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Nonlinear Schrödinger Equations

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Contents

1 The Schrödinger equation. Introduction **3**
1.1 Motivation 3
1.2 Standing waves 6
1.3 Semiclassical states 7

2 Spikes for the Nonlinear Schrödinger Equation **9**
2.1 The problem 9
2.2 The auxiliary equation 12
2.3 The reduced functional 15

3 Recent results **19**
3.1 Decaying potentials 19
3.2 The critical case $V(x_0) = 0$ 20
3.3 Multibump solutions around maxima of the potential 20
3.4 Radial solutions concentrating on spheres 21
3.5 Schrödinger-Poisson systems 21

Bibliography **23**

Chapter 1

The Schrödinger equation. Introduction

1.1 Motivation

The Schrödinger equation is a law that models the evolution in time of a quantum system, in the same way as Newton's laws predict the evolution of a classical system. Consider one particle (for instance, one electron) in a medium, and imagine that we want to know something about the position of that particle. This can be done by using the so-called wave function $\psi(x,t)$, $\psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$. With the wave function in hand, the probability of the particle to be in a certain region $\Omega \subset \mathbb{R}^N$ is:

$$\int_\Omega |\psi(x,t)|^2 \, dx.$$

Of course, when $\Omega = \mathbb{R}^N$, the probability is one: then, we have:

$$\int_{\mathbb{R}^n} |\psi(x,t)|^2 \, dx = 1.$$

The wave function satisfies the Schrödinger equation:

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + U(x,t)\psi, \tag{1.1}$$

where $\hbar = 1.054572 \cdot 10^{-34}$ is the (reduced) Planck constant, m is the mass and $U(x,t)$ is an external potential.

In order to determine a solution of the above equation we will need to know an initial condition, that is:

$$\psi(x,0) = \psi_0(x).$$

Let us consider first the case $U(x,t) = 0$. We briefly recall some properties of the Fourier transform. In this first chapter $L^2(\mathbb{R}^N)$ is the space of complex-valued functions, that is,

$$L^2(\mathbb{R}^N) = \{f : \mathbb{R}^N \to \mathbb{C} : \int_{\mathbb{R}^N} |f(x)|^2 \, dx < +\infty\}.$$

Lemma 1.1. *Given $f \in L^2(\mathbb{R}^N)$, we define the Fourier transform of f of the form:*

$$\mathcal{F}[f](\omega) = \hat{f}(\omega) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} f(x) e^{-ix\cdot\omega} dx \ \ \forall\, \omega \in \mathbb{R}^N.$$

With this definition, $\hat{f} \in L^2(\mathbb{R}^N)$ and $\mathcal{F}$ is an isometry of $L^2(\mathbb{R}^N)$, with inverse:

$$\mathcal{F}^{-1}[f](\omega) = \check{f}(\omega) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} f(x) e^{ix\cdot\omega} dx \ \ \forall\, \omega \in \mathbb{R}^N.$$

Clearly, $\check{f}(\omega) = \hat{f}(-\omega)$. Moreover, there holds:

1. *Suppose that $f \in L^2(\mathbb{R}^N)$. There holds that $f \in H^1(\mathbb{R}^N) \Leftrightarrow |\omega|\hat{f}(\omega) \in L^2(\mathbb{R}^N)$, and in such case we have:*

   $$\mathcal{F}[f_{x_i}](\omega) = i\omega_i \mathcal{F}[f](\omega).$$

   Moreover $\mathcal{F}$ is an isometry from $H^1(\mathbb{R}^N)$ into

   $$L = \{f \in L^2(\mathbb{R}^N); |x| f(x) \in L^2(\mathbb{R}^N)\}$$

   when considered with the norm: $\|f\|_L = \left(\int_{\mathbb{R}^N} |f(x)|^2 (1+|x|^2)\, dx\right)^{1/2}$.

2. *The dual space of $H^1(\mathbb{R}^N)$ (that is, this is the space of continuos linear maps from $H^1(\mathbb{R}^N)$ into $\mathbb{R}$) is denoted by $H^{-1}(\mathbb{R}^N)$. Through the Fourier transform, we can give an expression of $H^{-1}(\mathbb{R}^N)$. Actually, the dual of the space L is:*

   $$L' = \{\xi : \mathbb{R}^N \to \mathbb{C} \text{ measurable };\ \frac{\xi(x)}{1+|x|} \in L^2(\mathbb{R}^N)\},$$

   with norm $\|\xi\|_{L'} = \left(\int_{\mathbb{R}^N} \frac{|\xi(x)|^2}{(1+|x|^2)}\, dx\right)^{1/2}$. The duality between L and L' is given by the relation:

   $$\langle \xi, f \rangle = Re \int_{\mathbb{R}^N} \xi(x) \overline{f(x)}\, dx.$$

   The Fourier transform implies then an isometry between $H^{-1}(\mathbb{R}^n)$ and L' in the following form; given $\xi \in H^{-1}$, $\mathcal{F}[\xi]$ belongs to the dual of L and acts on L in the form:

   $$\mathcal{F}[\xi](g) = \langle \xi, \mathcal{F}^{-1}[g] \rangle \ \forall\, g \in L.$$

Proposition 1.2. *Let $\psi_0 \in H^1(\mathbb{R}^N)$. Then the unique solution of the problem:*

$$\begin{aligned} &i\psi_t + \Delta\psi = 0, \\ &\psi(x,0) = \psi_0(x), \end{aligned} \tag{1.2}$$

is given by the formula:

$$\psi(x,t) = \left(\frac{1}{4\pi i t}\right)^{N/2} \int_{\mathbb{R}^N} e^{\frac{i|x-y|^2}{4t}} \psi_0(y)\, dy. \tag{1.3}$$

In other words, $\psi \in C^1(\mathbb{R}, H^{-1}(\mathbb{R}^N))$ and the equality (1.2) *holds in H^{-1}.*

Proof. By applying the Fourier transform to (1.2) in the spatial variable x, we have:

$$i\hat{\psi}_t(\omega,t) - |\omega|^2\hat{\psi}(\omega,t) = 0,$$
$$\hat{\psi}(x,0) = \hat{\psi}_0(x).$$

This is an easy ODE that is solved:

$$\hat{\psi}(\omega,t) = e^{-i|\omega|^2 t}\hat{\psi}_0(\omega) := K(\omega,t)\hat{\psi}_0(\omega). \tag{1.4}$$

The idea now is to use the inverse Fourier transform to find out the value of ψ. As we know, the inverse Fourier transform turns products into convolutions. The proof is then finished since

$$\check{K}(x,t) = \left(\frac{1}{4\pi it}\right)^{n/2} e^{\frac{i|x|^2}{4t}}.$$

□

Actually, for any function $\psi_0 \in L^2(\mathbb{R}^N)$, the expression (1.3) can be seen as a solution, in a certain sense (actually, in $H^{-2}(\mathbb{R}^N)$) of the problem.

