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Nash-Moser theory and Hamiltonian PDEs

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LECTURE NOTES on

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Chapter 1

A tutorial in Nash-Moser theory

1.1 Introduction

The classical implicit function theorem is concerned with the solvability of the equation

$$\mathcal{F}(x, y) = 0 \tag{1.1}$$

where

$$\mathcal{F} : X \times Y \to Z$$

is a smooth map, X, Y, Z are Banach spaces, and there exists $(x_0, y_0) \in X \times Y$ such that

$$\mathcal{F}(x_0, y_0) = 0 \, .$$

If x is close to x_0 we want to solve (1.1) finding $y = y(x)$.

The main assumption of the classical implicit function theorem is that the partial derivative $(D_y \mathcal{F})(x_0, y_0) : Y \to Z$ possesses a *bounded* inverse

$$(D_y \mathcal{F})^{-1}(x_0, y_0) \in \mathcal{L}(Z, Y) \, .$$

Note that, if $(D_y \mathcal{F})(x_0, y_0) \in \mathcal{L}(Y, Z)$ is injective and surjective, by the open mapping theorem, the inverse operator $(D_y \mathcal{F})^{-1}(x_0, y_0) : Z \to Y$ is automatically continuous.

There are several situations where

$$(D_y \mathcal{F})(x_0, y_0) \text{ has an } \textit{unbounded} \text{ inverse}$$

(for example the image $(D_y \mathcal{F})(x_0, y_0)[Y]$ is only dense in Z).

An approach to these class of problems has been proposed by Nash in the pioneering paper [26], for proving that any Riemannian manifold can be isometrically embedded in $\mathbf{R}^N$ for N sufficiently large.

Subsequently, Moser [19] has highlighted the main features of the technique in an abstract setting, being able to cover problems arising from Celestial Mechanics and partial differential equations [20]-[21]-[22]. Further extensions and applications were made by Gromov [11], by Zehnder [33] to small divisors problems, by Hörmander [13] to problems in gravitation, by Sergeraert [29] to catastrophe theory, by Schaeffer [27] to free boundary problems in electromagnetics, by Beale [2] in water waves, by Hamilton [12] to foliations, by Klainermann [14] to Cauchy problems, by Craig-Wayne [8] and Bourgain [5]-[6] for periodic and quasi-periodic solutions in Hamiltonian PDEs, to mention just a few, showing the power and versatility of the technique.

The main idea is to replace the usual Picard iteration method with a modified Newton iteration scheme. Roughly speaking, the advantage is that, since this latter scheme is quadratic (see remark 1.2.1 and 1.3.2), the iterates shall converge to the expected solution at a super-exponential rate. This accelerated speed of convergence is sufficiently strong to compensate the divergences in the scheme due to the "loss of derivatives".

There are many ways to present the Nash-Moser theorems, according to the applications one has in mind. We shall prove first a very simple "analytic" Implicit Function Theorem (inspired to Theorem 6.1 by Zehnder in [26], see also [9]) to highlight the main features of the method in an abstract "analytic" setting (i.e. with estimates which can be typically obtained in Banach scales of analytic functions). In the application to the nonlinear wave equation [4], indeed, we shall be able to prove, with a variant of this scheme, existence of analytic (in time) solutions of the nonlinear wave equation for positive measure sets of frequencies.

Next, for completeness, we present also a Nash-Moser theorem in a differentiable setting (i.e. modeled for applications on spaces of functions with finite differentiability like, for example, Banach scales of Sobolev spaces). To avoid technicalities we present it in the form of an inversion type theorem as in Moser [19].

The present material follows the exposition in [3]

1.2 An analytic Nash-Moser Theorem

Consider three one parameter families of Banach spaces

$$X_\sigma\,,\ Y_\sigma\,,\ Z_\sigma\,, \qquad\qquad 0 \le \sigma \le 1$$

with norms $|\cdot|_\sigma$ such that (Banach scales)

$$\forall 0 \le \sigma \le \sigma' \le 1 \qquad |x|_\sigma \le |x|_{\sigma'} \quad \forall x \in X_{\sigma'}$$

(analogously for Y_σ, Z_σ) so that

$$\forall 0 \le \sigma \le \sigma' \le 1 \qquad X_1 \subseteq X_{\sigma'} \subseteq X_\sigma \subseteq X_0$$

(the same for Y_σ, Z_σ).

Example: The Banach spaces of analytic functions

$$X_\sigma := \Big\{ f : \mathbf{T}^d \to \mathbf{R}\,,\ f(\varphi) := \sum_k f_k e^{ik\cdot\varphi} \mid |f|_\sigma := \sum_k |f_k| e^{\sigma|k|} < +\infty \Big\}\,.$$

Let

$$\mathcal{F} : X_0 \times Y_0 \to Z_0$$

be a mapping defined on the largest spaces of the scales.

Suppose there exists $(x_0, y_0) \in X_1 \times Y_1$ (in the smallest spaces) such that

$$\mathcal{F}(x_0, y_0) = 0\,. \tag{1.2}$$

Assume that

$$\mathcal{F}(B_\sigma) \subset Z_\sigma \qquad \forall 0 \le \sigma \le 1 \tag{1.3}$$

where B_σ is the neighborhood of (x_0, y_0)

$$B_\sigma := B_R^\sigma(x_0) \times B_R^\sigma(y_0) \subset X_\sigma \times Y_\sigma$$

and

$$B_R^\sigma(x_0) := \Big\{ x \in X_\sigma |\ |x - x_0|_\sigma < R \Big\}$$

analogously for $B_R^\sigma(y_0) \subset Y_\sigma$.

We shall make the following hypotheses in which K and τ are fixed positive constants.

(H1) (Taylor Estimate) $\forall 0 < \sigma \le 1$, $\forall x \in B_R^\sigma(x_0)$ the map $\mathcal{F}(x, \cdot) : B_R^\sigma(y_0) \to Z_\sigma$ is differentiable and, $\forall (x, y), (x, y') \in B_\sigma$,

$$\Big| \mathcal{F}(x, y') - \mathcal{F}(x, y) - (D_y \mathcal{F})(x, y)[y' - y] \Big|_\sigma \le K |y' - y|_\sigma^2\,.$$

Condition (H1) is clearly satisfied if $\mathcal{F}(x, \cdot) \in C^2(B_R^\sigma(y_0), Z_\sigma)$ and $D_{yy}^2 \mathcal{F}(x, \cdot)$ is uniformly bounded for $x \in B_R^\sigma(x_0)$.

