NONLINEAR EIGENVALUE PROBLEMS INVOLVING THE p-LAPLACIAN

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Chapter 1

Variational eigenvalues of Δ_p

Let us consider the linear eigenvalue problem

$$\begin{cases} -\Delta u - \lambda\, u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}$ is a bounded domain. In the weak setting, to find an eigenvalue and an eigenfunction of (1.1) means to find $\lambda \in \mathbb{R}$ and $u \in W_0^{1,2}(\Omega),\ u \neq 0$, such that the integral identity

$$\int_\Omega \nabla u . \nabla v \,\mathrm{d}x - \lambda \int_\Omega uv \,\mathrm{d}x = 0$$

holds for all test functions $v \in W_0^{1,2}(\Omega)$. It is well known that all eigenvalues of (1.1) can be arranged into a sequence

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots \to +\infty,$$

the multiplicity of any eigenvalue is finite and the corresponding normalised eigenfunctions $\{u_1, u_2, u_3, \ldots\}$ form a complete orthonormal system.
Using the Courant-Weinstein variational principle, the eigenvalues can be expressed as follows:

$$\lambda_k = \inf_{\substack{u \perp \{u_1, \ldots, u_{k-1}\} \\ ||u||_{L^2} = 1}} \int_\Omega |\nabla u|^2 \,\mathrm{d}x, \quad k = 1, 2, \ldots \tag{1.2}$$

In connection with the forthcoming nonlinear problem, let us emphasise the fact that the sequence obtained in (1.2) *exhausts the set of all eigenvalues* of (1.2).

Let us consider now the *p-Laplacian* for $p > 1$, i.e. the quasilinear second order operator defined by

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u).$$

Clearly, $\Delta_2 u = \Delta u$, and for $p \neq 2$, this operator is $(p-1)$-*homogeneous* and *nonlinear*. Natural eigenvalue problem for the p-Laplacian, which generalises that of (1.1), reads as follows

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2}\, u = 0 & \text{in } \Omega, \\ \qquad\qquad\qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

In the weak setting:

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \,\mathrm{d}x - \lambda \int_\Omega |u|^{p-2} uv \,\mathrm{d}x = 0 \tag{1.4}$$

for any $v \in W_0^{1,p}(\Omega)$.

There are several analogues of the variational formula (1.2) which allows for construction of sequences of the so called *variational eigenvalues* of the p-Laplacian. However, with exception of the case $N = 1$, it is not clear if these sequences exhaust the *set of all eigenvalues.*

Before giving one of the definitions of the variational eigenvalues, we investigate some basic properties of the functional

$$I(u) = \frac{\int_\Omega |\nabla u|^p \,\mathrm{d}x}{\int_\Omega |u|^p \,\mathrm{d}x}, \qquad u \in W_0^{1,p}(\Omega) \setminus \{o\}.$$

Using the notation $\langle . \, , \, . \rangle$ for the duality between $(W_0^{1,p}(\Omega))^*$ and $W_0^{1,p}(\Omega)$, we have

$$\begin{aligned} &\langle I'(u), v\rangle \\ &= \frac{p\left(\int_\Omega |\nabla u|^{p-2}\nabla u . \nabla v \,\mathrm{d}x\right)\left(\int_\Omega |u|^p \,\mathrm{d}x\right) - \left(\int_\Omega |\nabla u|^p \,\mathrm{d}x\right) p \left(\int_\Omega |u|^{p-2} uv \,\mathrm{d}x\right)}{\left(\int_\Omega |u|^p \,\mathrm{d}x\right)^2} \end{aligned}$$

for any $v \in W_0^{1,p}(\Omega)$. If we restrict onto

$$\mathcal{S} := \left\{ u \in W_0^{1,p}(\Omega) : ||u||_{L^p} = 1 \right\},$$

we have

$$\langle I'(u), v\rangle = p\left[\langle A(u), v\rangle - I(u)\, \langle B(u), v\rangle\right], \tag{1.5}$$

where $A,\ B:\ W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ are defined as follows:

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\,\nabla u.\nabla v\,\mathrm{d}x,$$
$$\langle B(u), v\rangle = \int_\Omega |u|^{p-2}\,uv\,\mathrm{d}x,$$

u, $v \in W_0^{1,p}(\Omega)$. Both operators are odd, $(p-1)$-homogeneous and continuous. Moreover, A is continuously invertible and B is compact. Notice that $\langle A(u), u\rangle = I(u)$ and $\langle B(u), u\rangle = 1$ for $u \in \mathcal{S}$, and, hence for any $u \in \mathcal{S}$ we have

$$\langle I'(u), u\rangle = 0.$$

"Important observation": By the dual characterisation of the norm we then obtain that for any $u \in \mathcal{S}$,

$$||(I|_{\mathcal{S}})'(u)||_{(\mathcal{T}u)^*} = ||(I'(u)||_{(W_0^{1,p}(\Omega))^*},$$

where $\mathcal{T}u$ stands for the tangent space of $\mathcal{S}$ at the point $u \in \mathcal{S}$. In particular, we will use this fact in Lemma 1.1 and in Lemma 1.2 below. We also drop the indices of the norm of I' for the sake of brevity.

Let us assume now that $\lambda \in \mathbb{R}$ is a critical level of $I|_{\mathcal{S}}$ and $u_\lambda \in \mathcal{S}$ is the corresponding critical point. It follows from the Lagrange multiplier method that there is $\mu \in \mathbb{R}$ such that

$$\langle A(u_\lambda), v\rangle = \mu\,\langle B(u_\lambda), v\rangle$$

for all $v \in W_0^{1,p}(\Omega)$. But it follows immediately that $\mu = \lambda = I(u_\lambda) = \langle A(u_\lambda), u_\lambda\rangle$. In other words, (1.4) holds and so, the critical levels of $I|_{\mathcal{S}}$ and the critical points are in one-to-one correspondence with the eigenvalues and eigenfunctions of the p-Laplacian, respectively.