Remark 1.3. We now extract some consequences from (1.4). Observe that the L^2 norm of $\hat{\psi}$ is constant in time; hence, the norm L^2 of ψ is constant. Analogously, if $\psi_0(x) \in H^1(\mathbb{R}^N)$, the H^1 norm is constant.

From a physical point of view, the term $|\psi(x,t)|^2$ is called the density function, and $|\nabla\psi(x,t)|^2$ is the kinetic energy. Hence, this remark states the conservation of mass and energy.

The Schrödinger equation is usually studied when the action of a potential is also taken under consideration. Sometimes, we can also consider the action of some nonlinear terms. These terms appear mostly when we consider a system of various particles that interact with each other.

In general, the nonlinear Schrödinger equation can be written in the form:

$$\begin{aligned} &i\psi_t + \Delta\psi = g(x,t,\psi)\psi, \\ &\psi(x,0) = \psi_0(x), \end{aligned} \tag{1.5}$$

where $g : \mathbb{R}^N \times \mathbb{R} \times \mathbb{C} \to \mathbb{R}$. We will rapidly motivate why we impose $g(x,t,\psi) \in \mathbb{R}$.

Suppose that we have a solution of (1.5), $\psi(\cdot,t) \in H^1(\mathbb{R}^N)$ for all t. The equation is physically relevant if it preserves the total density, as before. A formal calculation yields:

$$\Lambda(t) = \int_{\mathbb{R}^N} |\psi(x,t)|^2\,dx = Re\int_{\mathbb{R}^N} \psi(x,t)\overline{\psi(x,t)}\,dx,$$

$$\Lambda'(t) = 2Re\int_{\mathbb{R}^N} \psi_t(x,t)\overline{\psi(x,t)}\,dx = 2Re\int_{\mathbb{R}^N} i\,(\Delta\psi\overline{\psi} - g(x,t,\psi)\psi\overline{\psi})\,dx =$$
$$2Re\; i\int_{\mathbb{R}^N} |\nabla\psi|^2 - g(x,t,\psi)|\psi|^2\,dx = 0.$$

The most usual type of nonlinearity is the form $g(x,t,|\psi|)\psi$. Among these, the case $g = |\psi|^{p-1}\psi$ has been very important because of its simplicity. The case $p = 3$ reveals to be specially relevant in Physics.

There is a huge literature on the time dependent Schrödinger equation (linear and nonlinear); for the interested reader we suggest [11]. As we shall see in the next section, our purpose is to look for special solutions; in this search we will be concerned with elliptic equations in $\mathbb{R}^N$.

1.2 Standing waves

Let us consider the nonlinear Schrödinger equation:

$$\begin{aligned} &i\psi_t + \Delta\psi + U(x)\psi = g(|\psi|)\psi, \\ &\psi(x,0) = \psi_0(x), \end{aligned} \tag{1.6}$$

where $U : \mathbb{R}^N \to \mathbb{R}$ is an external potential and g is the nonlinear term. We will be interested in a very specific type of solutions, the so-called standing waves, which are solutions of the form:

$$\psi(x,t) = e^{-iEt}u(x), \tag{1.7}$$

where $u(x)$ is a real function and $E \in \mathbb{R}$. These solutions correspond to static situations in the sense that the density $|\psi(x,t)|^2 = u(x)^2$ does not change in time.

However, from this static solutions we can obtain also other solutions: indeed, if $U(x)$ is invariant along the direction given by $\xi \in \mathbb{R}^N - \{0\}$, then (1.6) is invariant under the galilean transformation:

$$\psi(x,t) \mapsto \psi(x - \xi t, t) exp\left[\frac{i}{2}\left(\xi \cdot x - \frac{1}{2}|\xi|^2 t\right)\right].$$

Therefore, the standing waves also yield solutions that move in the direction given by ξ, preserving its form; those are called traveling waves.

Plugging (1.7) into (1.6) we obtain:

$$-\Delta u + (U(x) - E)u + g(x, |u|)u = 0.$$

This is the stationary nonlinear Schrödinger equation.

As we see, we have got an elliptic equation in $\mathbb{R}^N$. In order to study it, we can use topological degree or variational methods, that have already been introduced in other courses.

Let us briefly treat the easiest case: suppose that both V and g are equal to zero. Then, we have the problem:

$$-\Delta u = \lambda u, \ x \in \mathbb{R}^N. \tag{1.8}$$

Exercise 1.1. Suppose that $u \in H^1(\mathbb{R}^N)$ is a solution of (1.8) for some $\lambda \in \mathbb{R}$. Prove that $u = 0$.

Then, the linear Schrödinger equation without potentials does not admit standing waves. Let us see what happens for nontrivial potentials. We are concerned now with a linear equation in the form:

$$-\Delta u + V(x)u = \lambda u, \ x \in \mathbb{R}^N. \tag{1.9}$$

We will say that λ is an eigenvalue if this problem has a nontrivial solution in $H^1(\mathbb{R}^N)$. Such a solution is called eigenfunction.

Exercise 1.2. Suppose that $V : \mathbb{R}^N \to \mathbb{R}$ is continuous and $\lim_{|x|\to+\infty} V(x) = +\infty$. Prove that there exists an increasing sequence of eigenvalues $\{\lambda_k\}_{k\in\mathbb{N}}$ of (1.9).

Hint: Suppose that $\inf V \geq 0$, and consider the Hilbert space $H = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)u^2(x)\,dx < +\infty\}$, with the norm:

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u(x)|^2 + V(x)u^2(x)\,dx\right)^{1/2}.$$

A more difficult example is the following:

Example 1.4. (Kato theorem) Suppose that $V \geq 0$ is a continuous function satisfying that:

$$\lim_{|x| \to +\infty} |x| V(x) = 0.$$

Then there is no eigenvalue for problem (1.9).

As we see from the above examples, the existence or not of eigenvalues depends strongly on the behavior of the potential $V(x)$.

For more information on the eigenvalues and, in general, the spectrum of an operator in the form $-\Delta + V(x)$, we recommend [8].

