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Perturbation in Critical Point Theory

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PERTURBATION IN CRITICAL POINT THEORY

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Lectures based on the monograph

A. Ambrosetti and A. Malchiodi: *Perturbation methods and semilinear elliptic problems on* $\mathbb{R}^n$, Progress in Math. Vol. 240, Birkhäuser, 2005.

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1 Motivations

1.1 Sub-critical Elliptic equations on $\mathbb{R}^n$

To prove existence of solutions of elliptic problems on $\mathbb{R}^n$ one of the main difficulties is the lack of compactness.

For ex., the functional

$$I_0(u) := \int_{\mathbb{R}^n} \tfrac{1}{2}\left[|\nabla u|^2 + u^2\right] dx - \tfrac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1} dx, \quad u \in W^{1,2}(\mathbb{R}^n),$$

does not satisfy the Palais-Smale (PS) compactness condition.

Actually, it is easy to see that there exists a unique positive, radially symmetric function $U \in W^{1,2}(\mathbb{R}^n)$ satisfying

$$-\Delta U + U = U^p.$$

Then for every $\xi \in \mathbb{R}^n$, any $U(x-\xi)$ is a solution of

$$-\Delta u + u = u^p, \quad u \in W^{1,2}(\mathbb{R}^n),$$

and hence a critical point of I_0.

Remark. The lack of (PS) is closely related to the fact that the embedding of $W^{1,2}(\mathbb{R}^n)$ into $L^{p+1}(\mathbb{R}^n)$ is not compact, even if $p+1 < 2^*$.

On the other hand, a classical result by W. Strauss states that the subspace

$$W_r^{1,2}(\mathbb{R}^n) = \{u \in W^{1,2}(\mathbb{R}^n) : u \text{ is radial}\}$$

is compactly embedded in $L^q(\mathbb{R}^n)$ when $1 < q < 2^*$.

This allows us to show that I_0 restricted to $W_r^{1,2}$ satisfies the (PS) condition. Moreover, it is immediate to check that I_0 has the Mountain-Pass geometry, namely

(i) $u = 0$ is a strict local minimum of I_0, and

(ii) there exists $e \in W_r^{1,2}$ such that $I_0(e) < I_0(0) = 0$.

Thus I_0 has a M-P critical point which is nothing but U.
More in general, consider the b.v.p.

$$(P_b) \qquad -\Delta u + u = b(x)u^p, \quad u \in W^{1,2}(\mathbb{R}^n),$$

where we assume:

$$(1) \qquad \lim_{|x|\to\infty} b(x) = b_\infty > 0.$$

To simplify the notation we will take $b_\infty = 1$.
The corresponding functional is given by

$$I_b(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{p+1}\int_{\mathbb{R}^n} b(x)|u|^{p+1}dx.$$

Let us introduce its *limit at infinity*, obtained substituting b with $b_\infty = 1$, namely

$$I_0(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{p+1}\int_{\mathbb{R}^n} |u|^{p+1}dx.$$

Let c_0 denote the M-P critical level of I_0 (one has $c_0 = I_0(U)$) and let us set

$$S_{p+1} = \inf\{\|u\|^2 \,:\, u \in W^{1,2}(\mathbb{R}^n), \int_{\mathbb{R}^n} |u|^{p+1}dx = 1\}.$$

It is well known that $S_{p+1} > 0$ and is achieved at some u^* such that $\|u^*\|^2 = S_{p+1}$. Notice that S_{p+1} is the best Sobolev constant for the embedding $W^{1,2}(\mathbb{R}^n) \hookrightarrow L^{p+1}(\mathbb{R}^n)$ and hence

$$(2) \qquad \|u\|^2_{L^{p+1}} \leq S^{-1}_{p+1}\,\|u\|^2.$$

Moreover, we have that $U = S_{p+1}^{1/(p-1)}u^*$ satisfies $-\Delta U + U = U^p$ and hence

$$c_0 = I_0(U) = (\tfrac{1}{2} - \tfrac{1}{p+1})\|U\|^2 = (\tfrac{1}{2} - \tfrac{1}{p+1})S_{p+1}^{\frac{p+1}{p-1}}.$$

Lemma 1.1 *Suppose that b satisfies (1), with $b_\infty = 1$. Then I_b satisfies $(PS)_c$ for any $c < c_0$.*

It is easy to check that the assumption

$$(3) \qquad b(x) \geq b_\infty \ (=1) \qquad \forall x \in \mathbb{R}^n$$

implies that the M-P level c_b of I_b satisfies $c_b \leq c_0$, with strict inequality provided $b \not\equiv b_\infty \ (=1)$ (if $b \equiv 1$ one has that $I_b \equiv I_0$).

Then the previous Lemma implies that I_b satisfies $(PS)_c$ at $c = c_b$ and hence I_b has a M-P critical point. Thus

Theorem 1.2 *If (1) and (3) hold, I_b has a Mountain Pass critical point and hence the problem (P_b) has a (positive) solution.*

More in general, using the P.L. Lions *Concentration-Compactness Principle*, one can prove:

Theorem. (A. Bahri - P.L. Lions) *Let $1 < p < \frac{n+2}{n-2}$ and suppose that $b \in L^\infty(\mathbb{R}^n)$ satisfies*

(a) $b > 0$ and $\lim_{|x|\to\infty} b(x) = b_\infty > 0$;

(b) there exist $R, C, \delta > 0$ such that

$$b(x) \geq b_\infty - C\exp(-\delta x), \qquad \textit{for } |x| \geq R$$

Then the problem (P_b) has a positive solution.

Let us consider now the problem

$$(4) \qquad \begin{cases} -\Delta u + u = (1+\varepsilon h(x))u^p, \\ u \in W^{1,2}(\mathbb{R}^n),\ u > 0, \end{cases}$$

where $h(x)$ is a bounded function.

Question: Does equation (4) possess positive solutions for ε sufficiently small?

We will show that, under suitable, natural assumptions on h there exists $\bar\xi \in \mathbb{R}^n$ such that (4) has a solution $u_\varepsilon \sim U(\cdot - \bar\xi)$ for ε small enough.

Roughly, the new feature is that we do not need to compare h with its limit at infinity.

1.2 Equations with critical exponent

We will consider problems like

$$-\Delta u = (1+\varepsilon k(x))\, u^{\frac{n+2}{n-2}}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \quad u > 0. \tag{5}$$

Equations of this type arise in Differential Geometry.

The new feature is that the unperturbed problem

$$-\Delta u = u^{\frac{n+2}{n-2}}, \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^n), \quad u > 0 \tag{6}$$

is invariant by translation (like in the subcritical case) *and* by dilations.

