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Resonance Problems for the p-Laplacian

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Abstract

We consider resonance problems at an arbitrary eigenvalue of the p-Laplacian, and prove the existence of weak solutions assuming a standard Landesman-Lazer condition. We use variational arguments to characterize certain eigenvalues and then to establish the solvability of the given boundary value problem.
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1 Introduction

Consider the boundary value problem

$$
\begin{aligned}
-\Delta_p u - \lambda |u|^{p-2}u + f(x,u) = 0 \text{ in } \Omega, \\
u|_{\partial\Omega} = 0,
\end{aligned}
\tag{1}
$$

where $\Delta_p u := \nabla \cdot (|\nabla u|^{p-2}\nabla u)$, Ω is a bounded domain in R^N, $p > 1$, $\lambda \in R$, and $f : \Omega \times R \to R$ is a bounded Caratheodory function. This problem can be thought of as a perturbation of the homogeneous eigenvalue problem

$$
\begin{aligned}
-\Delta_p u - \lambda |u|^{p-2}u = 0 \text{ in } \Omega, \\
u|_{\partial\Omega} = 0,
\end{aligned}
\tag{2}
$$

where we say that λ is an eigenvalue of $-\Delta_p$ if (2) has a nontrivial solution, u, which is then called an eigenfunction.

For the case $p = 2$ the spectral properties of $-\Delta_2$ are well understood via the Spectral Theorem for Compact Self-adjoint Linear Operators and the Fredholm Alternative, and many variations of (1) have been studied since the pioneering work of Landesman and Lazer in [11]. In a nutshell these investigations have shown that if λ is not an eigenvalue then we can expect (1) to be solvable with no additional assumptions, and if λ is an eigenvalue then (1) will be solvable if we assume an appropriate orthogonality condition.

For the general case, $p > 1$, the spectral properties of $-\Delta_p$ are still being established and much work remains to be done. It is known that $-\Delta_p$ has a smallest eigenvalue, i.e. the *principal* eigenvalue, λ_1, which is simple and has an associated eigenfunction that is strictly positive in Ω (See [12]). Also, the properties of the next smallest eigenvalue, $\lambda_2 > \lambda_1$, have been investigated in [1], where it is shown that λ_2 has a variational characterization analogous to the usual characterization for the case $p = 2$. Beyond this, it is known that $-\Delta_p$ has a sequence of so-called *variational* eigenvalues, $\{\lambda_n\}$, satisfying a standard minimax characterization, but it is not known if this represents a complete list of the eigenvalues. It is interesting to note that some properties of the linear case do not carry over to the general case. For some interesting, and somewhat surprising, results regarding a generalized Fredholm Alternative see [3], [4] and [9].

In this paper we prove the following theorem.

Theorem 1 *Suppose that there is a function $\overline{f} \in L^q(\Omega)$, where $q = \dfrac{p}{p-1}$, such that $|f(x,t)| \leq \overline{f}(x)$ $\forall (x,t) \in \Omega \times R$. Further, assume that either*

$$(LL)^+_\lambda : \int_{v>0} f^+ v + \int_{v<0} f^- v > 0 \;\forall\, v \in \ker(-\Delta_p - \lambda) \setminus \{0\},$$

or

$$(LL)^-_\lambda : \int_{v>0} f^+ v + \int_{v<0} f^- v < 0 \;\forall\, v \in \ker(-\Delta_p - \lambda) \setminus \{0\},$$

where $f^\pm(x) := \lim_{t\to\pm\infty} f(x,t)$, a limit that is assumed to exist for a.e. $x \in \Omega$. Then (1) has a weak solution.

Notice that if λ is not an eigenvalue then the conditions $(LL)^{\pm}_{\lambda}$ are vacuously true.

This improves upon the recent work in [2], which examined resonance around the principal eigenvalue, and in [8], which examined resonance problems at arbitrary eigenvalues for the analogous ODE problem.

The proofs are variational in nature taking advantage of the structure provided by the variational eigenvalues of $-\Delta_p$ and applying a saddle point theorem for linked sets. (See [13] for standard details of the variational theory.)

