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Nodal Domain Theorem for the *p*-Laplacian

Pavel Drábek
Department of Mathematics
University of West Bohemia
Plzen, Czech Republic

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Pavel Drábek

Centre of Applied Mathematics, Faculty of Applied Sciences
University of West Bohemia, Pilsen

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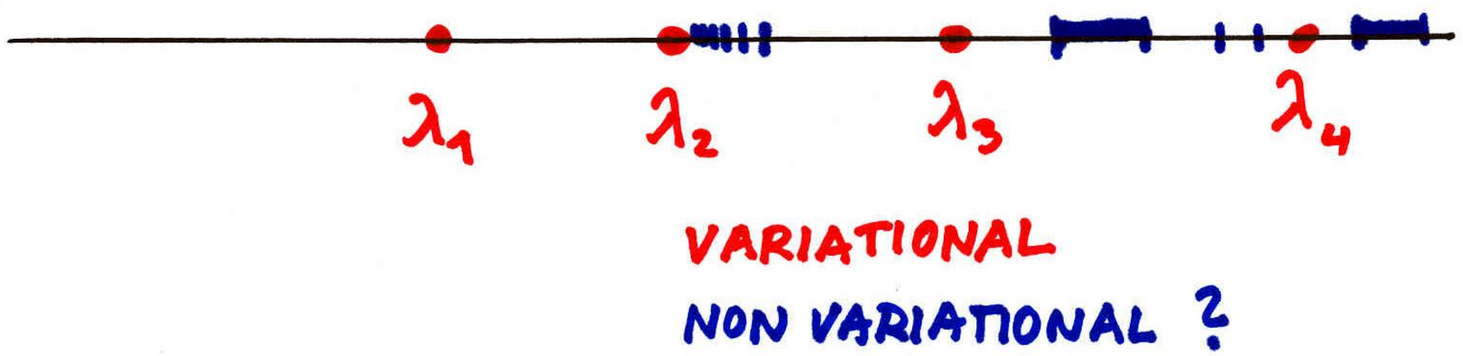
Eigenvalues, eigenfunctions

$$
\begin{aligned}
-\Delta_p u &= \lambda |u|^{p-2} u \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial\Omega,
\end{aligned}
\tag{1}
$$

$$
\int_\Omega |\nabla u_\lambda|^{p-2} \nabla u_\lambda \cdot \nabla v - \lambda \int_\Omega |u_\lambda|^{p-2} u_\lambda v = 0 \tag{2}
$$

Spectrum of the Δ_p



Courant Nodal Domain Theorem for Δ

Theorem 1 **(Courant Nodal Domain Theorem)**

Assume that u_{λ_n} is an eigenfunction associated with the n-th eigenvalue, λ_n, of $-\Delta_2$.
Then u_{λ_n} has at most n nodal domains.

Simple examples demonstrate that no similar lower bound is possible. (Consider $\Omega = [0, \pi] \times [0, L\pi]$ for large L.)

[1] P. Drábek and S.B. Robinson
On the Generalization of the Courant Nodal Domain Theorem,
Journal of Differential Equations 181 (2002), 58-71.

In **Drábek and Robinson** **[1]** we define a sequence of variational eigenvalues, $\{\lambda_k\}_{k=1}^{\infty}$, for the p-Laplacian and proceed to prove several theorems. We prove that previous theorem generalizes completely if we assume either that $-\Delta_p$ satisfies a unique continuation property or that $\lambda < \lambda_{n+1}$. For the general case we prove that, if u_{λ_n} is an eigenfunction associated with λ_n, then u_{λ_n} has at most $2n-2$ nodal domains. Also, if u_{λ_n} has $n+k$ nodal domains, then there is another eigenfunction corresponding to λ_n with at most $n-k$ nodal domains.