The following two assertions are important for our definition of variational eigenvalues.

Lemma 1.1 *$I|_{\mathcal{S}}$ satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\} \subset \mathcal{S}$ and $c > 0$ be such that

$$|I(u_n)| \le c \quad \text{for any } n \in \mathbb{N} \text{ and } p\,[A(u_n) - I(u_n)B(u_n)] = I'(u_n) \to 0 \quad (1.6)$$

in $(W_0^{1,p}(\Omega))^*$ as $n \to \infty$. Clearly, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, hence passing to subsequences if necessary, we can finally assume, that there exists $u \in W_0^{1,p}(\Omega)$, such that

$$u_n \rightharpoonup u \text{ in } W_0^{1,p}(\Omega),\ u_n \to u \text{ in } L^p(\Omega), \text{ and also } I(u_n) \to \bar{I} \text{ in } \mathbb{R}.$$

By compactness, we have $B(u_n) \to B(u)$ in $(W_0^{1,p}(\Omega))^*$. Making use of (1.6) and continuity of A^{-1} we arrive at

$$u_n \to A^{-1}(\bar{I}B(u)) \text{ in } W_0^{1,p}(\Omega),$$

which completes the proof.

Q.E.D.

Since $\mathcal{S}$ is a connected and complete C^1 Finsler manifold and $I|_{\mathcal{S}} \in C^1(\mathcal{S}, \mathbb{R})$, we can use Lemma 3.7 from [G, p.55]. We will state only special version of it, which is important for us.

Lemma 1.2 *Let $C \subset \mathcal{S}$ be a compact set. Assume that there exists $\varepsilon > 0$ such that for all $u \in C$ we have*

$$||I'(u)|| \geq 2\varepsilon > 0.$$

Then there exists a continuous one parameter family of homeomorphisms

$$\psi : \ \mathcal{S} \times [0,1] \to \mathcal{S}$$

such that

1. $\psi(u,0) = u$ *for every* $u \in \mathcal{S}$
2. $I(\psi(u,1)) \leq I(u) - \varepsilon$ *for every* $u \in \mathcal{S}$.

In particular, since $\mathcal{S} = -\mathcal{S}$ and also $I(u) = I(-u)$ for every $u \in \mathcal{S}$, then the homeomorphisms can be chosen to preserve the symmetry, i.e. we also have

3. $\psi(-u,t) = -\psi(u,t)$ *for every* $u \in \mathcal{S}$ *and* $t \in [0,1]$.

For any $k \in \mathbb{N}$ let

$$\mathcal{F}_k := \{\mathcal{A} \subset \mathcal{S} : \text{ there exists continuous odd surjection } h : \ \mathcal{S}^{k-1} \to \mathcal{A}\}.$$

Here $\mathcal{S}^{k-1}$ stands for the unit sphere in $\mathbb{R}^k$. Further, define

$$\lambda_k := \inf_{\mathcal{A} \in \mathcal{F}_k} \sup_{u \in \mathcal{A}} I(u), \quad k = 1, 2, \ldots. \tag{1.7}$$

Theorem 1.3 *For any $k \in \mathbb{N}$, λ_k is a critical level of I (and hence an eigenvalue of the p-Laplacian).*

Proof. Suppose that λ_k is not a critical level of I for some $k \in \mathbb{N}$. Then there exist $\varepsilon > 0$ and $\mathcal{A}_\varepsilon \in \mathcal{F}_k$ such that $\sup_{u \in \mathcal{A}_\varepsilon} \leq \lambda_k + \frac{\varepsilon}{2}$ and for any $u \in \mathcal{A}_\varepsilon$ we have $||I'(u)|| \geq 2\varepsilon$. For otherwise, for any $n \in \mathbb{N}$, we find $\mathcal{A}_n \in \mathcal{F}$ such that

$$\sup_{u \in \mathcal{A}_n} I(u) \leq \lambda_k + \frac{1}{2n} \quad \text{and} \quad ||I'(u_n)|| < \frac{2}{n}$$

for some $u_n \in \mathcal{A}_n$. But this together with Lemma 1.1 imply the existence of $u \in \mathcal{S}$, such that $I'(u) = 0$ and $I(u_n) = \lambda_k$, a contradiction to our assumption that λ_k is not a critical level.

Now, having $\varepsilon > 0$ and $\mathcal{A}_\varepsilon \in \mathcal{F}_k$ in hands, we can apply Lemma 1.2 with $C = \mathcal{A}_\varepsilon$. Denote by $h : \mathcal{S}^{k-1} \to \mathcal{A}_\varepsilon$ a continuous odd surjection. If ψ is an object from Lemma 1.2 then

$$\psi\,(h(.), 1) : \ \ \mathcal{S}^{k-1} \to \psi(\mathcal{A}_\varepsilon, 1) = \tilde{\mathcal{A}}_\varepsilon$$

is a continuous and odd surjection as well, i.e. $\tilde{\mathcal{A}}_\varepsilon \in \mathcal{F}_k$. But according to Lemma 1.2 we have

$$\sup_{u \in \tilde{\mathcal{A}}_\varepsilon} I(u) \leq \lambda_k - \frac{\varepsilon}{2}$$

which contradicts the definition of λ_k. Hence λ_k must be a critical level of I for all $k \in \mathbb{N}$.