2 The Variational Formulation of the Problem

Let

$$J_\lambda(u) := \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda}{p}\int_\Omega |u|^p + \int_\Omega F(x,u),\ \forall u \in W_0^{1,p}(\Omega),$$

where $F(x,u) := \displaystyle\int_0^u f(x,t)dt$. It is well known that $J_\lambda \in C^1(W_0^{1,p}(\Omega), R)$, such that

$$J'_\lambda(u)\cdot v = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v - \lambda\int_\Omega |u|^{p-2}uv + \int_\Omega f(x,u)v,\ \forall v \in W_0^{1,p}(\Omega),$$

and such that weak solutions of (1) correspond to critical points of J_λ.

In order to apply the standard methods of variational theory an important first step is to prove that, given the assumptions in Theorem 1, J_λ satisfies the Palais-Smale condition, i.e. if $\{u_n\} \subset W_0^{1,p}(\Omega)$ such that $\{J_\lambda(u_n)\}$ is bounded and $J'_\lambda(u_n) \to 0$ in $(W_0^{1,p}(\Omega))^*$, then $\{u_n\}$ has a subsequence that converges in $W_0^{1,p}(\Omega)$. This requires several preliminary lemmas which we state without proof. (See [7], Lemma 3.3, page 124 for a proof of Lemma 1, below. Lemmas 2 and 3 are straightforward.) In all that follows we use $||u||_{W_0^{1,p}(\Omega)} := \left(\int_\Omega |\nabla u|^p\right)^{\frac{1}{p}}$, which is equivalent to the usual $W^{1,p}(\Omega)$ norm on $W_0^{1,p}(\Omega)$.

Lemma 1 *Let* $A : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ *such that*

$$A(u)\cdot v := \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v.$$

Then A is continuous, odd, $(p-1)$-homogeneous and continuously invertible. Moreover, $||A(u)||_{(W_0^{1,p}(\Omega))^*} = ||u||^{p-1}_{W_0^{1,p}(\Omega)}\ \forall u \in W_0^{1,p}(\Omega)$.

Lemma 2 *Let* $B : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ *such that*

$$B(u)\cdot v := \int_\Omega |u|^{p-2}uv.$$

Then B is continuous, odd, $(p-1)$-homogeneous and compact.

Lemma 3 *Let* $C : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))^*$ *such that*

$$C(u)\cdot v := \int_\Omega f(x,u)v.$$

Then C is continuous and compact and $||C(u)||_{(W_0^{1,p}(\Omega))^*} \le ||\overline{f}||_{L^q(\Omega)}\ \forall u \in W_0^{1,p}(\Omega)$.

Theorem 2 *If the hypotheses of Theorem 1 are satisfied, then J_λ satisfies the Palais-Smale condition.*

Proof:

Suppose that $\{u_n\} \subset W_0^{1,p}(\Omega)$ and $c > 0$ such that $|J_\lambda(u_n)| \leq c\ \forall n$ and $J'_\lambda(u_n) \to 0$ in $(W_0^{1,p}(\Omega))^*$. First, we argue by contradiction to show that $\{u_n\}$ is bounded. Suppose $||u_n||_{W_0^{1,p}(\Omega)} \to \infty$. Let $v_n := \dfrac{u_n}{||u_n||_{W_0^{1,p}(\Omega)}}$ and assume, without loss of generality, that $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$, $v_n \to v$ in $L^p(\Omega)$, and $v_n \to v$ pointwise a.e. in Ω . Notice that $J'(u_n) = A(u_n) - \lambda B(u_n) + C(u_n)$, where A, B and C are defined in the lemmas above. Dividing through by $||u_n||^{p-1}_{W_0^{1,p}(\Omega)}$ and using the $(p-1)$-homogeneity of A and B we get

$$\frac{J'_\lambda(u_n)}{||u_n||^{p-1}_{W_0^{1,p}(\Omega)}} = A(v_n) - \lambda B(v_n) + \frac{C(u_n)}{||u_n||^{p-1}_{W_0^{1,p}(\Omega)}}.$$

