) \rangle + 2 \sum_{i=1}^{k} (\lambda_i - \lambda_i^*) \left\langle X(\lambda^*), \tilde{X}_i(\lambda^*) \right\rangle + O^2(\lambda - \lambda^*),$$

hence, since $1 = \langle X(\lambda^*), X(\lambda^*) \rangle$,

$$\forall i = 1, \dots, k, \quad \left\langle X(\lambda^*), \tilde{X}_i(\lambda^*) \right\rangle = 0, \quad \left\langle Y(\lambda^*), \tilde{Y}_i(\lambda^*) \right\rangle = 0. \tag{5.3}$$

Now we have

$$\alpha_1(\lambda) = O^2(\lambda - \lambda^*) + \left(Y(\lambda^*) + \sum_{i=1}^{k} (\lambda_i - \lambda_i^*) \tilde{Y}_i(\lambda^*) \right)'$$

$$\times \left(A + \sum_{i=1}^{k} \lambda_i^* B_i + \sum_{i=1}^{k} (\lambda_i - \lambda_i^*) B_i \right) \left(X(\lambda^*) + \sum_{i=1}^{k} (\lambda_i - \lambda_i^*) \tilde{X}_i(\lambda^*) \right),$$

or

$$\begin{aligned} \alpha_1(\lambda) = O^2(\lambda - \lambda^*) &+ Y(\lambda^*)' \left(A + \sum_{i=1}^{k} \lambda_i^* B_i \right) X(\lambda^*) \\ &+ Y(\lambda^*)' \left(\sum_{i=1}^{k} (\lambda_i - \lambda_i^*) B_i \right) X(\lambda^*) + I, \end{aligned}$$

where

$$\begin{aligned} I = &\left(\sum_{i=1}^{k} (\lambda_i - \lambda_i^*) \tilde{Y}_i(\lambda^*) \right)' \left(A + \sum_{i=1}^{k} \lambda_i^* B_i \right) X(\lambda^*) \\ &+ Y(\lambda^*)' \left(A + \sum_{i=1}^{k} \lambda_i^* B_i \right) \left(\sum_{i=1}^{k} (\lambda_i - \lambda_i^*) \tilde{X}_i(\lambda^*) \right). \end{aligned}$$

If we show that $I = 0$, it follows that

$$\forall i = 1, \dots, k, \quad Y(\lambda^*)' B_i X(\lambda^*) = 0, \tag{5.4}$$

since for all λ, we have

$$\alpha_1(\lambda) \geq \alpha_1(\lambda^*) = Y(\lambda^*)' \left(A + \sum_{i=1}^{k} \lambda_i^* B_i \right) X(\lambda^*).$$

Let us show that $I = 0$. By the definition of X and Y, we have

$$\left(A + \sum_{i=1}^{k} \lambda_i^* B_i \right) X(\lambda^*) = -\alpha_1(\lambda^*) Y(\lambda^*),$$
$$\left(A + \sum_{i=1}^{k} \lambda_i^* B_i \right) Y(\lambda^*) = \alpha_1(\lambda^*) X(\lambda^*).$$

This implies

$$I = \alpha_1(\lambda^*) \sum_{i=1}^{k} (\lambda_i - \lambda_i^*) \Big(- \tilde{Y}_i(\lambda^*)' Y(\lambda^*) - X(\lambda^*)' \tilde{X}_i(\lambda^*) \Big).$$

By (5.3), this is zero. □

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