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ON THE MOTION PLANNING PROBLEM, COMPLEXITY, ENTROPY, AND NONHOLONOMIC INTERPOLATION

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Abstract. We consider the sub-Riemannian motion planning problem defined by a sub-Riemannian metric (the robot and the cost to minimize) and a non-admissible curve to be ε-approximated in the sub-Riemannian sense by a trajectory of the robot. Several notions characterize the ε-optimality of the approximation: the "metric complexity" MC and the "entropy" E (Kolmogorov–Jean). In this paper, we extend our previous results. 1. For generic one-step bracket-generating problems, when the corank is at most 3, the entropy is related to the complexity by $E = 2\pi MC$. 2. We compute the entropy in the special 2-step bracket-generating case, modelling the car plus a single trailer. The ε-minimizing trajectories (solutions of the "ε-nonholonomic interpolation problem"), in certain normal coordinates, are given by Euler's periodic inflexional elastica. 3. Finally, we show that the formula for entropy which is valid up to corank 3 changes in a wild case of corank 6: it has to be multiplied by a factor which is at most 3/2.

1. Introduction, notation, statement of the problems

A general motion planning problem from robotics is defined by a triple (Δ, g, Γ):

(i) a distribution Δ over $\mathbb{R}^n$ of a certain corank p, which represents the admissible motion (the kinematic constraints) of the robot;

(ii) a Riemannian metric g over Δ providing a (sub-Riemannian) metric structure d to measure the length of admissible curves (actually realized by the robot);

(iii) a smooth nonadmissible curve $\Gamma : [0,1] \to \mathbb{R}^n$ which we want to approximate by an admissible one. In practice, the choice of Γ makes the robot avoid possible obstacles in $\mathbb{R}^n$.

For given dimension n of the ambient space and corank p of the distribution, the set of motion planning problems $\Sigma = (\Delta, g, \Gamma)$ is denoted by

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$\mathcal{S}$. We endow $\mathcal{S}$ with the standard C^∞-topology. [Since our problems are always local around a compact curve Γ, there is no need to control anything at infinity, and Whitney topology is not used].

Remark 1. Sometimes, we will impose extra assumptions (e.g., assuming that distribution is one-step bracket generating). We will still keep the notation $\mathcal{S}$ for the relevant smaller open set of motion planning problems. Assuming that the problem is real analytic, we restrict the C^∞-topology to the subset of analytic objects.

We are interested in *approximating* or *interpolating* the curve Γ by admissible curves, ε-close (in the sub-Riemannian sense), and we want to analyze what happens as ε tends to zero (or at least is very small). This corresponds to the practical situation of an ambient space $\mathbb{R}^n$ almost full of obstacles.

We will need "equivalents" of quantities as ε tends to zero. We say that two functions $f_1(\varepsilon)$ and $f_2(\varepsilon)$ tending to $+\infty$ as ε tends to zero are weakly equivalent ($f_1 \simeq_w f_2$) if, as usual,

$$k_1 f_1(\varepsilon) \le f_2(\varepsilon) \le k_2 f_1(\varepsilon)$$

for certain strictly positive constants k_1 and k_2.

We say that $f_1(\varepsilon)$ and $f_2(\varepsilon)$ are strongly equivalent ($f_1 \simeq_s f_2$) if

$$\lim_{\varepsilon\to 0} \frac{f_1(\varepsilon)}{f_2(\varepsilon)} = 1.$$

The notation $f_1 \ge_s f_2$ is also used; it means that

$$\liminf_{\varepsilon\to 0} \frac{f_1(\varepsilon)}{f_2(\varepsilon)} \ge 1.$$

Now we define two crucial concepts associated with a motion planning problem: *metric complexity* and *entropy*.

For a given motion planning problem $\Sigma = (\Delta, g, \Gamma)$, denote by T_ε (respectively, C_ε) the sub-Riemannian ε-tube (respectively, ε-cylinder) around Γ:

$$T_\varepsilon = \{x \in \mathbb{R}^n;\ d(x,\Gamma) \le \varepsilon\},$$
$$C_\varepsilon = \{x \in \mathbb{R}^n;\ d(x,\Gamma) = \varepsilon\}.$$

Let $\gamma_\varepsilon : [0, t_{\gamma_\varepsilon}] \to \mathbb{R}^n$ be a parametrized admissible (almost everywhere tangent to Δ) curve (moreover, we can assume that it is Lipschitz, and arclength parametrized) such that $\gamma_\varepsilon([0, t_{\gamma_\varepsilon}]) \subset T_\varepsilon$ and $\gamma_\varepsilon(0) = \Gamma(0)$ and $\gamma_\varepsilon(t_{\gamma_\varepsilon}) = \Gamma(1)$. The *metric complexity* $MC(\varepsilon)$ is the (weak or strong) equivalence class of the infimum of the length $l(\gamma_\varepsilon) = t_{\gamma_\varepsilon}$ of such curves γ_ε divided by ε:

$$MC(\varepsilon) = \frac{1}{\varepsilon} \inf l(\gamma_\varepsilon).$$

Therefore, the metric complexity measures asymptotically (as ε tends to zero) the minimum length of ε-*approximating* curves.

Now assume that γ_ε has an extra property (the *ε-nonholonomic interpolation property*): γ_ε is formed by a finite number of pieces connecting points of Γ and the length of each piece does not exceed ε. The *entropy* $E(\varepsilon)$ is again a weak or strong equivalent of the infimum of the total length of such curves γ_ε, divided by ε:

$$E(\varepsilon) = \frac{1}{\varepsilon} \inf\{l(\gamma_\varepsilon);\ \gamma_\varepsilon \text{ is interpolating}\}.$$

Note that the usual Kolmogorov's definition of the entropy deals with the asymptotics of the minimum number of ε-balls covering Γ. Therefore, it is one half of our entropy.

For both metric complexity and entropy, one is interested in "realizing" constructively the asymptotic optimal strategy. A parametrized family of curves γ_ε weakly or strongly realizing the minimum is called a (weak or strong) *asymptotic optimal synthesis* (for the complexity or entropy).

2. Some preliminary results

We need some results from our previous papers [6–9].

We omit the explicit asymptotic optimal syntheses constructed in these papers and recall only the expressions of the metric complexity.

Assume that Δ is one-step bracket-generating.

For a nonadmissible curve $\Gamma : [0,1] \to \mathbb{R}^n$ which is transversal to the distribution Δ, we define a field along Γ of $(p-1)$-dimensional affine spaces Ω_t, of linear skew symmetric (with respect to g) endomorphisms of $\Delta(\Gamma(t))$,

$$A_t = A_t^0 + \sum_{i=1}^{p-1} \lambda_i A_t^i, \quad \lambda_i \in \mathbb{R}, \quad t \in [0,1],$$

as follows. Consider 1-forms α which vanish on Δ and which take value 1 on the vector $\dfrac{d\Gamma}{dt}(t)$. Then we set

$$\langle A_t X, Y \rangle_g = d\alpha(X,Y) = \alpha([X,Y])$$

for any $X, Y \in \Delta(\Gamma(t))$.

The fact that Δ is one-step bracket-generating ensures that Ω_t is a well-defined $(p-1)$ dimensional affine space not containing the zero. Now we define the principal invariant χ of the motion planning problem:

$$\chi(t) = \inf_{\Omega_t} \|A_t\|_g. \tag{2.1}$$

The function $\chi(t)$ is strictly positive and continuous. If at some isolated point t_0, $\Gamma(t)$ is tangent to Δ, then $\chi(t)$ tends to infinity as $t \to t_0$. Generically, this happens only for corank $p = 1$.

Theorem 1 (see [6,8])**.** $p \leq 3$, the one-step bracket-generating case. *There exists an open dense subset $\mathcal{S}^* \subset \mathcal{S}$ of motion planning problems such that*

$$MC(\varepsilon) \simeq_s \frac{2}{\varepsilon^2} \int\limits_{\Gamma} \frac{dt}{\chi(t)}. \tag{2.2}$$

For corank 1, this formula is also true, despite the fact that $\chi(t)$ can be infinite at some isolated points.

Also, in the case where $p = 1$, omitting the one-step bracket-generating assumption, a single new generic situation can appear, and this can happen only for $n = 3$: isolated Martinet points of Δ along Γ. Generically, at these points χ vanishes but remains smooth. Let $\varrho(t) = \left|\frac{d\chi}{dt}\right|(t)$.

Theorem 2 (see [6,7])**.** $p = 1$, no bracket assumption. *There is an open dense subset $\mathcal{S}^* \subset \mathcal{S}$ for which, either*

1. *formula* (2.2) *holds or*
2. $n = 3$ *and*

$$MC(\varepsilon) \simeq_s \sum_{\substack{\text{Martinet points}\\ t_i \text{ on } \Gamma}} -\frac{4}{\varrho(t_i)} \frac{\log(\varepsilon)}{\varepsilon^2}.$$

In the one-step bracket-generating case, the situation changes in a subtle way when the corank $p \geq 4$. This will be illustrated in Sec. 7, but in [8, 9], we have proved the following two theorems.

Theorem 3 (see [8])**.** The one-step bracket-generating case, arbitrary corank p.

$$MC(\varepsilon) \geq_s \frac{2}{\varepsilon^2} \int\limits_{\Gamma} \frac{dt}{\chi(t)}. \tag{2.3}$$

Theorem 4 (see [9])**.** The generic situation, one-step bracket generating, $n = 10$, $p = 6$. *For problems from an open-dense subset $\mathcal{S}^* \subset \mathcal{S}$, the following holds*: *there exists a curve $\tilde{\Gamma}$, which is arbitrarily C^0-close to Γ, and the metric complexity of $\tilde{\Gamma}$ is $2/\varepsilon^2 A$ with the constant A arbitrarily close to $\int\limits_{\Gamma} \frac{dt}{\chi(t)}$ (where $\chi(t)$ is relative to Γ).*

Therefore, in fact, the metric complexity estimate (2.2) is "almost true."

To clarify what happens for $p \geq 4$ (especially for the 10–6 case), for $0 \leq t \leq 1$, consider the image $\tilde{B}_t$ of the product $B_t \times B_t \subset \Delta_{\Gamma(t)} \times \Delta_{\Gamma(t)}$ of two unit balls via the bracket mapping $[\cdot,\cdot]$ into the quotient tangent space $T_{\Gamma(t)}\mathbb{R}^n/\Delta(\Gamma(t))$. Recall that the mapping $[\cdot,\cdot] \operatorname{mod} \Delta$ is a tensor.

Definition 1. The set $\tilde{B}_t$ is said to be *strictly convex in the direction* $V_t \in T_{\Gamma(t)}\mathbb{R}^n/\Delta(\Gamma(t))$ if:

(P1) there exist $x^* = \lambda V_t \bmod \Delta_{\Gamma(t)} \in \tilde{B}_t$, $\lambda > 0$, and a form ω from the dual space $(T_{\Gamma(t)}\mathbb{R}^n/\Delta_{\Gamma(t)})^* \approx (\mathbb{R}^p)^*$ such that for all $y \in \tilde{B}_t$,

$$\omega(x^*) - \omega(y) \geq 0$$

or (equivalent condition),

(P2) if $V^* = \{\omega \in (\mathbb{R}^p)^*,\ \omega(V_t) = 1\}$, then there exist $\omega^* \in V^*$ and $x^* = \lambda V_t \in \tilde{B}_t$, $\lambda > 0$, with the property

$$\omega^*(x^*) = \sup_{x\in\tilde{B}_t} \omega^*(x) = \inf_{\omega\in V^*} \sup_{x\in\tilde{B}_t} \omega(x).$$

Properties (P1) and (P2) are equivalent since $\tilde{B}_t$ is not an arbitrary set: it is symmetric and star-shaped with respect to the origin, and by one-step bracket-generating assumption, it spans $\mathbb{R}^{n-p}$ as a vector space.