By Lemmas 2 and 3 we see that $B(v_n) \to B(v)$ and $\dfrac{C(u_n)}{||u_n||^{p-1}_{W_0^{1,p}(\Omega)}} \to 0$. Using Lemma 1 and the fact that $J'_\lambda(u_n) \to 0$, we get that $v_n \to A^{-1}(\lambda B(v))$ in $W_0^{1,p}(\Omega)$. It is clear that $||v||_{W_0^{1,p}(\Omega)} = 1$. Hence $v_n \to v \in \ker(-\Delta_p - \lambda) \setminus \{0\}$. Now observe that

$$pJ_\lambda(u_n) - J'_\lambda(u_n) \cdot u_n = p\int_\Omega F(x, u_n) - \int_\Omega f(x, u_n)u_n,$$

so

$$\frac{pJ_\lambda(u_n)}{||u_n||_{W_0^{1,p}(\Omega)}} - J'_\lambda(u_n) \cdot v_n = p\int_\Omega \frac{F(x, u_n)}{||u_n||_{W_0^{1,p}(\Omega)}} - \int_\Omega f(x, u_n)v_n.$$

By hypothesis, the left hand side of this equation has a limit of 0. However, for a.e. $x \in \{x : v(x) > 0\}$ we have $u_n(x) \to \infty$, so

$$\lim_{n\to\infty} f(x, u_n(x))v_n(x) = f^+(x)v(x),$$

and

$$\lim_{n\to\infty} \frac{F(x, u_n(x))}{||u_n||_{W_0^{1,p}(\Omega)}} = \lim_{n\to\infty} v_n(x)\frac{1}{u_n(x)}\int_0^{u_n(x)} f(x, t) = v(x)f^+(x),$$

where the last limit is justified by L'Hospital's Rule. Similarly, for a.e. $x \in \{x : v(x) < 0\}$ we have

$$\lim_{n\to\infty} f(x, u_n(x))v_n(x) = f^-(x)v(x),$$

and

$$\lim_{n\to\infty} \frac{F(x, u_n(x))}{||u_n||_{W_0^{1,p}(\Omega)}} = v(x)f^-(x).$$

Further, both integrands are bounded in absolute value by the quantity $\overline{f}|v_n|$, so they both have a pointwise limit of 0 on $\{x : v(x) = 0\}$, and we are justified in applying Lebesgue's Dominated Convergence Theorem to get

$$0 = \lim_{n\to\infty} \left[p \int_\Omega \frac{F(x, u_n)}{||u_n||_{W_0^{1,p}(\Omega)}} - \int_\Omega f(x, u_n) v_n \right] = (p-1) \left[\int_{v>0} f^+ v + \int_{v<0} f^- v \right],$$

which contradicts either $(LL)_\lambda^+$ or $(LL)_\lambda^-$. Hence $\{u_n\}$ is bounded and standard compactness arguments show that $\{u_n\}$ contains a convergent subsequence. Thus the theorem is proved. ∎

Now that the Palais-Smale condition has been verified we can state a deformation theorem which plays a fundamental role in proving that J_λ has critical points of saddle point type. (See [13], page 75.)

Theorem 3 *Suppose that J_λ satsifies the Palais-Smale condition. Let $\beta \in R$ be a regular value of J_λ and let $\overline{\epsilon} > 0$. Then there exists $\epsilon \in (0, \overline{\epsilon})$ and a continuous one-parameter family of homeomorphisms, $\phi : W_0^{1,p}(\Omega) \times [0, 1] \to W_0^{1,p}(\Omega)$, with the properties*

1. *$\phi(u, t) = u$, if $t = 0$ or if $|J_\lambda(u) - \beta| \geq \overline{\epsilon}$.*
2. *$J_\lambda(\phi(u, t))$ is non-increasing in t for any $u \in W_0^{1,p}(\Omega)$.*
3. *If $J_\lambda(u) \leq \beta + \epsilon$, then $J_\lambda(\phi(u, 1)) \leq \beta - \epsilon$.*

3 The Variational Eigenvalues, $\{\lambda_k\}_{k\in N}$

Consider the even functional

$$I(u) := \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p} \quad \forall u \in W_0^{1,p}(\Omega) \setminus \{0\},$$

and the manifold

$$\mathcal{S} := \{u \in W_0^{1,p}(\Omega) : ||u||_{L^p(\Omega)} = 1\}.$$

It is a straight forward task to verify that the eigenvalues and eigenfunctions of $-\Delta_p$ correspond to the critical values and critical points of $I|_\mathcal{S}$. (Observe that $I|_\mathcal{S}(u) = ||u||^p_{W_0^{1,p}(\Omega)}$ and $I|'_\mathcal{S}(u) = p(A(u) - I(u)B(u))$, where we use the notation introduced in the previous section.) As in the previous section we must first check that $I|_\mathcal{S}$ satisfies the Palais-Smale condition before standard minimax theorems can be applied.