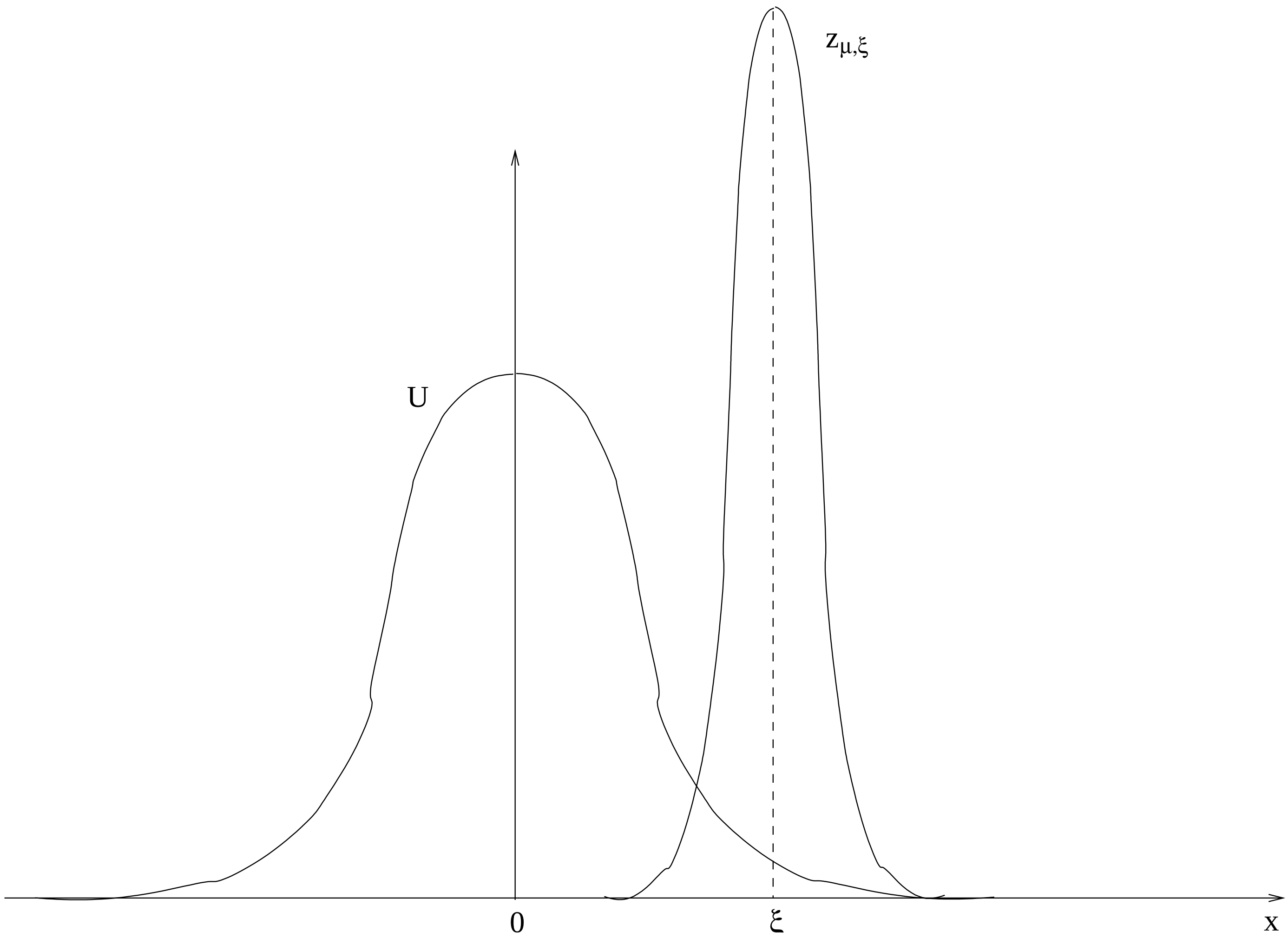
The *fundamental* solution U of (6) has the form (up to a constant)

$$U(x) = \left(\frac{1}{1+|x|^2}\right)^{\frac{n-2}{2}},$$

and for all $\xi \in \mathbb{R}^n$ and $\mu > 0$,

$$z_{\mu,\xi}(x) = \mu^{-\frac{n-2}{2}}\, U\left(\frac{x-\xi}{\mu}\right)$$

is a solution of (6).



Finding solutions of (5) is a delicate matter. For example, if $k(x)$ is *positive, radial*, has a unique maximum at $x = 0$ and decays to zero at infinity, (5) does not possess any positive solution in $\mathcal{D}^{1,2}(\mathbb{R}^n)$.

However, we will show that a solution exists provided k satisfies, in addition to some technical conditions, the following hypotheses

(a) k has a finite number of stationary points and $\Delta k(\xi) \neq 0, \ \forall \xi \in \mathbb{R}^n$ such that $\nabla k(\xi) = 0$.

(b) if $i(k', x)$ denotes the index (namely the local degree) of k' at x, there holds

$$\sum_{\nabla k(\xi)=0,\, \Delta k(\xi)<0} i(k', x) \neq (-1)^n.$$

1.3 Semiclassical standing waves of NLS

In Quantum Mechanics the behavior of a single particle is governed by the linear Schrödinger equation

$$i\hbar\,\frac{\partial\psi}{\partial t}=-\hbar^2\Delta\psi+Q(x)\psi,$$

where i is the imaginary unit, $\hbar$ is the Planck constant, $(t,x)\in\mathbb{R}\times\mathbb{R}^n$, Δ denotes the Laplace operator and $\psi=\psi(t,x)$ is a complex valued function. Differently, in the presence of many particles, one can try to simulate the mutual interaction effect by introducing a nonlinear term. Expanding this nonlinearity in odd power series

$$a_0\psi+a_1|\psi|^{p-1}\psi+\cdots,\quad (p\geq 3)$$

and keeping only the first nonlinear term, one is led to a nonlinear equation of the form

$$i\hbar\,\frac{\partial\psi}{\partial t}=-\hbar^2\Delta\psi+(a_0+Q(x))\psi+a_1|\psi|^{p-1}\psi. \tag{7}$$

A *stationary wave* of (7) is a solution of (7) of the form

$$\psi(t,x)=\exp\left(i\,\alpha\,\hbar^{-1}t\right)u(x)\quad u(x)\in\mathbb{R},\quad u>0.$$

Thus, looking for solitary waves of (7) is equivalent to find an $u>0$ satisfying

$$-\hbar^2\Delta u+(\alpha+a_0+Q(x))u=u^p. \tag{8}$$

Such an u will be called a *standing wave*. A particular interest is given to the so called *semiclassical states* that are standing waves existing for $\hbar\to 0$. Setting $\hbar=\varepsilon$ and $V(x)=\alpha+a_0+Q(x)$, we are finally led to

$$\begin{cases} -\varepsilon^2\Delta u+V(x)u=u^p,\\ u\in W^{1,2}(\mathbb{R}^n),\ u>0,\end{cases} \tag{9}$$

where the condition $u\in W^{1,2}(\mathbb{R}^n)$ is added in order to obtain *bound states*, namely solutions with finite energy.