When $p \leq 3$, for generic one-step bracket-generating problems, the set $\tilde{B}_t$ is always strictly convex in the direction of $\dot{\Gamma}(t)$ mod $\Delta(\Gamma(t))$ (this was shown in [8, 9]). On the contrary, for $p \geq 4$, $\tilde{B}_t$ is not strictly convex in general. In the 10–6 case, the situation is even much more interesting: generically, the direction of $\dot{\Gamma}(t)$ never meets $\tilde{B}_t \setminus \{0\}$ (except for some isolated points). This is justified by simple dimension arguments: projectivization of $\dot{\Gamma}(t) \bmod \Delta(\Gamma(t))$ lives in the (5-dimensional) projective space $\mathbb{RP}^5$, while the projectivization of $\tilde{B}_t$ is the image under the mapping $[\cdot,\cdot]/\Delta$ of the 4-dimensional Grassmannian of 2-planes in $\mathbb{R}^4$.

We also omit the results of [6–9] on the smoothness of the (leading term of the) metric complexity as a function of the endpoint of Γ.

3. Statement of the results and organization of the paper

This paper is mostly devoted to the *entropy* and its interactions with the *metric complexity*. We start from a result about the one-step bracket-generating case with $p \leq 3$.

Theorem 5. The one-step bracket-generating case, $p \leq 3$. *There exists an open-dense subset $\mathcal{S}^*$ of $\mathcal{S}$ such that the problems from $\mathcal{S}^*$ yield*

$$E(\varepsilon) \simeq_s \frac{4\pi}{\varepsilon^2} \int\limits_{\Gamma} \frac{dt}{\chi(t)}. \tag{3.1}$$

In other words, the entropy is equal to 2π times the metric complexity.

Remark 2. This theorem is also valid for $p > 3$ but for any t, $\tilde{B}_t$ is strictly convex in the direction of Γ.

The following result describes the generic case of a corank 6 distribution in $\mathbb{R}^{10}$ (wild 10–6 case).

Theorem 6. The one-step bracket-generating, analytic wild 10–6 case. *There exists an open dense subset $\mathcal{S}^*$ of $\mathcal{S}$ such that, for $\Sigma \in \mathcal{S}^*$, there exists another invariant $\varrho(t)$, $|\varrho(t)| \leq 1$, such that:*

1. *if* $\varrho(t) = \pm 1$ *identically, then* $\tilde{B}_t$ *is strictly convex in the direction of* Γ*, and the entropy is still given as follows*:
$$E(\varepsilon) \simeq_s \frac{4\pi}{\varepsilon^2} \int_\Gamma \frac{dt}{\chi(t)};$$
2. *otherwise, the entropy has the following property*:
$$\frac{4\pi}{\varepsilon^2} \int_\Gamma \frac{dt}{\chi(t)} \leq_s E(\varepsilon) \leq_s \frac{6\pi}{\varepsilon^2} \int_\Gamma \frac{dt}{\chi(t)}. \tag{3.2}$$
3. *In particular, if* ϱ *is a nonzero constant, then*
$$E(\varepsilon) \simeq_s \frac{2(3 - |\varrho|)\pi}{\varepsilon^2} \int_\Gamma \frac{dt}{\chi(t)}.$$

Hence, in the worst case, there is a ratio $3/2$ between the formula of entropy of the 10–6 case and formula (3.1) for entropy when $p \leq 3$.

We will also consider the unique 2-step bracket-generating generic 4–2 case, which corresponds to the *car with a trailer*.

In this case, we define the "normalized abnormal vector field." It is an intrinsic admissible vector field H on $\mathbb{R}^4$ defined as follows: if F and G are two orthonormal vector fields defining a sub-Riemannian metric, then $H = uF + vG$, where
$$\begin{gathered} u^2 + v^2 = 1, \\ \det(F, G, [F, G], u[F, [F, G]] + v[G, [F, G]]) = 0. \end{gathered} \tag{3.3}$$

In a generic (open, dense) case, this vector field H is well defined up to the sign in a neighborhood of Γ and is *completely intrinsic*. Denote by I an unitary (with respect to the metric g) vector field in the distribution orthogonal to H. We set $K = [I, H]$. K is well defined up to a sign, and in the generic situation, it never belongs to Δ and is never collinear to $\dot{\Gamma}$. Therefore, we have a canonical 3-frame (I, H, K) on $\mathbb{R}^4$, defining another sub-Riemannian metric on $\mathbb{R}^4$, of corank 1, which is not tangent to $\dot{\Gamma}$ except for some isolated points. The influence of these isolated points on the metric complexity and entropy is negligible (see [8] for the arguments in the corank-one case). The underlying distribution Δ' is just the derivative distribution of Δ. The metric over Δ' is denoted by g' (the frame (I, H, K) is orthonormal with respect to g').

Let γ be a one-form which vanishes on I, H, and K, and which is 1 on $\dot{\Gamma}$. (It is uniquely defined modulo a function which is 1 on Γ). Then, we define a field of skew-symmetric endomorphisms of $\Delta'(\Gamma(t))$ along Γ as follows:
$$\left\langle \hat{A}(t)X, Y \right\rangle_{g'} = d\gamma(X, Y) = \gamma([X, Y]) \quad \forall X, Y \in \Delta'(\Gamma(t)).$$

Also, we set

$$\delta(t) = \|\hat{A}(t)\|_{g'} \quad \forall t \in [0,1].$$

The function $\delta(t)$ is strictly positive and independent of the choice of signs for vector fields I, H, and K.

We prove the following theorem.

Theorem 7. The 2-step bracket-generating case, $n = 4$, $p = 2$. *There is an open-dense subset $\mathcal{S}^*$ of $\mathcal{S}$ such that the entropy has the following expression*:

$$E(\varepsilon) = \frac{3}{2\sigma\varepsilon^3} \int\limits_{\Gamma} \frac{dt}{\delta(t)}.$$

Here $\sigma \approx 0.00580305$ is a certain universal constant.

Theorems 5–7 are constructive. We provide either an explicit asymptotic optimal synthesis, or a method of constructing it.

The proofs are based on the notions of "normal coordinates" and "normal form" along Γ (introduced in Sec. 4). Then we define *nilpotent approximations along* Γ of a motion-planning problem and prove the following Theorem 8, reducing Theorems 5–7 to the consideration of respective nilpotent approximations only.

Theorem 8. *In all cases under consideration, the entropy of the motion planning problem is strongly equivalent to that of the nilpotent approximation along* Γ.

Hence, the organization of the paper is as follows. Section 4 presents all the tools we need: normal coordinates, normal forms, nilpotent approximations along Γ, and certain rough estimates. Section 5 contains the proof of Theorem 8. Section 6 gives the proof of Theorem 5 about the ratio 2π between the entropy and complexity for $p \leq 3$. Section 7 gives the proof of Theorem 6 about entropy in the wild 10–6 case. Section 8 gives the proof of Theorem 7 about entropy in the 4–2 case. The asymptotic optimal synthesis, in the normal coordinates, turns out to be the inflexional periodic Euler elastica. Finally, in the Appendix, we prove a few useful auxiliary facts.

4. Technical preliminaries

In this section, we recall main constructions of [1–4, 6–9], needed in the sequel.

4.1. **Normal coordinates.** Consider a motion planning problem $\Sigma = (\Delta, g, \Gamma)$, not necessarily one-step bracket-generating. Take a (germ along Γ of) *parametrized* p-dimensional surface S, transversal to Δ,

$$S = \{q(s_1, \dots, s_{p-1}, t) \in \mathbb{R}^n\}, \quad \text{where} \quad q(0, \dots, 0, t) = \Gamma(t).$$

Such a germ exists if Γ is not tangent to Δ. As was already mentioned in Sec. 3, excluding a neighborhood of an isolated point, where Γ is tangent to Δ, i.e., Γ becomes "almost admissible," does not affect the estimates.

Lemma 1 (normal coordinates with respect to S)**.** *There exist mappings $x : \mathbb{R}^n \to \mathbb{R}^{n-p}$, $y : \mathbb{R}^n \to \mathbb{R}^{p-1}$, and $w : \mathbb{R}^n \to \mathbb{R}$ such that $\xi = (x, y, w)$ is a coordinate system in some neighborhood of S in $\mathbb{R}^n$ such that*:

0. $S(y, w) = (0, y, w)$, $\Gamma = \{(0, 0, w)\}$;
1. $\Delta_{|S} = \ker dw \cap \bigcap\limits_{i=1,\dots,p-1} \ker dy_i$, $g_{|S} = \sum\limits_{i=1}^{n-p} (dx_i)^2$;
2. $\mathcal{C}_\varepsilon^S = \left\{ \xi \;\middle|\; \sum\limits_{i=1}^{n-p} {x_i}^2 = \varepsilon^2 \right\}$;
3. *geodesics of the Pontryagin maximum principle* [16] *satisfying the transversality conditions with respect to S are the straight lines through S contained in the planes $P_{y_0,w_0} = \{\xi \mid (y, w) = (y_0, w_0)\}$. Hence, they are orthogonal to S.*

These normal coordinates are unique up to changes of coordinates of the form

$$\tilde{x} = T(y, w)x, \quad (\tilde{y}, \tilde{w}) = (y, w), \tag{4.1}$$

where $T(y, w) \in O(n - p)$, the $(n - p)$-orthogonal group.

Here, $\mathcal{C}_\varepsilon^S$ denotes the cylinder $\{\xi;\ d(S, \xi) = \varepsilon\}$.

4.2. **Normal form.**

4.2.1. *Frames.* A motion planning problem can be specified by a couple (Γ, F), where $F = (F_1, \dots, F_{n-p})$ is a g-orthonormal frame of vector fields generating Δ. Hence we will also write $\Sigma = (\Gamma, F)$. If a global coordinate system (x, y, w), not necessarily normal, is given in a neighborhood of Γ in $\mathbb{R}^n$, where $x \in \mathbb{R}^{n-p}$, $y \in \mathbb{R}^{p-1}$, and $w \in \mathbb{R}$, then we write:

$$F_j = \sum_{i=1}^{n-p} \mathcal{Q}_{i,j}(x, y, w)\frac{\partial}{\partial x_i} + \sum_{i=1}^{p-1} \mathcal{L}_{i,j}(x, y, w)\frac{\partial}{\partial y_i} + \mathcal{M}_j(x, y, w)\frac{\partial}{\partial w}, \tag{4.2}$$

where $j = 1, \dots, n - p$. Hence, the sub-Riemannian metric is specified by the triple $(\mathcal{Q}, \mathcal{L}, \mathcal{M})$ of smooth (x, y, w)-dependent matrices, and we also write $\Sigma = (\Gamma, \mathcal{Q}, \mathcal{L}, \mathcal{M})$. If, in the chosen coordinates (e.g., in normal coordinates), $\Gamma(t) = (0, 0, t)$, then we write $\Sigma = (\mathcal{Q}, \mathcal{L}, \mathcal{M})$.

4.2.2. *The general normal form.* Fix a surface S as in Sec. 4.1 and a normal coordinate system $\xi = (x, y, w)$ for a problem Σ.

Theorem 9 (normal form)**.** *There exists a unique orthonormal frame $F = (\mathcal{Q}, \mathcal{L}, \mathcal{M})$ for (Δ, g) with the following properties*:

1. $\mathcal{Q}(x, y, w)$ *is symmetric,* $\mathcal{Q}(0, y, w) = \mathrm{Id}$ (*the identity matrix*);

2. $\mathcal{Q}(x, y, w)x = x$;
3. $\mathcal{L}(x, y, w)x = 0$, $\mathcal{M}(x, y, w)x = 0$;
4. *conversely, if* $\xi = (x, y, w)$ *is a coordinate system satisfying conditions* 1–3 *above, then* ξ *is a normal coordinate system for the sub-Riemannian metric defined by the orthonormal frame* F *with respect to the parametrized surface* $\{(0, y, w)\}$.

Clearly, this normal form is invariant with respect to the changes of normal coordinates (4.1).

