The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
and Cultural Organization

International Atomic
Energy Agency

SMR1777/12

# School on Nonlinear Differential Equations

**(9 - 27 October 2006)**

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# Robot motion planning: a wild case

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# ROBOT MOTION PLANNING: A WILD CASE

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Abstract. A basic problem from robotics is the constructive motion planning problem: given an arbitrary (nonadmissible) trajectory $\Gamma$ of the robot, find an admissible $\varepsilon$-approximation (in the subriemannian sense) $\gamma(\varepsilon)$ of $\Gamma$, of minimal subriemannian length. Then, the (asymptotic behavior of the) subriemannian length $L(\gamma(\varepsilon))$ is called the metric complexity of $\Gamma$ (in the sense of Jean, [13], [14]). We have solved this problem in the case of a SR metric of corank 3 at most. After this corank 3, there is a deep change of behavior in the problem.

But, the first really critical case is the 4-10 case (a 4 dimensional distribution in $\mathbb{R}^{10}$). Here, we address this critical case. We give partial, but constructive results, that generalize in some sense the results of our previous papers.

## 1. Introduction, Statement of Results

**1.1. Basic concepts, statement of the problems.** We consider here "motion planning problems" on an open subset $\Xi \subset \mathbb{R}^n$, that is, the data $\Sigma = (\Delta, g, \Gamma)$ of a smooth curve $\Gamma : [0,1] \to \Xi$, well parametrized, i.e. $\frac{d\Gamma}{dt} \neq 0$ for all $t$, without double points, and a sub-Riemannian metric $(\Delta, g)$ on $\Xi$. Here, $\Delta$ is assumed to be a one-step-bracket-generating distribution over $\Xi$, and $g$ is a Riemannian metric over $\Delta$.

In fact, the problems we consider depend only on the germ along $\Gamma$ of the SR metric $(\Delta, g)$. These problems come from robotics: the distribution $\Delta$ is the set of nonholonhomic constraints (the "dynamics") of the robot. The SR metric $g$ is the cost to be minimized, and the curve $\Gamma$ is a given nonadmissible curve (-i.e. not tangent to $\Delta$), and $\Gamma$ has to be approximated by a motion of the robot. Then, we denote by $\mathcal{S}$ the set of smooth ($C^\infty$) couples of a curve $\Gamma$ and a SR metric on $\Xi$, endowed with the $C^\infty$ topology

In this paper, we use exactly the same conventions and notations as in our previous paper [9]. As usual in this type of problems, $\varepsilon$ is a small parameter and $d$ denotes the SR distance function. Then $\mathcal{T}_\varepsilon$ is the $\varepsilon$-sub-Riemannian tube $\mathcal{T}_\varepsilon = \{q \in \mathbb{R}^n | d(q, \Gamma) \leq \varepsilon\}$, and $\mathcal{C}_\varepsilon = \{q \in \mathbb{R}^n | d(q, \Gamma) = \varepsilon\}$ is the corresponding cylinder. Two functions $f_1$, $f_2$ in $\varepsilon$, tending to $+\infty$ when $\varepsilon$ tends to zero, are said to be "strongly equivalent", $f_1 \approx_s f_2$ (resp.$f_1$ is "weakly equivalent" to $f_2$, $f_1 \approx_w f_2$) if $\lim_{\varepsilon\to 0} \frac{f_1(\varepsilon)}{f_2(\varepsilon)} = 1$ (resp. $k_1\ f_1(\varepsilon) \leq f_2(\varepsilon) \leq k_2 f_1(\varepsilon)$, for $\varepsilon$ small enough, for certain constants $k_1, k_2 > 0$). We also write $f_1 \geq_s f_2$ if

$$\liminf_{\varepsilon\to 0} \frac{f_1(\varepsilon)}{f_2(\varepsilon)} \geq 1.$$

*Date*: February, 2005.

1991 *Mathematics Subject Classification.* 53C17, 49J15, 34H05.

Let $\gamma : [0, \theta_\gamma] \to \mathcal{T}_\varepsilon$ be a (smooth, piecewise smooth, Lipschitz) arclength-parametrized admissible curve (i.e. almost everywhere tangent to $\Delta$) connecting $\Gamma(0)$ to $\Gamma(1)$ : $\gamma(0) = \Gamma(0)$, $\gamma(\theta_\gamma) = \Gamma(1)$. Then, the SR length of $\gamma$ is $\theta_\gamma$. The question is to find the minimum of this length $\theta_\gamma$ among all the curves $\gamma$. This minimum is denoted by $\theta^*(\varepsilon)$.

Then, the class of strong (resp. weak) equivalence of the function $MC_\Sigma(\varepsilon) = \frac{1}{\varepsilon}\theta^*(\varepsilon)$, when $\varepsilon$ tends to zero, is called the strong (resp. weak) metric complexity of the given motion planning problem. This notion has been introduced by F. Jean, see [13], [14], [15]. It is a natural and important notion, coming from practitioners in robotics. For $T < 1$, we also denote by $MC_\Sigma(\varepsilon, T)$ the metric complexity of the piece of the curve $\Gamma : \{\Gamma(t) | t \leq T\}$.

In the paper, we will consider only relevant motion planning problems: a motion planning problem $\Sigma = (\Delta, g, \Gamma)$ is said **relevant** if for all $t \in [0, 1]$, $\dot{\Gamma}(t) \notin \Delta(\Gamma(t))$. Clearly (standard transversality arguments), for a distribution of corank $p \geq 2$, this condition defines an open-dense subset of $\mathcal{S}$, that we still denote by $\mathcal{S}$. Note that for corank 1, this set is open, but not dense.

The case of a corank one distribution (eventually not one-step-bracket generating) has been addressed and solved (constructively) in [7], [8]: explicit expressions for the strong metric complexity were exhibited (at least in generic cases) in terms of the basic invariants of the given motion planning problem $\Sigma$. The case of a corank 2 and 3 distribution has been treated in [9], in the one-step-bracket-generating case, i.e. $\Delta_x + [\Delta, \Delta]_x = T_x\mathbb{R}^n$ for all $x \in \Gamma([0, 1])$.

Here, we still consider the one-step-bracket-generating case, but for a distribution of corank more than 3. More precisely, it will appear that the first really wild case is the case of a distribution of rank 4 in $\mathbb{R}^{10}$. We will specially address this case here.

### 1.2. The affine space of fundamental 2-forms, statement of previous results.

The crucial notion of fundamental-form comes from [7], [8], [9].