(**H2**) **(Right Inverse of loss τ)** $\forall 0 < \sigma \leq 1$, $\forall (x,y) \in B_\sigma$ there is a linear operator $L(x,y) \in \mathcal{L}(Z_\sigma, Y_{\sigma'})$, $\forall \sigma' < \sigma$, such that $\forall z \in Z_\sigma$

$$(D_y\mathcal{F})(x,y) \circ L(x,y)z = z$$

in $Z_{\sigma'}$ and

$$\Big| L(x,y)[z] \Big|_{\sigma'} \leq \frac{K}{(\sigma - \sigma')^\tau} |z|_\sigma \,. \tag{1.4}$$

The operator $L(x,y)$ is the right inverse of $(D_y\mathcal{F})(x,y)$ in the sense that

$$(D_y\mathcal{F})(x,y) \circ L(x,y) \text{ is the continuous injection } Z_\sigma \xrightarrow{i} Z_{\sigma'}$$

$\forall \sigma' < \sigma$.

Estimate (1.4) is a typical "Cauchy-type" estimate for operators acting somewhat as differential operators of order τ in scales of Banach spaces of analytic functions.

Theorem 1.2.1 *Let $\mathcal{F}$ satisfy (1.2),(1.3), (H1)-(H2). If $x \in B_R^\sigma(x_0)$ for some $\sigma \in (0,1]$ and $|\mathcal{F}(x,y_0)|_\sigma$ is sufficiently small*[1]*, then there exists a solution*

$$y(x) \in B_R^{\sigma/2}(y_0) \subset Y_{\sigma/2}$$

of the equation

$$\mathcal{F}(x, y(x)) = 0 \,.$$

PROOF. We define the Newton iteration scheme

$$\begin{cases} y_{n+1} = y_n - L(x,y_n)\mathcal{F}(x,y_n) \\ y_0 := y_0 \in Y_1 \subseteq Y_\sigma \end{cases} \tag{1.5}$$

for $n \geq 0$. Throughout the induction proof we will verify at each step (see the CLAIM below) that y_n belongs to the domain of $\mathcal{F}(x,\cdot)$, $L(x,\cdot)$ and therefore y_{n+1} is well defined.

Since the inverse operator $L(x,y_n)$ "loses analyticity" (hypothesis (H2)) the iterates y_n will belong to larger and larger spaces Y_{σ_n}.

To quantify this phenomenon, let us define the sequence

$$\sigma_0 := \sigma \in (0,1] \,, \ \sigma_{n+1} := \sigma_n - \delta_n$$

where the "loss of analyticity" at each step of the iteration is

$$\delta_n := \frac{\delta_0}{n^2+1}$$

[1] quantified in (1.7); this latter condition defines a neighborhood of x_0 in X_σ.

and $\delta_0 > 0$ is small enough so that the "total loss of analyticity"

$$\sum_{n\geq 0} \delta_n = \sum_{n\geq 0} \frac{\delta_0}{n^2+1} < \frac{\sigma}{2} \tag{1.6}$$

(therefore $\sigma_n > \sigma/2$, $\forall n \geq 0$).

We claim the following:

CLAIM: *Take* $\chi := 3/2$ *and define*[2]

$$\rho := \rho(K, R, \tau, \sigma) := \min\Big\{ \frac{\sqrt{e}}{K}, \min_{n\geq 0} \Big(\frac{\delta_{n+1}^{\tau}}{K^2} e^{(2-\chi)\chi^n} \Big), \frac{R/2}{\sum_{k=0}^{\infty} e^{-\chi^k}} \Big\} > 0 \,.$$

If

$$|\mathcal{F}(x, y_0)|_\sigma < \min\Big\{ \rho e^{-1}, \frac{\delta_0^\tau}{K} \rho e^{-1}, \frac{\delta_0^\tau}{K} \frac{R}{2} \Big\}, \tag{1.7}$$

then the following statements hold true for all $n \geq 0$:

$(n;1)$ $(x, y_n) \in B_{\sigma_n}$ *and* $|\mathcal{F}(x, y_n)|_{\sigma_n} \leq \rho e^{-\chi^n}$,

$(n;2)$ $|y_{n+1} - y_n|_{\sigma_{n+1}} \leq \rho e^{-\chi^n}$,

$(n;3)$ $|y_{n+1} - y_0|_{\sigma_{n+1}} < R/2$.

Before proving the Claim, let us conclude the proof of Theorem 1.2.1.

By $(n;2)$ the sequence $y_n \in Y_{\sigma/2}$ is a Cauchy sequence (in the largest space $Y_{\sigma/2}$). Indeed, for any $n > m$

$$\begin{aligned} |y_n - y_m|_{\sigma/2} &\leq \sum_{k=m}^{n-1} |y_{k+1} - y_k|_{\sigma/2} \leq \sum_{k=m}^{n-1} |y_{k+1} - y_k|_{\sigma_{k+1}} \\ &\stackrel{(k;2)}{\leq} \sum_{k=m}^{n-1} \rho e^{-\chi^k} \to 0 \ \text{ for } \ n, m \to +\infty \,. \end{aligned}$$

Hence y_n converges in $Y_{\sigma/2}$ to some $y(x) \in Y_{\sigma/2}$. Actually $y(x) \in \overline{B_{R/2}^{\sigma/2}(y_0)} \subset B_R^{\sigma/2}$ by $(n;3)$. Finally, by the continuity of $\mathcal{F}$ with respect to the second variable and $(n;1)$

$$\mathcal{F}(x, y(x)) = \lim_{n\to\infty} \mathcal{F}(x, y_n) = 0$$

[2] We have $\rho > 0$ because the sequence of positive numbers

$$\delta_{n+1}^\tau e^{(2-\chi)\chi^n} = \delta_0^\tau \frac{e^{\frac{1}{2}(3/2)^n}}{(1+(n+1)^2)^\tau} \to +\infty \quad \text{as} \quad n \to +\infty \,.$$

implying that $y(x)$ is a solution of $\mathcal{F}(x, y) = 0$.

Let's now prove the Claim. Its proof proceeds by induction. First, let us verify it for $n = 0$. It reduces to the smallness condition (1.7) for $|\mathcal{F}(x, y_0)|_\sigma$.

(0; 1) By assumption $x \in B_R^\sigma(x_0)$ so that $(x, y_0) \in B_{\sigma_0} := B_\sigma$. By (1.3) we have that $\mathcal{F}(x, y_0) \in Z_\sigma$ and $|\mathcal{F}(x, y_0)|_\sigma \leq \rho e^{-1}$ follows by (1.7).

(0; 2)-(0; 3) Since $(x, y_0) \in B_\sigma$, by (1.5) and $(H2)$,

$$|y_1 - y_0|_{\sigma_1} = |L(x, y_0)\mathcal{F}(x, y_0)|_{\sigma_1} \leq \frac{K}{(\sigma_0 - \sigma_1)^\tau} |\mathcal{F}(x, y_0)|_\sigma \, .$$

Under the smallness condition (1.7) we have verified both (0; 2)-(0; 3).

Now, suppose $(n; 1)$-$(n; 2)$-$(n; 3)$ are true. By $(n; 3)$,

$$y_{n+1} \in B_R^{\sigma_{n+1}}(y_0)$$

and so

$$(x, y_{n+1}) \in B_{\sigma_{n+1}} \, .$$

Hence $\mathcal{F}(x, y_{n+1}) \in Z_{\sigma_{n+1}}$ (by (1.3)) and, by (H2),

$$y_{n+2} := y_{n+1} - L(x, y_{n+1})\mathcal{F}(x, y_{n+1}) \in Y_{\sigma_{n+2}}$$

is well defined.