Unique Continuation Property

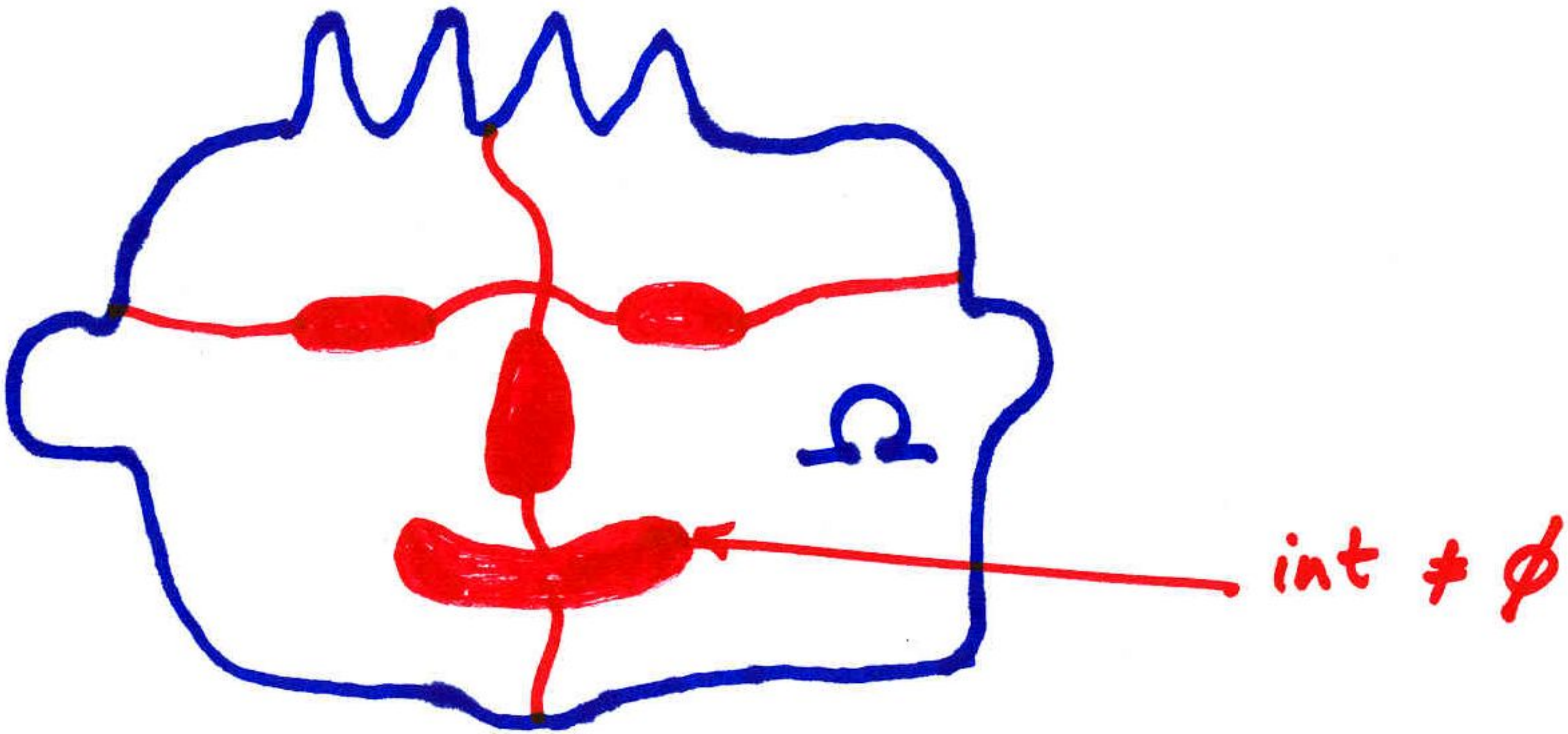
(UCP) If u_λ is a nontrivial eigenfunction of $-\Delta_p$, then $\{x : u_\lambda(x) = 0\}$ has empty interior.

It is well known that the unique continuation property holds for the case $p = 2$. The counterexample in Martio **[2]** may indicate that the unique continuation property does not extend, or is difficult to extend. On the other hand there are $p-1$ homogeneous quasilinear operators, such as $Lu := -\nabla \cdot (|u|^{p-2}\nabla u)$, that do satisfy the unique continuation property. See Ling **[3]**.