Q.E.D.

We will refer to the sequence $\{\lambda_k\}$ as to the sequence of *variational eigenvalues* of the p-Laplacian.

The following *basic question* is connected with the definition of variational eigenvalues:

"Does the sequence $\{\lambda_k\}$ defined by the minimax argument (1.7) represent a complete list of eigenvalues?"

Of course, the answer is YES if $p = 2$. The answer is the same if $p \neq 2$ and $N = 1$. In this case we have powerful one-dimensional machinery available to prove this result (shooting argument, uniqueness for the initial value problem, etc.). This is not the case if $p \neq 2$ and $N > 1$; the answer remains OPEN, and the problem seems to be rather difficult.

Let us close this section pointing out the relation between $\{\lambda_k\}$ defined by (1.7) and the other variational eigenvalues which are defined by means of *the*

Krasnoselski genus of the set. Recall that genus $\gamma(\mathcal{A})$ of the set $\mathcal{A}$ is defined as follows:

$$\gamma(\mathcal{A}) = \begin{cases} \inf \ [1999/05/25v2.5hStandardLaTeXfontdefinitions]\{m \in \mathbb{N}: \text{ there exists a con} \\ \infty \ \text{ if the infimum above does not exist.} \end{cases}$$

Set

$$\mathcal{F}_k^* := \{\mathcal{A} \subset \mathcal{S} : \ \bar{\mathcal{A}} = \mathcal{A}, \ -\mathcal{A} = \mathcal{A}, \ \gamma(\mathcal{A}) \geq k\}$$

and

$$\lambda_k^* := \inf_{\mathcal{A} \in \mathcal{F}_k^*} \sup_{u \in \mathcal{A}} I(u).$$

It is well known that $\{\lambda_k\}$ are eigenvalues of the p-Laplacian and $\lambda_k^* \to \infty$. Since there is no continuous odd map from $\mathcal{S}^{k-1}$ into $\mathbb{R}^m \setminus \{0\}$ with $m < k$, we have $\mathcal{F}_k \subset \mathcal{F}_k^*$, i.e. $\lambda_k \geq \lambda_k^*$, $k = 1, \ 2, \ , \dots$ In particular, we have also $\lambda_k \to \infty$. Since it is easy to see that $\lambda_1 = \lambda_1^*$, a natural question arises:

"Does $\lambda_k = \lambda_k^*$ hold for all $k \in \mathbb{N}$?"

The answer is YES if either $p = 2$ or else $p \neq 2$ and $N = 1$ by the same reasons as above. The answer is YES also if $N > 1$ and $k = 2$ (see the argument below). In case $N > 1$ and $k > 2$ the answer seems to be OPEN.

Proposition 1.4 We have $\lambda_2 = \lambda_2^*$.

Proof. It is sufficient to show that $\lambda_2 \leq \lambda_2^*$. It is proved in [AT] that

$$\lambda_2^* = \inf\{\lambda > \lambda_1^* : \ \lambda \text{ is an eigenvalue}\}$$

and if u_2 is some normalised eigenfunction associated with λ_2^*, then $u_2^+ \not\equiv 0$ and $u_2^- \not\equiv 0$ in Ω. Set

$$\mathcal{A} := \left\{ s\, u_2^+ + t\, u_2^- \ : \ s, t \in \mathbb{R}, \ |s|^p ||u_2^+||_{L^p}^p + |t|^p ||u_2^-||_{L^p}^p = 1 \right\}.$$

Then $\mathcal{A} \subset \mathcal{F}_2$ and for all $u \in \mathcal{A}$ we have

$$\int_\Omega |\nabla u|^p \,\mathrm{d}x = |s|^p \int_\Omega |\nabla u_2^+|^p \,\mathrm{d}x + |t|^p \int_\Omega |\nabla u_2^-|^p \,\mathrm{d}x$$

$$= |s|^p \lambda_2^* \int_\Omega |u_2^+|^p \,\mathrm{d}x + |t|^p \lambda_2^* \int_\Omega |u_2^-|^p \,\mathrm{d}x = \lambda_2^*.$$

The definition of λ_2 then yields $\lambda_2 \leq \lambda_2^*$.

Q.E.D.

Chapter 2

Landesman-Lazer type problem for Δ_p

Let us consider the boundary value problem

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2}u + f(x,u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory's function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ we have $|f(x,s)| \le \bar{f}(x)$ for some $\bar{f} \in L^{p'}(\Omega)$.
Using the degree argument it is not difficult to prove that (2.1) has at least one weak solution provided λ is not an eigenvalue of the p-Laplacian. In general, this is not the case if λ is an eigenvalue. So, the natural question arises:

"What are reasonable sufficient (or/and necessary) conditions on f for the existence of at least one weak solution of (2.1)?"

The answer goes back to works of Landesman and Lazer from the late sixties, who dealt with semilinear problem, i.e. (2.1) with $p = 2$.

Assume that there exist limits

$$f^{\pm}(x) = \lim_{s \to \pm\infty} f(x,s)$$

and either

$$(LL)^+_\lambda \qquad \int_{v>0} f^+(x)v(x)\,\mathrm{d}x + \int_{v<0} f^-(x)v(x)\,\mathrm{d}x > 0$$

or

$$(LL)^-_\lambda \qquad \int_{v>0} f^+(x)v(x)\,\mathrm{d}x + \int_{v<0} f^-(x)v(x)\,\mathrm{d}x < 0$$

is satisfied for all $v \in \mathrm{Ker}(-\Delta_p - \lambda) \setminus \{0\}$. Our goal is to show that under one of the conditions above, there exists at least one weak solution of problem (2.1) (cf. [DR1]).
Observe first, that both $(LL)^{\pm}_{\lambda}$ are vacuously true if λ is not an eigenvalue and hence the existence follows from our result as well.