Set for brevity

$$Q(y, y') := \mathcal{F}(x, y') - \mathcal{F}(x, y) - (D_y\mathcal{F})(x, y)[y' - y] \, . \tag{1.8}$$

By a Taylor expansion

$$\begin{aligned} |\mathcal{F}(x, y_{n+1})|_{\sigma_{n+1}} &= \Big|\mathcal{F}(x, y_n) + (D_y\mathcal{F})(x, y_n)[y_{n+1} - y_n] + Q(y_n, y_{n+1})\Big|_{\sigma_{n+1}} \\ &\overset{(1.5)}{=} |Q(y_n, y_{n+1})|_{\sigma_{n+1}} \overset{(H1)}{\leq} K|y_{n+1} - y_n|^2_{\sigma_{n+1}} \qquad (1.9) \\ &\overset{(n;2)}{\leq} K\rho^2 e^{-2\chi^n} \, . \qquad (1.10) \end{aligned}$$

By (1.10) the claim $(n + 1; 1)$ is verified whenever

$$K\rho^2 e^{-2\chi^n} < \rho e^{-\chi^{n+1}}$$

which holds true for any $n \geq 0$ if

$$\rho < \min_{n \geq 0} \Big(\frac{1}{K} e^{(2-\chi)\chi^n}\Big) = \frac{\sqrt{e}}{K} \, . \tag{1.11}$$

Now

$$
\begin{aligned}
|y_{n+2} - y_{n+1}|_{\sigma_{n+2}} &\overset{(1.5)}{=} |L(x, y_{n+1})\mathcal{F}(x, y_{n+1})|_{\sigma_{n+2}} \\
&\overset{(H2)}{\leq} \frac{K}{(\sigma_{n+1} - \sigma_{n+2})^{\tau}} |\mathcal{F}(x, y_{n+1})|_{\sigma_{n+1}} \\
&\overset{(1.9)}{\leq} \frac{K^2}{(\sigma_{n+1} - \sigma_{n+2})^{\tau}} |y_{n+1} - y_n|^2_{\sigma_{n+1}} \\
&\overset{(n;2)}{\leq} \frac{K^2}{(\sigma_{n+1} - \sigma_{n+2})^{\tau}} \rho^2 e^{-2\chi^n}
\end{aligned}
\tag{1.12}
$$

and therefore the claim $(n+1; 2)$ is verified whenever

$$
\frac{K^2}{(\sigma_{n+1} - \sigma_{n+2})^{\tau}} \rho^2 e^{-2\chi^n} < \rho e^{-\chi^{n+1}}
$$

which holds true, for any $n \geq 0$, if

$$
\rho < \min_{n \geq 0} \Big(\frac{\delta_{n+1}^{\tau}}{K^2} e^{(2-\chi)\chi^n} \Big) . \tag{1.13}
$$

Finally

$$
\begin{aligned}
|y_{n+2} - y_0|_{\sigma_{n+2}} &\leq \sum_{k=0}^{n+1} |y_{k+1} - y_k|_{\sigma_{n+2}} \leq \sum_{k=0}^{n+1} |y_{k+1} - y_k|_{\sigma_{k+1}} \\
&\overset{(k;2)}{\leq} \sum_{k=0}^{n+1} \rho e^{-\chi^k} < \rho \sum_{k=0}^{\infty} e^{-\chi^k}
\end{aligned}
$$

which implies $(n+1; 3)$ assuming

$$
\rho < \frac{R/2}{\sum_{k=0}^{\infty} e^{-\chi^k}} . \tag{1.14}
$$

In conclusion, if $\rho > 0$ is small enough (depending on K, τ, R, σ) according to (1.11)-(1.13)-(1.14) the claim is proved.

This completes the proof. ■

Remark 1.2.1 *The key point of the Nash-Moser scheme is the estimate*

$$
|y_{n+2} - y_{n+1}|_{\sigma_{n+2}} \leq \frac{K^2}{\delta_{n+1}^{\tau}} |y_{n+1} - y_n|^2_{\sigma_{n+1}} \tag{1.15}
$$

see (1.12). Even though $\delta_n \to 0$, this quadratic estimate ensures that the sequence of numbers $|y_{n+1} - y_n|_{\sigma_{n+1}}$ tends to zero at a super-exponential rate *(see (n; 2)) if $|y_1 - y_0|_{\sigma_1}$ is sufficiently small. Note that the Picard iteration scheme would yield just $|y_{n+2} - y_{n+1}|_{\sigma_{n+2}} \le C\delta_{n+1}^{-\tau}|y_{n+1} - y_n|_{\sigma_{n+1}}$, i.e. the divergence of the estimates.*

Clearly, the drawback to get (1.15) is to invert the linearized operators in a whole neighborhood of (x_0, y_0), see (H2). This is the most difficult step to apply the Nash-Moser method in concrete situations, see e.g. [4].

Remark 1.2.2 *The hypotheses in Theorem 1.2.1 could be considerably weakened, see [26]. For example in (H1) one could assume a loss of analyticity*[3] *also in the quadratic part of the Taylor expansion*

$$\Big|\mathcal{F}(x, y') - \mathcal{F}(x, y) - (D_y\mathcal{F})(x, y)[y' - y]\Big|_{\sigma'} \le \frac{K}{(\sigma - \sigma')^{\alpha}}|y' - y|_{\sigma}^2$$

$\forall \sigma' < \sigma$ and some $\alpha > 0$ (independent of σ).

Furthermore one could assume the existence of just an "approximate right inverse", namely $\forall z \in Z_\sigma$

$$\Big|\Big((D_y\mathcal{F})(x, y) \circ L(x, y) - I)[z]\Big)\Big|_{\sigma'} \le \frac{K}{(\sigma - \sigma')^{\tau}}|\mathcal{F}(x, y)|_{\sigma}|z|_{\sigma} \qquad (1.16)$$

(remark that $L(x, y)$ is an exact inverse at the solutions $\mathcal{F}(x, y) = 0$).

Furthermore in the statement of Theorem 1.2.1 it is possible to get better and quantitative estimates.

Since we have not assumed the existence of the left inverse of $(D_y\mathcal{F})(x, y)$ in the assumptions of Theorem 1.2.1, uniqueness of the solution $y(x)$ can not be expected (it could lack also in the linear problem).