Given $\Sigma = (\Gamma, \Delta, g) \in \mathcal{S}$, we consider the one forms $\alpha$ defined on $\Xi$ such that:

$$\alpha(\Delta) = 0, \ \alpha(\dot{\Gamma}) = 1.$$

This space of one-forms is invariant under multiplication by a function which is 1 on $\Gamma$. Now, let us consider the space $\Omega_t$ of 2-forms obtained by taking the exterior derivative of the $\alpha$'s, and by restricting to $\Delta(\Gamma(t))$ for all $t \in [0, 1]$.

Then, $\Omega_t$ is an affine space of 2-forms on $\Delta(\Gamma(t))$, and each $\omega_t \in \Omega_t$ may be written as:

$$\omega_t = \omega_t^0 + \sum_{i=1}^{p-1} \lambda_i \omega_t^i, \ \lambda_i \in \mathbb{R},$$

and we will see below (Lemma 6) that, due to the bracket-generating assumption, $\omega_t^0$, $\omega_t^i$ **are independent .**

We can also define $\bar{\Omega}_t$, for all $t \in [0, 1]$, an affine space of skew-symmetric endomorphisms $\bar{\omega}_t$ of $\Delta(\Gamma(t))$, by:

$$\bar{\Omega}_t = \{\bar{\omega}_t | < \bar{\omega}_t(X), Y >_g = \omega_t(X, Y), \ \ \forall X, Y \in \Delta(\Gamma(t)), \ \omega_t \in \Omega_t\}.$$

If moreover an orthonormal frame is specified on each $\Delta(\Gamma(t))$, we call $\tilde{\Omega}_t$ the affine space of matrices obtained by taking the matrices of the $\bar{\omega}_t \in \bar{\Omega}_t$ with respect to this frame. Then, $\tilde{\Omega} = \{\tilde{\Omega}_t,\ t \in [0,1]\}$ is a field along $\Gamma$ of $p$-dimensional affine spaces of skew symmetric matrices $A_t$.

This field is well defined and unique once an orthonormal frame is chosen along $\Gamma$.

This field $\Omega$ (resp. $\bar{\Omega}$, $\tilde{\Omega}$) of affine spaces of 2-forms, (resp. endomorphisms of $\Delta(\Gamma(t))$, resp. skew-symmetric matrices), is called the field of **fundamental** 2-forms (resp. fundamental endomorphisms, fundamental matrices).

Let $eig_t$ denote the infimum over the affine space $\bar{\Omega}_t$ of the norm of the $\bar{\omega}_t$ (the norm $||.||_2$ subordinated to the norm w.r.t. the metric $g$, of course). Then we have also:

$$eig_t = \inf_{\bar{\omega}_t \in \bar{\Omega}_t} \sup_{||X||,||Y|| \leq 1} \omega_t(X,Y) \tag{1.1}$$

The notation $eig_t$ comes from the fact that $||\bar{\omega}_t||_2$ is also the maximal modulus of eigenvalue of $\bar{\omega}_t$.

Our results in the paper [9] are the following:

**Theorem 1.** *(a.* ***minoration of the metric complexity****) For any relevant motion planning problem, the following inequality holds:*

$$MC_\Sigma(\varepsilon,T) \geq_s \frac{2}{\varepsilon^2}\int_0^T \frac{dt}{eig_t}, \tag{1.2}$$

*(b.* ***Corank 1, 2 or 3****) There is an open dense subset $\mathcal{S}^*$ of $\mathcal{S}$ such that, for a motion planning problem $\Sigma \in \mathcal{S}^*$ :*

$$MC_\Sigma(\varepsilon,T) \simeq_s \frac{2}{\varepsilon^2}\int_0^T \frac{dt}{eig_t}, \tag{1.3}$$

In fact, we can prove that in general, the estimation 1.3 **fails** starting from corank 4.

### 1.3. Presentation of the wild case.

Due to the following standard formula, for a 1-form $\alpha$ and vector fields $X, \tilde{X}$,

$$d\alpha(X,\tilde{X}) = \alpha([X,\tilde{X}]) - L_X\alpha(\tilde{X}) + L_{\tilde{X}}\alpha(X), \tag{1.4}$$

it follows that the mapping $[.,.]_\Delta$: $\Delta_q \times \Delta_q \to T_q\Xi/\Delta_q$, $(X,\tilde{X}) \to [X,\tilde{X}] + \Delta_q$ is well defined.

Then let us denote by $B_t$ the image by the bracket mapping $[.,.]_\Delta$of the product of two unit balls:

$$\begin{aligned} B_t &= \{[X,Y]_\Delta, ||X|| \leq 1, ||Y|| \leq 1,\ X,Y \in \Delta(\Gamma(t))\}, \\ B_t &\subset T_{\Gamma(t)}\Xi/\Delta_{\Gamma(t)}. \end{aligned} \tag{1.5}$$

**Definition 1.** *The set $B_t$ is called strictly-convex in the direction $\{V_t + \Delta_{\Gamma(t)}\} \in T_{\Gamma(t)}\Xi/\Delta_{\Gamma(t)}$, if any of the 2 following equivalent requirements holds:*

*(R1) there is $x^* = \lambda V_t \in B_t$, $\lambda > 0$, and $\omega \in (T_{\Gamma(t)}\Xi/\Delta_{\Gamma(t)})^* \approx (\mathbb{R}^p)^*$ (dual space of $T_{\Gamma(t)}\Xi/\Delta_{\Gamma(t)}$), such that for all $y \in B_t$,*

$$\omega(x^*) - \omega(y) \geq 0;$$

*(R2) If $V^* = \{\omega \in (\mathbb{R}^p)^*, \, \omega(V_t) = 1\}$, then, there exist $\omega^* \in V^*$, $x^* = \lambda V_t \in B_t$, $\lambda > 0$, with:*

$$\omega^*(x^*) = \sup_{x \in B_t} \omega^*(x) = \inf_{\omega \in V^*} \sup_{x \in B_t} \omega(x).$$

In fact, in the paper [9], we have shown the following:

**Theorem 2.** *Corank 1, 2, 3. For a generic (open dense) motion planning problem, for all $t \in [0,1]$, $B_t$ is strictly convex in the direction of $\frac{d\Gamma}{dt}$.*

This theorem is not stated in that way in [9], but, looking carefully, it is completely equivalent to the crucial property (4.2) in [9], and this property is itself crucial for the proof of Theorem 1, part (b).

But this convexity property ceases to be true in higher codimension. (This is very clear from the transversality arguments we used to prove it in [9]).