Lemma 4 *$I|_\mathcal{S}$ satisfies the Palais-Smale condition.*

Proof:
Let $\{u_n\} \subset \mathcal{S}$ and $c > 0$ such that $|I(u_n)| \leq c \ \forall n$ and $A(u_n) - I(u_n)B(u_n) \to 0$ in $(W_0^{1,p}(\Omega))^*$. It is clear that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$, so, without loss of generality, $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$. We may also assume that $I(u_n) \to \overline{I}$ in R.

By compactness we see that $B(u_n) \to B(u)$ in $(W_0^{1,p}(\Omega))^*$ and thus $u_n \to A^{-1}(\bar{I}B(u))$ in $W_0^{1,p}(\Omega)$. ■

Now we state a deformation theorem for $I|_{\mathcal{S}}$ similar to the one stated for J_λ. Observe that since $\mathcal{S}$ is symmetric about the origin and $I|_{\mathcal{S}}$ is even we obtain the additional property that the deformation preserves symmetry. An interesting technical difficulty arises in the justification of this theorem. For $p \geq 2$ we know that $\mathcal{S}$ is a $C^{1,1}$ manifold and a well-known deformation result can be used to prove the theorem. (See [13], page 79.) For $p < 2$ the situation is more delicate, but the appropriate theorem can be recovered using Ghoussoub's result in [10], page 55, which only requires $\mathcal{S}$ to be a C^1 manifold. A simple modification of Ghoussoub's proof yields the appropriate preservation of symmetry.

Theorem 4 *Let $\beta \in R$ be a regular value of $I|_{\mathcal{S}}$ and let $\bar{\epsilon} > 0$. Then there exists $\epsilon \in (0, \bar{\epsilon})$ and a continuous one-parameter family of homeomorphisms, $\psi : \mathcal{S} \times [0,1] \to \mathcal{S}$, with the properties*

1. *$\psi(u,t) = u$, if $t = 0$ or if $|I|_{\mathcal{S}}(u) - \beta| \geq \bar{\epsilon}$.*
2. *$I|_{\mathcal{S}}(\psi(u,t))$ is non-increasing in t for any $u \in \mathcal{S}$.*
3. *If $I|_{\mathcal{S}}(u) \leq \beta + \epsilon$, then $I|_{\mathcal{S}}(\psi(u,1)) \leq \beta - \epsilon$.*
4. *$\psi(-u,t) = -\psi(u,t)$ for any $t \geq 0$ and any $u \in \mathcal{S}$.*

For any $k \in N$ let $\mathcal{F}_k := \{\mathcal{A} \subset \mathcal{S} :$ there exists a continuous odd surjection $h : \mathcal{S}^{k-1} \to \mathcal{A}\}$, where $\mathcal{S}^{k-1}$ represents the unit sphere in R^k. Next define

$$\lambda_k := \inf_{\mathcal{A} \in \mathcal{F}_k} \sup_{u \in \mathcal{A}} I(u).$$

It is clear that λ_k is a well-defined finite value. Moreover, a standard argument shows that

Theorem 5 *λ_k is a critical value of $I|_{\mathcal{S}}$.*

Proof:
Suppose that λ_k is not a critical value, i.e. that it is a regular value. Using $\bar{\epsilon} = 1$ and $\beta = \lambda_k$, let $\epsilon \in (0,1)$ and ψ be the objects guaranteed by the deformation theorem above. By definition there is an $\mathcal{A} \in \mathcal{F}_k$ such that $\sup_{u \in \mathcal{A}} I(u) \leq \lambda_k + \epsilon$. But if $h : \mathcal{S}^{k-1} \to \mathcal{A}$ is a continuous odd surjection, then so is $\psi(h(\cdot), 1) : \mathcal{S}^{k-1} \to \psi(\mathcal{A}, 1)$. Thus $\psi(\mathcal{A}, 1) \in \mathcal{F}_k$ such that $\sup_{u \in \psi(\mathcal{A},1)} I(u) \leq \lambda_k - \epsilon$, which contradicts the definition of λ_k. ■