Local uniqueness follows assuming the existence of a left inverse:

(H2)$'$ $\forall 0 < \sigma \le 1$, $\forall (x, y) \in B_\sigma$ there is a linear operator $\xi(x, y) \in \mathcal{L}(Z_\sigma, Y_{\sigma'})$, $\forall \sigma' < \sigma$, such that, $\forall h \in Y_\sigma$

$$\xi(x, y) \circ (D_y\mathcal{F})(x, y)[h] = h$$

in $Y_{\sigma'}$ and $\forall z \in Z_\sigma$

$$\Big|\xi(x, y)[z]\Big|_{\sigma'} \le \frac{K}{(\sigma - \sigma')^{\tau}}|z|_{\sigma}\,. \qquad (1.17)$$

The operator $\xi(x, y)$ is the left inverse of $(D_y\mathcal{F})(x, y)$ in the sense that $\xi(x, y) \circ (D_y\mathcal{F})(x, y)$ is the continuous injection $Y_\sigma \xrightarrow{i} Y_{\sigma'}$, $\forall \sigma' < \sigma$.

---

[3]In the application considered in [4] the quadratic part Q satisfies (H1), i.e. it does not lose regularity.

Theorem 1.2.2 (Uniqueness) *Let $\mathcal{F}$ satisfy (1.3),* **(H1)**-**(H2)**$'$. *Let (x,y), $(x,y') \in B_\sigma$ be solutions of $\mathcal{F}(x,y)=0$, $\mathcal{F}(x,y')=0$. If $|y-y'|_\sigma$ is small enough (depending on K, τ, σ) then $y=y'$ in $Y_{\sigma/2}$.*

PROOF. Setting $h := y - y' \in Y_\sigma$ we have

$$
\begin{aligned}
|h|_{\sigma'} &\overset{(H2)'}{=} |\xi(x,y)\circ(D_y\mathcal{F})(x,y)[h]|_{\sigma'} \\
&\overset{(1.17)}{\le} \frac{K}{(\sigma-\sigma')^\tau}|(D_y\mathcal{F})(x,y)[h]|_\sigma \\
&\le \frac{K}{(\sigma-\sigma')^\tau}|Q(y,y')|_\sigma
\end{aligned}
\tag{1.18}
$$

since $\mathcal{F}(x,y)=0$, $\mathcal{F}(x,y')=0$ and recalling the definition of $Q(y,y')$ in (1.8).

By (1.18) and **(H1)** we get

$$
|h|_{\sigma'} \overset{(H1)}{\le} \frac{K^2}{(\sigma-\sigma')^\tau}|y'-y|_\sigma^2 = \frac{K^2}{(\sigma-\sigma')^\tau}|h|_\sigma^2\,, \quad \forall \sigma' < \sigma
$$

whence, for $\sigma' := \sigma_{n+1}$, $\sigma := \sigma_n$, $\delta_n := \sigma_n - \sigma_{n+1}$,

$$
|h|_{\sigma_{n+1}} \le K^2 \delta_n^{-\tau} |h|_{\sigma_n}^2\,, \quad \forall n \ge 0\,.
$$

These last estimates imply that if $|h|_\sigma = |y-y'|_\sigma$ ($\sigma = \sigma_0$) is sufficiently small (depending on K, τ, σ) then $h = y - y' = 0$ in $Y_{\sigma/2}$. ∎

1.3 A differentiable Nash-Moser Theorem

The iterative scheme (1.5) can not work to prove a Nash-Moser implicit function theorem in spaces, say, of class C^k, because, due to the loss of derivatives of the inverse linearized operators, after a fixed number of iterations all derivatives will be exhausted. The scheme has to be modified applying a sequence of "smoothing" operators which regularize $y_{n+1} - y_n$ at each step.

To avoid technicalities, we present the ideas of the Nash-Moser differentiable theory in the form of an inversion type theorem (as in Moser [19]) rather than an Implicit function type theorem.

To make it precise, consider a Banach scale $(Y_s)_{s\ge 0}$ satisfying

$$
Y_{s'} \subset Y_s \subset Y_0\,, \quad \forall s' \ge s \ge 0
$$

equipped with a family of "smoothing" linear operators

$$S(t) : Y_0 \to Y_\infty := \bigcap_{s \geq 0} Y_s \,, \qquad t \geq 0$$

such that

$$|S(t)u|_{s+r} \leq C_{s,r}\, t^r |u|_s \,, \qquad \forall u \in Y_s \tag{1.19}$$

$$|(I - S(t))u|_s \leq C_{s,r}\, t^{-r} |u|_{s+r} \,, \qquad \forall u \in Y_{s+r} \,, \tag{1.20}$$

for some positive constants $C_{s,r}$. For the construction of these smoothing operators for concrete Banach scales see for example Schwartz [28] or Zhender in [26].

Remark 1.3.1 *Estimates (1.19)-(1.20) are the usual ones in the Sobolev scale*

$$Y_s := \Big\{ f(\varphi) := \sum_k f_k e^{ik\cdot\varphi} \mid |f|_s^2 := \sum_k |f_k|^2 (1 + |k|^{2s}) < +\infty \Big\}$$

for the projector S_N on the first N Fourier-modes

$$S_N\Big(\sum_k f_k e^{ik\cdot x} \Big) := \sum_{|k| \leq N} f_k e^{ik\cdot x}$$

(when $t := N$ is an integer).

EXERCISE: *On a scale $(X_s)_{s\geq 0}$ equipped with smoothing operators $(S(t))_{t\geq 0}$, the following convexity inequality holds: for all $0 \leq \lambda_1 \leq \lambda_2$, $\alpha \in [0,1]$ and $u \in X_{\lambda_2}$:*

$$|u|_\lambda \leq K_{\lambda_1,\lambda_2} |u|_{\lambda_1}^{1-\alpha} |u|_{\lambda_2}^{\alpha} \,, \quad \lambda = (1-\alpha)\lambda_1 + \alpha\lambda_2 \,. \tag{1.21}$$

This implies the well known Gagliardo-Nirenberg-Moser interpolation estimates in Sobolev spaces, see [30] for a modern account.

We make the following assumptions where α, K, τ are fixed positive constants.

(H1) (Tame estimate) $\mathcal{F} : Y_{s+\alpha} \to Y_s$, $\forall s \geq 0$, satisfies[4]

$$|\mathcal{F}(y)|_s \leq K(1 + |y|_{s+\alpha}) \,, \quad \forall y \in Y_{s+\alpha} \,.$$

[4]Differential operators $\mathcal{F}$ of order α satisfy the "tame" property **(H1)**, i.e. $|\mathcal{F}(y)|_s$ grows at most linearly with the higher norm $|\cdot|_{s+\alpha}$. This apparently surprising fact follows by the interpolation inequalities (1.21), see [26], [12].

(**H2**) **(Taylor estimate)** $\mathcal{F} : Y_{s+\alpha} \to Y_s$, $\forall s \geq 0$, is differentiable and

$$\begin{cases} |(D\mathcal{F})(y)[h]|_s \leq K|h|_{s+\alpha}\,, \\ \Big|\mathcal{F}(y') - \mathcal{F}(y) - (D\mathcal{F})(y)[y'-y]\Big|_s \leq K|y'-y|^2_{s+\alpha}\,. \end{cases}$$

(**H3**) **(Inverse of loss τ)** $\forall y \in Y_\infty$ there is a linear operator $L(y) \in \mathcal{L}(Y_{s+\tau}, Y_s)$, $\forall s \geq 0$, i.e.

$$|L(y)[h]|_s \leq K|h|_{s+\tau}\,, \quad \forall h \in Y_{s+\tau}\,,$$

such that

$$D\mathcal{F}(y) \circ L(y)[h] = h\,.$$

Hypothesys **(H1)-(H2)-(H3)** state, roughly, that $\mathcal{F}$, $D\mathcal{F}$, respectively L, act somewhat as differential operators of order α, respectively τ.