1.3 Semiclassical states

Let us write the nonlinear Schrödinger equation in the form:

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi U(x)\psi - g(|\psi|)\psi. \tag{1.10}$$

Let us define $\varepsilon^2 = \frac{\hbar}{2m}$. In order to study standing waves for this problem, as we have seen above, we are led with the equation:

$$-\varepsilon^2\Delta u + V(x)u = g(|u|)u. \tag{1.11}$$

If we consider a problem with a mass m at a macroscopic scale, the value of ε will be then very small. That is, for normal masses the quantum effect is negligible and the motion is described by classical mechanics. On the other hand, if m is a very small quantity (for instance, if we deal with electrons or other microscopic particles), then ε becomes significant.

We will be interested not only in the existence of solutions for (1.11), but also in their behavior as $\varepsilon \to 0$. In a sense, this will give us the transition between classical and quantum mechanics. The solutions that are found for ε very small are called semiclassical states. Typically, the solutions may tend to concentrate around a certain point, as $\varepsilon \to 0$; in such case, the solutions are called *spikes*.

Chapter 2

Spikes for the Nonlinear Schrödinger Equation

2.1 The problem

In this chapter we will study the problem:

$$-\varepsilon^2 \Delta u + V(x)u = |u|^{p-1}u, \;\; u \in H^1(\mathbb{R}^N). \tag{2.1}$$

We assume that $1 < p < \frac{N+2}{N-2}$ and V satisfies $0 < a \leq V(x) < b$. For technical reasons, we also assume that:

[H] *$V \in C^2$ and $\nabla V(x)$, $D^2 V(x)$ are uniformly bounded.*

In general, the existence of solution of (2.1) is not an easy matter, and still there are important open problems. In the following examples, we show that the existence will depend strongly on the potential V:

Proposition 2.1. *Suppose that $\nabla V(x) \cdot \xi > 0$ for a. e. $x \in \mathbb{R}^N$ and some fixed $\xi \in \mathbb{R}^N - \{0\}$. Then the unique solution of* (2.1) *is $u = 0$.*

Proof. Let $u \in H^1(\mathbb{R}^N)$ be a solution of (2.1). Observe that $(V(x)-1)u$ also belongs to $H^1(\mathbb{R}^N)$. So, we can apply the bootstrap method to the problem:

$$-\varepsilon^2 \Delta u + u = |u|^{p-1}u - (V(x)-1)u, \;\; u \in H^1(\mathbb{R}^N).$$

In such way, we obtain that $u \in H^2(\mathbb{R}^N)$, and the above equation holds in $L^2(\mathbb{R}^N)$.

Suppose for simplicity that $\xi = (1, 0 \ldots 0)$, and denote $v = u_{x_1} \in H^1(\mathbb{R}^N)$. We multiply equation (2.1) by v and integrate to obtain:

$$\varepsilon^2 \int_{\mathbb{R}^N} \nabla u \cdot \nabla v + V(x)uv - |u|^{p-1}uv = 0.$$

But:

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v = \frac{d}{dt}\bigg|_{t=0} \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(x+t\xi)|^2 \, dx = 0,$$

$$\int_{\mathbb{R}^N} |u|^{p-1}uv = \frac{d}{dt}\Big|_{t=0} \frac{1}{2}\int_{\mathbb{R}^N} |u(x+t\xi)|^{p+1}\,dx = 0,$$

$$\int_{\mathbb{R}^N} V(x)uv = \frac{d}{dt}\Big|_{t=0} \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2(x+t\xi)\,dx = \frac{d}{dt}\Big|_{t=0} \frac{1}{2}\int_{\mathbb{R}^N} V(y-t\xi)u^2(y)\,dy =$$

$$-\frac{1}{2}\int_{\mathbb{R}^N} \nabla V(y)\cdot\xi\, u^2(y)\,dy.$$

By assumptions, u must be equal to zero.

□

Exercise 2.1. In the above proof we have used the derivation under the integral sign. Is it justified? Can you prove the result without using this?

Example 2.2. Suppose that $V(x) = \lambda > 0$, $\varepsilon = 1$. Then problem (2.1) has a positive radial function.

Proof. It suffices to consider the Sobolev space of radial functions:

$$H^1_r(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : \; u\circ g = u \; a.e. \; x \in \mathbb{R}^N, \; \forall g \in O(N)\},$$

where $O(N)$ is the group of linear isometric transformations of $\mathbb{R}^N$. A well-known result of Strauss [25] establish that the inclusion $H^1_r(\mathbb{R}^N) \subset L^p(\mathbb{R}^N)$ is compact for $2 < p < \frac{2N}{N-2}$. Let us consider the functional:

$$I : H^1_r(\mathbb{R}^N) \to \mathbb{R}, \;\; I(u) = \int \frac{1}{2}\left(|\nabla u|^2 + \lambda u^2\right) - \frac{1}{p+1}u_+^{p+1}.$$

Exercise 2.2. Check that the hypotheses of the mountain pass theorem are verified by I.

Once the exercise is done, the mountain pass theorem of Ambrosetti-Rabinowitz [6] implies the existence of a critical point of I. Then, there is a solution $u \in H^1_r(\mathbb{R}^N)$ of the problem:

$$-\Delta u + \lambda u = u_+^p.$$

Finally, the maximum principle implies that u is positive. □

Remark 2.3. From now on we denote by U_λ the solution obtained above, and $U = U_1$. It is easy to check that $U_\lambda(x) = \lambda^{\frac{1}{p-1}}U(\sqrt{\lambda}x)$. It is also well-known that U is C^∞ and it and its derivatives have an exponential decay (see [25]).

In this chapter we are interested in studying concentration phenomena for $\varepsilon \to 0$. In other words, we want to answer to the questions: are there solutions concentrating around a certain point? Which points are eligible?

In this chapter we will state and prove the following theorem, that answers to both questions. It was first stated by Floer and Weinstein [18] for $p = 3$ and $N = 1$; a general proof was given in [1, 15, 19, 21]. The proof we give is based on the monograph [3].