To obtain a perturbation problem like the preceding ones, it is convenient to make the change of variables $x\mapsto\varepsilon x+x_0$, where $x_0\in\mathbb{R}^n$ will be chosen in an appropriate way, that leads to

$$\begin{cases} -\Delta u+V(\varepsilon x+x_0)u=u^p,\\ u\in W^{1,2}(\mathbb{R}^n),\quad u>0.\end{cases} \tag{10}$$

The solutions of (10) are the critical points $u>0$ of the functional

$$I_\varepsilon(u)=\int_{\mathbb{R}^n}\tfrac{1}{2}\left[|\nabla u|^2+V(\varepsilon x+x_0)u^2\right]dx-\tfrac{1}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx,\ u\in W^{1,2}(\mathbb{R}^n).$$

This functional is perturbative in nature: the unperturbed functional is

$$I_0(u)=\int_{\mathbb{R}^n}\tfrac{1}{2}\left[|\nabla u|^2+V(x_0)u^2\right]dx-\tfrac{1}{p+1}\int_{\mathbb{R}^n}|u|^{p+1}dx$$

while the perturbation term is given by

$$\tfrac{1}{2}\int_{\mathbb{R}^n}\left[V(\varepsilon x+x_0)-V(x_0)\right]u^2dx.$$

The unperturbed equation $I_0'(u)=0$ becomes:

$$\left\{\begin{array}{l}-\Delta u+V(x_0)u=u^p,\\ u\in W^{1,2}(\mathbb{R}^n),\quad u>0.\end{array}\right. \tag{11}$$

If $V(x_0)>0$, (11) possesses as before a unique radial solution $U_0>0$. Moreover, any $U_0(\cdot-\xi)$, $\xi\in\mathbb{R}^n$, is also a solution of (11).

It will be shown that if x_0 is stationary point of the potential V which is *stable* (in a suitable sense specified later on), then (NLS) has for $\varepsilon\neq 0$ small a solution of the form

$$u_\varepsilon(x)\sim U_0\left(\frac{x-x_0}{\varepsilon}\right),$$

hence a solution that concentrates at x_0.

This kind of solutions are called *spike layers* or simply *spikes.*

From the physical point of view, spikes are important because they show that (focusing) NLS of the type (11) are not dispersive but the energy is localized in packets.

1.4 Neumann singularly perturbed problems

Another example is given by elliptic singularly perturbed problems with Neumann boundary conditions like

$$\left\{\begin{array}{l}-\varepsilon^2\Delta u+u=u^p,\quad \text{in }\Omega\\ u>0,\quad \text{in }\Omega,\\ \frac{\partial u}{\partial\nu}=0,\quad \text{on }\partial\Omega,\end{array}\right. \tag{12}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and ν denotes the unit outer normal at $\partial\Omega$. As before, we take $1<p<\frac{n+2}{n-2}$. Problems like (12) arise in the study of some reaction-diffusion systems in biology.

The specific feature of (12) is to possess spike layer solutions.

The role of the potential V in the NLS is played here by the curvature of the boundary, in the sense that there exist solutions concentrating at *stable* stationary points of the mean curvature H of $\partial\Omega$.

Final Remark. In all the preceding examples we look for critical points of a perturbed functional

$$I_\varepsilon(u)=I_0(u)+\varepsilon G(u),\qquad u\in\mathcal{H}$$

with the feature that the unperturbed functional I_0 has a finite dimensional manifold Z of critical points.

For example, for subcritical problems on $\mathbb{R}^n$

$$Z = \{U(\cdot - \xi) \,:\, \xi \in \mathbb{R}^n\} \simeq \mathbb{R}^n$$

while in the critical case

$$Z = \{\mu^{-\frac{n-2}{2}}\, U\left(\frac{\cdot - \xi}{\mu}\right) \,:\, \xi \in \mathbb{R}^n,\, \mu > 0\} \simeq \mathbb{R}^n \times \mathbb{R}^+.$$

2 Abstract setting: critical points of perturbed functionals

We consider a class of functionals of the form

$$I_\varepsilon(u) = I_0(u) + \varepsilon G(u).$$

where $\mathcal{H}$ is a Hilbert space, $I_0 \in C^2(\mathcal{H}, \mathbb{R})$ plays the role of the unperturbed functional and $G \in C^2(\mathcal{H}, \mathbb{R})$ is the perturbation.

We will always suppose that there exists a d-dimensional smooth, say C^2, manifold Z, $0 < d = dim(Z) < \infty$, such that all $z \in Z$ is a critical point of I_0. The set Z will be called a *critical manifold* (of I_0). Let T_z denote the tangent space to Z at z. If Z is a critical manifold then

$$I_0'(z) = 0, \qquad \forall\, z \in Z.$$

Differentiating the identity $I_0'(z) \equiv 0$, we get

$$(I_0''(z)[v]|\phi) = 0, \quad \forall\, v \in T_z,\ \forall\, \phi \in \mathcal{H}.$$

It follows that all $v \in T_z$ is a solution of the *linearized equation* $I_0''(z)[v] = 0$, namely $v \in Ker[I_0''(z)]$.

Thus

$$T_z \subseteq Ker[I_0''(z)].$$

In particular, $I_0''(z)$ has a non trivial Kernel (whose dimension is at least d) and hence all the $z \in Z$ are degenerate critical points of I_0. We shall require that this degeneracy is minimal. Precisely we will suppose that

(ND) $\qquad T_z = Ker[I_0''(z)], \quad \forall\, z \in Z.$

So, proving that Z satisfies (ND) is equivalent to show that $Ker[I_0''(z)] \subseteq T_z$, namely that every solution of the linearized equation $I_0''(z)[v] = 0$ belongs to T_z.

In addition to (ND) we will assume that

(Fr) $\qquad$ for all $z \in Z$, $I_0''(z)$ is an index 0 Fredholm map.