Let us concentrate on the situation when λ is an eigenvalue and point out that our result holds regardless λ is a variational or nonvariational eigenvalue. Before discussing the proof let us consider some special cases of $f = f(x,s)$ in order to understand better the meaning of Landesman-Lazer type conditions $(LL)^{\pm}_{\lambda}$. Let $f(x,s) = g(s) - h(x)$, i.e. (2.1) reads as

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2}u + g(u) = h & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Assume that $h \in L^{p'}(\Omega)$, $g : \mathbb{R} \to \mathbb{R}$ is bounded and continuous and there exist the limits

$$g(\pm\infty) = \lim_{s \to \pm\infty} g(s).$$

Then $(LL)^{+}_{\lambda}$ reads as

$$g(+\infty) \int_{v>0} v(x)\,\mathrm{d}x + g(-\infty) \int_{v<0} v(x)\,\mathrm{d}x > \int_{\Omega} h(x) v(x)\,\mathrm{d}x$$

for all $v \in \mathrm{Ker}(-\Delta_p - \lambda) \setminus \{0\}$, and similarly for $(LL)^{-}_{\lambda}$.

If we simplify even more and assume $\lambda = \lambda_1$ and $\varphi_1 > 0$ is the corresponding normalised eigenfunction (notice that $\lambda_1 > 0$ is simple eigenvalue even for the p-Laplacian), and $g(s) = \arctan s$, then $(LL)^{-}_{\lambda_1}$ reduces to

$$-\frac{\pi}{2} \int_{\Omega} \varphi_1(x)\,\mathrm{d}x < \int_{\Omega} h(x)\varphi_1(x)\,\mathrm{d}x < \frac{\pi}{2} \int_{\Omega} \varphi_1(x)\,\mathrm{d}x$$

which is the condition having simple "geometric interpretation".

Let us introduce the functional

$$J_{\lambda}(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p\,\mathrm{d}x - \frac{\lambda}{p} \int_{\Omega} |u|^p\,\mathrm{d}x + \int_{\Omega} \left(\int_0^{u(x)} f(x,s)\,\mathrm{d}s \right) \mathrm{d}x$$

and denote

$$F(x, u(x)) = \int_0^{u(x)} f(x,s)\,\mathrm{d}s.$$

Then

$$\begin{aligned}\langle J'_\lambda(u), v\rangle &= \int_\Omega |\nabla u|^{p-2}\,\nabla u.\nabla v\,\mathrm{d}x - \lambda\int_\Omega |u|^{p-2}\,u\,v\,\mathrm{d}x \\ &\quad + \int_\Omega f(x,u)\,v\,\mathrm{d}x,\ \ u,v\in W_0^{1,p}(\Omega)\end{aligned}$$

and it follows that $J'_\lambda(u) = 0$ if and only if u is a weak solution of (2.1). Note that the operator $C : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ defined by

$$\langle C(u), v\rangle = \int_\Omega f(x, u(x))\,v(x)\,\mathrm{d}x, \quad u, v \in W_0^{1,p}(\Omega)$$

is compact and bounded, i.e. for any $u \in W_0^{1,p}(\Omega)$ we have

$$||C(u)||_{(W_0^{1,p}(\Omega))^*} \le ||\bar{f}||_{L^{p'}}.$$

Our first observation about J_λ concerns the Palais-Smale condition.

Lemma 2.1 *Under the hypotheses* $(LL)^\pm_\lambda$ *the functional* J_λ *satisfies the* (Palais-Smale) *condition on* $W_0^{1,p}(\Omega)$.

Proof. Let $\{u_n\} \subset W_0^{1,p}(\Omega)$, $c > 0$ be such that for all $n \in \mathbb{N}$:

$$|J_\lambda(u_n)| \le c \ \text{ and } \ J'_\lambda(u_n) \to 0.$$

First, we show that $\{u_n\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$. Assume the contrary, i.e. $||u_n|| \to \infty$ (we drop the indices of norms for brevity). Set $v_n := \frac{u_n}{||u_n||}$. Passing to subsequence if necessary, we can assume that there are $v \in W_0^{1,p}(\Omega)$ and $\bar{g} \in L^p(\Omega)$ such that $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$, $v_n \to v$ in $L^p(\Omega)$, $v_n \to v$ a.e. in Ω, and $|v_n(x)| \le \bar{g}(x)$ for a.e. $x \in \Omega$. Consider

$$\frac{J'_\lambda(u_n)}{||u_n||^{p-1}} = A(v_n) - \lambda B(v_n) + \frac{C(u_n)}{||u_n||^{p-1}}.$$

By the assumption, the left-hand side tends to zero; by the compactness of B and boundedness of C, the second and the third terms on the right-hand side tend to $\lambda B(v)$ and 0, respectively. Employing the continuity of A^{-1} we get

$$v_n \to A^{-1}(\lambda B(v)).$$

Hence $||v|| = 1$ and $A(v) - \lambda B(v) = 0$, i.e. $v_n \to v \in \mathrm{Ker}(-\Delta_p - \lambda) \setminus \{0\}$. Next we have

$$pJ_\lambda(u_n) - \langle J'_\lambda(u_n), u_n\rangle = p\int_\Omega F(x, u_n)\,\mathrm{d}x - \int_\Omega f(x, u_n)u_n\,\mathrm{d}x,$$

i.e.