We will refer to $\{\lambda_k\}_{k \in N}$ as the *variational* eigenvalues of $-\Delta_p$. It is not known if this represents a complete list of eigenvalues. Fortunately, even without this knowledge, this portion of the spectrum provides enough structure for the saddle point arguments of the next section.

It is important to note that the given characterization of λ_k is not the same as the usual Ljusternik-Schnirrelman characterization involving a minimax over sets of genus greater than k, although it is not hard to argue that they share some important properties. Let

$\{\mu_k\}$ be the eigenvalues defined by the Ljusternik-Schnirrelman characterization. Since $\mathcal{F}_k$ is a subset of the sets of genus k, it follows that $\lambda_k \geq \mu_k$. Thus $\mu_k \to \infty$ implies $\lambda_k \to \infty$. Moreover, it is clear that $\lambda_1 = \mu_1$, and we can argue, as follows, that $\lambda_2 = \mu_2$. It is proved in [1], Proposition 2, that $\mu_2 = \inf\{\lambda > \mu_1 : \lambda \text{ is an eigenvalue of } -\Delta_p\}$. Let u_2 be some normalized eigenfunction associated with μ_2. Then u_2 must change sign in Ω,i.e. $u_2^+ \neq 0, u_2^- \neq 0$ (see [1]). Set $\mathcal{A} := \{su_2^+ + tu_2^- : s,t \in R \text{ and } |s|^p||u_2^+||_{L^p} + |t|^p||u_2^-||_{L^p} = 1\}$. Then $\mathcal{A}$ belongs to the class $\mathcal{F}_2$ and for any u from $\mathcal{A}$ we have $\int_\Omega |\nabla u|^2 = \mu_2$. Hence $\lambda_2 \leq \mu_2$, i.e. they are equal.

4 The Case $\lambda_k < \lambda < \lambda_{k+1}$

In this section we prove Theorem 1 for the case $\lambda_k < \lambda < \lambda_{k+1}$. Our proof will establish the existence of a critical value of the functional J_λ characterized as a minimax over *linked* sets. The following discussion should make this characterization more precise.

Let $\mathcal{A} \in \mathcal{F}_k$ be such that $\sup\limits_{u \in \mathcal{A}} I(u) = m \in (\lambda_k, \lambda)$. For any $u \in \mathcal{A}$ and $t > 0$ it is easy to see that

$$J_\lambda(tu) \leq \frac{t^p}{p}(m - \lambda) + t||\overline{f}||_{L^q(\Omega)}.$$

Now let $\mathcal{E}_{k+1} := \{u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p \geq \lambda_{k+1} \int_\Omega |u|^p\}$, and notice that for $u \in \mathcal{E}_{k+1}$ we have

$$J_\lambda(u) \geq \frac{1}{p}(\lambda_{k+1} - \lambda)||u||^p_{L^p(\Omega)} - ||\overline{f}||_{L^q(\Omega)}||u||_{L^p(\Omega)}.$$

Thus we can let $\alpha := \inf\limits_{u \in \mathcal{E}_{k+1}} J_\lambda(u)$ and set $T > 0$ such that $\max\limits_{u \in \mathcal{A}, t \geq T} J_\lambda(tu) = \gamma < \alpha$. Now let $\mathcal{B}_k$ represent the closed unit ball in R^k and let $T\mathcal{A} := \{tu : u \in \mathcal{A}, t \geq T\}$. Consider the family of mappings

$$\Gamma := \{h \in C^0(\mathcal{B}_k, W_0^{1,p}(\Omega)) : h|_{\mathcal{S}^{k-1}} \text{ is an odd map into } T\mathcal{A}\}.$$

We establish the necessary properties of Γ in the following lemmas.