[2] O. Martio
Counterexamples for unique continuation,
Manuscripta Math. 60 (1988), 21-47.

[3] J.Ling
Unique continuation for a class of degenerate elliptic operators,
J. Math. Anal. Appl. 168 (1992), no.2, 511-517.

Unique Continuation Property



Let λ be an eigenvalue and let $u \in W_0^{1,p}(\Omega)$ be an associated eigenfunction. Then

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v - \lambda \int_\Omega |u|^{p-2} uv = 0 \tag{3}$$

holds for any $v \in W_0^{1,p}(\Omega)$.

Lemma 2

Let u_λ be an eigenfunction associated with the eigenvalue λ and let Ω_λ be a nodal domain for u_λ. Define

$$\phi := \begin{cases} u_\lambda(x) : x \in \Omega_\lambda \\ 0 : x \notin \Omega_\lambda \end{cases}$$

Then $\phi \in W_0^{1,p}(\Omega)$. Moreover, $\int_\Omega |\nabla \phi|^p = \lambda \int_\Omega |\phi|^p$.

Lemma 3

Assume that there exists $K > 0$ such that $\lambda \in (0, K)$. Let Ω_λ be any nodal domain of u_λ. Then $|\Omega_\lambda| \geq c_1(K) > 0$, where $c_1 = c_1(K)$ is a constant depending only on K. ($|\Omega|$ denotes the Lebesgue measure of Ω.)

It follows from previous lemma that any eigenfunction of $-\Delta_p$ has a finite number of nodal domains.

Lemma 4

Let λ be an eigenvalue with a corresponding eigenfunction $u_\lambda \in W_0^{1,p}(\Omega)$. Then there exists an $\eta \in (0, 1)$ such that $u_\lambda \in C^{1,\eta}(\overline{\Omega})$.

Lemma 5

Let λ be an eigenvalue not equal to λ_1, and let u_λ be a nontrivial eigenfunction associated with λ. Let Ω_1 be any nodal domain for u_λ. Then there is another nodal domain Ω_2, a point $x_0 \in (\partial\Omega_1 \bigcap \partial\Omega_2) \setminus \partial\Omega$, and an $\epsilon > 0$ such that $B_\epsilon(x_0) \bigcap \partial\Omega_1 \bigcap \partial\Omega_2$ is a smooth manifold separating $B_\epsilon(x_0) \bigcap \Omega_1$ and $B_\epsilon(x_0) \bigcap \Omega_2$. Moreover, if $u_\lambda > 0$ in Ω_1 (respectively, $u_\lambda < 0$ in Ω_1), then $\frac{\partial u_\lambda}{\partial \nu} < 0$ (> 0), where ν represents the unit outward normal to $\partial\Omega_1$ at x_0.

When the (UCP) is not available then the following technical result is helpful:

Lemma 6

Let Ω_1 and Ω_2 be nodal domains for an eigenfunction u_λ, and let $x_0 \in (\partial\Omega_1 \bigcap \partial\Omega_2) \setminus \partial\Omega$ and $\epsilon > 0$ such that $B_\epsilon(x_0) \bigcap \partial\Omega_1 \bigcap \partial\Omega_2$ is a smooth manifold separating $B_\epsilon(x_0) \bigcap \Omega_1$ and $B_\epsilon(x_0) \bigcap \Omega_2$. Suppose that u^*_λ is another eigenfunction such that $u^*_\lambda = \gamma_i u_\lambda$ on Ω_i. Then $\gamma_1 = \gamma_2$.

Our variational arguments rely on the following deformation theorem:

Theorem 7

Let $\mathcal{X}$ be a C^1 Finsler manifold, and let $J \in C^1(\mathcal{X}, \Re)$. Let $\mathcal{C}$ be a compact subset of $\mathcal{X}$. Assume that there is an $\epsilon > 0$ such that $||J'(u)||_* \geq 2\epsilon > 0$ for all $u \in \mathcal{C}$. Then there exists a continuous one-parameter family of homeomorphisms $\psi : \mathcal{X} \times [0,1] \to \mathcal{X}$ such that

(i) $dist(\psi(u,t), u) \leq 2t$ *for every* $u \in \mathcal{X}$;

(ii) $J(\psi(u,t)) \leq J(u) - \epsilon t$ *for every* $u \in \mathcal{C}, t \in [0,1]$.