$$\frac{pJ_\lambda(u_n)}{||u_n||} - \langle J'_\lambda(u_n), v_n\rangle = p\int_\Omega \frac{F(x,u_n)}{||u_n||}\,\mathrm{d}x - \int_\Omega f(x,u_n)v_n\,\mathrm{d}x. \tag{2.3}$$

For a.e. $x \in \{x\in\Omega:\ v(x)>0\}$ we have $u_n(x)\to+\infty$ and so

$$\lim_{n\to\infty} f(x,u_n(x))v_n(x) = f^+(x)\,v(x),$$

$$\lim_{n\to\infty}\frac{F(x,u_n(x))}{||u_n||} = \lim_{n\to\infty}\frac{v_n(x)\int_0^{u_n(x)} f(x,s)\,\mathrm{d}s}{u_n(x)} = \text{(l'Hospital rule)} = v(x)\,f^+(x);$$

similarly, for $x\in\{x\in\Omega:\ v(x)<0\}$, we have $u_n(x)\to-\infty$ and

$$\lim_{n\to\infty} f(x,u_n(x))\,v_n(x) = f^-(x)\,v(x),$$

$$\lim_{n\to\infty}\frac{F(x,u_n(x))}{||u_n||} = v(x)\,f^-(x).$$

Both integrands on the right hand side of (2.3) have a majorant $\bar f\bar g\in L^1(\Omega)$. It then follows from the Lebesgue theorem, the assumptions of Palais-Smale condition and (2.3) that

$$\begin{aligned} 0 &= \lim_{n\to\infty}\left[p\int_\Omega\frac{F(x,u_n)}{||u_n||}\,\mathrm{d}x - \int_\Omega f(x,u_n)v_n\,\mathrm{d}x\right] \\ &= (p-1)\left[\int_{v>0} f^+\,v\,\mathrm{d}x + \int_{v<0} f^-\,v\,\mathrm{d}x\right], \end{aligned}$$

which contradicts $(LL)^{\pm}_\lambda$. This proves the boundedness of $\{u_n\}$ in $W_0^{1,p}(\Omega)$. Passing to a subsequence we may assume that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ for some $u\in W_0^{1,p}(\Omega)$. Since we also assume

$$0 \longleftarrow J'_\lambda(u_n) = A(u_n) - \lambda B(u_n) + C(u_n),$$

we can employ compactness of B and C, and continuity of A^{-1}, to conclude

$$u_n \to A^{-1}\left(\lambda B(u) - C(u)\right).$$

Q.E.D.

With Palais-Smale condition in hands we can apply Theorem 3.4 from [S, p.75]:

Lemma 2.2 *Let $\beta\in\mathbb{R}$ be a regular value of J_λ and let $\bar\varepsilon>0$. Then there exists $\varepsilon\in(0,\bar\varepsilon)$ and a continuous one-parameter family of homeomorphisms $\varphi:\ W_0^{1,p}(\Omega)\times[0,1]\to W_0^{1,p}(\Omega)$ such that*

1. *$\varphi(u,t) = u$ if $t = 0$ or if $|J_\lambda(u) - \beta| \geq \bar{\varepsilon}$;*

2. *$J_\lambda(\varphi(u,t))$ is non-increasing in t for any $u \in W_0^{1,p}(\Omega)$;*

3. *$J_\lambda(u) \leq \beta + \varepsilon$ implies $J_\lambda(\varphi(u,1)) \leq \beta - \varepsilon$.*

Now we state the main result of this section.

Theorem 2.3. *Let us assume that either $(LL)^+_\lambda$ or $(LL)^-_\lambda$ holds. Then (2.1) has at least one weak solution.*

Proof.

1. Assume that λ is not a variational eigenvalue. Since $\lambda_n \to \infty$, there exists $k \in \mathbb{N}$ such that $\lambda_k < \lambda < \lambda_{k+1}$. The plan is to show that there exists a critical value of J_λ which can be characterised as a minimax over *linked sets.*
   Observe that there exists $\mathcal{A} \in \mathcal{F}_k$ such that
   $$\sup_{u \in \mathcal{A}} I(u) = m \in (\lambda_k, \lambda).$$
   Then for all $t > 0$ and $u \in \mathcal{A}$ we have
   $$J_\lambda(tu) \leq \frac{t^p}{p}(m - \lambda) + t||\bar{f}||_{L^{p'}}$$
   and hence $J_\lambda(tu) \to -\infty$ as $t \to +\infty$. Set
   $$\mathcal{E}_{k+1} := \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p \,\mathrm{d}x \geq \lambda_{k+1} \int_\Omega |u|^p \,\mathrm{d}x \right\}.$$
   Then for all $u \in \mathcal{E}_{k+1}$ we have
   $$J_\lambda(u) \geq \frac{1}{p}(\lambda_{k+1} - \lambda)||u||^p_{L^p} - ||\bar{f}||_{L^{p'}}||u||_{L^p}$$
   and hence
   $$-\infty < \alpha = \inf_{u \in \mathcal{E}_{k+1}} J_\lambda(u).$$
   For $T > 0$ denote $T\mathcal{A} := \{tu;\ \ t \geq T,\, u \in \mathcal{A}\}$. Take T so large that
   $$\alpha > \gamma := \max_{u \in T\mathcal{A}} J_\lambda(u).$$
   Consider the family
   $$\Gamma := \left\{ h \in C^0(B_k, W_0^{1,p}(\Omega)) : \, h|_{\mathcal{S}^{k-1}} \text{ is an odd mapping into } T\mathcal{A} \right\},$$

where B_k stands for closed unit ball in $\mathbb{R}^k$. Then one can prove that $\Gamma \neq \phi$ ([DR1, Lemma 5]) and for any $h \in \Gamma : h(B_k) \cap \mathcal{E}_{k+1} \neq \phi$ ([DR1, Lemma 6]). This means that the sets $T\mathcal{A}$ and $\mathcal{E}_{k+1}$ are linked in a way that allows the application of standard minimax theorems. By means of Lemma 2.2 it can be proved ([DR1, Lemma 6]) that

$$c := \inf_{h\in\Gamma} \sup_{x\in B_k} J_\lambda(h(x))$$

is a critical value of J_λ and $c \geq \alpha$.