Lemma 5 *Γ is nonempty.*

Proof:

By definition there exists a surjective continuous odd map $h : \mathcal{S}^{k-1} \to \mathcal{A}$. Define $\overline{h} : \mathcal{B}_k \to W_0^{1,p}(\Omega)$ by $\overline{h}(ts) = tTh(s)$ for any $s \in \mathcal{S}^{k-1}$ and any $t \in [0, 1]$. Clearly, $\overline{h} \in \Gamma$. ■

Lemma 6 *If $h \in \Gamma$ then $h(\mathcal{B}_k) \bigcap \mathcal{E}_{k+1} \neq \emptyset$.*

Proof:

If $0 \in h(\mathcal{B}_k)$, then we are done. Otherwise we consider the map

$$\tilde{h} : \mathcal{S}^k \to \mathcal{S} : \tilde{h}(x_1, ..., x_{k+1}) = \begin{cases} \pi \circ h(x_1, ..., x_k) \text{ if } x_{k+1} \geq 0 \\ -\pi \circ h(-x_1, ..., -x_k) \text{ if } x_{k+1} < 0 \end{cases},$$

where π represents radial projection onto $\mathcal{S}$ in $W_0^{1,p}(\Omega) \setminus \{0\}$. It is straight forward to verify that $\tilde{h}(\mathcal{S}^k) \in \mathcal{F}_{k+1}$. Thus $I(u) \geq \lambda_{k+1}$ for some $u \in \tilde{h}(\mathcal{S}^k)$, i.e. $u \in \mathcal{E}_{k+1}$. But $\pi \circ h(x) \in \mathcal{E}_{k+1}$ implies $h(x) \in \mathcal{E}_{k+1}$. Thus $h(\mathcal{B}_k) \bigcap \mathcal{E}_{k+1} \neq \emptyset$. ■

The discussion above demonstrates that $T\mathcal{A}$ and $\mathcal{E}_{k+1}$ are *linked* in a way that allows the application of standard minimax theorems. In particular, using an argument by contradiction similar to that in the proof of Theorem 5, we can show that

Theorem 6

$$c := \inf_{h \in \Gamma} \sup_{x \in \mathcal{B}_k} J_\lambda(h(x))$$

is a critical value of J_λ with $c \geq \alpha$.

Proof:

Suppose that c is a regular value of J_λ. It is clear from previous estimates that $c \geq \alpha$. Using $\beta = c$ and $\bar{\epsilon} < c - \gamma$, we can apply Theorem 3 to get a deformation ϕ and a corresponding ϵ. Notice that if $u \in T\mathcal{A}$ then $J_\lambda(u) \leq \gamma < \beta - \bar{\epsilon}$, so ϕ leaves the set $T\mathcal{A}$ fixed. By definition of c there is an $h \in \Gamma$ such that $\sup_{x \in \mathcal{B}_k} J_\lambda(h(x)) \leq c + \epsilon$. Consider $\tilde{h}(\cdot) := \phi(h(\cdot), 1)$. If $x \in \mathcal{S}^{k-1}$ then $h(x) \in T\mathcal{A}$ and $\tilde{h}(x) = \phi(h(x), 1) = h(x)$, so $\tilde{h}|_{\mathcal{S}^{k-1}} = h|_{\mathcal{S}^{k-1}}$ is an odd mapping into $T\mathcal{A}$. Hence $\tilde{h} \in \Gamma$ and $\sup_{x \in \mathcal{B}_k} J_\lambda(\tilde{h}(x)) \leq c - \epsilon$, a contradiction. The theorem is proved. ■

5 The Case $\lambda = \lambda_k$

Our approach in this section is to find a critical point for the functional J_{λ_k} by taking the limit of a sequence of critical points for the functionals J_{μ_n}, where $\mu_n \to \lambda_k$. Our first argument assumes $(LL)^+_{\lambda_k}$.