Theorem 2.4. *Let x_0 be a nondegenerate critical point of V. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists a positive solution v_ε of the problem:*

$$-\varepsilon^2 \Delta v + V(x)v = |v|^{p-1}v, \;\; v \in H^1(\mathbb{R}^N). \tag{2.2}$$

Moreover v_ε is a spike at x_0, that is,

$$\|v_\varepsilon\|_{L^\infty} \in [c, C], \; 0 < c < C,$$

$$\forall \delta > 0, \; \| \, v_\varepsilon|_{\mathbb{R}^N - B(x_0,\delta)} \|_{L^\infty} \to 0 \;\; (\varepsilon \to 0).$$

For the proof, observe that we can assume $x_0 = 0$; if not, it suffices to make a convenient translation. By making the change of variables $u(x) = v(\varepsilon x)$, we are lead with the equation:

$$-\Delta u + V(\varepsilon x)u = u^p \;, \; u(x) > 0. \tag{2.3}$$

We use a variational perturbative scheme. The solutions of (2.3) correspond to critical points of the functional $I_\varepsilon : H^1(\mathbb{R}^N) \to \mathbb{R}$,

$$I_\varepsilon(u) = \int_{\mathbb{R}^N} \frac{1}{2}\left(|\nabla u|^2 + V(\varepsilon x)u^2\right) - \frac{1}{p+1}|u|^{p+1}.$$

A first attempt could be trying to obtain solutions u_ε by using the Implicit Function Theorem to the operator I_0', where $I_0 : H^1(\mathbb{R}^N) \to \mathbb{R}$ is the unperturbed functional:

$$I_0(u) = \int_{\mathbb{R}^N} \frac{1}{2}\left(|\nabla u|^2 + V(0)u^2\right) - \frac{1}{p+1}|u|^{p+1}.$$

The problem is that I_0 has a degenerate manifold of critical points due to its translation invariance; this manifold is:

$$\{U_{V(0)}(\cdot - \xi) : \xi \in \mathbb{R}^N\}.$$

So, the Implicit Function Theorem is not directly applicable. Instead, the idea is to use a Lyapunov-Schmidt reduction.

Roughly speaking, if a solution z of (2.3) takes its relevant values around a certain point ξ (that may depend on ε), then z should be similar to $z(x) = U_{V(\varepsilon\xi)}(x - \xi)$ when $\varepsilon \to 0$. Motivated by this idea, we define the manifold of "possible approximate solutions":

$$Z = Z_\varepsilon = \{z_{\varepsilon\xi} = U_{V(\varepsilon\xi)}(\cdot - \xi) : \xi \in B(0, \varepsilon^{-1})\}.$$

For every $z_{\varepsilon\xi} \in Z$, we define $W = W_z = T_z Z^\perp$, $P = P_z : H \to W$ the orthogonal projection onto W. Our approach is to find a pair $z \in Z$, $w \in W_z$ such that $I_\varepsilon'(z+w) = 0$, or equivalently:

$$\begin{cases} \text{a) } PI_\varepsilon'(z+w) = 0, \\ \text{b) } (\mathbb{I} - P)I_\varepsilon'(z+w) = 0. \end{cases} \tag{2.4}$$

The fist equation above is called auxiliary equation, and the second one receives the name of bifurcation equation.

The general idea of the proof is to find, for ε small and any $z \in Z_\varepsilon$, a function $w_z \in W_z$ solving the auxiliary equation; this is possible because in this equation we do not have the degenerate effect of the translations. After that we will try to find a solution for the bifurcation equation among the set of pairs $\{z + w_z : z \in Z\}$.

2.2 The auxiliary equation

As we said before, we will be able to solve the auxiliary equation because, working on W, we do not have the invariance given by the translations. The proof does not follow directly from the Implicit Function Theorem, but the application of the ideas of its proof will give us the result.

We first give an abstract result, that may be applied in many problems of this kind.

Consider a Hilbert space H, $z \in H$, and let $\Phi \in C^1(H,H)$ be an operator. Suppose that for some fixed $\delta > 0$, there holds:

$$\|\Phi(z)\| < \delta, \tag{A1}$$

$$\Phi'(z) : H \to H \text{ is invertible and } \|\Phi'(z)^{-1}\| \leq c, \;\; c > 0. \tag{A2}$$

Take $\rho \geq 2c$ and define:

$$B = \{u \in H : \; \|u\| \leq \rho\delta\}.$$

We further assume that

$$\|\Phi'(z+u) - \Phi'(z)\| < \frac{1}{\rho} \;\forall u \in B. \tag{A3}$$

Theorem 2.5. *Under the assumptions (A1), (A2) and (A3), there exists a unique $u \in B$ such that $\Phi(z+u) = 0$.*

Remark 2.6. Theorem 2.5 holds for any $\delta > 0$; however, condition (A3) is more likely to be satisfied when δ is a small quantity.

Proof. Let us define the map $S : B \mapsto H$ by setting

$$S(w) = w - [\Phi'(z)]^{-1}(\Phi(z+w)). \tag{2.5}$$

Clearly, a fixed point u of S will give rise to a solution of the equation $\Phi(z+u) = 0$. We apply now the Banach contraction theorem to the operator S.

For any $v \in H$, $w \in B$, one has

$$S'(w)[v] = v - [\Phi'(z)]^{-1}(\Phi'(z+w)[v]) = [\Phi'(z)]^{-1}(\Phi'(z)[v] - \Phi'(z+w)[v]).$$

Thus we find

$$\|S'(w)[v]\| \leq \frac{c}{\rho}\|v\| \leq \frac{1}{2}\|v\|. \tag{2.6}$$

Therefore we conclude that S is a contraction. We finish the proof if we show that S maps B into itself. With that purpose, let us compute:

$$\|S(0)\| = \|[\Phi'(z)]^{-1}(\Phi(z))\| \leq c\delta.$$

On the other hand, for any $w \in B$ we can use (2.6) to deduce:

$$\|S(w) - S(0)\| \leq \frac{c}{\rho}\|w\| \leq c\delta.$$

By using the triangular inequality of the norm, we obtain:

$$\|S_\varepsilon(w)\| \leq 2c\delta \leq \rho\delta.$$

This shows that $S(B) \subset B$ and completes the proof of the lemma.

□

Let us fix $z = z_{\varepsilon,\xi} \in Z_\varepsilon$. We intend to apply Theorem 2.5 to find a zero for the operator $\Phi : W \to W$, $\Phi(w) = PI'_\varepsilon(z+w)$. Let us verify that the conditions (A1), (A2) and (A3) are satisfied:

(A1) $\|PI'_\varepsilon(z)\| \leq C_1\varepsilon$ for all $z \in Z$.