Let us write:

$$\begin{aligned}\mathcal{Q}(x, y, w) &= \mathrm{Id} + Q_1(x, y, w) + Q_2(x, y, w) + \dots,\\ \mathcal{L}(x, y, w) &= 0 + L_1(x, y, w) + L_2(x, y, w) + \dots,\\ \mathcal{M}(x, y, w) &= 0 + M_1(x, y, w) + M_2(x, y, w) + \dots,\end{aligned}$$

where Q_k, L_k, and M_k are matrices depending on ξ, whose coefficients have order k with respect to x (i.e., they are in the kth power of the ideal of $C^\infty(x, y, w)$ generated by the functions x_r, $r = 1, \dots, n-p$). In particular, Q_1 is linear in x, Q_2 is quadratic, etc. Set $u = (u_1, \dots, u_{n-p}) \in \mathbb{R}^{n-p}$. Then

$$\sum_{j=1}^{p-1} L_{1_j}(x, y, w)u_j = L_{1,y,w}(x, u)$$

is quadratic in (x, u), and $\mathbb{R}^{p-1}$-valued. Its ith component is a quadratic expression denoted by $L_{1,i,y,w}(x, u)$. Similarly,

$$\sum_{j=1}^{p-1} M_{1_j}(x, y, w)u_j = M_{1,y,w}(x, u)$$

is a quadratic form in (x, u). The corresponding matrices are denoted by $L_{1,i,y,w}$, $i = 1, \dots, p-1$, and $M_{1,y,w}$.

The following proposition was proved in [2, 3] for corank 1.

Proposition 1. 1. $Q_1 = 0$;
2. $L_{1,i,y,w}$, $i = 1, \dots, p-1$, *and* $M_{1,y,w}$ *are skew symmetric matrices.*

4.2.3. *Special* 4–2*case.* In the two-step bracket-generating 4–2 case, there exists an important canonical choice of both the surface S and the rotation $T(y, w)$ from (4.1). We still use the notation of Sec. 3.

First, we reparametrize the curve $\Gamma(t)$ by setting

$$d\tau = \frac{3}{2}\frac{dt}{\delta(t)}. \tag{4.3}$$

From now on, we will work with the new parameter, keeping the initial one in the statement of the final results only. Thus, from now on, $\delta(t) = 3/2$.

Second, choose the surface S and its parametrization as follows:

$$S(s, t) = \exp sK(\Gamma(t)).$$

Third, choose the rotation $T(y,w)$ (see (4.1)) to make the normalized abnormal vector field H equal to $\partial/\partial x_2$ at S.

4.3. **Cylinders in normal coordinates.** The standard "ball-box theorem" (see [10]) and the properties of the normal form imply the following estimates.

Let $\xi = (x, y, w)$ be a normal coordinate system and let $F = (\mathcal{Q}, \mathcal{L}, \mathcal{M})$ be the associated normal form. Assume that either Δ is one-step bracket generating, or Σ is 4–2 generic (in particular, it is two-step bracket generating).

Theorem 10 (normal cylinder-box theorem [8])**.**

1. *If* $\xi = (x, y, w) \in T_\varepsilon$, *then*

$$\|x\|_2 \le \varepsilon, \quad \|y\|_2 \le k_2\varepsilon^2$$

for some $k_2 > 0$.

2. *Take* $0 < \omega < 1$ *and set*

$$K_\varepsilon^{k_1,\omega} = \{\xi = (x, y, w) \mid \|x\|_2 \le \omega\varepsilon,\ \|y\|_2 \le k_1\varepsilon^2\}.$$

Then for sufficiently small k_1, $K_\varepsilon^{k_1,\omega} \subset T_\varepsilon$.

Obviously, Theorem 10 (stated in [8] for the one-step bracket-generating case) holds also in the 4–2 case with the special choice of normal coordinates, as above.

4.4. **Nilpotent approximations along Γ.** Fix a normal coordinate system (according to the rules of Sec. 4.2.3 in the 4–2 case) and the corresponding normal form.

Restricting to the tubes $T_\varepsilon \subset T_{\varepsilon_0}$ for sufficiently small $\varepsilon \le \varepsilon_0$, assign the weights 1, 2, and 0 to the variables x_i, y_i, and w, respectively (according to their orders in ε in Theorem 10). Then the vector field $\partial/\partial x_i$ has the weight -1, $\partial/\partial y_i$ has the weight -2, and (to agree with the "local effect" in the direction of Γ) for $\partial/\partial w$, we set the weight -2 in the one-step bracket-generating case, and -3 in the 4–2 case.

Definition 2. The nilpotent approximation $\hat{\Sigma}$ of Σ along Γ consists of the sub-Riemannian metric obtained by keeping only the terms of order -1 in the normal form (i.e., an orthonormal frame for $\hat{\Sigma}$ is the normal frame of Σ truncated at order -1).

The nilpotent approximation in the one-step bracket-generating case has the following form (using the control system notation):

$$\begin{aligned} \dot{x} &= u, \\ \dot{y}_i &= \frac{1}{2}x' L^i(w)u, \quad i = 1, \dots, p-1; \\ \dot{w} &= \frac{1}{2}x' M(w)u. \end{aligned} \tag{4.4}$$

Here x' is the transpose of x and w is the coordinate along Γ. The matrices L^i and M depending on w are skew symmetric.

In fact, the (y,w)-space is identified with the surface S, and with the quotient $T_{\Gamma(w)}\mathbb{R}^n/\Delta(\Gamma(w))$, and the mapping

$$(x,u) \to (x'L^1(w)u, \dots, x'L^{p-1}(w)u, x'M(w)u)$$

is just the coordinate form of the bracket mapping $[\cdot,\cdot]/\Delta$.

The properties of the normal form imply the following lemma.

Lemma 2. *For an admissible trajectory of Σ which remains in T_ε, we have*

$$\begin{aligned} \dot{x} &= u + O(\varepsilon^2), \\ \dot{y}_i &= \frac{1}{2}x'L^i(w)u + O(\varepsilon^2), \quad i = 1, \dots, p-1, \\ \dot{w} &= \frac{1}{2}x'M(w)u + O(\varepsilon^2), \end{aligned} \tag{4.5}$$

where $O(\varepsilon^2)$ denote smooth functions bounded by $C\varepsilon^2$ for some appropriate positive constant C.

Respectively, in the 4–2 generic case, the nilpotent approximation takes the form

$$\begin{aligned} \dot{x}_1 = u_1; \dot{x}_2 &= u_2, \\ \dot{y} &= \frac{1}{2}(x_2u_1 - x_1u_2), \\ \dot{w} &= \frac{1}{2}x_1(x_2u_1 - x_1u_2), \end{aligned} \tag{4.6}$$

which implies the following lemma.

Lemma 3 (the two-step bracket-generating 4–2 case)**.** *For an admissible trajectory of Σ which remains in T_ε, we have*

$$\begin{aligned} \dot{x} &= u + O(\varepsilon^2), \\ \dot{y} &= \frac{1}{2}(x_2u_1 - x_1u_2) + O(\varepsilon^2), \\ \dot{w} &= \frac{1}{2}x_1(x_2u_1 - x_1u_2) + O(\varepsilon^3), \end{aligned} \tag{4.7}$$

where $O(\varepsilon^i)$ are smooth functions bounded by $C\varepsilon^i$ for some appropriate positive constant C.

Remark 3. Note that in the 4–2 case, due to the canonical normalizations, the nilpotent approximation is unique: it does not depend on any parameter.

4.5. **A rough estimate.** We need to estimate sub-Riemannian balls with centers on Γ. Denote by $B_t(\varepsilon)$ a ε-sub-Riemannian ball centered at the point $(0,0,t)$ of Γ and by $\hat{B}_t(\varepsilon)$ the corresponding ball for the nilpotent approximation along Γ. The following lemma is an immediate consequence of the definition of normal coordinates and of the standard ball-box theorem from [10].

We still fix a normal coordinate system and the associated normal form, according to Sec. 4.2.

Lemma 4. *There exists a positive constant k such that the balls $B_{w_0}(\varepsilon)$ and $\hat{B}_{w_0}(\varepsilon)$ contain the following set, for all $(0,0,w_0) \in \Gamma$:*

1. *one-step bracket-generating case*:
$$\{(x,y,w);\ \|x\| \le k\varepsilon,\ \|y\| \le k\varepsilon^2,\ |w-w_0| \le k\varepsilon^2\};$$
2. *the generic* 4–2 *case, two-step bracket-generating*:
$$\{(x,y,w);\ \|x\| \le k\varepsilon,\ \|y\| \le k\varepsilon^2,\ |w-w_0| \le k\varepsilon^3\}.$$

4.6. **Nilpotent approximation in the** 10–6 **case.** Now we will improve the general expression (4.4) of the nilpotent approximation in the one-step bracket-generating case, using a change of coordinates on S and an appropriate change of normal coordinates (4.1). Note that the final form of the nilpotent approximation is independent of the choice of S itself.

Decompose the Lie algebra $so(4)$ in pure quaternions and pure skew-quaternions:
$$so(4) = P \oplus \hat{P},$$
where P is the vector space of pure quaternions, generated by i, j, and k:
$$i = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
and $\hat{P}$ is generated by $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, where
$$\hat{\imath} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \hat{\jmath} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \hat{k} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$
The following theorem holds for generic 10-6 analytic problems.

Theorem 11. *Outside an arbitrarily small neighborhood of a finite subset of Γ, there exists a choice of the normal coordinates, and a parametrization of S (preserving the parametrization of Γ), such that the nilpotent approximation takes the form*
$$\begin{gathered} \dot{x} = u, \qquad \dot{y}_1 = \frac{1}{2}x'(\hat{\imath} + \varrho(w)i)u, \qquad \dot{y}_2 = \frac{1}{2}x'ju, \\ \dot{y}_3 = \frac{1}{2}x'ku, \qquad \dot{y}_4 = \frac{1}{2}x'\hat{\jmath}u, \qquad \dot{y}_5 = \frac{1}{2}x'\hat{k}u, \\ \dot{w} = \frac{1}{2}\chi(w)x'iu. \end{gathered} \tag{4.8}$$

Here x' is the transpose of x and $\varrho(w)$ is a certain invariant of the motion planning problem, $-1 \le \varrho(w) \le 1$. The value $\varrho(w) = \pm 1$ if $\tilde{B}_w$ is strictly convex in the direction of Γ. Otherwise, the direction of $\dot{\Gamma}_w$ avoids $\tilde{B}_w$.

Proof. First, reparametrize Γ setting $d\tilde{w} = \dfrac{2}{\chi(w)} dw$. For brevity, below we omit tilde in the notation. Then, in the nilpotent approximation we can assume that $\dot{w} = x'M(w)u$, where $M(w)$ is skew-symmetric with double eigenvalues of modulus 1. (This was shown in [8] and is repeated in Corollary 1, Appendix 2; the "modulus one" claim comes from the reparametrization of Γ.) Now, a change of normal coordinates of the form $T(w)$ (see (4.1)), with the matrix $T(w)$ from $O(4, \mathbb{R})$ (not necessarily from $SO(4, \mathbb{R})$), yields $M(w) = i$. Finally, an appropriate change of parametrization of S identical on Γ yields

$$
\begin{aligned}
&\dot{x} = u, &&\dot{y}_1 = x'(\hat{\imath} + \varkappa_1(w)i)u, &&\dot{y}_2 = x'(j + \varkappa_2(w)i)u,\\
&\dot{y}_3 = x'(k + \varkappa_3(w)i)u, &&\dot{y}_4 = x'(\hat{\jmath} + \varkappa_4(w)i)u, &&\dot{y}_5 = x'(\hat{k} + \varkappa_5(w)i)u,\\
& &&\dot{w} = x'iu.
\end{aligned}
$$

Note that $\varkappa_2 = \varkappa_3 = 0$ since

$$\|i\| = \inf_{\lambda} \|i + \lambda_1(j + \varkappa_2 i) + \lambda_2(k + \varkappa_3 i)\|$$

(we can assume this from the very beginning, applying if needed an admissible change of coordinates on S of the form $\tilde{w} = w + \sum \mu_i(w) y_i$ preserving the parametrization on Γ). Note that the standard norm of quaternions or the Hilbert–Schmidt norm of the matrix coincide with the L_2-norm of the matrix up to some constant factors depending on conventions. Also, generically outside a finite subset of Γ, the coefficients μ_i can be chosen smooth (see [8]).