Theorem 1.3.1 *Let $\mathcal{F}$ satisfy (H1)-(H2)-(H3) and fix any $s_0 > \alpha + \tau$. If $|\mathcal{F}(0)|_{s_0+\tau}$ is sufficiently small (depending on α, τ, K, s_0) then there exists a solution $y \in Y_{s_0}$ of the equation $\mathcal{F}(y) = 0$.*

Proof. Consider the iterative scheme

$$\begin{cases} y_{n+1} = y_n - S(N_n)L(y_n)\mathcal{F}(y_n) \\ y_0 := 0 \end{cases} \tag{1.22}$$

where

$$N_n := e^{\lambda\chi^n}\,, \quad N_{n+1} = N_n^\chi\,, \quad \chi := 3/2$$

for some λ large enough, depending on α, τ, K, s_0, to be chosen later.

By (1.22), the increment $y_{n+1} - y_n \in Y_\infty$, $\forall n \geq 0$, and, therefore, $y_n \in Y_\infty$, $\forall n \geq 0$ (because $y_0 := 0 \in Y_\infty$). Furthermore

$$\begin{aligned} |y_{n+1} - y_n|_{s_0} &\stackrel{(1.22)}{=} |S(N_n)L(y_n)\mathcal{F}(y_n)|_{s_0} \stackrel{(1.19)}{\leq} C_0 N_n^{\alpha+\tau}|L(y_n)\mathcal{F}(y_n)|_{s_0-\alpha-\tau} \\ &\stackrel{(H3)}{\leq} C_0 N_n^{\alpha+\tau} K|\mathcal{F}(y_n)|_{s_0-\alpha} \end{aligned} \tag{1.23}$$

where $C_0 := C_{s_0-\alpha-\tau,\alpha+\tau}$ is the constant from (1.19).

By a Taylor expansion, for $n \geq 1$, setting for brevity $Q(y; y') := \mathcal{F}(y') - \mathcal{F}(y) - D\mathcal{F}(y)[y'-y]$,

$$\begin{aligned} |\mathcal{F}(y_n)|_{s_0-\alpha} &\leq |\mathcal{F}(y_{n-1}) + D\mathcal{F}(y_{n-1})[y_n - y_{n-1}]|_{s_0-\alpha} + |Q(y_{n-1}, y_n)|_{s_0-\alpha} \\ &\stackrel{(1.22)}{=} |D\mathcal{F}(y_{n-1})(I - S(N_{n-1}))L(y_{n-1})\mathcal{F}(y_{n-1})|_{s_0-\alpha} \\ &+ |Q(y_{n-1}, y_n)|_{s_0-\alpha} \\ &\stackrel{(H2)}{\leq} K|(I - S(N_{n-1}))L(y_{n-1})\mathcal{F}(y_{n-1})|_{s_0} + K|y_n - y_{n-1}|^2_{s_0} \\ &\stackrel{(1.20)}{\leq} KC_{s_0,\beta}N_{n-1}^{-\beta}B_{n-1} + K|y_n - y_{n-1}|^2_{s_0} \end{aligned} \tag{1.24}$$

where $B_{n-1} := |L(y_{n-1})\mathcal{F}(y_{n-1})|_{s_0+\beta}$.

By (1.23) and (1.24) we deduce

$$|y_{n+1} - y_n|_{s_0} \le C_1 N_n^{\alpha+\tau} N_{n-1}^{-\beta} B_{n-1} + C_1 N_n^{\alpha+\tau} |y_n - y_{n-1}|_{s_0}^2 \tag{1.25}$$

for some positive $C_1 := C(\alpha, \tau, s_0, K)$.

To prove, by (1.25), the super-exponential smallness of $|y_{n+1} - y_n|_{s_0}$, the main issue is to give an *a-priori* estimate for the divergence of the B_n independent of β.

For $n \ge 0$ we have

$$B_n \;:=\; |L(y_n)\mathcal{F}(y_n)|_{s_0+\beta} \overset{(H3)}{\le} K|\mathcal{F}(y_n)|_{s_0+\beta+\tau} \tag{1.26}$$

and, for $n \ge 1$, writing $y_n = \sum_{k=1}^n (y_k - y_{k-1})$,

$$\begin{aligned} B_n &\overset{(H1)}{\le} K^2(1 + |y_n|_{s_0+\beta+\tau+\alpha}) \le K^2\Big(1 + \sum_{k=1}^n |y_k - y_{k-1}|_{s_0+\beta+\tau+\alpha}\Big) \\ &\overset{(1.22)}{=} K^2\Big(1 + \sum_{k=1}^n |S(N_{k-1})L(y_{k-1})\mathcal{F}(y_{k-1})|_{s_0+\beta+\tau+\alpha}\Big) \\ &\overset{(1.19)}{\le} K^2\Big(1 + \sum_{k=1}^n C_2 N_{k-1}^{\tau+\alpha} |L(y_{k-1})\mathcal{F}(y_{k-1})|_{s_0+\beta}\Big) \\ &\le C_3\Big(1 + \sum_{k=0}^{n-1} N_k^{\tau+\alpha} B_k\Big)\,. \end{aligned} \tag{1.27}$$

where $C_2 := C_{s_0+\beta, r+\alpha}$ is the constant from (1.19) and $C_3 := K^2 \max\{1, C_2\}$.

We claim the following:

Claim: *Take $\beta := 15(\alpha+\tau)$ and suppose*

$$|\mathcal{F}(0)|_{s_0+\tau} < e^{-\lambda 4(\alpha+\tau)}/KC_{s_0,0}\,. \tag{1.28}$$

There is $\lambda := \lambda(\tau, \alpha, K, s_0) \ge 1$, such that the following statements hold true for all $n \ge 0$:

- *(n;1)* $B_n \le N_n^{\nu} = e^{\lambda \chi^n \nu}$, $\quad \nu := 4(\tau+\alpha)$,
- *(n;2)* $|y_{n+1} - y_n|_{s_0} \le N_n^{-\nu} = e^{-\lambda\chi^n \nu}$.

Statement $(0;1)$ is verified by

$$B_0 := |L(0)\mathcal{F}(0)|_{s_0+\beta} \overset{(H3)}{\le} K|\mathcal{F}(0)|_{s_0+\beta+\tau} \le e^{\lambda\nu}$$

which holds true for $\lambda := \lambda(s_0, \alpha, \tau, K)$ large enough.