We compute:

$$I'_\varepsilon(z)(v) = \int_{\mathbb{R}^N} \nabla z \nabla v + V(\varepsilon x)zv - |z|^{p-1}zv = \int_{\mathbb{R}^N} V(\varepsilon x) - V(\varepsilon\xi)zv \leq C\varepsilon \int_{\mathbb{R}^N} |x-\xi|zv \leq$$

$$C\varepsilon \left(\int_{\mathbb{R}^N} |x-\xi|^2 |U_{V(\varepsilon\xi)}(x-\xi)|^2 \, dx \right)^{1/2} \|v\|_{L^2} \leq C\varepsilon.$$

In the last inequality we have used the exponential decay of the function U, see Remark 2.3.

(A2) $\|[PI''_\varepsilon(z)]^{-1}\| \leq C_2$ for all $z \in Z$.

This is the reason why we have had to restrict ourselves to W.

First of all, let us compute the second derivative of I_ε:

$$I''_\varepsilon(z)[\varphi_1, \varphi_2] = \int_{\mathbb{R}^N} \nabla\varphi_1 \cdot \nabla\varphi_2 + V(\varepsilon x)\varphi_1\varphi_2 - pz^{p-1}\varphi_1\varphi_2. \tag{2.7}$$

The following nondegeneracy result of the limiting problem is well-known, and will be the main tool in our arguments. For a proof, see for instance [3].

Proposition 2.7. *Let $\lambda > 0$ and $U_\lambda \in H^1(\mathbb{R}^N)$ the unique positive radial solution of the problem:*

$$-\Delta U + \lambda U = U^p.$$

Define the quadratic operator $Q : H^1(\mathbb{R}^N) \to \mathbb{R}$ in the form:

$$Q_\lambda[v] = \int_{\mathbb{R}^N} |\nabla v|^2 + \lambda v^2 - pU_\lambda^{p-1} v^2.$$

We denote $(U_\lambda)_k = \frac{\partial U}{\partial x_k}$. Then there holds:

1. $Q_\lambda[U_\lambda] = (1-p)\|U_\lambda\|^2 < 0$.

2. $Q_\lambda[(U_\lambda)_k] = 0$, $k = 1 \ldots N$.

3. $Q_\lambda[v] \geq c\|v\|^2$ *for all* $v \perp U_\lambda$, $v \perp (U_\lambda)_k$, $k = 1 \ldots N$.

Recall that: $z(x) = U_{V(\varepsilon\xi)}(x-\xi) = V(\varepsilon\xi)^{\frac{1}{p-1}} U\left(\sqrt{V(\varepsilon\xi)}(x-\xi)\right)$. The space $T_z Z$ is spanned by the functions:

$$z_k = \frac{\partial z_{\varepsilon,\xi}}{\partial \xi_k} = -V(\varepsilon\xi)^{\frac{1}{p-1}} \frac{\partial U}{\partial x_k}\left(\sqrt{V(\varepsilon\xi)}(x-\xi)\right) + O(\varepsilon). \tag{2.8}$$

So, for ε very small, the space $T_z Z$ becomes "very close" to the space spanned by $\{\frac{\partial U_{V(\varepsilon\xi)}}{\partial x_k}(x-\xi)\}$.

We now decompose $W = (T_z Z)^{\perp}$ as an orthogonal sum of the spaces $W = A \oplus B$, where A is the space spanned by z and $B = A^{\perp} \cap W$. By using Proposition 2.7, one can prove that for ε small enough, there exists $c > 0$ such that:

1. $PI''_{\varepsilon}(z)[z,z] \geq c > 0$.
2. $PI''_{\varepsilon}(z)[u,u] \leq -c < 0$ for all $u \in B$.

We conclude the proof of (A2) thanks to the following result, which is left as an exercise:

Exercise 2.3. Let H be a Hilbert space and $L : H \to H$ a linear self-adjoint operator (that is, $\langle Lu, v\rangle = \langle u, Lv\rangle$ for all $u,\ v \in H$). We define its associated quadratic form $Q : H \times H \to \mathbb{R}$, $Q(u,v) = \langle Lu, v\rangle$.

Suppose that there exists $E \subset H$ a finite dimensional space, $c > 0$ such that:

1. $Q[u,u] \leq -c\|u\|^2$ for all $u \in E$,
2. $Q[u,u] \geq c\|u\|^2$ for all $u \in E^{\perp}$.

Prove that L is invertible.

Remark 2.8. *(and hint!)* When $E = \{0\}$, the above result is nothing but the well-known Lax-Milgram Theorem. Conversely, the Lax-Milgram Theorem is the key for the proof of the exercise.

(A3) For any fixed $\delta > 0$, we can take ε small enough so that $\|PI''_{\varepsilon}(z) - PI''_{\varepsilon}(z+u)\| \leq \delta$ for all $z \in Z$, $u \in B = B(0, C\varepsilon)$, where $C = 2C_1C_2$, C_k being the constants given in the previous steps.

Here this property is clearly satisfied: it suffices to observe that I''_{ε} is uniformly continuous in bounded sets.

Hence, the Proposition 2.7 yields the existence of a unique $w \in B$ solving the auxiliary equation. Observe that $\|w\| \leq C\varepsilon$ for certain positive constant $C > 0$.

We can define $\bar{Z} = \{z + w_z : \ z \in Z\}$, where w_z is the unique solution mentioned above.

In the next section we are concerned with the problem of finding solutions to the bifurcation equation on $\bar{Z}$.

2.3 The reduced functional

Before trying to solve the bifurcation equation, we consider again the set $\bar{Z}_\varepsilon$. In the next result, among other things, we obtain that $\bar{Z}_\varepsilon$ is a C^1 manifold.

Proposition 2.9. *For ε sufficiently small, the application $w : B(0,\varepsilon^{-1}) \subset \mathbb{R}^N \to W$, $w(\xi) = w(z_{\varepsilon,\xi})$ is of class C^1. Moreover, $\left|\frac{\partial w}{\partial \xi_k}\right| \to 0$ (uniformly in $z \in Z$ as $\varepsilon \to 0$).*

Proof. The fact that $w_{\varepsilon,\xi}$ is C^1 with respect to ξ follows from the Uniform Contraction Theorem studied in the mini-course of Ernest Fontich. Recall that w has been found by applying the Banach contraction principle to an operator that depends on ξ in a C^1 form.

In order to estimate the derivatives of $w(\xi)$, recall that

$$PI'_\varepsilon(z+w) = 0 \Rightarrow I'_\varepsilon(z+w) = \sum_{k=1}^{N} \langle I'_\varepsilon(z+w)\,,\, z_k\rangle \frac{z_k}{\|z_k\|^2},$$

where $z = z_{\varepsilon,\xi}$, $w = w_{\varepsilon,\xi}$ and z_k is the partial derivative with respect to ξ_k, see (2.8).