Moreover, the real analyticity implies that either all $\varkappa_i$ vanish identically or some of them are nonzero outside a finite set. (Recall that these special isolated points cause no trouble in the estimates of metric complexity or entropy; see [8].)

Now we make a change of normal coordinates (4.1), where $T(w)$ is a certain skew-quaternion of norm 1, which acts by conjugation on the matrices from the normal form. Then, a permutation of y_1, y_4, y_5 makes $\varkappa_1(w)$ nonzero. (Note that the problem becomes trivial if $\varkappa_j$ is identically zero for all j.)

Now we set

$$\tilde{y}_4 = y_4 - \frac{\varkappa_4}{\varkappa 1}(w) y_1, \quad \tilde{y}_5 = y_5 - \frac{\varkappa 5}{\varkappa 1}(w) y_1$$

to obtain the following pre-normal form:

$$\begin{aligned} &\dot{x} = u, && \dot{y}_1 = x'(\hat{\imath} + \varkappa_1(w)i)u, && \dot{y}_2 = x'ju, \\ &\dot{y}_3 = x'ku, && \dot{y}_4 = x'(\hat{\jmath} + \lambda_4(w)\hat{\imath})u, && \dot{y}_5 = x'(\hat{k} + \lambda_5(w)\hat{\imath})u, \\ & && \dot{w} = x'iu. \end{aligned} \tag{4.9}$$

A change of normal coordinates (4.1) with the help of a skew-quaternion matrix $T(w)$ of norm 1 reduces the equations for y_4 and y_5 to the form

$$\dot{y}_4 = a(w)x'\hat{\jmath}u + b(w)x'\hat{k}u, \quad \dot{y}_5 = c(w)x'\hat{\jmath}u + d(w)x'\hat{k}u.$$

In fact, an appropriate conjugation with a unit skew-quaternion maps a certain 2-plane in skew-quaternions, to the plane orthogonal to $\hat{\imath}$.

Finally, the change

$$\begin{pmatrix} \tilde{y}_4 \\ \tilde{y}_5 \end{pmatrix} = \begin{pmatrix} a(w) & b(w) \\ c(w) & d(w) \end{pmatrix}^{-1} \begin{pmatrix} y_4 \\ y_5 \end{pmatrix}$$

of variables y_4 and y_5 and a suitable renormalization of $\dot{y}_1$ provide the required normal form for nilpotent approximation.

The fact that $\varrho(w) = \varkappa_1(w)$ is an invariant of the structure follows from the results of Sec. 7: its values characterize the entropy of the given motion planning problem.

An easy argument shows that $|\varrho(w)| \leq 1$: the transformations we have made do not affect the equality

$$1 = \|i\| = \inf_{\lambda}(\|i + \lambda(\hat{\imath} + \varrho i)\|),$$

where the norm is the L_2-norm.

Remark 4. The calculation showing that $|\varrho(w)| \leq 1$ should be made using the L_2-norm, and are of a different nature than the calculation showing that $\varkappa_2 = \varkappa_3 = 0$, which can be made using the standard norm of quaternions.

In particular, the case $|\varrho(w)| = 1$ corresponds to the strict convexity of $\tilde{B}_w$ in the direction of Γ. □

5. Reduction to the nilpotent approximation

In this section, we prove Theorem 8.

Let $\xi(t) = (x(t), y(t), w(t))$ and $\hat{\xi}(t) = (\hat{x}(t), \hat{y}(t), \hat{w}(t))$, $t \in [0, \varepsilon]$ be arclength parametrized trajectories of the motion planning problem Σ and its nilpotent approximation $\hat{\Sigma}$, respectively, and let a normal coordinate system and the corresponding normal form be fixed.

Both trajectories correspond to the same control $u(t)$ and the same initial condition $r = (0, 0, w_0) \in \Gamma$. Then, according to (4.5) and (4.7), we have

$$\begin{aligned} &\dot{x} = u + O(\varepsilon^2), \\ &\dot{y}_i = \frac{1}{2}x'L^i(w)u + O(\varepsilon^2), \quad i = 1, \dots, p-1, \end{aligned} \tag{5.1}$$

and

$$\dot{w} = \frac{1}{2} x' M(w) u + O(\varepsilon^2)$$

or, respectively,

$$\dot{w} = \frac{1}{2} x_1 (x_2 u_1 - x_1 u_2) + O(\varepsilon^3).$$

These relations imply

$$\begin{aligned} \|x(\varepsilon) - \hat{x}(\varepsilon)\| &\leq K\varepsilon^3, \\ \|y(\varepsilon) - \hat{y}(\varepsilon)\| &\leq K\varepsilon^3, \\ \|w(\varepsilon) - \hat{w}(\varepsilon)\| &\leq K\varepsilon^3 \quad (\text{or } K\varepsilon^4) \end{aligned} \tag{5.2}$$

for a certain constant $K > 0$.

Now we assume that one of the trajectories $\xi(\dots)$ or $\hat{\xi}(\dots)$ returns to Γ in time ε. Let $q = (0, 0, w_1)$ denote the corresponding point of Γ. According to Lemma 4, the endpoint ($\hat{\xi}(\varepsilon)$ or $\xi(\varepsilon)$) of the other trajectory belongs to both balls $B_{w_1}(\varepsilon^{5/4})$ and $\hat{B}_{w_1}(\varepsilon^{5/4})$ for sufficiently small ε. Therefore, we can modify the last trajectory to make it interpolate the same points and have the length smaller than or equal to $\varepsilon(1 + \varepsilon^{1/4})$.

Let γ_1 and γ_2 be two trajectories, where γ_1 is a trajectory of Σ (respectively, $\hat{\Sigma}$), ε-interpolating Γ, and γ_2 is a trajectory of $\hat{\Sigma}$ (respectively, Σ) obtained from γ_1 by the previous construction: for any interpolating piece of γ_1 of length $a \leq \varepsilon$, we obtain the corresponding interpolating piece of γ_2 of length $b \leq a(1 + a^{1/4})$, interpolating the same points. Therefore,

$$l(\gamma_1) = \sum_{i=1}^{N} a_i, \quad l(\gamma_2) = \sum_{i=1}^{N} b_i \leq \sum_{i=1}^{N} a_i (1 + \varepsilon^{1/4}) \leq (1 + \varepsilon^{1/4}) l(\gamma_1).$$

Then

$$\frac{l(\gamma_2)}{\varepsilon(1 + \varepsilon^{\frac{1}{4}})} \leq \frac{l(\gamma_1)}{\varepsilon}.$$

Now we assume that γ_1 is optimal among ε-interpolating curves (Lemma 9 from the Appendix states that such a curve does exist), then

$$\frac{l(\gamma_2)}{\varepsilon(1 + \varepsilon^{\frac{1}{4}})} \leq E_1(\varepsilon),$$

where E_1 is the entropy of Σ (respectively, $\hat{\Sigma}$). Since γ_2 is $\varepsilon(1 + \varepsilon^{1/4})$-interpolating, we obtain

$$E_2(\varepsilon(1 + \varepsilon^{1/4})) \leq E_1(\varepsilon),$$

where E_2 is the entropy of $\hat{\Sigma}$ (respectively, Σ).

Then we obtain

$$\hat{E}(\varepsilon(1 + \varepsilon^{1/4})) \leq E(\varepsilon), \tag{5.3a}$$

$$E(\varepsilon(1 + \varepsilon^{1/4})) \leq \hat{E}(\varepsilon). \tag{5.3b}$$

On the other hand, for all cases considered in this paper, we show independently that $\hat{E}(\varepsilon) \simeq_s A/\varepsilon^p$, where either $p = 2$ or $p = 3$. Hence, inequality (5.3a) implies that

$$E(\varepsilon)\frac{\varepsilon^p}{A} \geq \frac{\varepsilon^p(1+\varepsilon^{1/4})^p}{A}\hat{E}(\varepsilon(1+\varepsilon^{1/4}))\frac{1}{(1+\varepsilon^{1/4})^p}.$$

Passing to the lim inf, we obtain

$$\liminf\left(E(\varepsilon)\frac{\varepsilon^p}{A}\right) \geq 1, \quad E(\varepsilon) \geq_s \hat{E}(\varepsilon).$$

Similarly, inequality (5.3b) implies

$$E(\varepsilon(1+\varepsilon^{1/4}))\frac{\varepsilon^p(1+\varepsilon^{1/4})^p}{A} \leq \hat{E}(\varepsilon)\frac{\varepsilon^p(1+\varepsilon^{1/4})^p}{A}.$$

Passing to the limit, we obtain

$$\limsup_{\varepsilon\to 0}\left(E(\varepsilon(1+\varepsilon^{1/4}))\frac{\varepsilon^p(1+\varepsilon^{1/4})^p}{A}\right) \leq 1.$$

Now, setting $\tilde{\varepsilon} = \varepsilon(1+\varepsilon^{1/4})$, we obtain

$$\limsup_{\tilde{\varepsilon}\to 0}\left(E(\tilde{\varepsilon})\frac{\tilde{\varepsilon}^p}{A}\right) \leq 1$$

and, consequently,

$$E(\varepsilon) \leq_s \hat{E}(\varepsilon).$$

6. The case of corank $p \leq 3$

In this section, we prove Theorem 5.

According to Theorem 8, we work with the nilpotent approximation, which takes the following form in the normal coordinates on a tube T_ε:

$$\begin{aligned} \dot{x} &= u, \\ \dot{y}_i &= \frac{1}{2}x' L^i(w)u, \quad i = 1, \dots, p-1, \\ \dot{w} &= \frac{1}{2}x' M(w)u. \end{aligned} \tag{6.1}$$

Assume that we have made a change of normal coordinates of the form $(x, y, w) \to (x, y, \tilde{w})$, where

$$\tilde{w} = w + \sum_{i=1}^{p-1}\lambda_i^*(w)y_i,$$

$$\left\|M(w) + \sum_{i=1}^{p-1}\lambda_i^*(w)L^i(w)\right\|_g = \inf_\lambda \left\|M(w) + \sum_{i=1}^{p-1}\lambda_i(w)L^i(w)\right\|_g.$$

This is a reparametrization of the surface S, which preserves the curve Γ. The new coordinates remain normal. It was shown in [8] that such a smooth vector $\lambda^*(w)$ exists, except for a finite subset of Γ. This finite set of special isolated points is treated exactly as in [8] and does not affect the result. Hence we can assume that in (6.1),

$$\|M(w)\|_g = \inf_\lambda \left\| M(w) + \sum_{i=1}^{p-1} \lambda_i(w) L^i(w) \right\|_g .$$

Using a change of normal coordinates of the form (4.1), we can assume that $M(w)$ has the following block-diagonal form:

$$M(w) = \begin{pmatrix} 0 & \alpha_1 & 0 & \dots & \dots & \dots & \dots & 0 \\ -\alpha_1 & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\ \dots & 0 & 0 & \alpha_2 & 0 & \dots & \dots & \dots \\ \dots & \dots & -\alpha_2 & 0 & 0 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 & \alpha_l & \dots \\ \dots & \dots & \dots & \dots & \dots & -\alpha_l & 0 & \dots \\ 0 & \dots & \dots & \dots & \dots & \dots & \dots & 0 \end{pmatrix},$$

where $\alpha_1, \dots, \alpha_l$ are smooth functions of w such that for all w,

$$\alpha_1(w) > \cdots > \alpha_l(w) > 0.$$

For odd $n - p$, the zero eigenvalue should be added in the bottom right corner.