Statement $(0;2)$ follows by

$$\begin{aligned} |y_1 - y_0|_{s_0} &\overset{(1.22)}{=} |S(N_0)L(0)\mathcal{F}(0)|_{s_0} \overset{(1.19)}{\leq} C_{s_0,0}|L(0)\mathcal{F}(0)|_{s_0} \\ &\overset{(H3)}{\leq} C_{s_0,0}K|\mathcal{F}(0)|_{s_0+\tau} \overset{(1.28)}{<} e^{-\lambda\nu}. \end{aligned}$$

Now suppose $(n;1)$-$(n;2)$ are true. To prove $(n+1;1)$ write

$$\begin{aligned} B_{n+1} &\overset{(1.27)}{\leq} C_3\Big(1 + \sum_{k=0}^{n} N_k^{\tau+\alpha} B_k\Big) \overset{(n;1)}{\leq} C_3\Big(1 + \sum_{k=0}^{n} e^{(\tau+\alpha+\nu)\lambda\chi^k}\Big) \\ &= C_3\Big(1 + e^{(\tau+\alpha+\nu)\lambda\chi^n} \sum_{k=0}^{n} e^{-(\tau+\alpha+\nu)\lambda(\chi^n-\chi^k)}\Big) \\ &\leq C_3\Big(1 + e^{(\tau+\alpha+\nu)\lambda\chi^n} \sum_{k=0}^{n} e^{-4(\tau+\alpha)(\chi^n-\chi^k)}\Big) \\ &\leq C_4 e^{(\tau+\alpha+\nu)\lambda\chi^n} < e^{\nu\lambda\chi^{n+1}} \end{aligned}$$

for some $C_4 := C_4(\alpha, \tau, K, s_0) > 0$ and $\lambda := \lambda(\alpha, \tau, K, s_0) \geq 1$ sufficiently large (because $\nu(\chi - 1) > \tau + \alpha$).

Remark 1.3.2 *The main novelty w.r.t to the analytic scheme -compare (1.25) with (1.15)- is to prove that the term $N_n^{\alpha+\tau} N_{n-1}^{-\beta} B_{n-1}$ in (1.25) is super-exponentially small. This follows, for β large, by (n;1), implying that $|y_{n+1} - y_n|_{s_0}$ still converges to zero at a super-exponential rate if $|y_1 - y_0|_{s_0}$ is sufficiently small, statement $(n;2)$.*

Let us prove $(n+1;2)$. Recalling that $N_n := e^{\lambda\chi^n}$ we have

$$\begin{aligned} |y_{n+2} - y_{n+1}|_{s_0} &\overset{(1.25)}{\leq} C_1 e^{\lambda(\alpha+\tau)\chi^{n+1}} e^{-\lambda\beta\chi^n} B_n + C_1 e^{\lambda(\alpha+\tau)\chi^{n+1}} |y_{n+1} - y_n|_{s_0}^2 \\ &\overset{(n;1),(n;2)}{\leq} C_1 e^{\lambda(\alpha+\tau)\chi^{n+1}} e^{-\lambda\beta\chi^n} e^{\lambda\nu\chi^n} + C_1 e^{\lambda(\alpha+\tau)\chi^{n+1}} e^{-2\nu\lambda\chi^n} \\ &\leq e^{-\lambda\chi^{n+1}\nu} \end{aligned}$$

once we impose

$$C_1 e^{\lambda\chi^n(\chi(\alpha+\tau)-\beta+\nu)} < \frac{e^{-\lambda\chi^{n+1}\nu}}{2}, \quad C_1 e^{\lambda\chi^n(\chi(\alpha+\tau)-2\nu)} < \frac{e^{-\lambda\chi^{n+1}\nu}}{2}.$$

These inequalities are satisfied, for λ large enough depending on α, τ, K, s_0, because

$$\beta - \nu(1 + \chi^2) - \chi(\alpha + \tau) > 0 \quad \text{and} \quad (2 - \chi)\nu - \chi(\alpha + \tau) > 0$$

for $\beta := 15(\alpha+\tau)$, $\nu := 4(\alpha+\tau)$, $\chi = 3/2$.

This concludes the proof of the Claim.

By $(n;2)$ the sequence y_n is a Cauchy sequence in Y_{s_0} and therefore $y_n \to y \in Y_{s_0}$. By (1.24), $(n;1)$-$(n;2)$, $|\mathcal{F}(y_n)|_{s_0-\alpha} \to 0$ and therefore $\mathcal{F}(y) = 0$. ∎

Remark 1.3.3 *Clearly much weaker conditions could be assumed. First of all conditions (H1)-(H2)-(H3) need to hold just on a neighborhood of* $y_0 = 0$. *Next, we could allow the constant* $K := K(|\ |_{s_0})$ *to depend on the weaker norm* $|\cdot|_{s_0}$. *The inverse could be substitute by an approximate right inverse as in (1.16).*

Chapter 2

Hamiltonian PDEs

We want to show how to extend the local bifurcation theory of periodic solutions close to elliptic equilibria (nonlinear normal modes) developed for finite dimensional dynamical systems by Lyapunov [18], Fadell-Rabinowitz [10], and Weinstein [32]-Moser [23] (see [3]-[24]), to infinite dimensional Hamiltonian PDEs (free vibrations). This requires the use of a Nash-Moser type implicit function theorem to solve the range equation after a Lyapunov-Schmidt decomposition usual in bifurcation theory.

As other applications of the Nash-Moser techniques to the problem of forced vibrations we refer to [1].

2.1 Introduction

Let consider the autonomous nonlinear wave equation

$$\begin{cases} u_{tt} - u_{xx} + a_1(x)u = a_2(x)u^2 + a_3(x)u^3 + \dots \\ u(t,0) = u(t,\pi) = 0 \end{cases} \tag{2.1}$$

which possesses the equilibrium solution $u \equiv 0$.

We pose the following

- Question: there exist periodic solutions of (2.1) close to $u = 0$?

The first step is to study the linearized equation

$$\begin{cases} u_{tt} - u_{xx} + a_1(x)u = 0 \\ u(t,0) = u(t,\pi) = 0\,. \end{cases} \tag{2.2}$$

The Sturm-Liouville operator $-\partial_{xx} + a_1(x)$ possesses a basis $\{\varphi_j\}_{j\geq 1}$ of eigenvectors with real eigenvalues λ_j

$$(-\partial_{xx} + a_1(x))\varphi_j = \lambda_j \varphi_j\,, \qquad \lambda_j \to +\infty\,. \tag{2.3}$$

The φ_j are orthonormal with respect to the L^2 scalar product.