Lemma 7 *If $(LL)^+_{\lambda_k}$ is satisfied, then there is a $\delta > 0$ such that $(LL)^+_\mu$ is satisfied for all $\mu \in (\lambda_k - \delta, \lambda_k + \delta)$.*

Proof:

If not, then there is a sequence $\{\mu_n\}$ with $\mu_n \to \lambda_k$, and a corresponding sequence $\{v_n\}$ with $v_n \in \ker(-\Delta_p - \mu_n) \bigcap \mathcal{S}$, such that

$$\int_{v_n > 0} f^+ v_n + \int_{v_n < 0} f^- v_n \leq 0 \; \forall n.$$

Note that $I|_\mathcal{S}(v_n) = \mu_n$ is bounded and $I|'_\mathcal{S}(v_n) \equiv 0$, so $\{v_n\}$ is a Palais-Smale sequence for $I|_\mathcal{S}$ and thus, without loss of generality, $v_n \to v$ in $W_0^{1,p}(\Omega) \bigcap \mathcal{S}$. It follows easily that $v \in \ker(-\Delta_p - \lambda_k) \bigcap \mathcal{S}$ such that

$$\int_{v > 0} f^+ v + \int_{v < 0} f^- v \leq 0,$$

a contradiction of $(LL)^+_{\lambda_k}$. ■

For convenience in all that follows we assume that $\lambda_{k-1} < \lambda_k - \delta$. Let $\{\mu_n\} \subset (\lambda_k - \delta, \lambda_k)$ be an increasing sequence such that $\mu_n \to \lambda_k$. By the results in the previous section we know that J_{μ_n} has at least one critical point for each $n \in N$. More specifically, we can prove the following.

Lemma 8 *There is a decreasing sequence of critical values, $\{c_n\}$, associated with the functionals J_{μ_n}.*

Proof:
Select $\mathcal{A} \in \mathcal{F}_{k-1}, T_1 > 0, \mathcal{E}_k$ and Γ_1, as in the previous section, such that

$$c_1 := \inf_{h\in\Gamma_1} \sup_{x\in\mathcal{B}_{k-1}} J_{\mu_1}(h(x))$$

is a critical value of J_{μ_1}. To determine c_2 we can use the same sets $\mathcal{A}$ and $\mathcal{E}_k$, but we might need to choose $T_2 > T_1$ which leads to a corresponding choice of Γ_2. Since $T_2\mathcal{A} \subset T_1\mathcal{A}$, it is clear that $\Gamma_2 \subset \Gamma_1$, and thus

$$\inf_{h\in\Gamma_2} \sup_{x\in\mathcal{B}_{k-1}} J_{\mu_1}(h(x)) \geq \inf_{h\in\Gamma_1} \sup_{x\in\mathcal{B}_{k-1}} J_{\mu_1}(h(x)) = c_1.$$

On the other hand, any $h_1 \in \Gamma_1$ can be transformed into an element of Γ_2 in the following way.

$$h_2(x) := \begin{cases} h_1(2x) \text{ for } |x| \leq \frac{1}{2} \\ h_1(\frac{x}{|x|})[1 + 2(|x| - \frac{1}{2})T_2] \text{ for } |x| > \frac{1}{2} \end{cases}$$

Observe that $h_2(x) \in T_1\mathcal{A}$ for all $|x| \geq \frac{1}{2}$, and thus $J_{\mu_1}(h_2(x)) \leq \gamma < \alpha \leq c_1$ for all $|x| \geq \frac{1}{2}$, where we are using the notation of the previous section. It follows that the maximum is achieved on $\{x : |x| \leq \frac{1}{2}\}$, so

$$\sup_{x\in\mathcal{B}_{k-1}} J_{\mu_1}(h_2(x)) = \sup_{|x|\leq\frac{1}{2}} J_{\mu_1}(h_2(x)) = \sup_{x\in\mathcal{B}_{k-1}} J_{\mu_1}(h_1(x)).$$

Thus

$$c_1 = \inf_{h\in\Gamma_1} \sup_{x\in\mathcal{B}_{k-1}} J_{\mu_1}(h(x)) = \inf_{h\in\Gamma_2} \sup_{x\in\mathcal{B}_{k-1}} J_{\mu_1}(h(x))$$

Next observe that

$$J_{\mu_2}(u) = J_{\mu_1}(u) + \frac{1}{p}(\mu_1 - \mu_2) \int_\Omega |u|^p \leq J_{\mu_1}(u) \;\; \forall u \in W_0^{1,p}(\Omega),$$