Definition. *A critical manifold Z will be called non degenerate, ND in short, if (ND) and (Fr) hold.*

2.1 A finite dimensional reduction

Let $W = (T_z)^\perp$ and let $P : \mathcal{H} \to W$ denote the orthogonal projection onto W

We look for critical points of I_ε in the form $u = z + w$ with $z \in Z$ and $w \in W$.

The equation $I'_\varepsilon(z+w) = 0$ is equivalent to the following system

$$(13) \qquad \begin{cases} PI'_\varepsilon(z+w) = 0, & \text{(the } \textit{auxiliary equation}\text{)} \\ (Id - P)I'_\varepsilon(z+w) = 0, & \text{(the } \textit{bifurcation equation}\text{)} \end{cases}$$

Let first solve the auxiliary equation, namely

$$(14) \qquad PI'_0(z+w) + \varepsilon PG'(z+w) = 0,$$

by means of the Implicit Function Theorem.

Let $F : \mathbb{R} \times Z \times W \to W$ be defined by setting

$$F(\varepsilon, z, w) = PI'_0(z+w) + \varepsilon PG'(z+w).$$

F is of class C^1 and one has $F(0, z, 0) = 0$, for every $z \in Z$.

Lemma 2.1 *If (ND) and (Fr) hold, then $D_w F(0, z, 0)$ is invertible as a map from W into itself.*

PROOF. The map $D_w F(0, z, 0)$ is given by

$$D_w F(0, z, 0) : v \mapsto PI''_0(z)[v].$$

Since $PI''_0(z)[v] = I''_0(z)[v]$, the equation $D_w F(0, z, 0)[v] = 0$ becomes

$$I''_0(z)[v] = 0.$$

Thus $v \in Ker[I''_0(z)] \cap W$ and from (ND) it follows that $v = 0$, namely that $D_w F(0, z, 0)$ is injective. Using (Fr) we then deduce that $D_w F(0, z, 0) : W \to W$ is invertible. ■

Let Z_c be a compact subset of Z. Lemma 2.1 allows us to apply the Implicit Function Theorem to $F(\varepsilon, z, w) = 0$ yielding:

Lemma 2.2 $\exists\ \varepsilon_0 > 0$ *such that* $\forall\ |\varepsilon| < \varepsilon_0$, $\forall\ z \in Z_c$, *the auxiliary equation* (14) *has a unique solution $w_\varepsilon = w_\varepsilon(z) \in W$, with*
(a) $w_\varepsilon(z) \in W = (T_z Z)^\perp$ and $w_\varepsilon(z) \to 0$, as $|\varepsilon| \to 0$;
(b) w_ε is of class C^1 w.r.t. to $z \in Z_c$ and $w'_\varepsilon \to 0$ as $|\varepsilon| \to 0$;
(c) $\|w_\varepsilon(z)\| = O(\varepsilon)$ as $\varepsilon \to 0$, for all $z \in Z_c$.

Proof of (b). w'_ε satisfies

$$PI''_0(z + w_\varepsilon)[q + w'_\varepsilon] + \varepsilon PG''(z + w_\varepsilon)[q + w'_\varepsilon] = 0, \quad (q \in T_z Z)$$

Then for $\varepsilon = 0$ we get $PI''_0(z)[q + w'_0] = 0$. Since $q \in T_z Z \subseteq Ker[I''_0(z)]$, then $PI''_0(z)[q] = 0$, and this implies $w'_0 = 0$. ■

2.2 Existence of critical points

To solve the bifurcation equation, let us define the *reduced functional* $\Phi_\varepsilon : Z \to \mathbb{R}$ by setting

$$\Phi_\varepsilon(z) = I_\varepsilon(z + w_\varepsilon(z)). \tag{15}$$

Theorem 2.3 *Let $I_0, G \in C^2(\mathcal{H}, \mathbb{R})$ and suppose that I_0 has a smooth ND critical manifold Z. If Φ_ε has, for $|\varepsilon|$ sufficiently small, a critical point $z_\varepsilon \in Z_c$, then $u_\varepsilon = z_\varepsilon + w_\varepsilon(z_\varepsilon)$ is a critical point of $I_\varepsilon = I_0 + \varepsilon G$.*

Sketch of the proof. Consider the manifold $Z_\varepsilon = \{z + w_\varepsilon(z)\}$. Since z_ε is a critical point of Φ_ε, it follows that $u_\varepsilon \in Z_\varepsilon$ is a critical point of I_ε constrained on Z_ε and thus u_ε satisfies $I'_\varepsilon(u_\varepsilon) \perp T_{u_\varepsilon} Z_\varepsilon$. Moreover from $PI'_\varepsilon(z + w_\varepsilon) = 0$, it follows that $I'_\varepsilon(z + w_\varepsilon(z)) \in T_z Z$. In particular, $I'_\varepsilon(u_\varepsilon) \in T_{z_\varepsilon} Z$. Since, for $|\varepsilon|$ small, $T_{u_\varepsilon} Z_\varepsilon$ and $T_{z_\varepsilon} Z$ are close, see Lemma 2.2, it follows that $I'_\varepsilon(u_\varepsilon) = 0$. ■

When Z is compact the preceding result immediately implies

Corollary 2.4 *If, in addition to the assumptions of Theorem 2.3, the critical manifold Z is compact, then for $|\varepsilon|$ small enough, I_ε has at least $Cat(Z)$ (the Lusternik-Schnierelman category of Z) critical points.*

In order to use Theorem 2.3 it is convenient to expand Φ_ε.

Lemma 2.5 *One has:*

$$\Phi_\varepsilon(z) = c_0 + \varepsilon G(z) + o(\varepsilon), \quad \textit{where } c_0 = I_0(z).$$

Proof. Recall that

$$\Phi_\varepsilon(z) = I_0(z + w_\varepsilon(z)) + \varepsilon G(z + w_\varepsilon(z)).$$

Let us evaluate separately the two terms above. First we have

$$I_0(z + w_\varepsilon(z)) = I_0(z) + (I'_0(z) \,|\, w_\varepsilon(z)) + o(\|w_\varepsilon(z)\|).$$

Since $I'_0(z) = 0$ we get

$$I_0(z + w_\varepsilon(z)) = c_0 + o(\|w_\varepsilon(z)\|). \tag{16}$$

Similarly, one has

$$\begin{aligned} G(z + w_\varepsilon(z)) &= G(z) + (G'(z) \,|\, w_\varepsilon(z)) + o(\|w_\varepsilon(z)\|) \\ &= G(z) + O(\|w_\varepsilon(z)\|). \end{aligned} \tag{17}$$