We compute the derivatives with respect to ξ_j, and obtain:

$$I''_\varepsilon(z+w)[z_j+w_j] = \Bigg(\sum_{k=1}^{N} \langle I''_\varepsilon(z+w)[z_j+w_j],\, z_k\rangle \frac{z_k}{\|z_k\|^2} +$$
$$\langle I'_\varepsilon(z+w),\, z_{j,k}\rangle \frac{z_k}{\|z_k\|^2} + \langle I'_\varepsilon(z+w),\, z_k\rangle \frac{\partial}{\partial \xi_j} \frac{z_k}{\|z_k\|^2} \Bigg).$$

In order to obtain estimates about the differential, we recall the condition (A1) to obtain:

$$I'_\varepsilon(z+w) = I'_\varepsilon(z) + I''_\varepsilon(\chi)(w) = O(\varepsilon) \text{ (for some } \chi \in [z, z+w]).$$

Moreover, taking into account (2.8), we have

$$I''_\varepsilon(z+w)[z_j] - I''_\varepsilon(z)[z_j] \le \|I''_\varepsilon(z+w) - I''_\varepsilon(z)\|\; \|z_j\| = o(1) \text{ (independently of } z \in Z)$$

Finally, from (2.7), (2.8) and Proposition 2.7 we have the estimate:

$$I''_\varepsilon(z)[z_j] = O(\varepsilon). \tag{2.9}$$

Hence, we arrive to the conclusion that $PI''_\varepsilon(z+w)[w_j] = o(1)$. We use now (A2) to conclude that $\|w_j\| = o(1)$.

□

Remark 2.10. Recall that by the arguments of the previous section, the set $\bar{Z}$ is very close to Z. The Proposition 2.9 implies that $\bar{Z}$ varies very little from Z in a C^1 sense.

In next proposition we show that $\bar{Z}$ is a *natural constraint* for I_ε; in other words:

Proposition 2.11. *The critical points of I_ε that are located in $\bar{Z}$ are exactly the critical points of $I_\varepsilon|_{\bar{Z}}$.*

Proof. Because of the definition of $\bar{Z}$, $I_\varepsilon'(z+w)$ is orthogonal to W.

If w is a critical point of $I_\varepsilon|_{\bar{Z}}$, it means that $\langle I_\varepsilon'(z+w), v_k\rangle = 0$ for all $v_k \in T_{z+w}\bar{Z}$.

Because of Proposition 2.9, we have that $H^1(\mathbb{R}^N) = W_z \oplus T_{z+w}\bar{Z}$ is a continuous decomposition (even if it is not orthogonal). Therefore, $I_\varepsilon'(z+w) = 0$ and we are done. □

Definition 2.12. In general, the restriction of the functional I_ε to the natural constraint $\bar{Z}$ receives the name of *reduced functional*. Its critical points yield solutions of the problem under consideration.

So, in order to conclude the proof, we just need to estimate the reduced functional:

$$\phi_\varepsilon : \bar{Z} \to \mathbb{R}$$

$$\phi_\varepsilon(z+w) = I_\varepsilon(z_{\varepsilon,\xi} + w_{\varepsilon,\xi}).$$

By using the parametrization of Z, we need to compute the critical points of the function:

$$\alpha_\varepsilon : B(0,\varepsilon^{-1}) \to \mathbb{R},\ \alpha_\varepsilon(\xi) = I_\varepsilon(z_{\varepsilon,\xi} + w_{\varepsilon,\xi}).$$

Recall that w is an unknown function but is small (actually, of order ε). We try then to to approximate the values of α_ε by computing:

$$\beta_\varepsilon : B(0,\varepsilon^{-1}) \to \mathbb{R},\ \beta_\varepsilon(\xi) = I_\varepsilon(z_{\varepsilon,\xi}).$$

First of all, we compute the difference by using a Taylor expansion:

$$I_\varepsilon(z_{\varepsilon,\xi} + w_{\varepsilon,\xi}) - I_\varepsilon(z_{\varepsilon,\xi}) = I_\varepsilon'(z_{\varepsilon,\xi})(w) + \frac{1}{2} I_\varepsilon''(\chi)[w^2],$$

with $\chi \in [z, z+w]$. By using (A1) and (A2), we get:

$$\alpha_\varepsilon(\xi) - \beta(\xi) = I_\varepsilon(z_{\varepsilon,\xi} + w_{\varepsilon,\xi}) - I_\varepsilon(z_{\varepsilon,\xi}) = O(\varepsilon^2) \tag{2.10}$$

We now compute the difference between the derivatives; in the computations below we denote z_k, w_k to the derivatives of z, w with respect to ξ_k:

$$\frac{\partial}{\partial \xi_k}\alpha_\varepsilon(\xi) = I_\varepsilon'(z+w)(z_k + w_k),\ \ \frac{\partial}{\partial \xi_k}\beta_\varepsilon(\xi) = I_\varepsilon'(z)(z_k);$$

$$\frac{\partial}{\partial \xi_k}(\alpha_\varepsilon(\xi) - \beta_\varepsilon(\xi)) = I_\varepsilon'(z+w)(w_k) + I_\varepsilon'(z+w)(z_k) - I_\varepsilon'(z)(z_k).$$

Clearly, for some $\chi \in [z, z+w]$, $I_\varepsilon'(z+w)(w_k) - I_\varepsilon'(z)(w_k) = I_\varepsilon''(\chi)[w, w_k] = o(\varepsilon)$. On the other hand,

$$I_\varepsilon'(z+w)(z_k) - I_\varepsilon'(z)(z_k) = I_\varepsilon''(\chi)[z_k, w] =$$

$$I_\varepsilon''(\chi)[z_k, w] - I_\varepsilon''(z)[z_k, w] + I_\varepsilon''(z)[z_k, w] \le$$

$$\|I_\varepsilon''(\chi) - I_\varepsilon''(z)\|\ \|z_k\|\ \|w\| + \|I_\varepsilon''(z)(z_k)\|\ \|w\| \Rightarrow$$

$$\frac{\partial}{\partial \xi_k}(\alpha_\varepsilon(\xi) - \beta_\varepsilon(\xi)) = o(\varepsilon). \tag{2.11}$$

In the last estimate we have used (2.9) and the fact that I_ε is uniformly C^2 in bounded sets.