The following crucial property was also proved in [8]:

$$L^i_{1,2}(w) = L^i_{2,1}(w) = 0 \quad \text{for all} \;\; i = 1, \dots, p-1. \tag{6.2}$$

Remark 5. The last property means exactly that $\tilde{B}_w$ is strictly convex in the direction of $\dot{\Gamma}$ (a generic property for $p < 4$, as we have said): here $\dot{\Gamma}$ is collinear to $\partial/\partial w$, the bracket modulo Δ of $\partial/\partial x_1$ and $\partial/\partial x_2$ has a component $L^i_{1,2}(w)$ in the direction of $\partial/\partial y_i$. Therefore, property (P2) of Definition 1 holds with $\omega^* = dw$ and $x^* = \left[\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}\right] + \Delta_w$.

Now let an admissible arclength parametrized path $\gamma : [0, a] \to \mathbb{R}^n$, $a \leq \varepsilon$, connect points $(0, 0, w_1)$ and $(0, 0, w_2)$.

Along γ we have

$$\frac{\dot{w}}{\alpha_1(w)} \leq \frac{1}{2}(x_1 u_2 - x_2 u_1) + \frac{1}{2} \frac{\alpha_2(w)}{\alpha_1(w)}(x_3 u_4 - x_4 u_3) + \dots .$$

Hence, denoting by $\gamma_1, \dots, \gamma_l$ the projections of the curve γ to the coordinate 2-planes (x_1, x_2), (x_3, x_4), …, we obtain

$$\int_{w_1}^{w_2} \frac{dw}{\alpha_1(w)} \leq \frac{1}{2} \int_{\gamma_1} (x_1\dot{x}_2 - x_2\dot{x}_1)dt + \frac{1}{2} \int_{\gamma_2} \varphi_2(w)(x_3\dot{x}_4 - x_4\dot{x}_3)dt + \dots,$$

where $\varphi_2, \dots, \varphi_l$ are smooth functions of w not exceeding 1. Moreover,

$$\varphi_i(w) = \varphi_i(w_1) + (w - w_1)\psi_i(w),$$

where ψ_i is smooth and

$$|w - w_1| \leq k\varepsilon^2.$$

Since $\|x\| \leq \varepsilon$ (according to the cylinder box theorem, since the curve belongs to T_ε) and $t \leq a \leq \varepsilon$, we obtain

$$\int_{w_1}^{w_2} \frac{dw}{\alpha_1(w)} \leq \frac{1}{2} \int_{\gamma_1} (x_1\dot{x}_2 - x_2\dot{x}_1)dt$$

$$+ \frac{1}{2}\varphi_2(w_1) \int_{\gamma_2} (x_3\dot{x}_4 - x_4\dot{x}_3)dt + \dots + Ka\varepsilon^3$$

for some $K > 0$. Denoting by $\Omega_1, \dots, \Omega_l$ the domains encircled by $\gamma_1, \dots, \gamma_l$ in the corresponding 2-planes, we obtain

$$\int_{w_1}^{w_2} \frac{dw}{\alpha_1(w)} \leq \frac{1}{2} \int_{\Omega_1} 2dx_1 \wedge dx_2 + \frac{1}{2}\varphi_2(w_1) \int_{\Omega_2} 2dx_3 \wedge dx_4 + \dots + Ka\varepsilon^3$$

$$\leq A(\Omega_1) + \dots + A(\Omega_l) + Ka\varepsilon^3,$$

where $A(\Omega_i)$ denotes the area of Ω_i. Now the isoperimetric inequality on the plane implies that

$$\int_{w_1}^{w_2} \frac{dw}{\alpha_1(w)} \leq \frac{1}{4\pi}((P_1)^2 + \dots + (P_l)^2) + Ka\varepsilon^3,$$

where P_i is the perimeter of Ω_i and

$$\int_{w_1}^{w_2} \frac{dw}{\alpha_1(w)} \leq \frac{1}{4\pi} \left[\left(\int_0^a (u_1^2 + u_2^2)^{1/2} dt \right)^2 + \left(\int_0^a (u_3^2 + u_4^2)^{1/2} dt \right)^2 + \dots \right]$$

$$+ Ka\varepsilon^3 \leq \frac{\varepsilon}{4\pi} \int_0^a (u_1^2 + u_2^2 + \dots + u_{2l}^2)dt + Ka\varepsilon^3 \leq \frac{\varepsilon a}{4\pi} + Ka\varepsilon^3.$$

If several (say, m) steps are needed to go from $w = 0$ to the other endpoint $w = \hat{w}$ of Γ, then

$$\int_0^{\hat{w}} \frac{dw}{\alpha_1(w)} \le \sum_{i=1}^{m} a_i \left(\frac{\varepsilon}{4\pi} + K\varepsilon^3 \right),$$

$$\operatorname{length}(\gamma) = \sum_{i=1}^{m} a_i \ge \frac{4\pi}{\varepsilon} \int_0^{\hat{w}} \frac{dw}{\alpha_1(w)} - \tilde{K}\varepsilon^2.$$

This implies that

$$E(\varepsilon) \ge_s \frac{4\pi}{\varepsilon^2} \int_0^{\hat{w}} \frac{dw}{\alpha_1(w)}. \tag{6.3}$$

To prove the inverse inequality, construct an asymptotic optimal synthesis for the nilpotent approximation. The cylinder in $\mathbb{R}^n$

$$C_\varepsilon^1 = \left\{ \left(x_1 - \frac{\varepsilon}{2\pi} \right)^2 + x_2^2 + \cdots + x_{n-p}^2 = \frac{\varepsilon^2}{4\pi^2},\ x_3 = \cdots = x_{n-p} = 0 \right\}$$

of the perimeter ε has dimension $p+1$ and is transversal to Δ for sufficiently small ε (since Γ is transversal to Δ). The intersections of tangent planes to the cylinder with Δ define a single vector field X on C_ε^1 of norm one such that the coordinate w increases along its trajectories. The corresponding controls (easily computed with the normal form) are as follows:

$$u_1 = 2\pi \frac{x_2}{\varepsilon}, \quad u_2 = -2\pi \frac{(x_1 \varepsilon / 2\pi)}{\varepsilon}, \quad u_3 = \cdots = 0.$$

The trajectory of this field X starting from $(0,0,0)$ has the components

$$x_1(t) = \frac{\varepsilon}{2\pi} \left(1 - \cos \frac{2\pi t}{\varepsilon} \right), \quad x_2(t) = \frac{\varepsilon}{2\pi} \sin \frac{2\pi t}{\varepsilon}, \quad x_3(t) = \cdots = 0,$$

while by crucial property (6.2),

$$y_i(t) = 0 \ \text{ for } i = 1, \ldots, p-1.$$

Hence at the time $t = \varepsilon$ the trajectory intersects Γ again.

The component $w(t)$ satisfies the equation

$$\dot{w} = \frac{1}{2} \alpha_1(w)(x_1 u_2 - x_2 u_1) = \alpha_1(w) \frac{x_1(t)}{2} = \alpha_1(w) \frac{\varepsilon}{4\pi} \left(1 - \cos \frac{2\pi}{\varepsilon} t \right).$$

Therefore, the time T of passing from the origin to the coordinate $\hat{w}$ of the other endpoint of Γ satisfies the relation

$$\frac{4\pi}{\varepsilon} \int_0^{\hat{w}} \frac{dw}{\alpha_1(w)} = T - \frac{\varepsilon}{2\pi} \sin \frac{2\pi}{\varepsilon} T$$

and, therefore,

$$T \leq \frac{4\pi}{\varepsilon} \int_0^{\hat{w}} \frac{dw}{\alpha_1(w)} + \frac{\varepsilon}{2\pi}.$$

The trajectory does not arrive exactly at the endpoint $\Gamma(1)$, but at some nearby point $\tilde{w}$ on Γ satisfying $|\hat{w} - \tilde{w}| \leq \Delta\varepsilon^2$ for some $\Delta > 0$. Hence the remaining piece to arrive at $\Gamma(1)$ has a length less than $\delta\varepsilon$ for a certain $\delta > 0$, by Lemma 4.

Thus, the whole entropy satisfies the inequality

$$E(\varepsilon) \leq_s \frac{4\pi}{\varepsilon^2} \int_0^{\hat{w}} \frac{dw}{\alpha_1(w)},$$

as required.

7. The wild 4–10 case

In this section, we prove Theorem 6. According to Theorem 8, we start from the normal form (4.8) for the nilpotent approximation, which after a suitable reparametrization of Γ can be written as follows:

$$\begin{aligned} &\dot{x} = u, \qquad && \dot{y}_1 = x'(\hat{\imath} + \varrho i)u, \qquad && \dot{y}_2 = x'ju, \\ &\dot{y}_3 = x'ku, && \dot{y}_4 = x'\hat{\jmath}u, && \dot{y}_5 = x'\hat{k}u, \\ & && \dot{w} = x'iu. && \end{aligned} \tag{7.1}$$

Formula (3.2) for the entropy in the case where ϱ is a function of w, easily follows from the formula when ϱ is a constant: the dependence of ϱ on w at each ε-interpolation step produces a small deviation of the component y_1, which can be compensated during an extra time interval of higher order than ε. Therefore, it does not affect the estimates. Hence we consider only the case where $\varrho = \text{const}$.

Clearly (see Lemmas 9 and 10), for sufficiently small ε, the interpolating pieces joining two points of Γ should be minimal length geodesics of the sub-Riemannian metric joining these points. Therefore, we calculate the optimal geodesics of length ε of the sub-Riemannian metric. By the one-step bracket-generating assumption, it suffices to consider normal geodesics.

Note that, similarly to the Riemannian geometry, calculating the geodesics we can minimize the energy instead of the length. We take the arclength-parametrized geodesics.

Proposition 2. *The equations of geodesics are*

$$\dot{x} = u, \tag{7.2}$$

$$\frac{du}{dt} = Au,$$

$$\frac{dy_j}{dt} = x' L_j u, \quad \frac{dw}{dt} = x' i u, \tag{7.3}$$

where L_j are given above in (7.1) *and A is an arbitrary skew-symmetric matrix.*

Proof. If $\Psi(t)$ is the adjoint vector, then $u_i = \Psi(t) \cdot X_i(x(t))$, $i = 1, \ldots, n$, where X_i are the components of the control vector field. But the functions $\Psi(t)[X_i, X_j]$ are constant since the second brackets $[[X_i, X_j], X_k]$ are zero. $\square$

We are looking for a parametrized minimal geodesic

$$\xi(t) = (x(t), y(t), w(t), u(t)), \quad t \in [0, \varepsilon],$$

connecting the point $(x, y, w) = (0, 0, 0)$ with the point $(0, 0, W)$. Let $A = H' \Lambda H$, where H is orthogonal, $\det(H) = 1$, and Λ is skew symmetric and (2×2)-block-diagonal. We set $V = Hu$ and $V_0 = Hu_0$. Denote by $BD(\alpha, \beta)$ the block-diagonal (4×4)-matrix with the (2×2)-blocks α and β.

To make the proof more clear, we split it into several cases. We reproduce completely the arguments in each case despite their similarity. On the contrary, we omit some long but routine computational details.

Case 1. Noninvertible matrix Λ.

Lemma 5. *A minimal interpolating geodesic has a noninvertible matrix Λ if and only if $\varrho = \pm 1$.*

Proof. The noninvertible matrix Λ has the form $\Lambda = BD(0, \alpha J)$, where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Setting $\tilde{\xi} = Hx$, we obtain that $V = BD(\mathrm{Id}, e^{\alpha J t}) \cdot V_0$ and

$$\tilde{\xi} = BD\left(t \, \mathrm{Id}, \ -\frac{1}{\alpha} J (e^{\alpha J t} - \mathrm{Id})\right) V_0.$$

By the nonholomic interpolation assumption $\tilde{\xi}(\varepsilon) = 0$, therefore, the matrix $(e^{\alpha J \varepsilon} - \mathrm{Id})$ is noninvertible. This happens for the smallest value of $\alpha = 2\pi/\varepsilon$. Hence $V_0 = (0, 0, \cos(\varphi), \sin(\varphi))$ for some φ. We set $\tilde{V}_0 = (\cos \varphi, \sin \varphi)$.