Putting together (16) and (17) we infer that

$$\Phi_\varepsilon(z) = c_0 + \varepsilon \Big[G(z) + O(\|w_\varepsilon(z)\|) \Big] + o(\|w_\varepsilon(z)\|). \tag{18}$$

Since $\|w_\varepsilon(z)\| = O(\varepsilon)$, see Lemma 2.2-($c$), the result follows. ■

The preceding lemma, jointly with Theorem 2.3 yields

Theorem 2.6 *Let $I_0, G \in C^2(\mathcal{H}, \mathbb{R})$. Suppose that I_0 has a ND smooth critical manifold Z. Moreover, setting $\Gamma := G_{|Z}$, we assume and that there exists a critical point $\bar{z} \in Z$ of $\Gamma = G_{|Z}$ satisfying*

(G') $\quad \exists\, \mathcal{N} \subset \mathbb{R}^d$ *open bounded such that the topological degree $d(\Gamma', \mathcal{N}, 0) \neq 0$.*

Then for $|\varepsilon|$ small the functional I_ε has a critical point u_ε and there exists $\hat{z} \in \mathcal{N}$, $\Gamma'(\hat{z}) = 0$, such that $u_\varepsilon \to \hat{z}$ as $\varepsilon \to 0$. Therefore if, in addition, $\mathcal{N}$ contains only an isolated critical point $\bar{z}$ of Γ', then $u_\varepsilon \to \bar{z}$ as $\varepsilon \to 0$.

Remark. Examples in which condition (G') holds are:
(i) $\bar{z}$ is a strict local maximum (or minimum),

(ii) $\bar{z}$ is any non-degenerate critical point $\bar{z}$.
In both cases, $u_\varepsilon \to \bar{z}$ as $\varepsilon \to 0$.
If $G(z) \equiv 0$, Theorem 2.6 is useless and we need to evaluate the further terms in the expansion of Φ_ε.

However, it is possible to show that the preceding results still hold true, provided we substitute Φ_ε and Γ with, resp.

$$\widetilde{\Phi}_\varepsilon(z) = c_0 - \tfrac{1}{2}\,\varepsilon^2 (G'(z) \,|\, L_z G'(z)) + o(\varepsilon^2),$$

and

$$\widetilde{\Gamma}(z) = \tfrac{1}{2}\,(G'(z) \,|\, L_z G'(z)),$$

where $L_z = (P I_0''(z))^{-1}$.

3 Applications

3.1 Subcritical Problems

We will consider the elliptic problem

$$(P_\varepsilon) \qquad -\Delta u + u = (1 + \varepsilon h(x)) u^p, \quad u \in W^{1,2}(\mathbb{R}^n),\ u > 0,$$

where $n \geq 3$ and p is a subcritical exponent, namely

$$1 < p < \frac{n+2}{n-2}.$$

In order to use the techniques discussed before we set $\mathcal{H} = W^{1,2}(\mathbb{R}^n)$ and

$$I_\varepsilon(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{p+1}\int_{\mathbb{R}^n} u_+^{p+1} dx - \varepsilon \cdot \tfrac{1}{p+1}\int_{\mathbb{R}^n} h(x)\, u_+^{p+1} dx,$$

where, for simplicity, we assume that $h \in L^\infty(\mathbb{R}^n)$.

Here $\mathcal{H} = W^{1,2}(\mathbb{R}^n)$ is the usual Sobolev space, endowed with the standard scalar product, resp. norm,

$$(u|v) = \int_{\mathbb{R}^n} (\nabla u \cdot \nabla v + uv)dx, \qquad \|u\|^2 = \int_{\mathbb{R}^n} (|\nabla u|^2 + u^2)dx.$$

Plainly, $I_\varepsilon \in C^2(\mathcal{H}, \mathbb{R})$ and solutions of (P_ε) are critical points of I_ε.

I_ε has the form

$$I_\varepsilon(u) = I_0(u) + \varepsilon G(u),$$

where the *unperturbed functional* I_0 is given by

$$I_0(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{p+1}\int_{\mathbb{R}^n} u_+^{p+1} dx,$$

and the perturbation is

$$G(u) = -\tfrac{1}{p+1}\int_{\mathbb{R}^n} h(x)\, u_+^{p+1} dx.$$

The unperturbed problem $I_0'(u) = 0$ is equivalent to the elliptic equation

$$-\Delta u + u = u^p, \quad u \in \mathcal{H}, \quad u > 0 \tag{19}$$

which has a unique positive radial solution U which decays exponentially to zero at infinity. Moreover, since (19) is translation invariant, it follows that any $z_\xi(x) := U(x - \xi)$ is also a solution of (19). In other words, I_0 has a (non-compact) critical manifold given by

$$Z = \{z_\xi(x) : \xi \in \mathbb{R}^n\} \simeq \mathbb{R}^n.$$

Lemma 3.1 *Z is non-degenerate.*

Proof (Sketch). $v \in \mathcal{H}$ belongs to $Ker[I_0''(U)]$ iff

$$-\Delta v + v = pU^{p-1}(x)v, \quad v \in \mathcal{H}. \tag{20}$$

We set

$$r = |x|, \qquad \vartheta = \frac{x}{|x|} \in S^{n-1}$$

and let Δ_r, resp. $\Delta_{S^{n-1}}$ denote the Laplace operator in radial coordinates, resp. the Laplace-Beltrami operator. To find solutions of (20) we recall that every $v \in \mathcal{H}$ can be written in the form

$$v(x) = \sum_{k=0}^{\infty} \psi_k(r) Y_k(\vartheta), \qquad \text{where } \psi_k(r) = \int_{S^{n-1}} v(r\vartheta) Y_k(\vartheta) d\vartheta \in W^{1,2}(\mathbb{R}),$$

and $Y_k(\vartheta)$ are the spherical harmonics satisfying

$$-\Delta_{S^{n-1}} Y_k = \lambda_k Y_k. \tag{21}$$

The eigenvalues (and their multiplicity) of (21) are known. In particular,

$\lambda_0 = 0$ has multiplicity 1,

$\lambda_1 = n-1$, has multiplicity n.

The rest of eigenvalues are given by $\lambda_k = k(k+n-2),\ k = 2,3\dots$

Substituting $v = \sum \psi_k Y_k$ into (20) we get the following equations for ψ_k:

$$A_k(\psi_k) := -\psi_k^{''} - \frac{n-1}{r}\psi_k' + \psi_k + \frac{\lambda_k}{r^2}\psi_k - pU^{p-1}\psi_k = 0, \quad k = 0,1,2,\dots$$

If $k=0$, $\lambda_0 = 0$ and thus ψ_0 satisfies

$$A_0(\psi_0) = -\psi_0^{''} - \frac{n-1}{r}\psi_0' + \psi_0 - pU^{p-1}\psi_0 = 0.$$

It is possible to show that all the solutions of $A_0(u) = 0$ are unbounded. Since we are looking for solutions $\psi_0 \in W^{1,2}(\mathbb{R})$, it follows that $\psi_0 = 0$.