In the 4-10 case (a rank 4 distribution in $\mathbb{R}^{10}$), the situation is even worse than this nonconvexity property: the mapping $[.,.]_\Delta$ has a projectivisation (still denoted by $[.,.]_\Delta$) from the Grassmannian $G_{2,4}$ of 2-planes in $\mathbb{R}^4$, to the projective space $P\mathbb{R}^6$. But $\dim(G_{24}) = 4 < 5 = \dim(P\mathbb{R}^6)$. Hence, in generic situation (and at generic points), the direction of $\frac{d\Gamma}{dt}$ never meets $B_t$. In particular, in generic situation, $B_t$ is never strictly convex in the direction of $\frac{d\Gamma}{dt}$.

**This generic nonintersection property happens for the first time in these dimensions $4 - 10$:**

-By the results of [9], it does not happen in corank 1, 2, 3;

-For a distribution $\Delta$ of rank 2 or 3, due to the One-Step-Bracket-Generating assumption, corank may be 3 at most.

-For a distribution of rank 4, if corank is 5, there is generic intersection.

Therefore, this $4 - 10$ case is the first really wild case, and, we will restrict to this case in the remaining of the paper.

Nevertheless, in the $4 - 10$ case, the following property holds:

**Lemma 1.** *(4-10 case, One-Step-Bracket-Generating assumption) $B_t$ is strictly convex in the direction of its (nonzero) own points.*

1.4. **Statement of the results, organization of the paper.** In the remaining part of the paper, we consider only the 4-10 case. We consider the (open and dense) set $\mathcal{S}$ of relevant, one-step-bracket-generating motion planning problems $\Sigma = (\Gamma, \Delta, g)$.

The purpose of this paper is to prove the following theorem:

**Theorem 3.** *For a generic (open dense) set $\mathcal{S}^*$ of motion planning problems $\Sigma = (\Gamma, \Delta, g) \in \mathcal{S}$, the following holds:*

*There are arbitrarily close to $\Gamma$ parametrized curves $\hat{\Gamma}$ (arbitrarily close in the $C^0$ topology) such that the metric complexity of the curve $\hat{\Gamma}$ is of the form $\frac{2}{\varepsilon^2}A$, where the constant $A$ is arbitrarily close to $\int_0^1 \frac{dt}{eig_t}$ (where $eig_t$ is relative to $\Gamma$).*

This result means that, generically, the usual formula for the metric complexity "almost holds".

In the next section 2, we recall a few technical results (normal coordinates, normal forms) that we need, and that come from our previous papers. In the section 3, we shortly sketch the proof of Theorem 3. Many technical details are similar to those in the proof of Theorem 1 in [9]. Therefore, for details, we refer to this paper.

## 2. Preliminaries:

All the results, notations, are similar to those in our previous papers. For proofs, see [9].

### 2.1. **Frames.**

- A motion planning problem may be specified by a couple $(\Gamma, F)$, where $F = (F_1, ..., F_{n-p})$ is a frame of vector fields that generate $\Delta$ and that are orthonormal for $g$. Hence, we will also write $\Sigma = (\Gamma, F)$. If a global coordinate system $(x, y, w)$ is given on $\Xi$, with $x \in \mathbb{R}^{n-p}$, $y \in \mathbb{R}^{p-1}$, $w \in \mathbb{R}$, then we write:

$$F_j = \sum_{i=1}^{n-p} \mathcal{Q}_{i,j}(x,y,w)\frac{\partial}{\partial x_i} + \sum_{i=1}^{p-1} \mathcal{L}_{i,j}(x,y,w)\frac{\partial}{\partial y_i} + \mathcal{M}_j(x,y,w)\frac{\partial}{\partial w}, \quad j = 1, ..., n-p. \tag{2.1}$$

  Then, the SR metrics is specified by the triple $(\mathcal{Q}, \mathcal{L}, \mathcal{M})$ of smooth $x, y, w$-dependant matrices, and we write also $\Sigma = (\Gamma, \mathcal{Q}, \mathcal{L}, \mathcal{M})$. It will often happen that $\Gamma$, in coordinates, will be the curve: $\Gamma(t) = (0, 0, t)$. In that case, we will write $\Sigma = (\mathcal{Q}, \mathcal{L}, \mathcal{M})$.

### 2.2. **Normal coordinates.**

Consider $\Sigma \in \mathcal{S}$, and fix a (well) parametrized surface $S$ in $\Xi$, $(y, w) \to S(y, w)$, with the following properties: $y, w$ are coordinates on $S$, that are global on a neighborhood of $\Gamma$ (and we restrict $S$ to this neighborhood). Also, $y \in \mathbb{R}^{p-1}$, $w \in \mathbb{R}$ and $S(0, w) = \Gamma(w)$, $\forall w \in [0, 1]$. Moreover, we require that $S$ is transversal to $\Delta$. This is always possible, since $\dot{\Gamma} \notin \Delta$ ($\Sigma$ is relevant). Let us define $\mathcal{T}_\varepsilon^S = \{q \in \mathbb{R}^n | d(q, S) \leq \varepsilon\}$ and $\mathcal{C}_\varepsilon^S = \{q \in \mathbb{R}^n | d(q, S) = \varepsilon\}$, the subriemannian $S$-tube and cylinder.

**Lemma 2.** *(Normal coordinates with respect to $S$) There are mappings $x : \Xi \to \mathbb{R}^{n-p}$, $y : \Xi \to \mathbb{R}^{p-1}$, $w : \Xi \to \mathbb{R}$, such that $\xi = (x, y, w)$ is a coordinate system on $\Xi$ (possibly restricting $\Xi$ to some neighborhood of $S$), such that:*

*0. $S(y, w) = (0, y, w)$,*

*1. $\Delta_{|S} = \ker dw \cap_{i=1,..p-1} \ker dy_i$, $g_{|S} = \sum_{i=1}^{n-p}(dx_i)^2$,*

*2. $\mathcal{C}_\varepsilon^S == \{\xi | \sum_{i=1}^{n-p} x_i{}^2 = \varepsilon^2\}$,*

*3. geodesics (from the Pontryagin's maximum principle [16]) meeting the transversality conditions w.r.t. $S$ are the straight lines through $S$, contained in the planes $P_{y_0,w_0} = \{\xi | (y, w) = (y_0, w_0)\}$ (hence, they are orthogonal to $S$).*

*These normal coordinates are unique up to changes of coordinates of the form*

$$\tilde{x} = T(y, w)x, (\tilde{y}, \tilde{w}) = (y, w), \tag{2.2}$$

*where $T(y, w) \in O(n-p)$, the $n-p$ orthogonal group.*

2.3. **Normal form.** Consider $\Sigma \in \mathcal{S}$, and fix a surface $S$ just as in Section 2.2. Fix a normal coordinate system $\xi = (x, y, w)$ given by Lemma 2.