In particular, if $\mathcal{X}$ is a submanifold of a Banach space satisfying $-\mathcal{X} = \mathcal{X}$ and if $J(-u) = J(u)$ for $u \in \mathcal{X}$, then the deformation can be chosen to preserve the symmetry, i.e. we also have

(iii) $\psi(-u,t) = -\psi(u,t)$ *for every* $u \in \mathcal{X}$ *and* $t \in [0,1]$.

There are several ways to characterize a sequence of variational eigenvalues for $-\Delta_p$. In **Drábek and Robinson** **[4]** we define a sequence of variational eigenvalues as follows:

Let us consider the even functional

$$I(u) := \frac{\int_\Omega |\nabla u|^p}{\int_\Omega |u|^p}, \ \forall u \in W_0^{1,p}(\Omega) \setminus \{0\},$$

and the symmetric manifold $\mathcal{S} := \{u \in W_0^{1,p}(\Omega) : \int_\Omega |u|^p = 1\}$. Clearly, the eigenvalues and eigenfunctions of $-\Delta_p$ correspond to the critical values and critical points of $I|_\mathcal{S}$. Using a standard compactness argument we proved in **Drábek and Robinson** **[4]**, Lemma 4, that $I|_\mathcal{S}$ satisfies the Palais-Smale condition.

[4] P. Drábek and S.B. Robinson
Resonance problems for the p-Laplacian,
Journal of Functional Analysis 169 (1999), 189-200.

For $k \in \mathcal{N}$ let

$$\mathcal{F}_k := \{\mathcal{A} \subset \mathcal{S} : \mathcal{A} \text{ is the image of a continuous odd function } h : \mathcal{S}^{k-1} \to \mathcal{S}\},$$

where $\mathcal{S}^{k-1}$ represents the unit sphere in $\Re^k$. Define

$$\lambda_k := \inf_{\mathcal{A} \in \mathcal{F}_k} \sup_{u \in \mathcal{A}} I(u).$$

It is straightforward to check that $\{\lambda_k\}_{k=1}^{\infty}$ is a sequence of eigenvalues for $-\Delta_p$. Moreover, $\lambda_k \to \infty$ as $k \to \infty$.

Theorem 8

Suppose that $-\Delta_p$ satisfies $\mathrm{(UCP)}$ and suppose that u_{λ_n} is an eigenfunction associated with λ_n. Then u_{λ_n} has at most n nodal domains.

Assume that u_{λ_n} has $n+k$ nodal domains where $k \geq 1$. Call them $\{\Omega_1, ..., \Omega_{n+k}\}$. Let $u_i := u_{\lambda_n} \cdot \chi_{\Omega_i}$, where χ_{Ω_i} is the characteristic function over the set Ω_i. Let

$$\mathcal{A} := \{\sum_{i=1}^{n} \gamma_i u_i : \sum_{i=1}^{n} |\gamma_i|^p \int_{\Omega_i} |u_i|^p = 1\}.$$

It is easy to see that $\mathcal{A}$ is symmetric and homeomorphic to $\mathcal{S}^{n-1}$. If $u \in \mathcal{A}$, then

$$\begin{aligned} \int_\Omega |u|^p &= \sum_{i=1}^n \int_{\Omega_i} |u|^p \\ &= \sum_{i=1}^n |\gamma_i|^p \int_{\Omega_i} |u_i|^p \\ &= 1, \end{aligned}$$