Finally, let us compute the function β_ε and its derivatives:

$$\beta_\varepsilon(\xi) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla z_{\varepsilon,\xi}|^2 + V(\varepsilon x) z_{\varepsilon,\xi}^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^N} z_{\varepsilon,\xi}^{p+1} =$$

$$\frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon x) - V(\varepsilon\xi)) z_{\varepsilon,\xi}^2 + \frac{1}{2}\int_{\mathbb{R}^N} |\nabla z_{\varepsilon,\xi}|^2 + V(\varepsilon\xi) z_{\varepsilon,\xi}^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^N} z_{\varepsilon,\xi}^{p+1} =$$

$$\frac{1}{2}\int_{\mathbb{R}^N} |\nabla U_\lambda|^2 + \lambda U_\lambda^2\, dx - \frac{1}{p+1}\int_{\mathbb{R}^N} U_\lambda^{p+1} + O(\varepsilon),$$

where $\lambda = V(\varepsilon\xi)$. We now recall that $U_\lambda(x) = \lambda^{\frac{1}{p-1}} U(\sqrt{\lambda}x)$; after some computations, we obtain that:

$$\beta_\varepsilon(\xi) = CV(\varepsilon\xi)^\theta + \frac{1}{2}\int_{\mathbb{R}^N} (V(\varepsilon(x+\xi)) - V(\varepsilon\xi)) U_{V(\varepsilon\xi)}(x)^2\, dx =$$

$$CV(\varepsilon\xi)^\theta + O(\varepsilon),$$

where $C = \left(\frac{1}{2} - \frac{1}{p+1}\right) \|U\|^2 > 0$ and $\theta = \frac{p+1}{p-1} - \frac{N}{2} > 0$. Moreover:

$$\frac{\partial}{\partial \xi_k}\beta_\varepsilon(\xi) = C\theta V(\varepsilon\xi)^{\theta-1}\varepsilon \frac{\partial V}{\partial \xi_k}(\varepsilon\xi) + o(\varepsilon).$$

In conclusion, we observe that α_ε is very close to the function $CV(\varepsilon\xi)^\theta$ in a C^1 sense; then, if V has a nondegenerate critical point at $0 \in B(0,1)$, also α_ε will have a critical point at ξ_ε such that $\varepsilon\xi_\varepsilon \to 0$. And this corresponds to a critical point of I_ε, and hence a solution of (2.3).

Moreover, the solution u_ε that we have found is of the form $u_\varepsilon \sim U_{V(\varepsilon\xi_\varepsilon)}(x - \xi_\varepsilon)$. Recalling the change of variables $u(x) = v(\varepsilon x)$, we get a solution v_ε of (2.2) in the form:

$$v_\varepsilon(x) \sim U_{V(0)}\left(\frac{x}{\varepsilon}\right).$$

Remarks 2.13.

In Theorem 2.4 we stated that there are spikes around nondegenerate critical points of V. The condition of nondegeneracy can be relaxed: actually, it suffices that V has a critical point which is preserved under small C^1 perturbations of V. For instance, this is verified by any strict minimum or maximum.

Observe that we have not proved that ξ_ε converges to zero, but that $\varepsilon\xi_\varepsilon$ does. Actually, the convergence of ξ_ε will depend on the behavior of V near x_0.

There is a different approach to the study of semiclassical states, that have been developed by Del Pino and Felmer, see [15, 16, 17]. It consists of finding critical points of a conveniently penalized energy functional. The method is more complicated, specially if the critical point of V is not a minimum, but allows us to study situations in which there is no information available about the linearized operator (in our case, Proposition 2.7).

If V has several critical points one can prove the existence of multibump solutions, that is, solutions that concentrate around several points (see [21]). Actually, for any set

$A = \{x_1 \ \dots x_m\}$ of nondegenerate critical points of V, and ε small enough, there exists a solution v_ε of (2.2) such that:

$$v_\varepsilon(x) \sim \sum_{k=1}^{m} U_{V(x_k)}\left(\frac{x - x_k}{\varepsilon}\right).$$

Chapter 3

Recent results

In this chapter we just give a short overview on some recent results in the study of semiclassical states for the Nonlinear Schrödinger equation. For convenience of the reader, we state again the problem under consideration:

$$-\varepsilon^2\Delta u + V(x)u = |u|^{p-1}u, \quad u \in H^1(\mathbb{R}^N). \tag{3.1}$$

We assume that $1 < p < \frac{N+2}{N-2}$ and V satisfying the [H] condition:

[H] *$V \in C^2$ and $\nabla V(x)$, $D^2V(x)$ are uniformly bounded.*

3.1 Decaying potentials

In Theorem 2.4 some hypotheses were assumed on V. Some of them are of technical type (namely [H]), but some are not. In particular, the condition:

$$0 < a \le V(x) < b$$

reveals to be essential at many points of the proofs. Some recent studies have been concerned with the case of potentials that decay to zero at infinity, see [2, 5]. The following result has been proved in [5]:

Theorem 3.1. *Suppose that there exists a, b positive constants such that:*

$$b \ge V(x) \ge \frac{a}{1+|x|^2}.$$

Let x_0 be a nondegenerate critical point of V. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists a positive solution u_ε of (2.1) satisfying that:

$$\|u_\varepsilon\|_{L^\infty} \in [c, C], \; 0 < c < C,$$

$$\forall \delta > 0, \; \| u_\varepsilon|_{\mathbb{R}^N - B(x_0,\delta)}\|_{L^\infty} \to 0.$$

In order words, Theorem 2.4 holds if we permit V to have a decay like $|x|^{-2}$ at infinity. In a sense this result is optimal: if V has a decay of order $|x|^{-\alpha}$ for some $\alpha > 2$, solutions in $H^1(\mathbb{R}^N)$ of (3.1) are not to be expected.

One of the main difficulties is that the solutions we are trying to find do not have an exponential decay. Similar results have also been obtained in [9].