**Theorem 4.** *(Normal form) There is a unique orthonormal frame $F = (\mathcal{Q}, \mathcal{L}, \mathcal{M})$ for $(\Delta, g)$ with the following properties:*
*1. $\mathcal{Q}(x, y, w)$ is symmetric, $\mathcal{Q}(0, y, w) = Id$ (the identity matrix),*
*2. $\mathcal{Q}(x, y, w)x = x$,*
*3. $\mathcal{L}(x, y, w)x = 0$, $\mathcal{M}(x, y, w)x = 0$.*
*4. Conversely if $\xi = (x, y, w)$ is a coordinate system such that 1, 2, 3 above are satisfied, then, $\xi$ is a normal coordinate system for the SR metric defined by the orthonormal frame $F$, with respect to the parametrized surface $\{(0, y, w)\}$.*

Clearly, this normal form is invariant under the changes of normal coordinates (2.2).

Let us write:

$$\mathcal{Q}(x, y, w) = Id + Q_1(x, y, w) + Q_2(x, y, w) + ...,$$
$$\mathcal{L}(x, y, w) = 0 + L_1(x, y, w) + L_2(x, y, w) + ...,$$
$$\mathcal{M}(x, y, w) = 0 + M_1(x, y, w) + M_2(x, y, w) + ...,$$

where $Q_i, L_i, M_i$ are matrices depending on $\xi$, the coefficients of which have order $i$ w.r.t. $x$ (i.e. they are in the $i^{th}$ power of the ideal of $C^\infty(x, y, w)$ generated by the $x_r$, $r = 1, ..., n-p$). Then in particular, $Q_1$ is linear in $x$, $Q_2$ is quadratic. Set $u = (u_1, ..., u_{n-p}) \in \mathbb{R}^{n-p}$. Then $\sum_{j=1}^{p-1} L_{1_j}(x, y, w)u_j = L_{1,y,w}(x, u)$ is quadratic in $(x, u)$, and $\mathbb{R}^{p-1}$-valued. Its $i^{th}$ component is the quadratic expression denoted by $L_{1,i,y,w}(x, u)$. Similarly $\sum_{j=1}^{p-1} M_{1_j}(x, y, w)u_j = M_{1,y,w}(x, u)$ is a quadratic expression in $(x, u)$. The corresponding matrices are denoted by $L_{1,i,y,w}$, $i = 1, ..., p-1$, and $M_{1,y,w}$.

We have the following proposition:

**Proposition 1.** *1. $Q_1 = 0$,*
*2. $L_{1,i,y,w}$, $i = 1, ..., p-1$, and $M_{1,y,w}$ are skew symmetric matrices.*

2.4. **Cylinder-box Theorem in Normal coordinates.** The following theorem is a result that can be easily obtained from the normal form and the ball-box Theorem of SR geometry (see for instance Gromov, [11]).

Let $\xi = (x, y, w)$ be a normal coordinate system, and $F = (\mathcal{Q}, \mathcal{L}, \mathcal{M})$ be the associated normal form. Assume that $\Delta$ is one-step-bracket-generating.

**Theorem 5.** *(Normal cylinder-box Theorem)*
*1. Let $\xi = (x, y, w) \in \mathcal{T}_\varepsilon$, then:*

$$||x||_2 \leq \varepsilon,$$
$$||y||_2 \leq k_2\ \varepsilon^2,$$

*for some $k_2 > 0$.*
*2. Take $0 < \omega < 1$ and set $K_\varepsilon^{k_1,\omega} = \{\xi = (x, y, w) |\ ||x||_2 \leq \omega\varepsilon, ||y||_2 \leq k_1\ \varepsilon^2\}$. Then, for $k_1$ small enough, $K_\varepsilon^{k_1,\omega} \subset \mathcal{T}_\varepsilon$.*

The item 1 of this theorem, together with Theorem 4 and Proposition 1, implies immediately the following lemma:

**Lemma 3.** *In normal coordinates restricted to the tubes $\mathcal{T}_\varepsilon$, the normal form of linear combinations of $F = (F_1, ...F_{n-p})$ is as follows, with $u = (u_1, ..., u_{n-p}) \in \mathbb{R}^{n-p}$:*

$$\sum_{j=1}^{n-p} F_j u_j = \sum_{j=1}^{n-p} u_j \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{i=1}^{p-1} L_w^i(x,u)\frac{\partial}{\partial y_i} + \frac{1}{2}M_w(x,u)\frac{\partial}{\partial w} + O^2(\varepsilon), \tag{2.3}$$

*where $L_w^i(x,u)$, and $M_w(x,u)$ are skew symmetric bilinear forms depending smoothly on $w$, and $O^2(\varepsilon)$ is a smooth vector field, the components of which are bounded by $K\varepsilon^2$, for some $K > 0$, independent of $\varepsilon$.*

In matrix form, we will write:

$$\begin{aligned} L_w^i(x,u) &= x' L_w^i u, \quad i = 1, ..., p-1, \\ M_w(x,u) &= x' M_w u, \end{aligned} \tag{2.4}$$

where the ' denotes transposition, and $L_w^i$, $M_w$ are smooth one-parameter families of skew-symmetric matrices.

In this notation, $w$ is not an index, and one should not confuse the notations $M_w$, $M_i$ : The matrix $M_w$ is equal to $M_{1,0,w}$ in the notations of Section 2.3.

In the following lemma and corollary, we give useful rough estimates that are also consequences of the standard SR ball-box Theorem.

**Lemma 4.** *In normal coordinates $\xi = (x, y, w)$ on a compact neighborhood $N$ of the curve $\Gamma$ : for all $0 < \omega < 1$, there are positive constants, $k_1(\omega)$, $k_2$, such that the balls $B_{\varepsilon,w_0}$ of radius $\varepsilon$ centered at points $(0, 0, w_0)$ of $\Gamma$, satisfy:*
*a. $B_{\varepsilon,w_0} \subset \{\xi / \ ||x||_2 \le \varepsilon;\ |y_i| \le k_2\varepsilon^2,\ i = 1, ..., p-1;\ |w - w_0| \le k_2\varepsilon^2\}$,*
*b. $\{\xi / \ ||x||_2 \le \omega\varepsilon;\ |y_i| \le k_1(\omega)\varepsilon^2,\ i = 1, ..., p-1;\ |w - w_0| \le k_1(\omega)\varepsilon^2\} \subset B_{\varepsilon,w_0}$.*

The lemma 4 implies the following:

**Corollary 1.** *In normal coordinates $\xi = (x, y, w)$ on a compact neighborhood $N$ of the curve $\Gamma$ : there are constants, $h_1, h_2 > 0$, such that , if $\xi_1$, $\xi_2 \in \mathcal{T}_\varepsilon$, then, the time (or the arclength) to go from $\xi_1$ to $\xi_2$, remaining inside $\mathcal{T}_\varepsilon$, is less than $(w_2 - w_1)\frac{h_1}{\varepsilon} + h_2\varepsilon$.*

This lemma is the key lemma to treat roughly certain isolated points that appear generically along $\Gamma$. For details, see [9].