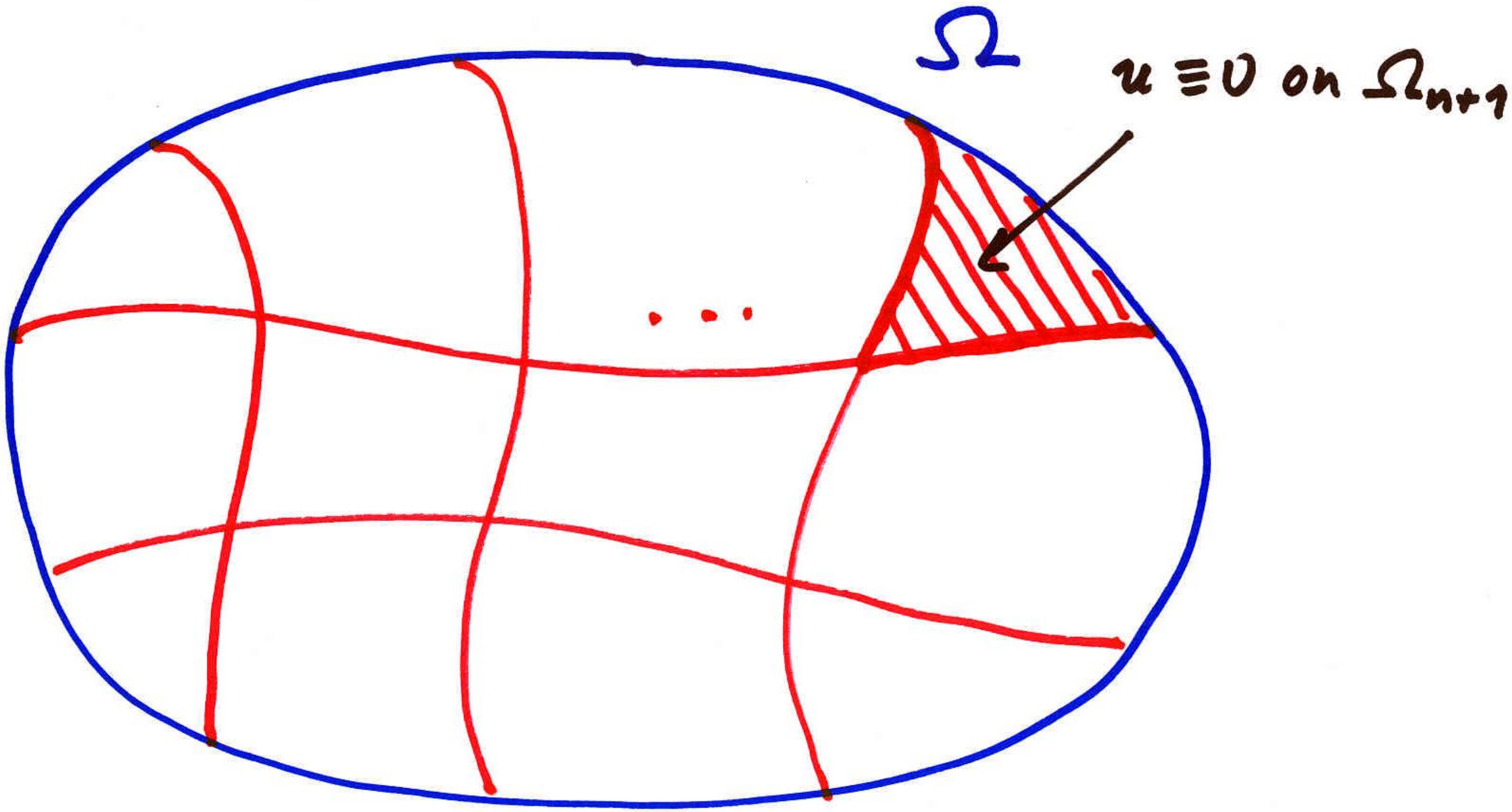
so $\mathcal{A} \subset \mathcal{S}$. Hence $\mathcal{A} \in \mathcal{F}_n$.