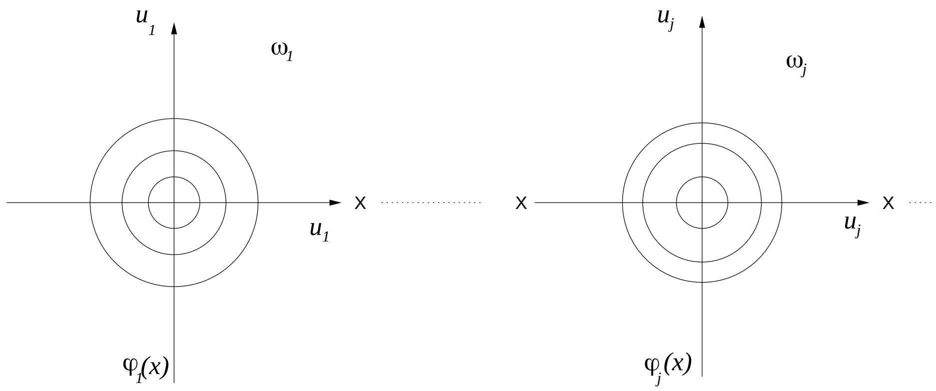

Figure 2.1: The basis of eigenvectors

In this basis equation (2.2) reduces to infinitely many decoupled linear oscillators: $u(t,x) = \sum_j u_j(t)\varphi_j(x)$ is a solution of (2.2) iff

$$\ddot{u}_j + \lambda_j u_j = 0 \qquad j = 1, 2, \ldots . \tag{2.4}$$

If $-\partial_{xx} + a_1(x)$ is positive definite, all its eigenvalues $\lambda_j > 0$ are positive[1] and $u = 0$ looks like an "infinite dimensional elliptic equilibrium" for (2.2) with linear frequencies of oscillations

$$\omega_j := \sqrt{\lambda_j}\,,$$

see figure 2.1. The quadratic Hamiltonian which generates (2.2),

$$H_2(u,p) = \int_0^\pi \frac{p^2}{2} + \frac{u_x^2}{2} + a_1(x)\frac{u^2}{2}\,dx\,,$$

where $p := u_t$, is positive definite and, in coordinates, writes

$$H_2 = \sum_{j\ge 1} \frac{p_j^2 + \lambda_j u_j^2}{2}$$

where $p_j := \dot{u}_j \in l^2$ (Plancharel Theorem).

The general solution of (2.2) is therefore given by the linear superposition of infinitely many oscillations of amplitude a_j, frequency ω_j and phase θ_j on the normal modes φ_j:

$$u(t,x) = \sum_{j\ge 1} a_j \cos(\omega_j t + \theta_j)\varphi_j(x)\,.$$

[1] If $\lambda_j < 0$ (there are at most finitely many negative eigenvalues) then the corresponding linear equation (2.4) describes an harmonic repulsor (hyperbolic directions).

Hence all solutions of (2.2) are either *periodic* in time, either *quasi-periodic*, either *almost-periodic*.

A solution u is periodic when each of the frequencies ω_j for which the amplitude a_j is nonzero (active frequencies) is an integer multiple of a basic frequency ω_0:

$$\omega_j = l_j \omega_0 \,, \qquad l_j \in \mathbf{Z} \,.$$

In this case u is $2\pi/\omega_0$ periodic in time.

The solution u is quasi-periodic with a m-dimensional frequency base if there is a m-dimensional frequency vector $\omega_0 \in \mathbf{R}^m$ with rationally independent components (i.e. $\omega_0 \cdot k \neq 0$, $\forall k \in \mathbf{Z}^m \setminus \{0\}$) such that the active frequencies satisfy

$$\omega_j := l_j \cdot \omega_0 \,, \qquad l_j \in \mathbf{Z}^m \,.$$

A solution is called almost periodic otherwise, namely if there is not a finite number of base frequencies.

It is a natural question to ask whether some of these periodic, quasi-periodic, or almost periodic solutions of the linear equation (2.2) persists in the non-linear equation (2.1).

2.2 Outiline of results

The first existence results were obtained by Kuksin [15] and Wayne [31] extending KAM theory, and by Craig-Wayne [8] via a Lyapunov-Schmidt reduction and Nash-Moser theory.

We start describing the Craig-Wayne result [8] which is an extension of the Lyapunov Center Theorem to the nonlinear wave equation (2.1). The main difficulty to overcome is the appearance of a (i) *"small divisors" problem* (which in finite dimension arises only for the search of quasi-periodic solutions).

To explain how it arises, we recall the key non-resonance hypothesys in the Lyapunov Center Theorem (see e.g. [24])

$$\omega_j - l\omega_1 \neq 0 \,, \quad \forall l \in \mathbf{Z} \,, \quad \forall j = 2, \ldots, n \,.$$

Hence, in finite dimension, for *any* ω sufficiently close to ω_1, the same condition $\omega_j - l\omega \neq 0$, $\forall l \in \mathbf{Z}$, $\forall j = 2, \ldots, n$, holds and the standard implicit function theorem can be applied.

In contrast, the eigenvalues of the Sturm-Liouville problem (2.3) grow polynomially[2] like $\lambda_j \approx j^2 + O(1)$ for $j \to +\infty$ (as it is seen by lower and

[2]For example the eigenvalues of $-\partial_{xx} + m$ are $\lambda_j = j^2 + m$ with eigenvectors $\sin(jx)$.

upper comparison with the operator with constant coefficients), and therefore $\omega_j = j + o(1)$. As a consequence, in infinite dimensions, the set

$$\{\omega_j - l\omega_1\,, \quad \forall l \in \mathbf{Z}\,,\ j = 2, 3, \dots\}$$

accumulates to zero and the non-resonance condition

$$\omega_j - l\omega_1 \neq 0\,,\ \forall l \in \mathbf{Z}\,,\ j = 2, 3, \dots \tag{2.5}$$

is not sufficient to apply the standard implicit function theorem.

This is the "small divisors" problem (this name is due by the fact that such quantities appears as denominators).

Nevertheless, replacing (2.5) with some stronger condition, persistence of a large Cantor like set of small amplitude periodic solutions of (2.1) can be ensured using a Nash-Moser iteration scheme.

Theorem 2.2.1 (Craig-Wayne [8]) *Let*

$$f(x, u) := a_1(x)u - a_2(x)u^2 - a_3(x)u^3 + \dots$$

be a function analytic in the region $\{(x, u) \mid |\mathrm{Im}\, x| < \sigma\,, |u| < 1\}$ *and odd* $f(-x, -u) = -f(x, u)$. *Among this class of nonlinearities there is an open dense set* $\mathcal{F}$ *(in* C^0*-topology) such that,* $\forall f \in \mathcal{F}$*, there exist a Cantor like set* $\mathcal{C} \subset [0, r_*)$ *of positive measure and a* C^∞ *function* $\Omega(r)$ *with* $\Omega(0) = \omega_1$ *such that* $\forall r \in \mathcal{C}$*, there exists a periodic solution* $u(t, x; r)$ *of (2.1) with frequency* $\Omega(r)$. *These solutions are analytic in* (x, t) *and satisfy*

$$|u(t, x; r) - r\cos(\Omega(r)t)\varphi_1(x)| \le Cr^2\,, \qquad |\Omega(r) - \omega_1| < Cr^2\,.$$

The Lyapunov solutions $u(t, x; r)$ are parametrized with the amplitude r, but also the corresponding set of frequencies $\Omega(r)$, $r \in \mathcal{C}$, has positive measure.