Simple computations, which we omit, prove the following:

**Lemma 5.** *(fundamental matrices in normal coordinates) Let $\xi = (x, y, w)$ be a normal coordinate system. The field of fundamental matrices is:*

$$\tilde{\Omega}_t = \{M_t + \sum_{i=1}^{p-1} \lambda_i L_t^i, \quad \lambda_i \in \mathbb{R}\}, \tag{2.5}$$

where $M_t$, $L_t^i$, $i = 1, ..., p-1$, are the matrices appearing in the normal form 2.3, 2.4.

**Remark 1.** *(Important) Notice that the matrices $M_t$, $L_t^i$ themselves depend on the surface $S$, and even on its parametrization. They depend also on the normal coordinates. But, the associated field $\bar{\Omega}_t$ of affine spaces of skew symmetric (w.r.t. g) endomorphisms of $\Delta(\Gamma(t))$ does not. It depends only on the given $\Sigma \in \mathcal{S}$.*

2.5. **Brackets.** Fix $\Sigma \in \mathcal{S}$, together with a normal coordinate system $\xi = (x, y, w)$.

Then at a point $\xi_0 = (0, 0, w_0) \in \Gamma$, the tangent plane $T_{\xi_0}S \approx S$, with coordinates $(y, w)$, identifies with $T_{\xi_0}\Xi/\Delta_{\xi_0}$, and $\Delta_{\xi_0}$ identifies with the horizontal plane $P_{0,w_0} = \{\xi | y = 0, w = w_0\}$, finally, the mapping $[.,.]_\Delta$ is just the mapping:

$$(x, \tilde{x}) \mapsto (x' L^1_w \tilde{x}, ..., x' L^{p-1}_w \tilde{x}, x' M_w \tilde{x}), \tag{2.6}$$

where $M_w$, $L^i_w$, $i = 1, ..., p-1$ are the skew symmetric matrices appearing in Formulas 2.3, 2.4.

**Lemma 6.** *The distribution $\Delta$ being one-step-bracket-generating then, the skew symmetric matrices $M_w$, $L^i_w$, $i = 1, ..., p-1$ are independent, for all $w$.*

2.6. **Genericity.** Consider the mapping $\Lambda$: $\mathcal{S} \times [0, 1] \to \mathcal{A}(p, so(n-p))$, $(\Sigma = (F, \Gamma), t) \to \tilde{\Omega}_t$, where $\tilde{\Omega}_t$ is the affine space of skew symmetric matrices corresponding to the choice of $F(\Gamma(t))$ for an orthonormal frame in $\Delta(\Gamma(t))$. Here, $so(n-p)$ denotes as usual the set of skew symmetric matrices of size $(n-p)$, and $\mathcal{A}(p, so(n-p))$, denotes the set of affine spaces (of dimension $p-1$) of skew symmetric matrices of size $n-p$.

**Proposition 2.** *The mapping $\Lambda$ is a surjective submersion.*

Let $\tilde{\omega} \in \mathcal{A}(p, so(n-p))$, and let $eig(\tilde{\omega})$ denote the minimum over the affine space $\tilde{\omega}$ of the maximum moduli of eigenvalues of the skew symmetric matrices $A \in \tilde{\omega}$ (or of the norm $||A||_2$ of such matrices).

**Lemma 7.** *The function $eig : \mathcal{A}(p, so(n-p)) \to \mathbb{R}_+$ is semialgebraic, continuous.*

Let $\mathcal{A}^+(p, so(n-p))$ denote the set of affine spaces that are not vector subspaces. The following is obvious:

**Lemma 8.** *The function $eig$ is bounded from below by a strictly positive number $a_\Omega$, in restriction to any compact subset of $\mathcal{A}^+(p, so(n-p))$.*

For $\omega \in \mathcal{A}(p, so(n-p))$, denote by $\Lambda(\omega)$ the set of $A \in \omega$ reaching the minimum $eig(\omega)$. Let $L$ be the set of all triples $(\omega, A, e)$, $A \in \Lambda(\omega)$, $e = eig(\omega)$, and let $\Pi$ be the projection on the first component, $\Pi : L \to \mathcal{A}(p, so(n-p))$. It is not hard to see that $L$ is semialgebraic, closed, and that $L$ is bounded vertically ($\Pi$ is proper). Then, by the theorems on stratification of mappings ([12]), there is an analytic section $s : U \subset \mathcal{A}(p, so(n-p)) \to L$, where $U$ is open, dense. Let $\mathcal{E}$ denote the complement of $U$ in $\mathcal{A}(p, so(n-p))$. It is subanalytic, closed, of codimension at least one. As a consequence, we have the following lemma.

**Lemma 9.** *A smooth curve $\gamma : [0, 1] \to \mathcal{A}(p, so(n-p))$ transversal to $\mathcal{E}$ has the following property ($\mathcal{P}_1$), by construction: there is a (bounded) section $s$ over $\gamma$, $\Pi \circ s = Id_\gamma$, $s(\gamma(t)) \in \Lambda(\gamma(t))$, which is smooth except for a finite number of points. Moreover, $eig(\gamma(t))$ is also smooth out of this finite set.*

It follows from Lemma 9, and from the theorems of transversality to closed Whitney-stratified subsets (see [10]), that there is an open-dense subset $\mathcal{S}^* \subset \mathcal{S}$ of $\Sigma = (\Gamma, F)$ such that the mapping $t \to \bar{\Omega}_t$ is transversal to the closure of the subsets $\mathcal{E}$ and $\dot{\Gamma}(t)$ avoids $B_t$ except for some finite set of isolated points.

To summarize, for $\Sigma \in \mathcal{S}^*$ the following holds, except for a finite set of time values from $[0,1]$ :
-the mapping $t \to eig_t$ is smooth;
-there is a smooth mapping $t \to \Lambda_t \in \bar{\Omega}_t$, with $eig_t = ||\Lambda_t||$,
-$\dot{\Gamma}_t$ avoids $B_t$,
-$\Lambda_t$ has double eigenvalues, for all $t \in [0,T]$.