For $k=1$, one has that $\lambda_1 = n-1$ and we find

$$A_1(\psi_1) = -\psi_1^{''} - \frac{n-1}{r}\psi_1' + \psi_1 + \frac{n-1}{r^2}\psi_1 - pU^{p-1}\psi_1 = 0.$$

Let $\widehat{U}(r)$ denote the function such that $U(x) = \widehat{U}(|x|)$. Since $U(x)$ satisfies $-\Delta U + U = U^p$, then $\widehat{U}$ solves

$$-\widehat{U}'' - \frac{n-1}{r}\widehat{U}' + \widehat{U} = \widehat{U}^p.$$

Differentiating, we get

$$-(\widehat{U}')'' - \frac{n-1}{r}(\widehat{U}')' + \frac{n-1}{r^2}\widehat{U}' + \widehat{U}' = p\widehat{U}^{p-1}\widehat{U}'. \tag{22}$$

In other words, $\widehat{U}'(r)$ satisfies $A_1(\widehat{U}') = 0$, and $\widehat{U}' \in W^{1,2}(\mathbb{R})$.

Let us look for a second solution of $A_1(\psi_1) = 0$ in the form $\psi_1(r) = c(r)\widehat{U}'(r)$. By a straight calculation, we find that $c(r)$ solves

$$-c''\widehat{U}' - 2c' \cdot (\widehat{U}')' - \frac{n-1}{r}c'\widehat{U}' = 0.$$

If $c(r)$ is not constant, it follows that

$$-\frac{c''}{c'} = 2\frac{\widehat{U}''}{\widehat{U}'} + \frac{n-1}{r},$$

and hence

$$c'(r) \sim \frac{1}{r^{n-1}\widehat{U}'^2}\ , \qquad (r \to +\infty).$$

This and $U(r) \sim e^{-|r|}|r|^{-\frac{n-1}{2}}$ imply that $c(r) \sim e^{2r}$ and therefore $c(r)\widehat{U}'(r) \sim -e^r r^{(1-n)/2}$ as $r \to +\infty$. From this we infer that $c(r)\widehat{U}'(r) \notin W^{1,2}(\mathbb{R})$, unless $c(r) = cst.$. Then $\psi_1(r) = \overline{c}\widehat{U}'(r)$, for some $\overline{c} \in \mathbb{R}$.

Finally, one shows that the equation $A_k(\psi_k) = 0$ has only the trivial solution in $W^{1,2}(\mathbb{R})$, provided that $k \geq 2$.

Conclusion. Any $v \in Ker[I_0''(U)]$ has to be a constant multiple of $\widehat{U}'(r)Y_1(\vartheta)$. Here Y_1 is such that

$$-\Delta_{S^{n-1}} Y_1 = \lambda_1 Y_1.$$

Recalling that λ_1 has multiplicity n and letting $Y_1 = \sum_1^n a_i Y_{1,i}$, we find that

$$v \in span\{\widehat{U}' Y_{1,i} : 1 \leq i \leq n\} = span\{U_{x_i} : 1 \leq i \leq n\} = T_U Z.$$

This proves that (ND) holds.

Theorem 3.2 (P_ε) *has a solution for* $|\varepsilon|$ *is small enough, provided one of the following conditions is fullfilled*

(h_1) $\quad h \in L^s$ *with* $s = \frac{2^*}{2^*-(p+1)}$ *and* $\int_{\mathbb{R}^n} h(x)U^{p+1}(x) \neq 0$;

(h_2) $\quad \exists\, r \in [1,2]$ *such that* $h \in L^s \cap L^r$.

(h_3) $\quad h \in L^\infty$ *and* $\lim_{|x|\to\infty} h(x) = 0$

The proof in the cases $(h_1 - h_2)$ is based on the following lemma

Lemma 3.3 *Suppose that* $h \in L^s$. *Then*

$$\lim_{|\xi|\to\infty} \Gamma(\xi) = 0, \quad \left(\Gamma(\xi) = \int_{\mathbb{R}^n} h(x)U^{p+1}(x-\xi)dx\right).$$

To prove the lemma we write, for a suitable $\rho > 0$,

$$\Gamma(\xi) = \int_{|x|<\rho} h(x)U^{p+1}(x-\xi)dx + \int_{|x|>\rho} h(x)U^{p+1}(x-\xi)dx$$

Let us evaluate separately the two terms (I), (II), in the preceding eq.

$$\begin{aligned}
|(I)| &\leq \left(\int_{|x|<\rho} |h(x)|^s dx\right)^{1/s} \left(\int_{|x|<\rho} U^{s'(p+1)}(x-\xi)dx\right)^{1/s'} \\
&= \left(\int_{|x|<\rho} |h(x)|^s dx\right)^{1/s} \left(\int_{|y+\xi|<\rho} U^{s'(p+1)}(y)dy\right)^{1/s'} \\
&\leq c_1 \left(\int_{|x+\xi|<\rho} U^{s'(p+1)}(x)dx\right)^{1/s'}.
\end{aligned}$$

Since U decays exponentially to zero as $|x| \to \infty$, the last integral tends to zero as $|\xi| \to \infty$ $\forall\, \rho > 0$ and hence

$$\lim_{|\xi|\to\infty} \int_{|x|<\rho} h(x)U^{p+1}(x-\xi)dx = 0, \qquad \forall\, \rho > 0.$$

Moreover,

$$
\begin{aligned}
|(II)| &\leq \left(\int_{|x|>\rho} |h(x)|^s dx\right)^{1/s} \left(\int_{|x+\xi|>\rho} U^{s'(p+1)}(x)dx\right)^{1/s'} \\
&\leq \left(\int_{|x|>\rho} |h(x)|^s dx\right)^{1/s} \left(\int_{\mathbb{R}^n} U^{s'(p+1)}(x)dx\right)^{1/s'} \\
&\leq c_2 \left(\int_{|x|>\rho} |h(x)|^s dx\right)^{1/s}.
\end{aligned}
$$

Thus, given any $\eta > 0$ there exists $\rho > 0$ large enough, in such a way that $|(II)| \leq \eta$.

Thus $(I) + (II) \to 0$, proving the lemma.

The condition (h_1) says that $\Gamma(0) = \int_{\mathbb{R}^n} h(x)U^{p+1}(x) \neq 0$, hence $\Gamma \not\equiv 0$.