Also, using the results of Lemma 2 as well as the fact that $\{x : u_i(x) \neq 0\} \bigcap \{x : u_j(x) \neq 0\}$ has measure zero for $i \neq j$, we get

$$\begin{aligned} \int_\Omega |\nabla u|^p &= \sum_{i=1}^n \int_{\Omega_i} |\nabla u|^p \\ &= \sum_{i=1}^n |\gamma_i|^p \int_{\Omega_i} |\nabla u_i|^p \\ &= \sum_{i=1}^n |\gamma_i|^p \lambda_n \int_{\Omega_i} |u_i|^p \\ &= \lambda_n. \end{aligned}$$

Thus $I \equiv \lambda_n$ on $\mathcal{A}$. Observe that if $u \in \mathcal{A}$ then $u \equiv 0$ on Ω_{n+1}, so u cannot be an eigenfunction, else (UCP) would be contradicted. Thus I has no critical points on $\mathcal{A}$. Since $\mathcal{A}$ is compact, there is an $\epsilon > 0$ such that $||I'(u)||_* \geq 2\epsilon > 0$ for $u \in \mathcal{A}$. Apply Theorem 7 with $\mathcal{C} = \mathcal{A}$ to obtain a symmetry preserving flow ψ. Let $\mathcal{A}^* := \psi(A, 1)$. Now we have $\mathcal{A}^* \in \mathcal{F}_n$ with $\sup\limits_{u \in \mathcal{A}^*} I(u) < \lambda_n$, a contradiction.

Proof of Theorem 8



Theorem 9

Suppose $\lambda < \lambda_{n+1}$ is an eigenvalue with associated eigenfunction u_λ. Then u_λ has at most n nodal domains.

Suppose u_λ has nodal domains $\{\Omega_1, ..., \Omega_{n+k}\}$ for some $k \geq 1$. Let $u_i := u_\lambda \cdot \chi_{\Omega_i}$, as in the previous proof. Let

$$\mathcal{A} := \{\sum_{i=1}^{n+k} \gamma_i u_i : \sum_{i=1}^{n+k} |\gamma_i|^p \int_{\Omega_i} |u_i|^p = 1\}.$$

As in the previous proof we can verify that $\mathcal{A} \in \mathcal{F}_{n+k}$ and that $I(u) = \lambda$ for $u \in \mathcal{A}$. But the characterization of λ_{n+k} implies that $\lambda = \sup_{u \in \mathcal{A}} I(u) \geq \lambda_{n+k} \geq \lambda_{n+1} > \lambda$, a contradiction.

In the previous theorem λ is not required to be a variational eigenvalue.

Let us assume that all variational eigenvalues are simple. Then $\lambda_n < \lambda_{n+1}$ for all n and the estimates above give a direct generalization of the Courant Nodal Domain Theorem.

Theorem 10

Let u_{λ_n} be an eigenfunction associated with λ_n. Then u_{λ_n} has at most $2n-2$ nodal domains.