**Note**: The last claim (double eigenvalues) holds because of the following: if $\Lambda_t$ has simple eigenvalues for some $t$, then $\Lambda_t$ has simple eigenvalues on some nontrivial interval $[t_1, t_2]$. Then, we may apply the proof of [9], to show that in fact, on this interval, $B_t$ is strictly convex in the direction of $\dot{\Gamma}$. This is impossible.

In normal coordinates, it means that there is a smooth mapping $w \to \Lambda^*(w)$, (defined except for a finite number of values of $w$), such that, except for these $w$ :

$$M_w^* = M_w + \sum_{i=1}^{p-1} \Lambda_i^*(w) L_w^i = eig_w \bar{M}_w, \tag{2.7}$$

$$eig_w = \inf_{\Lambda} ||M_w + \sum_{i=1}^{p-1} \Lambda_i L_w^i||,$$

where $\bar{M}_w$ is a (skew symmetric) matrix conjugate to $J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

## 3. Proof of Theorem 3

The proof is divided in two steps, the main idea being like that:
In a first step, we modify the curve $\Gamma$ into a curve $\hat{\Gamma}$ which is $C^0$ close, and such that the modified motion planning problem is strictly convex in the direction of $\frac{d\hat{\Gamma}}{dt}$.
Then, the second step is very similar to the proof in [9]. We will only sketch it, for the sake of completeness.

3.1. **Proof of Theorem 3, part 1.** The purpose of this step is to modify the curve $\Gamma$ slightly.

We start with the generic motion planning problem $\Sigma$ in normal coordinates (with respect to an arbitrary surface $S$) and normal form given by theorem 4. With $||x|| \leq \varepsilon$, (or on the tube $\mathcal{T}_\varepsilon^S$),

$$\begin{aligned} \dot{x} &= u + O^2(\varepsilon), \\ \dot{y}_i &= \frac{1}{2} x' L_i(y,w) u + O^2(\varepsilon), \quad i = 1, ..., p-1, \\ \dot{w} &= \frac{1}{2} x' M(y,w) u + O^2(\varepsilon). \end{aligned} \tag{3.1}$$

By the genericity arguments of Lemma 9, Section 2.6, we may assume that there is a smooth mapping $\Lambda(w)$, such that, for all $w$, if $\tilde{M}(0,w) = M(0,w) + \sum_{i=1}^{p-1} \Lambda_i(w) L_i(0,w)$, then $||\tilde{M}(0,w)|| = eig_w$, $\tilde{M}(0,w) = eig_w \bar{M}(0,w)$, and $\bar{M}(0,w)$ is unitary, conjugate to $J$.

Hence, making the following change of coordinates in the surface $S$,

$$\tilde{y} = y, \tag{3.2}$$
$$\tilde{w} = w + \sum_{i=1}^{p-1} \Lambda_i(w) y_i,$$

we may assume that $\Sigma$ is in the following normal form, on $\mathcal{T}_\varepsilon^S$:

$$\begin{aligned} \dot{x} &= u + O^2(\varepsilon), \\ \dot{y}_i &= \frac{1}{2} x' L_i(y,w) u + O^2(\varepsilon), \quad i = 1, ..., p-1, \\ \dot{w} &= \frac{eig_w}{2} x' M(y,w) u + O^2(\varepsilon), \end{aligned} \tag{3.3}$$

where $M(0,w)$ is a unitary matrix, reducible to $J$ by conjugation (depending on $w$), and for all $w$ :

$$1 = ||M(0,w)|| = \inf_\lambda ||M(0,w) + \sum_{i=1}^{p-1} \lambda_i L_i(0,w)||$$

There is no obstruction here to consider that $eig_w = 1 \ \forall w$ : this is just making a change of $w$ coordinate in $S$, setting $d\tilde{w} = \frac{dw}{eig_w}$ (a reparametrization of the curve $\Gamma$). This reparametrization being done, the parameter $t$ of the curve $\Gamma$ belongs to some interval $[0,T]$, $T > 0$.

Under this assumption (3.3), we forget about the (finite number of) special isolated points that appear in the genericity results (Section 2.6). They are of no importance and are subject to the same treatment as in the paper [9]. The (rough) estimate, crucial for this treatment is given by Corollary 1.

Then,

$$\begin{aligned} 1 &= \inf_\lambda X(w)'(M(0,w) + \sum_{i=1}^{p-1} \lambda_i L_i(0,w)) Y(w), \\ &= \hat{X}(w)' M(0,w) \hat{Y}(w), \end{aligned}$$

where $\hat{X}$ is arbitrary with $||\hat{X}(w)|| = 1$, and

$$\hat{Y}(w) = -M(0,w)\hat{X}(w). \tag{3.4}$$

Now, whatever $w \in [0,T]$, $1 = \inf_\lambda \sup_{(a,b)\in B_w} (a + \sum_{i=1}^{p-1} \lambda_i b_i)$, where $B_w$, as in Section 1.3, is the image of the product of two unit balls in $\Delta_w$, by the mapping $[.,.]_\Delta$.

It is easy to see that:

$$1 = \inf_\lambda \sup_{(a,b)\in Co(B_w)} (a + \sum_{i=1}^{p-1} \lambda_i b_i),$$

where $Co(B_w)$ is the convex hull of $B_w$.

On the other hand, it is known that (See [17] for instance), the inf sup of a convex-concave function $\varphi(x,y)$, where $x \in X$ convex compact, is equal to the sup inf . Hence:

$$1 = \sup_{(a,b)\in Co(B_w)} \inf_{\lambda}(a + \sum_{i=1}^{p-1} \lambda_i b_i),$$

and then, the $\inf_\lambda(a + \sum_{i=1}^{p-1} \lambda_i b_i)$ has to be reached for $b_i = 0$, $i = 1, ..., p-1$.

Therefore, for all $w \in [0,T]$,

$$1 = \sup_{(a,0)\in Co(B_w)} a.$$

This shows that $(1,0) \in Co(B_w)$ $\forall w \in [0,T]$.

By a standard result of Caratheodory, it means that:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sum_{j=1}^{p+1} \omega_j \begin{pmatrix} y_j' L_i(0,w)x_j \\ y_j' M(0,w)x_j \end{pmatrix}, \ ||y_j||, ||x_j|| \leq 1, \text{ with } \sum_{j=1}^{p+1} \omega_j = 1.$$

Then, for all $j$, we must have $y_j' M(0,w)x_j = 1$, which implies $||x_j|| = 1$, $y_j = M(0,w)x_j$.