2. Assume that λ is a variational eigenvalue and $(LL)^+_\lambda$ holds. Since $\lambda_n \to \infty$, there exists $k \in \mathbb{N}$ such that $\lambda = \lambda_k$ and $\lambda_{k-1} < \lambda_k$. Choosing an increasing sequence $\mu_n \in (\lambda_{k-1}, \lambda_k)$, $\mu_n \nearrow \lambda_k$, by the previous step we know that for any $n \in \mathbb{N}$ there is at least one critical point of J_{μ_n} and the corresponding critical levels form a decreasing sequence ([DR1, Lemma 8]). This fact together with $(LL)^+_\lambda$ then guarantee that the sequence of critical points of J_{μ_n} possesses strongly convergent subsequence to a critical point of J_{λ_k} ([DR1, Lemma 9]). Similar argument works also for $(LL)^-_\lambda$, where we have to choose $\mu_n \searrow \lambda_k$.

Q.E.D.

Chapter 3

Courant nodal domain theorem for Δ_p

In this section we return to the eigenvalues and eigenfunctions of problem (1.3). Let us recall the so called *Courant nodal domain theorem* for the linear Laplacian:

Theorem 3.1. *Assume, that u_{λ_n} is an eigenfunction associated with the n-th eigenvalue λ_n, of (1.1). Then u_{λ_n} has at most n nodal domains (maximal connected sets on which u_{λ_n} is of constant sign).*

Note that simple examples demonstrate that no similar lower bound is possible. Consider e.g. $\Omega = (0, \pi) \times (0, L\pi)$, where L is large.

In [DR2] we show that Theorem 3.1 generalises completely to (1.3) if we assume that the p-Laplacian satisfies a *unique continuation property* (defined below) or that $\lambda < \lambda_{n+1}$. For the general case, we prove that, if u_{λ_n} is an eigenfunction associated with λ_n, u_{λ_n} has at most $2n - 2$ nodal domains. Also, if u_{λ_n} has $n + k$ nodal domains, then there is another eigenfunction corresponding to λ_n with at most $n - k$ nodal domains.

Recall also that for the p-Laplacian it is now well known that λ_1 and λ_2 are both variational eigenvalues, λ_1 is simple and corresponding eigenfunctions are of constant sign. On the other hand, any eigenfunction associated with λ_2 changes sign exactly once, i.e. it has exactly two nodal domains. An interesting feature of our estimates consists in the fact that they hold for both variational and nonvariational eigenvalues.

Definition 3.2. We say that $-\Delta_p$ satisfies the *unique continuation property* (UCP) if for any eigenfunction u_λ of (1.3) the set $\{x \in \Omega : u_\lambda(x) = 0\}$ has empty interior.

It is well known, that (UCP) holds for the case $p = 2$ but it is not clear if it holds also for $p \neq 2$.

Let us formulate some useful properties of the eigenfunctions $u_\lambda \in W_0^{1,p}(\Omega)$ of (1.3). The reader is kindly asked to find the proofs and other references in [DR2].

Lemma 3.3. *Let u_λ be an eigenfunction associated with the eigenvalue λ and let Ω_λ be a nodal domain for u_λ. Define*

$$w := \begin{cases} u_\lambda, & x \in \Omega_\lambda, \\ 0, & x \notin \Omega_\lambda. \end{cases}$$

Then $w \in W_0^{1,p}(\Omega)$ and $\int_\Omega |\nabla w|^p \, \mathrm{d}x = \lambda \int_\Omega |w|^p \, \mathrm{d}x$.

Lemma 3.4. *Assume that there exists $K > 0$ such that $\lambda \in (0, K)$. Let Ω_λ be a nodal domain of u_λ. Then $\operatorname{meas} \Omega_\lambda \geq c_1(K) > 0$, where $c_1 = c_1(K)$ is a constant depending only on K.*

In particular, it follows from Lemma 3.4 that any *eigenfunction* of the p-Laplacian has a *finite number of nodal domains.*

Lemma 3.5. *For any eigenfunction $u_\lambda \in W_0^{1,p}(\Omega)$ there exists $\eta \in (0, 1)$ such that $u_\lambda \in C^{1,\eta}(\bar{\Omega})$.*

Lemma 3.6. *Let λ be an eigenvalue different from λ_1, and let u_λ be an eigenfunction associated with λ. Let Ω_1 be a nodal domain for u_λ. Then there is another nodal domain Ω_2, a point $x_0 \in (\partial\Omega_1 \cap \partial\Omega_2) \setminus \partial\Omega$, and an $\varepsilon > 0$ such that $B_\varepsilon(x_0) \cap \partial\Omega_1 \cap \partial\Omega_2$ is a smooth manifold separating $B_\varepsilon(x_0) \cap \Omega_1$ and $B_\varepsilon(x_0) \cap \Omega_2$ (here $B_\varepsilon(x_0)$ stands for the ball centered at x_0 with radius ε). Moreover, if $u_\lambda > 0$ in Ω_1 (respectively, $u_\lambda < 0$ in Ω_1), then $\frac{\partial u}{\partial \nu} < 0$ (> 0), where ν represents the unit outward normal to $\partial\Omega_1$ at x_0.*