Decomposing the skew-symmetric matrix M into a linear combination of basic matrices i, j, k, $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, denote by y_M the corresponding linear combination of the coordinate functions y. Therefore, y_M satisfies the equation

$$\dot{y}_M = x' M u = \xi' H M H' e^{\Lambda t} V_0.$$

Setting

$$\tilde{M} = HMH' = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

we see that

$$\dot{y}_M = \frac{1}{\alpha}\tilde{V}_0' J(e^{-\alpha Jt} - \mathrm{Id}) M_4 e^{\alpha Jt}\tilde{V}_0.$$

But $M_4 = m_4 J$ and, therefore,

$$\dot{y}_M = \frac{1}{\alpha}\tilde{V}_0' J(\mathrm{Id} - e^{\alpha Jt}) M_4 \tilde{V}_0.$$

At $t = \varepsilon$ this gives $y_M(\varepsilon) = -\dfrac{\varepsilon^2}{2\pi}m_4$. This value must be zero for $M = j$, k, $\hat{\jmath}$, $\hat{k}$, and $\hat{\imath} + \varrho i$ (the geodesic intersects Γ at $t = \varepsilon$). Any H can be written in the form $H = e^q e^{\hat{q}}$, where q (respectively, $\hat{q}$) is a quaternion (respectively, skew). Now for $M = j$ and $M = k$, the condition $m_4 = 0$ implies that $e^q j e^{-q}$ and $e^q k e^{-q}$ are orthogonal to i. Hence the element $e^q i e^{-q}$ orthogonal to them must be $\pm i$:

$$e^q i e^{-q} = \omega_1 i, \quad \omega_1 = \pm 1.$$

Taking $M = \hat{\jmath}$ and $M = \hat{k}$, we similarly obtain

$$e^{\hat{q}}\hat{\imath}e^{-\hat{q}} = \omega_2\hat{\imath}, \quad \omega_2 = \pm 1.$$

Applying this argument also to $M = \hat{\imath} + \varrho i$, we have

$$\omega_1 \varrho y_i(\varepsilon) + \omega_2 y_{\hat{\imath}}(\varepsilon) = 0.$$

Hence

$$-\varrho\omega_1\frac{\varepsilon^2}{2\pi} + \omega_2\frac{\varepsilon^2}{2\pi} = 0.$$

Then $\varrho = 1$ or $\varrho = -1$.

The case where we take $\Lambda = BD(\alpha J, 0)$ is similar.

For $M = i$, we obtain $w(\varepsilon) = -\varepsilon^2/2\pi$, providing the entropy estimation for $\rho = \pm 1$. □

Remark 6. Finally, let us summarize our conclusions at the end of this case.

1. If $\varrho = \pm 1$, the entropy is still

$$E(\varepsilon) = \frac{4\pi}{\varepsilon^2}\int_\Gamma \frac{dw}{\chi(w)};$$

the factor 2 missing apparently comes from the reparametrization of Γ in (7.1): we have set $d\tilde{w} = 2dw/\chi(w)$. This is we already known since the case $\varrho = \pm 1$ is the case where $\tilde{B}_w$ is strictly convex in the direction of Γ and in tis case, the proof of Theorem 5 (see Sec. 6) still applies.

2. If $\varrho \neq \pm 1$, then the matrices A and Λ in the equation of our interpolating geodesics are invertible.

Case 2. Now we assume that $\varrho \neq \pm 1$ and Λ is invertible.

Lemma 6. *A geodesic defined by the matrix* $\Lambda = BD(\alpha_1 J, \alpha_2 J)$, $\alpha_1, \alpha_2 \neq 0$, *returns to* Γ *in time* ε *only if* $\alpha_1 = 2\pi k_1/\varepsilon$ *and* $\alpha_2 = 2\pi k_2/\varepsilon$ *for some integers* $k_1, k_2 \neq 0$.

Proof. We have, with the same notation as above:

$$V = BD(e^{\alpha_1 Jt}, e^{\alpha_2 Jt}) \cdot V_0,$$
$$\tilde{\xi} = BD\left(-\frac{1}{\alpha_1} J(e^{\alpha_1 Jt} - \mathrm{Id}),\ -\frac{1}{\alpha_2} J(e^{\alpha_2 Jt} - \mathrm{Id})\right) V_0.$$

Because of the nonholonomic interpolation at time ε, the matrix

$$BD\left[-\frac{1}{\alpha_1} J(e^{\alpha_1 Jt} - \mathrm{Id}),\ -\frac{1}{\alpha_2} J(e^{\alpha_2 Jt} - \mathrm{Id})\right]$$

must be noninvertible at $t = \varepsilon$.

This shows that either $\varepsilon = 2\pi k_1/\alpha_1$, or $\varepsilon = 2\pi k_2/\alpha_2$, or both.

Assume that $\alpha_2 = 2\pi k_2/\varepsilon$ for some integer k_2 but $\alpha_1 \neq 2\pi k_1/\varepsilon$ for any integer k_1.

Again, $V_0 = (0, 0, \cos\varphi, \sin\varphi)$ for $\tilde{\xi}(\varepsilon) = 0$, and, to obtain the geodesics parametrized by arclength, we set also $\tilde{V}_0 = (\cos\varphi, \sin\varphi)$.

Again, taking a skew-symmetric matrix M, the same calculation as in Lemma 5 for the equation $\dot{y}_M = xMu$ shows that

$$y_M(\varepsilon) = -\frac{\varepsilon^2}{2\pi} m_4.$$

The same reasoning as in Lemma 5 leads to the following result: if $\varrho \neq \pm 1$, such geodesics cannot realize nonholonomic interpolation. Therefore, we must have $\alpha_2 = 2\pi k_2/\varepsilon$ and $\alpha_1 = 2\pi k_1/\varepsilon$, for some integers $k_1, k_2 \neq 0$,

$$\Lambda = BD\left(\frac{2\pi k_1}{\varepsilon} J,\ \frac{2\pi k_2}{\varepsilon} J\right).$$

The lemma is proved. □

Case 3. The moduli k_1 and k_2 are equal. This case never happens for $|\rho| < 1$.

Lemma 7. *An* ε*-interpolating geodesic must have* $k_1 \neq \pm k_2$ *if* $|\rho| < 1$.

Proof. Assume that $k_1 = k_2 = k$ (the case $k_1 = -k_2$ is similar).

Now

$$\Lambda = BD\left(\frac{2\pi k}{\varepsilon} J,\ \frac{2\pi k}{\varepsilon} J\right)$$

and $\tilde{\xi}(\varepsilon) = 0$ for any $V_0 = (x, y, z, w)$. For a skew-symmetric matrix $M = q + \hat{q}$ and the equation $\dot{y}_M = \xi' M e^{\Lambda t} V_0$, the same computations as before show that

$$q = ai + \tilde{q}_1 j + \tilde{q}_2 k, \quad \hat{q} = b\hat{\imath} + q_1\hat{\jmath} + q_2\hat{k}, \tag{7.4a}$$

$$y_q(\varepsilon) = -\frac{a\varepsilon^2}{2k\pi}(x^2 + y^2 + z^2 + w^2), \tag{7.4b}$$

$$y_{\hat{q}}(\varepsilon) = \frac{\varepsilon^2}{2k\pi}\Big[2(q_1(yz - wx) + q_2(xz + wy)) + b(x^2 + y^2 - w^2 - z^2)\Big]. \tag{7.4c}$$

Write $H = e^r e^{\hat{r}}$ for some quaternion r and skew-quaternion $\hat{r}$. If we want to realize nonholonomic interpolation at the time $t = \varepsilon$, we must find a unit vector $V_0 = (x, y, z, w)$ such that $y_M(\varepsilon)$ vanishes for $M = e^r j e^{-r}$, $e^r k e^{-r}$, $e^{\hat{r}}\hat{\jmath}e^{-\hat{r}}$, $e^{\hat{r}}\hat{k}e^{-\hat{r}}$, and $\varrho e^r i e^{-r} + e^{\hat{r}}\hat{\imath}e^{-\hat{r}}$.

Again, the equalities $y_{e^r j e^{-r}}(\varepsilon) = 0$, $y_{e^r k e^{-r}}(\varepsilon) = 0$ and (7.4b) imply that $e^r j e^{-r}$ and $e^r k e^{-r}$ are orthogonal to i. Hence, $e^r i e^{-r} = \omega_1 i$, $\omega_1 = \pm 1$, and

$$y_{e^{\hat{r}}\hat{\jmath}e^{-\hat{r}}}(\varepsilon) = 0, \quad y_{e^{\hat{r}}\hat{k}e^{-\hat{r}}}(\varepsilon) = 0, \quad \frac{-\omega_1\varrho\varepsilon^2}{2k\pi} + y_{e^{\hat{r}}\hat{\imath}e^{-\hat{r}}}(\varepsilon) = 0.$$

Therefore, by (7.4c), the vector

$$W = (2(yz - wx), 2(xz + wy), x^2 + y^2 - w^2 - z^2) \in \mathbb{R}^3$$

must be orthogonal to two independent vectors V_1 and V_2, and its scalar product with the unit vector orthogonal to V_1 and V_2 must be $\omega_1\varrho$. Then, the norm of W must be $|\varrho|$. But a simple calculation shows that it is 1. (In the case $k_2 = -k_1$, we find $1 = 1/|\varrho|$.) □

Case 4. The remaining case $\alpha_2 = 2\pi k_2/\varepsilon$ and $\alpha_1 = 2\pi k_1/\varepsilon$, $k_1, k_2 \neq 0$, $k_1 \neq \pm k_2$.

Lemma 8. *An interpolating trajectory in the time ε is minimal only if $k_1 = -2$, $k_2 = 1$, or $k_1 = -1$, $k_2 = 2$.*

Proof. We will directly calculate $y_M(\varepsilon)$, where $\dot{y}_M = \xi' M e^{\Lambda t} V_0$, for an arbitrary initial vector $V_0 = (x, y, z, w)$ of norm 1.

Again, we take $M = q + \hat{q}$, $q = ai + \tilde{q}_1 j + \tilde{q}_2 k$, and $\hat{q} = b\hat{\imath} + q_1\hat{\jmath} + q_2\hat{k}$.

After a simple but tedious computation (verified with Mathematica), we find:

$$y_q(\varepsilon) = \frac{a\varepsilon^2}{2\pi k_1 k_2}\big(-k_2(x^2 + y^2) + k_1(w^2 + z^2)\big), \tag{7.5a}$$

$$y_{\hat{q}}(\varepsilon) = \frac{-b\varepsilon^2}{2\pi k_1 k_2}\big(k_2(x^2 + y^2) + k_1(w^2 + z^2)\big). \tag{7.5b}$$

Note that if our trajectory is optimal, then the value $y_q(\varepsilon)$ cannot vanish for all quaternions q: it must be nonzero on $e^r i e^{-r}$ for some quaternion r.