Hence $0 \in Co(\hat{B}_w)$, where $Co(\hat{B}_w)$ is the convex hull of the set $\hat{B}_w \subset \mathbb{R}^{p-1}$ :

$$\hat{B}_w = \{(-U'M(0,w)L_1(0,w)U, ..., -U'M(0,w)L_{p-1}(0,w)U), \quad ||U|| = 1\}.$$

Now, standard results from nonsmooth analysis (see [6]) say that the differential inclusion:

$$\dot{y} \in F(\{U; ||U|| = 1\}) = \hat{B}_w, \tag{3.5}$$

has an absolutely continuous solution $y(t)$, arbitrarily close to 0, corresponding to a measurable control $U(t), ||U(t)|| = 1$ $\forall t$.

More or less straightforward approximation results (approximation of measurable bounded controls by smooth controls), show that we may consider that $y(t), U(t)$ are smooth. If $U(t)$denotes this control (for the differential inclusion 3.5), set $X^*(t) = M(0,t)U(t), t \in [0,T]$.

Now, after this "approximation step", we come to some "perturbation step". Let us consider the Cauchy problem on $\mathbb{R}^{p-1}$ :

$$\begin{aligned} \dot{y}_i &= \frac{1}{U(t)'M(0,t)M(y,t)U(t)} U(t)'M(0,t)L_i(y,t)U(t), \ y_i(0) = 0, \\ i &= 1, ..., p-1, \quad t \in [0,T] \end{aligned} \tag{3.6}$$

Note that, whatever the value of $U(t)$, $||U(t)|| = 1$, the denominator $U(t)'M(0,t)M(y,t)U(t)$ is close to $-1$, provided that $y$ is small enough. Then, on a neighborhood of 0 for $y$, and for $0 \leq t \leq T$, the time dependant vector fields (3.6) is well defined.

Everything is smooth in (3.6). Moreover, this differential equation is a perturbation of 3.5: it can be rewritten as follows:

$$\begin{aligned} \dot{y}_i &= -U(t)'M(0,t)L_i(0,t)U(t) + G(y(t),t,U(t);y(t)) = 0, \\ i &= 1, ..., p-1, \quad t \in [0,T], \end{aligned} \tag{3.7}$$

where $G(y(t),t,U(t);z)$ is smooth and linear in $z$. Therefore, since $y(t)$ in 3.5 can be made small, it is an easy consequence of Gronwall's inequality and of the smoothness of all terms in 3.7, that the solution $y^*(t)$ for the regularized control $U(t)$ defined above, can be made arbitrarily small, uniformly on $[0,T]$.

The next step is the modification of the curve $\Gamma$: We will consider inside the surface $S$, the new (nonadmissible) curve $\hat{\Gamma}(t)=(0,y^*(t),t)$. The corresponding new motion planning problem will be denoted by $\hat{\Sigma}$.

Let us also make a change of parametrization of the surface $S$, for the curve $\hat{\Gamma}(t)$ becomes $(0,0,t)$. For this, we will set:

$$\hat{y}=y-y^*(w). \tag{3.8}$$

As a last step, we will make a change of normal coordinates according to (2.2), in such a way that, in the new coordinates, along $\hat{\Gamma}$, $\frac{\partial}{\partial x_1}=X^*(t)$, $\frac{\partial}{\partial x_2}=U(t)$, $\forall t\in[0,T]$.

Note that the surface $S$ does not change (only its parametrization changes), and these new coordinates are still normal coordinates for $\hat{\Sigma}$ around $\hat{\Gamma}$.

Thus, on $\mathcal{T}_\varepsilon^S$ we have:

$(a)$ $\dot{x}=u+O^2(\varepsilon)$,

$(b)$ $i=1,...,p-1:$ $\dfrac{d\hat{y}_i}{dt}=\dot{y}_i-\dfrac{dy_i^*}{dw}\dot{w}=\dfrac{1}{2}x'L_i(y,w)u-\dfrac{dy_i^*}{dw}\dot{w}+O^2(\varepsilon)$,

$(c)$ $2\dot{w}=x'M(y,w)u+O^2(\varepsilon)$.

Let us expand $(b)$ and $(c)$, modulo $O^2(\varepsilon)$ :

$$\begin{aligned}\frac{d\hat{y}_i}{dt}&=\frac{1}{2}x'L_i(\hat{y}+y^*(w),w)u+\frac{1}{U(w)'M(0,w)M(y^*(w),w)U(w)}X^*(w)'L_i(y^*(w),w)U(w)\dot{w}\\&=\frac{1}{2}x'L_i(\hat{y}+y^*(w),w)u+\\&\frac{1}{U(w)'M(0,w)M((y^*(w),w)U(w)}X^*(w)'L_i(y^*(w),w)U(w)\frac{1}{2}x'M(\hat{y}+y^*(w),w)u\\&=x'\tilde{L}_i(\hat{y},w)u=x'\tilde{L}_i(0,w)u+x'\hat{L}_i(\hat{y},w;\hat{y})u,\end{aligned}$$

where $\hat{L}_i(\hat{y},w;z)$ are linear in $z$.

Now it is clear that, for $x=X^*(w)$, $u=U(w)$, and $\hat{y}=0$, we get for all $i=1,...,p-1$,

$$\frac{d\hat{y}_i}{dt}=O^2(\varepsilon),\quad i=1,...,p-1.$$

This means exactly that:

$$\tilde{L}_i(0,w)_{1,2}=\tilde{L}_i(0,w)_{2,1}=0 \tag{3.9}$$

We have also:

$$\dot{w}=\frac{1}{2}x'M(\hat{y}+y^*(w),w)u=\frac{1}{2}x'M(y^*(w),w)u+x'\hat{M}(\hat{y},w;\hat{y})u,$$

where $\hat{M}(\hat{y},w;z)$ is linear in $z$, and $M(y^*(w),w)$ is arbitrarily close to $M(0,w)$, which is reducible to $J$ by conjugation, depending on $w$. In particular, the entries $M(y^*(w),w)_{1,2}$ and $M(y^*(w),w)_{2,1}$ can be made arbitrarily close to 1 and -1.