Lemma 3.7. *Let Ω_1 and Ω_2 be nodal domains for an eigenfunction u_λ, and let $x_0 \in (\partial\Omega_1 \cap \partial\Omega_2) \setminus \partial\Omega$ be such that $B_\varepsilon(x_0) \cap \partial\Omega_1 \cap \partial\Omega_2$ is a smooth manifold separating $B_\varepsilon(x_0) \cap \Omega_1$ and $B_\varepsilon(x_0) \cap \Omega_2$. Suppose that u_λ^* is another eigenfunction such that $u_\lambda^* = \gamma_i u_\lambda$ on Ω_i, $i = 1, 2$. Then $\gamma_1 = \gamma_2$.*

Theorem 3.8. *Suppose that $-\Delta_p$ satisfies (UCP) and suppose that u_{λ_n} is an eigenfunction associated with λ_n. Then u_{λ_n} has at most n nodal domains.*

Proof. Assume that u_{λ_n} has $(n+k)$ nodal domains, where $k \geq 1$. Call them $\{\Omega_1, \Omega_2, \ldots, \Omega_{n+k}\}$. Let $u_i := u_{\lambda_n}.\chi_{\Omega_i}$, where χ_{Ω_i} is the characteristic function over the set Ω_i. Let

$$\mathcal{A} := \left\{ \sum_{i=1}^{n} \gamma_i u_i : \sum_{i=1}^{n} |\gamma_i|^p \int_{\Omega} |u_i|^p \, \mathrm{d}x = 1 \right\}.$$

i.e. $\mathcal{A} \subset \mathcal{S}$. Hence $\mathcal{A} \in \mathcal{F}_n$.
Also, using Lemma 3.3 as well as the fact that $\{x \in \Omega : u_i(x) \neq 0\} \cap \{x \in \Omega : u_j(x) \neq 0\}$ has measure zero for $i \neq j$, we get

$$\begin{aligned} \int_{\Omega} |\nabla u|^p \, \mathrm{d}x &= \sum_{i=1}^{n} \int_{\Omega_i} |\nabla u|^p \, \mathrm{d}x = \sum_{i=1}^{n} |\gamma_i|^p \int_{\Omega_i} |\nabla u|^p \, \mathrm{d}x \\ &= \sum_{i=1}^{n} |\gamma_i|^p \lambda_n \int_{\Omega_i} |u|^p \, \mathrm{d}x = \lambda_n, \end{aligned}$$

i.e. $I \equiv \lambda_n$ on $\mathcal{A}$. Observe that if $u \in \mathcal{A}$ then $u \equiv 0$ on Ω_{n+1}, so u cannot be an eigenfunction, else (UCP) would be contradicted. Thus I has no critical points on $\mathcal{A}$. Since $\mathcal{A}$ is compact, there is an $\varepsilon > 0$ such that

$$||I'(u)|| \geq 2\varepsilon > 0$$

for $u \in \mathcal{A}$. Hence, we can apply Lemma 1.2 with $C = \mathcal{A}$ to obtain a symmetry preserving flow ψ. Let $\mathcal{A}^* := \psi(\mathcal{A}, 1)$. Then $\mathcal{A}^* \in \mathcal{F}_n$ with $\sup_{u \in \mathcal{A}^*} I(u) < \lambda_n$, a contradiction to the definition of λ_n.

Q.E.D

Theorem 3.9. *Suppose $\lambda < \lambda_{n+1}$ is an eigenvalue and u_λ has at most n nodal domains.*

Proof. Suppose u_λ has nodal domains $\{\Omega_1, \Omega_2, \ldots, \Omega_{n+k}\}$ for some $k \geq 1$. Let $u_i := u_\lambda.\chi_{\Omega_i}$ be as in the previous proof. Let

$$\mathcal{A} := \left\{ \sum_{i=1}^{n+k} \gamma_i u_i : \sum_{i=1}^{n+k} |\gamma_i|^p \int_{\Omega} |u_i|^p \, \mathrm{d}x = 1 \right\}.$$

As in the previous proof we can verify that $\mathcal{A} \in \mathcal{F}_{n+k}$ and that $I(u) = \lambda$ for $u \in \mathcal{A}$. But the characterisation of λ_{n+k} implies that

$$\lambda = \sup_{u \in \mathcal{A}} I(u) \geq \lambda_{n+k} \geq \lambda_{n+1} > \lambda,$$

a contradiction.

Q.E.D.

Notice that in the previous theorem λ is not required to be a variational eigenvalue.

Corollary 3.10. *Let us assume that all variational eigenvalues are simple. Then $\lambda_n < \lambda_{n+1}$ for all n and the estimate above gives a direct generalisation of Theorem 3.1.*

Theorem 3.11. *Let u_{λ_n} be an eigenfunction associated with λ_n. Then u_{λ_n} has at most $2n-2$ nodal domains.*