Thus

$$\inf_{h\in\Gamma_2} \sup_{x\in\mathcal{B}_{k-1}} J_{\mu_1}(h(x)) \geq \inf_{h\in\Gamma_2} \sup_{x\in\mathcal{B}_{k-1}} J_{\mu_2}(h(x)) := c_2,$$

and hence $c_1 \geq c_2$. Continue by induction to create a decreasing sequence of critical values. ■

Let $\{u_n\}$ be the sequence of critical points associated with the critical values $\{c_n\}$. If this sequence is bounded then it is a simple matter to show that, by passing to a subsequence, we obtain a critical point of J_{λ_k} in the limit. Thus it remains to rule out the possibility that $\{u_n\}$ is unbounded. We achieve this result using arguments similar to those in the proof of Theorem 2.

Lemma 9 *If* $||u_n||_{W_0^{1,p}(\Omega)} \to \infty$, *then* $\dfrac{u_n}{||u_n||_{W_0^{1,p}(\Omega)}} \to v \in \ker(-\Delta_p - \lambda_k) \setminus \{0\}$.

Proof:

Let $v_n := \frac{u_n}{||u_n||_{W_0^{1,p}(\Omega)}}$. Without loss of generality we have $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$ and $v_n \to v$ in $L^p(\Omega)$. Also $0 = J'_{\mu_n}(u_n)\ \forall n$, so

$$0 = A(v_n) - \mu_n B(v_n) + \frac{C(u_n)}{||u_n||^{p-1}_{W_0^{1,p}(\Omega)}} \quad \forall n.$$

But $B(v_n) \to B(v)$, $\mu_n \to \lambda_k$ and $\dfrac{C(u_n)}{||u_n||^{p-1}_{W_0^{1,p}(\Omega)}} \to 0$. Hence $v_n \to A^{-1}(\lambda_k B(v))$, so $v = A^{-1}(\lambda_k B(v))$. ■

Finally, observe that

$$pc_n = pJ_{\mu_n}(u_n) - J'_{\mu_n}(u_n)\cdot u_n = p\int_\Omega F(x,u_n) - \int_\Omega f(x,u_n)u_n.$$

However,

$$\lim_{n\to\infty}\left(p\int_\Omega \frac{F(x,u_n)}{||u_n||_{W_0^{1,p}(\Omega)}} - \int_\Omega f(x,u_n)v_n\right) = \int_{v>0} f^+ v + \int_{v<0} f^- v > 0,$$

so $\lim_{n\to\infty} \dfrac{pc_n}{||u_n||_{W_0^{1,p}(\Omega)}} > 0$, which contradicts the fact that $\{c_n\}$ is bounded above. Theorem 1 is proved subject to the condition $(LL)^+_{\lambda_k}$.

The proof assuming $(LL)^-_{\lambda_k}$ is similar in most respects. It is clear that the result of Lemma 7 holds for $(LL)^-_{\lambda_k}$ as well. Thus we can begin with a sequence $\{\mu_n\}$ that decreases to λ_k and a corresponding sequence of critical values characterized by

$$c_n := \inf_{h\in\Gamma_n}\sup_{x\in\mathcal{B}_k} J_{\mu_n}(h(x)).$$

In this case we can not rely upon the same $\mathcal{A} \in \mathcal{F}_k$ to characterize each Γ_n. However, we observe that the estimate

$$J_{\mu_n}(u) \geq \frac{1}{p}(\lambda_{k+1} - \mu_n)||u||^p_{L^p(\Omega)} - ||\overline{f}||_{L^q(\Omega)}||u||_{L^p(\Omega)} \ \ \forall u \in \mathcal{E}_{k+1}$$

has a uniform lower bound, and so the sequence $\{c_n\}$ is bounded below. If the corresponding sequence of critical points, $\{u_n\}$, is unbounded, then, precisely as in Lemma 9 and the comments that follow the lemma, we can use $(LL)^-_{\lambda_k}$ to show that $\lim_{n\to\infty} \frac{pc_n}{||u_n||_{W_0^{1,p}(\Omega)}} < 0$, a contradiction. Thus there will be a subsequence of critical points that converges to the desired solution.

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