The conditions on the terms $a_1(x)$, $a_2(x)$, $a_3(x)$, etc. are, roughly, the followings: first a condition on $a_1(x)$ to avoid primary resonances on the linear frequencies ω_j (which depend on a_1), see the non-resonance condition (2.5); next a condition of genuine nonlinearity placed upon $a_2(x)$, $a_3(x)$ is required to solve the 2-dimensional bifurcation equation. We refer to [7] for further discussions.

Remark 2.2.1 *To prove existence of quasi-periodic solutions with* m*-frequencies*

$$u(t, x) = U(\omega t, x)\,, \quad \omega \in \mathbf{R}^m\,,$$

where $U(\cdot, x) : \mathbf{T}^m \to \mathbf{R}$, the main difficulty w.r.t. the periodic case relies in a more complicated geometry of the numbers $\omega \cdot l - \omega_j$, $l \in \mathbf{Z}^m$, $j \in \mathbf{N}$. Existence of quasi-periodic solutions with the Lyapunov-Schmidt approach has been proved by Bourgain [5]. For existence results via the KAM approach see e.g. [17], [16] and references therein.

The "completely resonant" case

$$a_1(x) \equiv 0$$

where

$$\omega_j = j\,, \qquad \forall j \in \mathbf{N} \tag{2.6}$$

(infinitely many resonance relations among the linear frequencies) was left an open problem. In this case *all* the solutions of (2.2) are 2π-periodic. For infinite dimensional Hamiltonian PDEs, aside the small divisor problem (i), this leads to the further complication of an *infinite dimensional bifurcation phenomenon.*

In the paper [4] attached below we show how to deal with it. The results contained in [4] can be seen as an extension to Hamiltonian PDEs of the results of Weinstein-Moser and Fadell-Rabinowitz.

For further results and open problems concerning small divisors problem in Hamiltonian PDEs we refer to [7].

Bibliography

[1] Baldi P., Berti M., *Forced vibrations of a nonhomogeneous string*, preprint 2006; and *Periodic solutions of wave equations for asymptotically full measure sets of frequencies*, Rend. Mat. Acc. Naz. Lincei, v. 17, 2006.

[2] Beale T. *The existence of solitary water waves*, Comm. Pure Appl. Math. 30, 1977, 373-389.

[3] Berti M., *Nonlinear oscillations and Hamiltonian PDEs*, Lecture Notes 2006.

[4] Berti M., Bolle P., *Cantor families of periodic solutions for completely resonant nonlinear wave equations*, Duke Mathematical Journal, 134, 2, 359-419, 2006.

[5] Bourgain J., *Quasi-periodic solutions of Hamiltonian perturbations of 2D linear Schrödinger equations*, Ann. of Math., 148, 363-439, 1998.

[6] Bourgain, J. *Construction of periodic solutions of nonlinear wave equations in higher dimension*, Geom. Funct. Anal. 5 (1995), no. 4, 629–639.

[7] Craig W., *Problèmes de petits diviseurs dans les équations aux dérivées partielles*, Panoramas et Synthèses, 9, Société Mathématique de France, Paris, 2000.

[8] Craig W., Wayne E., *Newton's method and periodic solutions of nonlinear wave equation*, Comm. Pure and Appl. Math, vol. XLVI, 1409-1498, 1993.

[9] Deimling K., *Nonlinear functional analysis*, Springer-Verlag, Berlin, 1985.

[10] Fadell E., Rabinowitz P., *Generalized cohomological index theories for the group actions with an application to bifurcation questions for Hamiltonian systems*, Inv. Math. 45, 139-174, 1978.

[11] Gromov M.L., *Smoothing and inversion of differential operators*, Math USSr Sbornik 17, 1972, 381-434.

[12] Hamilton, R.S., *The inverse function theorem of Nash and Moser*, Bull. A.M.S., 7, 1982, 65-222.

[13] Hörmander L., *The boundary problems of physical geodesy*, Arch. Rat. Mech. Anal., 62, 1976, 1-52.

[14] Klainermann S., *Global existence for nonlinear wave equations*, Comm. Pure Appl. Math., 33, 1980, 43-101.

[15] Kuksin S., *Hamiltonian perturbations of infinite-dimensional linear systems with imaginary spectrum*, Funktsional. Anal. i Prilozhen. 21, no. 3, 22–37, 95, 1987.

[16] Kuksin S., *Analysis of Hamiltonian PDEs*, Oxford Lecture Series in Mathematics and its Applications, 19. Oxford University Press, 2000.

[17] Kuksin S., Pöschel J., *Invariant Cantor manifolds of quasi-periodic oscillations for a nonlinear Schrödinger equation*, Ann. of Math, 2, 143, no. 1, 149-179, 1996.

[18] Lyapunov A.M., *Problème général de la stabilité du mouvement*, Ann. Sc. Fac. Toulouse, 2, 203-474, 1907.

[19] Moser J., *A new technique for the construction of solutions of nonlinear differential equations*, Proc. Nat. Acad. Sci., 47, 1961, pp. 1824-1831.

[20] Moser J., *On invariant curves of area preserving mappings on an annulus*, Nachr. Akad. Göttingen Math. Phys., n. 1, 1962, pp. 1-20.

[21] Moser J., *A rapidly convergent iteration method and non-linear partial differential equations*, Ann. Scuola Normale Sup., Pisa, 3, 20, 1966, 499-535.

[22] Moser J., *Convergent series expansions for quasi-periodic motions*, Math. Ann., 169, 1967, 136-176.

[23] Moser J., *Periodic orbits near an Equilibrium and a Theorem by Alan Weinstein*, Comm. Pure Appl. Math., vol. XXIX, 1976.

[24] Moser J., Zehnder E., *Dynamical Systems*, ICTP Lecture Notes, 1991.

[25] Nash J., *The embedding problem for Riemannian manifolds*, Annals of Math. 63, 1956, pp. 20-63.

[26] Nirenberg J., *Topics in Nonlinear Functional Analysis*, Courant Institute of Mathematical Sciences, New York University.

[27] Schaeffer D., *A stability theorem for the obstacle problem*, Adv. in Math. 17, 1975, 34-47.

[28] Schwartz J., *On Nash's implicit function theorem*, Comm. Pure Appl. Math., 13, 1960, 509-530.

[29] Sergeraert F. *Un théorèm de fonctions implicites sur certains espaces de Fréchet et quelques applications*, Ann. Sci. École Norm. Sup., 5, 1972, 599-660.

[30] Taylor, *Partial Differential equations*, Springer.

[31] Wayne E., *Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory*, Commun. Math. Phys. 127, No.3, 479-528, 1990.

[32] Weinstein A., *Normal modes for Nonlinear Hamiltonian Systems*, Inv. Math, 20, 47-57, 1973.

[33] Zehnder E. *Generalized implicit function theorems with applications to some small divisors problems I-II*, Comm. Pure Appl. Math., 28, 1975, 91-140; 29, 1976, 49-113.