Then, if $\mathcal{T}_\varepsilon^{\hat{}}$ denotes the $\varepsilon - SR$-tube around the curve $\hat{\Gamma}$, we get the following normal form, in normal coordinates relative to the same surface $S:$

$$\begin{aligned}
\dot{x} &= u + O^2(\varepsilon), \\
\frac{d\hat{y}_i}{dt} &= \frac{1}{2}x'\tilde{L}_i(0,w)u + O^2(\varepsilon), \quad i = 1, ..., p-1, \\
\dot{w} &= \frac{1}{2}x'M(y^*(w), w)u + O^2(\varepsilon), \\
\text{with}: \ & \tilde{L}^i(0,w)_{1,2} = \tilde{L}^i(0,w)_{2,1} = 0, \\
\text{and}: \ & |M(y^*(w), w)_{1,2}|, \ |M(y^*(w), w)_{2,1}|, \text{ close to } 1.
\end{aligned} \tag{3.10}$$

This is due to Theorem 5 (normal cylinder box Theorem, which states that, on $\mathcal{T}_\varepsilon^{\hat{}}$, $\hat{y} = O^2(\varepsilon)$.

Now, the end of the proof will be similar to the proof for coranks 2, 3 in [9], which was crucially based on this normal form. Let us sketch this proof for the sake of completeness.

3.2. **Proof of Theorem 3, part 2.** We start with a motion planning problem $\Sigma$

in normal form (3.10), i.e.:

$$\begin{aligned}
\dot{x} &= u + O^2(\varepsilon), \\
\frac{d\hat{y}_i}{dt} &= \frac{1}{2}x'L_i(0,w)u + O^2(\varepsilon), \quad i = 1, ..., p-1, \\
\dot{w} &= \frac{1}{2}x'M(w)u + O^2(\varepsilon), \\
\text{with}: \ & L^i(0,w)_{1,2} = L^i(0,w)_{2,1}. = 0, \\
\text{and}: \ & M(w)_{1,2}, \ M(w)_{2,1}, \text{ close to1 and -1}.
\end{aligned} \tag{3.11}$$

As we said previously, we forget about the isolated points, that can be treated roughly with the estimate of Corollary 1. We set $\alpha_1(w) = |M(w)_{1,2}| = |M(w)_{2,1}|$

Consider the tube $\mathcal{T}_\varepsilon$, relative to $\Gamma = \{(0,0,w)|0 \leq w \leq h_1\}$.

The following constructive procedure will provide the estimation we need. This procedure is described in several recurrent steps:

We fix $\omega$, $0 < \omega < 1$,

***Step 1**: Start from $(x, y, w) = (0, 0, 0)$, and go to $(x = x_1, 0, 0)$, with $x_1 = (\omega\varepsilon, 0, ..., 0)$. This step costs a length (or a time) less than $\varepsilon$;

***Step2**: For $\varepsilon > 0$, consider the $(p+1)$-dimensional cylinder:

$$\mathcal{C}^{\varepsilon,\omega} = \{\xi | \sqrt{(x_1)^2 + (x_2)^2} = \varepsilon\omega, (x_3)^2 + ... + (x_{n-p})^2 = 0\}. \tag{3.12}$$

For $\varepsilon$ small enough $\mathcal{C}^{\varepsilon,\omega}$ is transversal to $\Delta$. The intersection of $\Delta$ with the tangent space to $\mathcal{C}^{\varepsilon,\omega}$ defines a field of tangent lines to $\mathcal{C}^{\varepsilon,\omega}$. Then it defines two opposite unitary vector fields. It is easily computed that the $w$-component of these vector fields is $\pm\frac{\omega\varepsilon}{2}\alpha_1(w)\frac{\partial}{\partial w} + O^2(\varepsilon)$. Again $O^2(\varepsilon)$ is a function bounded by $\varepsilon^2 K$, for some $K > 0$. Denote by $X$ the vector field along which $w$ increases.

- **Step 2.a.** Now, follow the flow of $X$, till getting out of $\mathcal{T}_\varepsilon$. It costs a certain length (or time) $T_1$, that we will majorize in a moment, to reach a certain height $w = W_{1,}$ that we will bound from below later.

- **Step 2.b.** Follow a piece of radial admissible curve and return back to $\tilde{\Gamma}$. This can be done in time (or length) $\varepsilon$ (since we are still inside $\mathcal{T}_\varepsilon$), and this makes eventually $w$ decrease of $h\varepsilon^2$, for a certain $h > 0$, as we read on the normal form. This $h\varepsilon^2$ can be compensated in time $h'\varepsilon > 0$, by Corollary 1. Finally the step 2.b. costs $h''\varepsilon$ for some $h'' > 0$.

Repeat steps 1-2 until the end of the curve $\tilde{\Gamma}$, $w = h_1$.

Now we estimate the time $T_1$ and the height $W_1$.

**Lemma 10.** $T_1 \leq \frac{2}{\omega\varepsilon}\int_0^{W_1} \frac{dw}{\alpha_1(w)}(1+\varepsilon H)$, *for some constant* $H > 0$, *and the time* $T_1 > 0$ *to stay inside* $\mathcal{T}_\varepsilon$ *can be taken independent of* $\varepsilon$.

The proof is easy, and may be found in [9]

Then, if we repeat step 2 $k$-times, with $k = \frac{A}{\varepsilon}$, we get a time $T$ satisfying, for some $A' > 0$:

$$T \leq \frac{2}{\omega\varepsilon}\int_0^{h_1} \frac{dw}{\alpha_1(w)} + A'.$$

Then, $\inf_{\substack{\gamma\subset\mathcal{T}_\varepsilon \\ \gamma(0)=\Gamma(0) \\ \gamma(T_\gamma)=\Gamma(w_2)}} (T(\gamma)) \leq \frac{2}{\omega\varepsilon}\int_0^{h_1} \frac{dw}{\alpha_1(w)} + A'$, and,

$$\frac{\varepsilon MC_\Sigma(\varepsilon, h_1)}{\frac{2}{\varepsilon}\int_0^{h_1} \frac{dw}{\alpha_1(w)}} \leq \frac{1}{\omega} + \varepsilon K_1,$$

for some $K_1 > 0$. Hence,

$$\limsup_{\varepsilon\to 0} \frac{\varepsilon MC_\Sigma(\varepsilon, w_{h_1})}{\frac{2}{\varepsilon}\int_0^{h_1} \frac{dw}{\alpha_1(w)}} \leq \frac{1}{\omega}.$$

Since this is true for all $0 < \omega < 1$, we get:

$$MC_\Sigma(\varepsilon, w_2) \leq_s \frac{2}{\varepsilon^2}\int_0^{h_1} \frac{dw}{\alpha_1(w)}.$$

Note that $\alpha_1(w)$ is close to one, which shows that the metric complexity is of the form

$\frac{2}{\varepsilon^2}A$, where $A$ is close to $\int_0^{h_1} dw$, which is what was needed to prove.

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