Proof. We begin the proof by dividing Ω into *nodal neighbourhoods*. Let Ω_1 be a nodal domain for u_{λ_n}. By Lemma 3.6. there is another nodal domain, Ω_2, a point $x_0 \in (\partial\Omega_1 \cap \partial\Omega_2) \setminus \partial\Omega$, and an $\varepsilon > 0$ such that $B_\varepsilon(x_0) \cap \partial\Omega_1 \cap \partial\Omega_2$ is a smooth manifold separating Ω_1 and Ω_2 in a neighbourhood of x_0. For convenience we will write $\Omega_1 \sim \Omega_2$, and say that these sets are *neighbours*. The nodal domain neighbourhood for Ω_1 will refer to the collection of nodal domains, Ω_k, such that Ω_1 and Ω_k are connected by a finite sequence of neighbours, i.e. there is a set of nodal domains $\{\Omega_1', \Omega_2', \ldots, \Omega_j'\}$ where $\Omega_i' \sim \Omega_{i+1}'$ for each $1 \le i \le j-1$, $\Omega_1 = \Omega_1'$, and $\Omega_j' = \Omega_k$. Using this notation we can organise all of the nodal domains for u_{λ_n} into neighbourhoods

$$\{\Omega_{11}, \Omega_{12}, \ldots, \Omega_{1j_1}\}$$
$$\{\Omega_{21}, \Omega_{22}, \ldots, \Omega_{2j_2}\}$$
$$\cdots$$
$$\{\Omega_{m1}, \Omega_{m2}, \ldots, \Omega_{Nj_m}\}.$$

Now suppose that u_{λ_n} has $n+k \ge 2n-1$ nodal domains. Let N represent the cardinality of $\mathcal{I} := \{(i,j) : 1 \le i \le m,\, j > 1\}$. Notice that $\{\Omega_{ij} : (i,j) \in \mathcal{I}\}$ includes all the nodal domains except first in each neighbourhood. Since each nodal domain neighbourhood contains at least two members, we have $N \ge \frac{1}{2}(2n-1)$, so $N \ge n$. Let $u_{ij} = u_{\lambda_n}\chi_{\Omega_{i,j}}$ and define

$$\mathcal{A} := \left\{ \sum_{\mathcal{I}} \gamma_{ij} u_{ij} : \sum_{\mathcal{I}} |\gamma_{ij}|^p \int_{\Omega_{ij}} |u_{ij}|^p \, \mathrm{d}x = 1 \right\}.$$

As in the previous proofs, it is straightforward to check that $\mathcal{A} \in \mathcal{F}_N$ and that $I(u) = \lambda_n$ for $u \in \mathcal{A}$. Suppose that $u_{\lambda_n}^* \in \mathcal{A}$ is a critical point for I, and thus an eigenfunction. Notice that $u_{\lambda_n}^* \equiv 0$ on the nodal domains Ω_{i1} for $1 \le i \le m$. By Lemma 3.7. it follows that $u_{\lambda_n}^* \equiv 0$ on every nodal domain that can be connected to an Ω_{i1} by a finite sequence of neighbours. Therefore $u_{\lambda_n}^* \equiv 0$ in Ω, which is a contradiction because $0 \notin \mathcal{A} \subset \mathcal{S}$. Hence $\mathcal{A}$ contains no critical points. Now the proof can be finished exactly as the proof of Theorem 3.8.

Q.E.D.

Remark 3.12. In particular, an eigenfunction associated with the second eigenfunction has at most 2 nodal domains. Since it has to change sign, it has exactly 2 nodal domains.

Theorem 3.13. *Let u_{λ_n} be an eigenfunction associated with λ_n such that u_{λ_n} has $n+k$ $(\leq 2n-2)$ nodal domains. Then there exists another eigenfunction $u^*_{\lambda_n}$ associated with λ_n, with at most $n-k$ $(\leq n)$ nodal domains.*

Proof. Divide Ω into nodal domain neighbourhoods exactly as in the proof of Theorem 3.11. Notice that there must be at least $k+1$ neighbourhoods, else the cardinality of $\{\Omega_{ij} : 1 \leq i \leq m,\, j > 1\}$ will be at least n, and we can apply the proof of Theorem 3.11 to obtain a contradiction. Now we define an index set

$$\mathcal{I} := \{(i,j) : 1 \leq i \leq k,\, j > 1\} \cup \{(i,j) : i \geq k+1,\, j \geq 1\},$$

so that $\{\Omega_{ij} : (i,j) \in \mathcal{I}\}$ omits one nodal domain in each of the first k nodal domain neighbourhoods, but includes all of the nodal domains from remaining neighbourhoods. Thus $\mathcal{I}$ has cardinality n.

Let

$$\mathcal{A} := \left\{ \sum_{\mathcal{I}} \gamma_{ij} u_{ij} : \sum_{\mathcal{I}} |\gamma_{ij}|^p \int_{\Omega_{ij}} |u_{ij}|^p \,\mathrm{d}x = 1 \right\}.$$

As in previous proofs we can show that $\mathcal{A} \in \mathcal{F}_n$ with $I \equiv \lambda_n$ on $\mathcal{A}$. The set $\mathcal{A}$ must contain a critical point of I, else we could derive a contradiction similarly as in the previous proofs. Let $u^*_{\lambda_n} \in \mathcal{A}$ be a critical point of I, i.e. another eigenfunction associated with λ_n. Since $u^*_{\lambda_n} \in \mathcal{A}$, we know that $u^*_{\lambda_n} \equiv 0$ in Ω_{i1} for $1 \leq i \leq k$. As in the proof of Theorem 3.11., it follows that $u^*_{\lambda_n} \equiv 0$ on each of the first k nodal domain neighbourhoods. Notice that the nodal domains for $u^*_{\lambda_n}$ are a subset of the nodal domains in the remaining nodal domain neighbourhoods. By removing the first k nodal domain neighbourhoods we have removed at least $2k$ nodal domains from consideration. Hence there are at most $n-k$ remaining nodal domains where $u^*_{\lambda_n}$ can be nontrivial.

Q.E.D.

The previous theorem yields the following *weaker form* of the Courant nodal domain theorem.

Corollary 3.14. *For each n there is an eigenfunction, $u^*_{\lambda_n}$, associated with λ_n, such that $u^*_{\lambda_n}$ has at most n nodal domains.*

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