From the lemma it follows that Γ achieves a strict (global) maximum or minimum.

From the abstract setting it follows that I_ε has a critical point which gives rise to a solution of (P_ε) for $|\varepsilon|$ is small enough.

Condition (h_2) replaces (h_1) and allows us to show that $\Gamma \not\equiv 0$ whenever $h \not\equiv 0$.

The case (h_3) is handled by proving that $\lim_{|\xi|\to\infty} \Phi_\varepsilon(\xi) = 0$. Then Φ_ε has a critical point and we can use once more the abstract setting.

3.2 The case of the critical exponent

Consider

$$
-\Delta u = (1 + \varepsilon k(x))u^{(n+2)/(n-2)}, \quad u > 0, \tag{23}
$$

We will work in $\mathcal{H} := \mathcal{D}^{1,2}(\mathbb{R}^n)$, the space of $u \in L^{2^*}(\mathbb{R}^n)$ such that $\nabla u \in L^2(\mathbb{R}^n)$, endowed with scalar product and norm, respectively

$$
(u|v) = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v dx, \qquad \|u\|^2 = \int_{\mathbb{R}^n} |\nabla u|^2 dx.
$$

Solutions of (23) are the critical points of $I_\varepsilon : \mathcal{H} \to \mathbb{R}$,

$$
I_\varepsilon(u) = \tfrac{1}{2}\|u\|^2 - \tfrac{1}{2^*}\int_{\mathbb{R}^n} u_+^{2^*} dx - \varepsilon\, \tfrac{1}{2^*}\int_{\mathbb{R}^n} k(x)u_+^{2^*} dx,
$$

where u_+ denotes the positive part of u.

As before, we need to consider the unperturbed problem

$$
-\Delta u = u^{(n+2)/(n-2)}, \quad u > 0, \quad u \in \mathcal{H},
$$

which possesses the following family of solutions, depending on $(n+1)$ parameters $\xi \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^+$,

$$z_{\mu,\xi}(x) = \mu^{-(n-2)/2} U\left(\frac{x-\xi}{\mu}\right),$$

where

$$U(x) = [n(n-2)]^{(n-2)/4} \left(\frac{1}{1+|x|^2}\right)^{(n-2)/2}.$$

Correspondingly, we have an (n+1)-dimensional manifold of solutions given by

$$Z = \{z = z_{\mu,\xi} : \mu > 0,\ \xi \in \mathbb{R}^n\}.$$

It is possible to show that Z is ND.

According to the general theory, we have to study the finite dimensional functional

$$\Gamma(\mu,\xi) := \int_{\mathbb{R}^n} k(x) z_{\mu,\xi}^{2^*}(x)dx.$$

We will make the following assumptions on $k(x)$. Let $Cr[k]$, denote the set of critical points of k.

$(k.0)$ $k \in L^\infty(\mathbb{R}^n) \cap C^2(\mathbb{R}^n)$;

$(k.1)$ $Cr[k]$ is finite and $\Delta k(x) \neq 0,\ \forall\, x \in Cr[k]$.

$(k.2)$ $\exists\, \rho > 0$ such that $\langle k'(x), x\rangle < 0,\ \forall\, |x| \geq \rho$

$(k.3)$ $\langle k'(x), x\rangle \in L^1(\mathbb{R}^n),\ \int_{\mathbb{R}^n} \langle k'(x), x\rangle dx < 0$;

From $(k.1)$ it follows that for every $x \in Cr[k]$ the index $i(k', x)$ (namely the local degree) of k' at x is well defined.

Theorem 3.4 *Let $(k.1-3)$ hold and suppose that*

$$\sum_{x \in Cr[k],\, \Delta k(x) < 0} i(k', x) \neq (-1)^n. \tag{24}$$

Then (23) has at least a solution, provided $|\varepsilon| \ll 1$.

Γ takes the form

$$\Gamma(\mu,\xi) = \mu^{-n} \int_{\mathbb{R}^n} k(x) U^{2^*}\left(\frac{x-\xi}{\mu}\right) dx = \int_{\mathbb{R}^n} k(\mu y + \xi) U^{2^*}(y) dy.$$

By a straight calculation we find

$$\lim_{\mu \downarrow 0} \Gamma(\mu,\xi) = a_0 k(\xi), \qquad a_0 = \int_{\mathbb{R}^n} U^{2^*}(y) dy.$$

Moreover, from $D_\mu\Gamma(\mu,\xi)=\int_{\mathbb{R}^n}\langle k'(\mu y+\xi),y\rangle U^{2^*}(y)dy$ and since $\int_{\mathbb{R}^n} y_iU^{2^*}(y)dy = 0$, it follows
$$\lim_{\mu\downarrow 0} D_\mu\Gamma(\mu,\xi)=0.$$

As a consequence, we can extend Γ to all of $\mathbb{R}^n$ by setting $\widetilde{\Gamma}(0,\xi)=a_0k(\xi)$ and $\widetilde{\Gamma}(\mu,\xi)=\Gamma(-\mu,\xi)$ if $\mu<0$. The extended function is of class C^1 and satisfies

$$D_\mu\widetilde{\Gamma}(0,\xi)=0, \qquad \forall\,\xi\in\mathbb{R}^n. \tag{25}$$

In particular,

$$\xi\in Cr[k] \quad\Longleftrightarrow\quad (0,\xi)\in Cr[\widetilde{\Gamma}], \tag{26}$$

Next, we evaluate the second derivatives of $\widetilde{\Gamma}$. We find
$$D^2_{\mu\mu}\widetilde{\Gamma}(\mu,\xi)=\int_{\mathbb{R}^n}\sum D^2_{ij}k(\mu y+\xi)y_iy_jU^{2^*}(y)dy.$$

Since $\int_{\mathbb{R}^n} y_iy_jU^{2^*}(y)dy=0 \iff i\neq j$, we infer

$$D^2_{\mu\mu}\widetilde{\Gamma}(0,\xi)=a_1\Delta k(\xi), \qquad a_1=\int_{\mathbb{R}^n}|y|^2U^{2^*}(y)dy. \tag{27}$$

Furthermore, differentiating (25) with respect to ξ_i we infer

$$D^2_{\mu\xi_i}\widetilde{\Gamma}(0,\xi)=0, \qquad i=1,\dots,n. \tag{28}$$

Putting together (27) and (28) one finds that the Hessian matrix $\widetilde{\Gamma}''(0,\xi)$ at any $\xi\in\mathbb{R}^n$ has the form

$$\widetilde{\Gamma}''(0,\xi) \;=\; \begin{pmatrix} a_0k''(\xi) & 0 \\ 0 & a_1\,\Delta k(\xi)\end{pmatrix}. \tag{29}$$