3.2 The critical case $V(x_0) = 0$

Suppose now that $\liminf_{|x|\to+\infty} V(x) = a > 0$, $V(x) \geq 0$ and $V(x_0) = 0$. Obviously, x_0 is a minimum for V, but its critical value is zero. A paper by Byeon and Wang ([10]) deals with this problem

This case is more complicated, and the asymptotic behavior of the solutions will depend much on how V approaches to zero at x_0. Actually, in [10] it is proved that:

Theorem 3.2. *We have the following results:*

1. *Suppose that $x_0 = 0$ is an isolated minimum of V and that $V(x) \sim |x|^{2m}$ for $x \to 0$ and $m \in \mathbb{N}$. Then for ε small enough there exists a solution u_ε of* (3.1) *such that:*

$$\varepsilon^{-\frac{2m}{(p-1)(m+2)}}\, u_\varepsilon\left(\varepsilon^{\frac{2}{m+2}}x\right) \to w,$$

where $w \in H^1\mathbb{R}^N$ is a positive radial solution of the problem:

$$-\Delta w(x) + |x|^{2m} w(x) = w(x)^p.$$

2. *Suppose now that $A = \{x \in \mathbb{R}^N : \ V(x) = 0\}$ is such that $int\,A$ is nonempty and connected. Then for ε small enough there exists a solution u_ε of* (3.1) *such that:*

$$\varepsilon^{-\frac{2}{p-1}} u_\varepsilon \to w$$

in a certain sense, where $w \in H^1(\mathbb{R}^N)$, $w \geq 0$ satisfies that:

$$\begin{array}{ll} -\Delta w(x) = w^p(x) & x \in int\,A, \\ w(x) = 0 & x \in \mathbb{R}^N - int\,A. \end{array}$$

The proofs of these results do not use the perturbative scheme shown in Chapter 2, but the approach by Del Pino and Felmer [15].

3.3 Multibump solutions around maxima of the potential

In the previous chapter we gave a comment about multibump solutions. Typically, multibumps appear around different critical points of the potential V. However, in [20] a quite surprising result is given; under some hypotheses, Kang and Wei prove the existence of multibumps whose maxima converge to a single point. In this case the interaction among the bumps is not negligible, as it is in the work of Li [21].

Actually, one has the following:

Theorem 3.3. *Suppose that $0 < a \leq V(x) < b$ and that V attains a strict relative maximum at x_0. Then, for any $k \in \mathbb{N}$ there exists $\varepsilon_k > 0$ such that for $0 < \varepsilon < \varepsilon_k$ there exists a solution u_ε of* (3.1) *with the following property:*

u_ε has exactly k local maxima at points $\{p_1^\varepsilon \dots p_k^\varepsilon\}$ such that $p_i^\varepsilon \to x_0$ as $\varepsilon \to 0$. Moreover,

$$\frac{|p_i^\varepsilon - p_j^\varepsilon|}{\varepsilon} \geq V(x_0)^{-1/2} \ln\left(\frac{c}{\varepsilon}\right).$$

It is quite remarkable that this kind of solutions do not exist around minima of V, for instance.

3.4 Radial solutions concentrating on spheres

So far we have considered spikes or multibumps, that is, solutions concentrating on one or several points. Suppose now that we have a radial potential $V(x) = V(|x|)$. One could think of other type of concentration, namely, solutions concentrating on certain spheres. This phenomenon has been studied by Ambrosetti, Malchiodi and Ni in [4]. The main result is as follows:

Theorem 3.4. *We denote $r = |x|$, and suppose that $0 < a \le V(r) < b$. Let $\bar{r}$ be a strict maximum or minimum of*

$$M(r) = r^{N-1} V(r)^{\theta},$$

where $\theta = \frac{p+1}{p-1} - \frac{1}{2}$. Then, for $\varepsilon > 0$ small enough, (3.1) *has a radial solution u_ε which concentrates near the sphere $|x| = \bar{r}$. Actually, one has:*

$$u_\varepsilon(r) \sim W_{V(\bar{r})}\left(\frac{r-\bar{r}}{\varepsilon}\right),$$

where W_λ is the unique even positive solution of the ODE problem:

$$-W''(r) + \lambda W(r) = W(r)^p. \tag{3.2}$$

Several comments are in order. First, observe that here the concentration does not hold at critical values of V. Actually, they hold at critical values of the auxiliary potential $M(r)$ defined above. The role of $M(r)$ is to balance the *volume energy* and the energy due to the potential V.

Secondly, observe that the profile of the solution is similar to the solution of the *ODE* problem.

Solutions concentrating around spheres for vanishing potentials have also been obtained in [7]

3.5 Schrödinger-Poisson systems

The Schrödinger-Poisson system considers a Nonlinear Schrödinger equation in which the potential is not fixed, but is given by the own wave function. Generally, the charge of the wave function ψ is $|\psi|^2$, and hence the Coulomb potential generated $\phi(x)$ satisfyies the Poisson equation:

$$-\Delta\phi = |\psi|^2.$$

If we are looking for standing waves, the model is:

$$\begin{cases} -\varepsilon^2 \Delta u + u + \phi(x) u = u^p \\ -\Delta \phi = \frac{1}{\varepsilon} u^2(x). \end{cases} \tag{3.3}$$

In the above equation $\varepsilon > 0$, $u \in H^1(\mathbb{R}^3)$, $\phi \in D^{1,2}(\mathbb{R}^3)$ and $1 < p < 5$. Observe that the problem is invariant under translations.

The system (3.3) (for $\varepsilon = 1$) has been studied in [12, 13, 24]. However, in this section we are interested in semiclassical states, that is, in concentration of solutions for $\varepsilon \to 0$.

The existence of spikes is easy, and is a direct consequence of the Implicit Function Theorem:

Proposition 3.5. *Let U be the unique positive radial solution in $\mathbb{R}^3$ for the problem:*

$$-\Delta u + u = u^p.$$

Then, for ε small enough there exists a solution of the problem (3.3) such that $u_\varepsilon \sim U(\frac{x}{\varepsilon})$.

Recently the existence of solutions concentrating on spheres has been studied, see [14, 23]. The following result has been obtained:

Theorem 3.6. *For any $p \in (1, 11/7)$ and ε small, there exist solutions u_ε of (3.3) concentrating around a sphere of radius $\bar{r}$, where*

$$\bar{r} = \frac{1}{M_0} \frac{a}{(a+1)^{\frac{5-p}{2(p-1)}}} \tag{3.4}$$

with $M_0 = \int_{\mathbb{R}} W_1^2$ and $a = \frac{8(p-1)}{11-7p}$. More precisely, we have the asymptotic:

$$u_\varepsilon(r) \sim W_{a+1}\left(\frac{r-\bar{r}}{\varepsilon}\right).$$

As we see, again the profile of the solutions is given by the functions W_λ defined in (3.2). Observe also that if $p \to \frac{11}{7}$, then $\bar{r} \to 0$.

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