But it must be zero on $e^r je^{-r}$ and $e^r ke^{-r}$. Then $e^r je^{-r}$ and $e^r ke^{-r}$ are orthogonal to i. It follows again that $e^r ie^{-r} = \omega_1 i$, $\omega_1 = \pm 1$. Hence

$$y_{e^r ie^{-r}} = \frac{\omega_1 \varepsilon^2}{2\pi k_1 k_2}\big(- k_2(x^2+y^2) + k_1(w^2+z^2)\big).$$

We consider two possibilities. First, assume that

$$k_2(x^2+y^2) + k_1(w^2+z^2) = 0.$$

Then we obtain $\varrho = 0$ since $\varrho y_i(\varepsilon) + y_{e^{\hat r}\hat\imath e^{-\hat r}} = 0$, $y_i(\varepsilon)$ is nonzero. Therefore,

$$k_2(x^2+y^2) + k_1(w^2+z^2) = 0, \quad (x^2+y^2) + (w^2+z^2) = 1. \tag{7.6}$$

Now let

$$k_2(x^2+y^2) + k_1(w^2+z^2) \neq 0.$$

Then by the nonholonomic interpolation, $y_{e^{\hat r}\hat\jmath e^{-\hat r}}(\varepsilon) = 0$ and $y_{e^{\hat r}\hat k e^{-\hat r}}(\varepsilon) = 0$. Hence, according to (7.5b), the quaternions $e^{\hat r}\hat\jmath e^{-\hat r}$ and $e^{\hat r}\hat k e^{-\hat r}$ are orthogonal to $\hat\imath$. Hence, the element $e^{\hat r}\hat\imath e^{-\hat r}$ orthogonal to them must be equal to $\omega_2\hat\imath$, $\omega_2 = \pm 1$. Since $y_M(\varepsilon) = 0$ also for $M = \varrho e^r ie^{-r} + e^{\hat r}\hat\imath e^{-\hat r}$, it follows that

$$\varrho\omega_1\big(- k_2(x^2+y^2) + k_1(w^2+z^2)\big) - \omega_2\big(k_2(x^2+y^2) + k_1(w^2+z^2)\big) = 0.$$

We treat only the case where $\omega_1 = \omega_2 = \pm 1$. The other case is similar. We obtain

$$-k_2(x^2+y^2)(\varrho+1) + k_1(w^2+z^2)(\varrho-1) = 0,$$

which implies

$$\begin{gathered} k_2(x^2+y^2)(\varrho+1) + k_1(w^2+z^2)(1-\varrho) = 0, \\ (x^2+y^2) + (w^2+z^2) = 1. \end{gathered} \tag{7.7}$$

This equation coincides with Eq. (7.6) for $\varrho = 0$. Therefore, it suffices to consider this case. Solving Eq. (7.7) provides:

$$\begin{aligned} (x^2+y^2) &= \frac{-k_1(1-\varrho)}{k_2(\varrho+1) - k_1(1-\varrho)}, \\ (w^2+z^2) &= \frac{k_2(\varrho+1)}{k_2(\varrho+1) - k_1(1-\varrho)}, \\ y_{e^r ie^{-r}}(\varepsilon) = y_i(\varepsilon) &= \frac{\varepsilon^2}{\pi}\frac{1}{k_2(\varrho+1) - k_1(1-\varrho)}. \end{aligned}$$

The denominator does not vanish since $|\varrho| < 1$ and k_1 and k_2 have opposite signs.

Actually, one can explicitly verify that these values realize nonholonomic interpolation at the time ε.

To maximize $w(\varepsilon) = y_i(\varepsilon)$, for $k_2, -k_1 > 0$, we set $\hat{k}_1 = -k_1$ and $\hat{k}_2 = k_2$. The maximum of $y_i(\varepsilon)$ corresponds to the minimum of the convex combination

$$\hat{k}_2 \frac{1+\varrho}{2} + \hat{k}_1 \frac{1-\varrho}{2}$$

of $\hat{k}_1$ and $\hat{k}_2$ with positive and distinct integers $\hat{k}_1$ and $\hat{k}_2$. Clearly, the maximum value of $y_i(\varepsilon)$ is obtained for $(\hat{k}_1, \hat{k}_2) = (2,1)$ or $(1,2)$. Then either $k_2 = 2$ and $k_1 = -1$ or $k_2 = 1$ and $k_1 = -2$. Respectively, we obtain

$$w(\varepsilon) = y_i(\varepsilon) = \frac{\varepsilon^2}{\pi} \frac{1}{\varrho + 3}, \quad w(\varepsilon) = y_i(\varepsilon) = \frac{\varepsilon^2}{\pi} \frac{1}{3-\varrho}.$$

If $\varrho > 0$, then the largest value is $w(\varepsilon) = y_i(\varepsilon) = \dfrac{\varepsilon^2}{\pi} \dfrac{1}{3-\varrho}$. □

End of the proof. Taking into account the factor 2 in the reparametrization $d\tilde{w} = 2dw/\chi(w)$ imposed at the beginning, we obtain the following formula for $d\tilde{w} = dw/\chi(w)$:

$$\tilde{w}(\varepsilon) = \frac{\varepsilon^2}{\pi} \frac{1}{2(3-|\varrho|)}.$$

Finally, Lemma 12 from the appendix implies that

$$E(\varepsilon) \simeq_s \frac{2(3-|\rho|)\pi}{\varepsilon^2} \int_\Gamma \frac{dw}{\chi(w)},$$

which proves the theorem.

8. The car with a trailer

In this section, we prove Theorem 7. According to Theorem 8, we work with the nilpotent approximation. To use the standard notation (see [14]) we permute the variables x_1 and x_2 and the variables u_1 and u_2. We reverse the signs of y and w and denote the phase variables by (x, y, z, w) and the control variables by (u, v). After reparametrization of Γ, Eqs. (4.6) takes the form $\dot{\xi} = F(\xi)u + G(\xi)v$ or, in coordinates,

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{z} = \frac{1}{2}(yu - xv), \quad \dot{w} = \frac{1}{2}y(yu - xv).$$

We want to find an admissible curve going from $(0,0,0,0)$ to $(0,0,0,\hat{w})$ in a fixed time ε and minimizing $\hat{w}$. To do this, we apply the Pontryagin maximum principle in a fixed time. The tranversality condition and the independence of the Hamiltonian on w imply the following form of the Hamiltonian:

$$H = \frac{s_0}{2} y(yu - xv) + pu + qv + \frac{r}{2}(yu - xv).$$

Here, s_0 is the auxiliary adjoint variable, $s_0 \leq 0$. In fact, by setting $\Psi(t) = (p, q, r, s_0)$, the Hamiltonian can be written as $H = \Psi \cdot Fu_1 + \Psi \cdot Gu_2$. The variable s_0 plays formally the same role as the adjoint variable of w. Also, r is a constant.

The Hamiltonian cannot be zero: the abnormal extremals do not satisfy the requirements.

Since the Hamiltonian is nonzero, the normal extremals (arclength parametrized) belong to the level-1 hypersurface of the Hamiltonian:

$$H = \sqrt{(\Psi \cdot F)^2 + (\Psi \cdot G)^2}.$$

Therefore, in particular, $p^2(0) + q^2(0) = 1$.

The corresponding canonical equations provide:

$$\frac{d(\Psi \cdot F)}{dt} = \Psi[F, G]\Psi \cdot G, \quad \frac{d(\Psi \cdot G)}{dt} = -\Psi[F, G]\Psi \cdot F.$$

If we set $h = \Psi[F, G]$, $u = \cos\varphi$, and $v = \sin\varphi$, we obtain $\dot{\varphi} = -h$.

Also,

$$\dot{h} = -\Psi \cdot [F, [F, G]]u - \Psi \cdot [G, [F, G]]v = 0 + \frac{3}{2}s_0 v = \frac{3}{2}s_0 \sin\varphi.$$

Then

$$\ddot{\varphi} = -\dot{h} = -\frac{3}{2}s_0 \sin\varphi.$$

Since $s_0 \leq 0$, we change φ by $\varphi + \pi$ and obtain

$$\ddot{\varphi} = -\frac{3}{2}s\sin(\varphi), \quad s \geq 0. \tag{8.1}$$

This equation arises in the standard treatment of Euler's elastica (see, e.g., [14]).

The projection of the extremal curve to the (x, y)-plane satisfies the equations

$$\dot{x} = u = -\cos\varphi, \quad \dot{y} = v = -\sin\varphi.$$

Therefore, following [14], it has to be an elastica: a curve which is a static equilibrium of an homogeneous elastic bar, constrained in the plane. Moreover, this (x, y)-curve must be a closed curve (joining the origin with the origin) and smooth (owing to the smoothness of normal extremals). The curve must encircle a domain with 0 area (the variable z, which is the area according to $\dot{z} = \frac{1}{2}(y\dot{x} - x\dot{y})$, vanishes at endpoints). Therefore, the single possibility is the periodic inflexional elastica. The inflexion occurs at the origin. This implies that

$$0 = \dot{\varphi}(0) = -h(0) = r + \frac{3}{2}sy(0).$$

Therefore, $r = 0$.

The function $\frac{1}{2}\dot{\varphi}^2 - \frac{3}{2}s\cos\varphi$ is an integral of the motion, hence Eq. (8.1) can be written as

$$\frac{1}{2}\dot{\varphi}^2 = -\frac{3s}{2}(\cos(\varphi_0) - \cos(\varphi)). \tag{8.2}$$

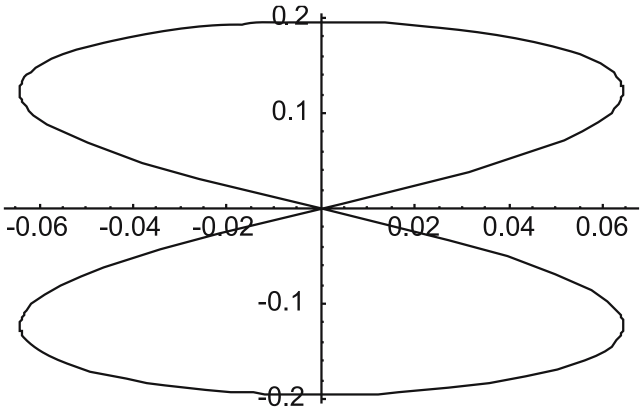

Fig. 1. The dance of the minimum entropy

Hence, if $k = \sin\frac{\varphi_0}{2}$ and $K(k)$ is the quarter period of the Jacobi elliptic functions of argument ν and modulus k, the period in $\nu = t\sqrt{\frac{3s}{2}}$ is $4K$, and the period in t is $4K\sqrt{2/3s} = \varepsilon$. The case of this periodic inflexional elastica corresponds to a value of K such that $2\,\mathrm{Eam}(K) = K$. This corresponds to approximately $\varphi_0 = 130°$ following [14, p. 403], and $\varphi_0 = 130.692°$ following Mathematica® (see [15]).

Finally, we obtain

$$x(t) = -\frac{\varepsilon}{4K}\left[-\frac{4Kt}{\varepsilon} + 2\left(\mathrm{Eam}\left(\frac{4Kt}{\varepsilon} + K\right) - \mathrm{Eam}(K)\right)\right],$$
$$y(t) = k\frac{\varepsilon}{2K}\,\mathrm{cn}\left(\frac{4Kt}{\varepsilon} + K\right),$$

where $K = 2\,\mathrm{Eam}(K) \approx 2.32073$ and the controls are determined as follows:

$$u = 1 - 2\,\mathrm{dn}\left(K\left(1 + \frac{4t}{\varepsilon}\right)\right)^2,$$
$$v = -2\,\mathrm{dn}\left(K\left(1 + \frac{4t}{\varepsilon}\right)\right)\mathrm{sn}\left(K\left(1 + \frac{4t}{\varepsilon}\right)\right)\sin\frac{\varphi_0}{2}.$$

In particular, note that the asymptotic optimal synthesis for the nilpotent approximation is smooth.

One can say that the projection (in normal coordinates) of the curve providing the minimum entropy to the (x, y)-horizontal planes represents a kind of a smooth dance (see Fig. 1).

For $t = \varepsilon$, the computations yields

$$w = 0.00580305\varepsilon^3.$$

Applying now Lemma 12 from the Appendix, we obtain the required formula from Theorem 7 for the entropy.

9. Appendix

9.1. **Appendix 1.** Estimating an entropy of a motion planning problem $\Sigma = (\Gamma, \Delta, g)$, we consider the following problem.

Problem (Q_ω) (ω is small). Find $l^* = \inf\{\text{length}(\gamma);\ \gamma$ is admissible, ω-interpolating, $\text{dom}(\gamma) = [0, t_\gamma]$, $\gamma(0) = \Gamma(0)$, $\gamma(t_\gamma) = \Gamma(1)\}$.

Problem (Q_ω) has a subproblem ($\mathrm{P}_\omega^{w_0}$).