In particular, $(0,\xi)$ is an isolated critical point of $\tilde{\Gamma}$ and, by the multiplicative property of the degree, we have $i(\tilde{\Gamma}',(0,\xi))=sgn(\Delta K(\xi))i(k',\xi)$. Let us collect the above results in the following Lemma

Lemma 3.5 *Let $(k.0)-(k.1)$ hold. Then $(0,\xi)$ is an isolated critical point of $\widetilde{\Gamma}$ if and only if $\xi\in Cr[k]$. Moreover one has*
$$i(\widetilde{\Gamma}',(0,\xi))=\begin{cases} i(k',\xi) & if \quad \Delta k(\xi)>0\\ -i(k',\xi) & if \quad \Delta k(\xi)<0\end{cases}$$

Furthermore, one proves

Lemma 3.6 *Let $(k.2)-(k.3)$ hold. Then $\exists\,R>0$ such that*
$$\langle\widetilde{\Gamma}'(\mu,\xi),(\mu,\xi)\rangle<0, \qquad \forall\,(\mu,\xi)\in\mathbb{R}^{n+1},\ \mu^2+|\xi|^2\geq R^2.$$
Therefore, $deg(\widetilde{\Gamma}',B_R^{n+1},0)=(-1)^{n+1}$.

Proof of Theorem 3.4 Letting C_+ denote the set of points of $Cr[\widetilde{\Gamma}]$ with $\mu > 0$, $C_- := \{(-\mu, \xi) : (\mu, \xi) \in C_+\}$ and $C_0 = \{(0, \xi) : \xi \in Cr[k]\}$, one checks that $Cr[\widetilde{\Gamma}] = C_+ \cup C_0 \cup C_-$.

Remark that C_0 and $C_\pm$ are compact.

In order to apply the abstract setting, we will show that for any open bounded set $\mathcal{N} \subset]0, \infty) \times \mathbb{R}^n$ with $C_+ \subset \mathcal{N}$ one has that $deg(\Gamma', \mathcal{N}, 0) \neq 0$.

Let us argue by contradiction. Let $\mathcal{O} \subset]0, \infty) \times \mathbb{R}^n$ be an open bounded set with $C_+ \subset \mathcal{O}$ and such that $deg(\Gamma', \mathcal{O}, 0) = 0$. Let us introduce the following notation:

$$\mathcal{O}_- = \{(-\mu, \xi) : (\mu, \xi) \in \mathcal{O}\}, \quad \mathcal{O}' = \mathcal{O} \cup \mathcal{O}_- \ .$$

Since $\Gamma = \tilde{\Gamma}$ in $]0, \infty) \times \mathbb{R}^n$, using Lemma 3.6 we deduce

$$deg(\widetilde{\Gamma}', B_R^{n+1} \setminus \mathcal{O}', 0) = (-1)^{n+1}. \tag{30}$$

Since the only critical points of $\widetilde{\Gamma}'$ in $B_R^{n+1} \setminus \mathcal{O}'$ are those in C_0 and taking into account that C_0 consists of isolated points, we get

$$\begin{aligned} deg(\widetilde{\Gamma}', B_R^{n+1} \setminus \mathcal{O}', 0) &= \sum_{\xi \in Cr[k]} i(\widetilde{\Gamma}', (0, \xi)) \\ &= \sum_{\xi \in Cr[k], \Delta k(\xi) > 0} i(\widetilde{\Gamma}', (0, \xi)) + \sum_{\xi \in Cr[k], \Delta k(\xi) < 0} i(\widetilde{\Gamma}', (0, \xi)). \end{aligned}$$

Then

$$deg(\widetilde{\Gamma}', B_R^{n+1} \setminus \mathcal{O}', 0) = \sum_{\xi \in Cr[k], \Delta k(\xi) > 0} i(k', \xi) - \sum_{\xi \in Cr[k], \Delta k(\xi) < 0} i(k', \xi).$$

By (30) we have that $deg(\widetilde{\Gamma}', B_R^{n+1} \setminus \mathcal{O}', 0) = (-1)^{n+1}$ whence

$$\sum_{\xi \in Cr[k], \Delta k(\xi) > 0} i(k', \xi) - \sum_{\xi \in Cr[k], \Delta k(\xi) < 0} i(k', \xi) = (-1)^{n+1}. \tag{31}$$

On the other hand, from $(k.2)$ it immediately follows that $deg(k', B_R^n, 0) = (-1)^n$ and hence

$$\sum_{\xi \in Cr[k]} i(k', \xi) = \sum_{\xi \in Cr[k], \Delta k(\xi) > 0} i(k', \xi) + \sum_{\xi \in Cr[k], \Delta k(\xi) < 0} i(k', \xi) = (-1)^n$$

This and (31) imply

$$\sum_{\xi \in Cr[k], \Delta k(\xi) < 0} i(k', \xi) = (-1)^n,$$

a contradiction to the assumption (24).

This proves that, for any open bounded set $\mathcal{N} \subset]0,\infty) \times \mathbb{R}^n$ such that $C_+ \subset \mathcal{N}$, one has

$$deg(\Gamma', \mathcal{N}, 0) \neq 0.$$

Now we can apply the abstract results yielding a critical point of I_ε and hence a solution of (23).

Further results can be found in

- Perturbation of $\Delta u + u^{(N+2)/(N-2)} = 0$, the scalar curvature problem in $\mathbb{R}^N$ and related topics, J. Funct. Analysis, 165 (1999), 117-149 (with J. Garcia Azorero and I. Peral)
- Elliptic variational problems in $\mathbb{R}^N$ with critical growth, J. Diff. Equat. 168-1 (2000), 10-32 (with J. Garcia Azorero and I. Peral)
- Remarks on a class of semilinear elliptic equations on $\mathbb{R}^n$, via perturbation methods, Advanced Nonlin. Studies, 1 (2001), 1-13 (with J. Garcia Azorero and I. Peral)

Elliptic equations with critical exponent arise in Differential conformal geometry. For some results on this topic, see

- A multiplicity result for the Yamabe problem on S^n, J. Funct. Analysis, 168-2 (1999), 529-561 (with A. Malchiodi)
- On the symmteric scalar curvature problem on S^n , J. Diff. Equat., 170-1 (2001), 228-245 (with A. Malchiodi)
- Yamabe and Scalar Curvature problem under boundary conditions, Math. Annalen, 322 (2002), 667-699 (with Y.Y. Li and A. Malchiodi)