Problem ($\mathrm{P}_\omega^{w_0}$) (ω is small). Given $\Gamma(w_0)$, find $w^* = \sup\{w;\ \exists\gamma$ admissible, $\exists a \leq \omega$, $\gamma(0) = \Gamma(w_0)$, $\gamma(a) = \Gamma(w_0 + w)\}$.

We state (and sketch the proof of) certain basic properties of these problems.

Lemma 9. *In problem* (Q_ω), *the minimum l^* is attained at a ω-interpolating curve γ_*, which is a* (*finitely*) *piecewise minimizing geodesic* (*minimizing pieces have length* $\leq\omega$).

Note that the proof of Lemma 9 contains the proof of the following known result.

Lemma 10. *In problem* ($\mathrm{P}_\omega^{w_0}$), *the maximum is attained at a curve which is a minimizing geodesic, and $a = \omega$.*

Sketch of the proof of Lemma 9. In these purely local constructions (around Γ), we can assume that the orthonormal frame $\mathcal{F} = (F_1, \ldots, F_{n-p})$ has compact support. Thus, we can assume that the trajectories under consideration are defined up to $t = +\infty$. Let u_n be a minimizing sequence of controls, and let x_n be the corresponding sequence of trajectories. Reparametrizing by a constant factor $dt = \alpha d\tau$ the trajectory which was initially arclength parametrized, we can assume that the domain of u_n and x_n is $[0, 1]$ and u_n are uniformly bounded in $L^\infty[0, 1]$ and in $L^2[0, 1]$. Also, denoting the L^2-norm by $\|\cdot\|$, we obtain that the sequence $\|u_n\| = \text{length}(x_n)$ is bounded. Thus, we see that u_n weakly converges in L^2 to some weak limit u^*, and the corresponding limit trajectory is x^*. The input-output mapping $u(\cdot) \to x(\cdot)$ is a continuous mapping from the space ($L^2[0, 1]$, weak) to the space ($C^0[0, 1]$, uniform). Moreover,

$$\text{length}(x^*) \leq \|u^*\| \leq \lim \|u_n\| = l^*.$$

Therefore, (u^*, x^*) is a minimizing trajectory and $x^*(0) = \Gamma(0)$ and $x^*(1) = \Gamma(1)$.

Let us show that x^* is ω-interpolating; assume the contrary. Then Lemma 11 implies that x^* contains a piece-segment P of the length l exceeding ω, with the corresponding control Q (defined on the same interval) such that P does not intersect Γ. Denote by (P_n, Q_n) the segments of the trajectories and controls (x_n, u_n) restricted to the same domain of the length l.

The uniform convergence of x_n implies that for sufficiently large n, P_n does not intersect Γ. Also,

$$\lim \operatorname{length}(P_n) = \sqrt{l} \lim \|Q_n\| \geq \sqrt{l}\|Q\| \geq \operatorname{length}(P) > \omega.$$

Therefore, (u_n, x_n) is not ω-interpolating, a contradiction.

If an interpolating piece of x^* is not a minimizing geodesic, then it is easy to see that x^* is not optimal. $\square$

Lemma 11. *If $\gamma(0)$, $\gamma(1) \in \Gamma$ and each segment of γ of the length $>\omega$ contains a point of Γ, then Γ is ω-interpolating.*

This is obvious, as well as the fact (which we use extensively throughout the paper) that γ is ω-interpolated with a *finite* number of pieces of length $\leq\omega$.

Now, given a motion planning problems Σ (in the normal form, with respect to certain normal coordinates), assume that for all sufficiently small ω, the solutions w^*_ω of Problems (P^w_ω) from Lemma 10 satisfy the inequality

$$A\omega^p \leq |w^*_\omega - w| \leq B\omega^p \tag{9.1}$$

for $p = 2$ or $p = 3$.

Assume that $\Gamma : [0, W] \to \mathbb{R}^n$ (keeping in mind certain reparametrizations, we do not restrict ourselves to the case $W = 1$).

Lemma 12. *Under these assumptions,*

$$\frac{W}{B\omega^p} \leq E(\omega) \leq \frac{W}{A\omega^p} + k, \quad k > 0.$$

Proof. First, solving repeatedly problems (P^w_ω) (for which the extremum is attained) we construct an ω-interpolating curve γ_* such that

$$\operatorname{length}(\gamma_*) \leq \frac{W}{A\omega^{p-1}} + k\omega, \quad k > 0.$$

The term $k\omega$ is introduced to compensate the upper-boundary effect. This can easily be done using Lemma 4.

Then, by definition,

$$E(\omega) \leq \frac{\operatorname{length}(\gamma_*)}{\omega} \leq \frac{W}{A\omega^p} + k.$$

Second, let γ_ω be any ω-interpolating curve, arclength parametrized. Let $t_0 = 0 < t_1 < \cdots < t_n = t_{\gamma_\omega}$ be the interpolation points. We have, by assumption (applying (9.1) for $\omega = t_{i+1} - t_i$), the following estimates:

$$|w_{i+1} - w_i| \leq B|t_{i+1} - t_i|^p,$$

$$\frac{|w_{i+1} - w_i|}{\omega^p} \leq B(\frac{|t_{i+1} - t_i|}{\omega})^p \leq B\frac{|t_{i+1} - t_i|}{\omega},$$

since $|t_{i+1} - t_i| \leq \omega$. This implies

$$|t_{i+1} - t_i| \geq \frac{|w_{i+1} - w_i|}{B\omega^{p-1}} \geq \frac{w_{i+1} - w_i}{B\omega^{p-1}},$$

and since $l(\gamma_\omega) = \sum |t_{i+1} - t_i|$, we obtain

$$l(\gamma_\omega) \geq \frac{W}{B\omega^{p-1}}.$$

Hence

$$\frac{l(\gamma_\omega)}{\omega} \geq \frac{W}{B\omega^p},$$

which consequently implies that taking the infimum over all ω-interpolating curves γ_ω, the required estimate holds:

$$E(\omega) \geq \frac{W}{B\omega^p}.$$

The lemma is proved. □

9.2. **Appendix 2.** Below, $\|\cdot\|$ means the L_2-norm.

Let M, L_j, $j = 1, \ldots, p-1$, be linearly independent skew-symmetric matrices (note that this independence is due to the one-step bracket-generating assumption). We use the abbreviated notation $N(\lambda) = M + \lambda N$ for $M + \sum_{j=1}^{p-1} \lambda_j L_j$. let λ^* be such that

$$\alpha = \|N(\lambda^*)\| = \inf_\lambda \|N(\lambda)\|.$$

Such λ^* automatically exists by the assumption of the independence, and $\|N(\lambda^*)\| > 0$.

Assume that $\|N(\lambda)\|$ is a smooth function in λ at least in a neighborhood V_0 of λ^*. This holds, in particular, when the maximum modulus eigenvalue $\sqrt{-1}\alpha$ $(= \sqrt{-1}\|N(\lambda^*)\|)$ of $N(\lambda^*)$ is simple. (In fact, in this case, $\alpha(\lambda) = \|N(\lambda)\|$ is an analytic function in λ on V_0).

Under these assumptions, the following crucial property, used throughout the paper, holds (we recall its proof from [8]).

Lemma 13. *The set*

$$B = \Big\{(x'My,\ x'L_1y,\ \ldots,\ x'L_{p-1}y);\ \|x\|, \|y\| \leq 1\Big\} \subset \mathbb{R}^p$$

is strictly convex in the direction $(1, 0, \ldots, 0)$.

Proof. Without loss of generality, we can assume that $\lambda^* = 0$. Then on V_0 we have

$$\alpha(\lambda) = \sup_{\|x\|=\|y\|=1} x'N(\lambda)y = \sup_{\|x\|,\|y\|\leq 1} x'N(\lambda)y = X(\lambda)'N(\lambda)Y(\lambda),$$
$$\|X(\lambda)\| = \|Y(\lambda)\| = 1.$$

Obviously, we can find smooth in λ vectors $X(\lambda)$ and $Y(\lambda)$ satisfying these relations.

Hence we can write

$$\begin{aligned} X(\lambda) &= X + \lambda\tilde{X} + O^2(\lambda), \\ Y(\lambda) &= Y + \lambda\tilde{Y} + O^2(\lambda), \\ \alpha(\lambda) &= \alpha + \lambda\tilde{\alpha} + O^2(\lambda). \end{aligned}$$

Then the equality $1 = \langle X(\lambda), X(\lambda)\rangle$ implies

$$1 = \langle X, X\rangle + 2\lambda\langle X, \tilde{X}\rangle + O^2(\lambda).$$

This and a similar equation for Y imply

$$\langle X, \tilde{X}\rangle = 0, \quad \langle Y, \tilde{Y}\rangle = 0. \tag{9.2}$$

We also have

$$\alpha(\lambda) = X(\lambda)' N(\lambda) Y(\lambda) \geq \alpha \quad \forall\lambda \in V_0$$

and

$$\begin{aligned} \alpha + \lambda\tilde{\alpha} + O^2(\lambda) &= \big(X + \lambda\tilde{X} + O^2(\lambda)\big)'(M + \lambda L)\big(Y + \lambda\tilde{Y} + O^2(\lambda)\big) \\ &= X'MY + \lambda\big(\tilde{X}'MY + X'M\tilde{Y} + X'LY\big) + O^2(\lambda) \\ &= \alpha + \lambda\big(\tilde{X}'MY + X'M\tilde{Y} + X'LY\big) + O^2(\lambda) \geq \alpha, \quad \forall\alpha \in V_0. \end{aligned}$$

This implies that $\tilde{X}'MY + X'M\tilde{Y} + X'LY = 0$. Since, by definition, $MX = -\alpha Y$ and $MY = \alpha X$, we have $\tilde{X}'MY = \alpha\tilde{X}'X = 0$ by (9.2). Since $X'M\tilde{Y} = -\tilde{Y}'MX = \alpha Y'\tilde{Y} = 0$, we obtain $X'LY = 0$.

This means that

$$(X'MY, X'L_1Y, \dots, X'L_{p-1}Y) = (\|N(\lambda^*)\|, 0, \dots, 0) \in B,$$

$$\inf_{\lambda} \sup_{(w,z)\in B} (w + \lambda z) = \|N(\lambda^*)\| = (1, \lambda_1^*, \dots, \lambda_{p-1}^*) \cdot \begin{pmatrix} \|N(\lambda^*)\| \\ 0 \\ \dots \\ \dots \\ 0 \end{pmatrix},$$

which is exactly condition (P_2) from Definition 1. Hence, B is strictly convex in the direction $(1, 0, \dots, 0)$. $\square$

Corollary 1 (in the notation of Sec. 2)**.** *If $\tilde{B}_t$ is not strictly convex in the direction of $\dot{\Gamma}_t \operatorname{mod} \Delta_{\Gamma(t)}$, then $M_t +\ \lambda^* L_t$ has a nonsimple maximum modulus eigenvalue.*

Recall another fact (see [8, Lemma 9, item 3]) used throughout the paper.

Lemma 14. *For a generic one-step bracket-generating Σ, except for finite subset of Γ, there exists a smooth mapping $w \to \lambda^*(w)$ such that $\left\| M(w) + \sum_{j=1}^{p-1} \lambda_j^*(w) L_j(w) \right\|$ is the minimum of $\|N(\lambda)\|$.*

The proof uses a standard transversality argument.

This lemma implies an important fact that, for nilpotent approximation (4.4), we can (excluding a finite subset of Γ without any influence on the estimates of the entropy and complexity) make the change of variables: $\tilde{w} = w + \sum_{j=1}^{p-1} \lambda_j^*(w) y_j$. This is a change of the parametrization of S, which preserves the parametrization of Γ and yields normal coordinates $(x, y, \tilde{w})$, provided that (x, y, w) are normal. In these new coordinates,

$$\|M(\tilde{w})\| = \inf_{\lambda} \left\| M(\tilde{w}) + \sum_{j=1}^{p-1} \lambda_j L_j(\tilde{w}) \right\|.$$

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