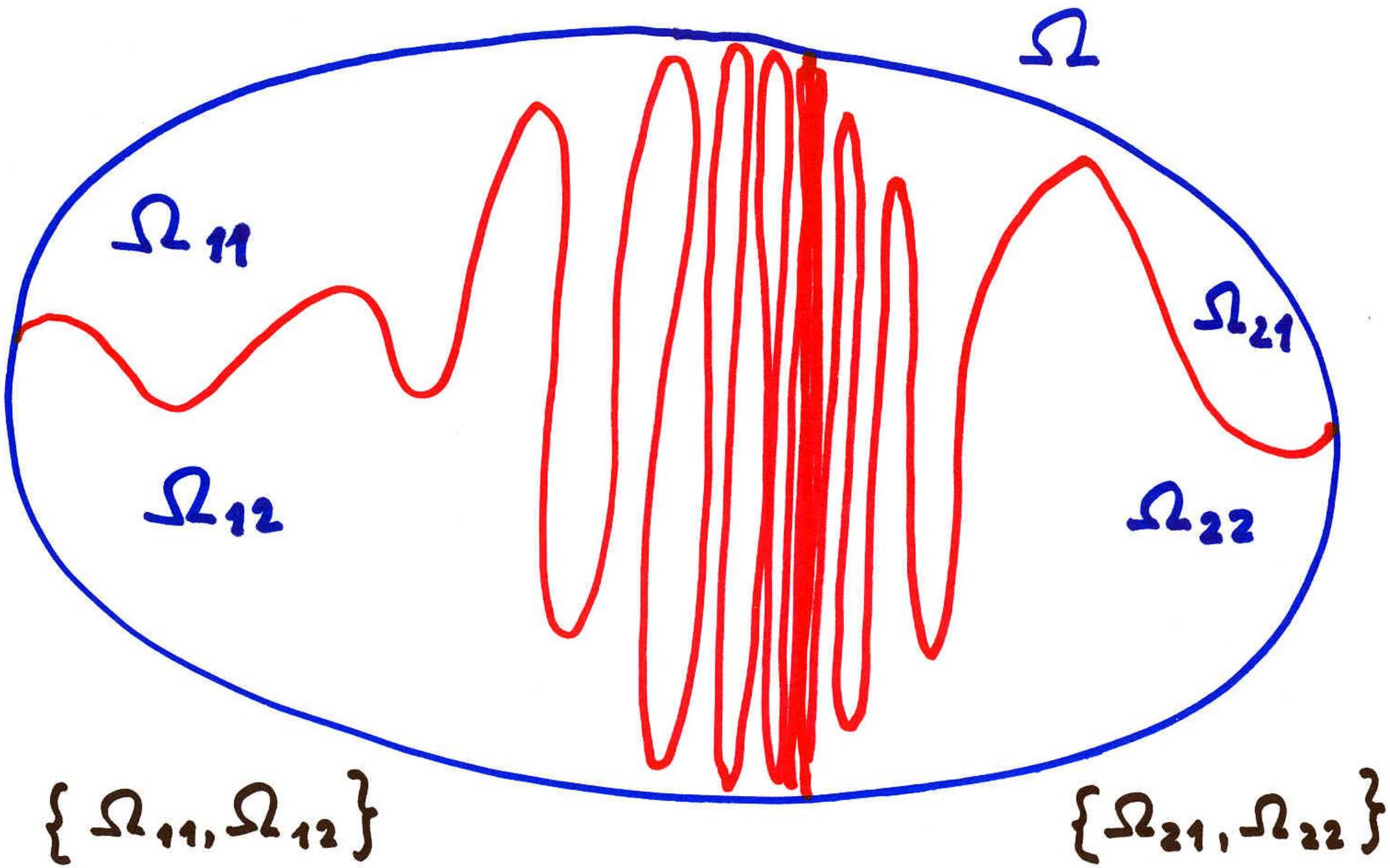
We begin the proof by dividing Ω into nodal domain *neighborhoods*. Let Ω_1 be a nodal domain for u_{λ_n}. By Lemma 5 there is another nodal domain, Ω_2, a point $x_0 \in (\partial\Omega_1 \bigcap \partial\Omega_2) \setminus \partial\Omega$, and an $\epsilon > 0$ such that $B_\epsilon(x_0) \bigcap \partial\Omega_1 \bigcap \partial\Omega_2$. For convenience we will write $\Omega_1 \sim \Omega_2$, and say that these sets are *neighbors*. The nodal domain *neighborhood* for Ω_1 will refer to the collection of nodal domains, Ω_k, such that Ω_1 and Ω_k are connected by a finite sequence of neighbors, i.e. there is a set of nodal domains $\{\Omega'_1, ..., \Omega'_j\}$ where $\Omega'_i \sim \Omega'_{i+1}$ for each $1 \leq i \leq j-1$, $\Omega'_1 = \Omega_1$, and $\Omega'_j = \Omega_k$. Using this notation we can organize all of the nodal domains for u_{λ_n} into neighborhoods

$$\{\Omega_{11}, ..., \Omega_{1j_1}\},$$
$$\{\Omega_{21}, ..., \Omega_{2j_2}\},$$
$$\dots$$
$$\{\Omega_{m1}, ..., \Omega_{mj_m}\},$$

Now suppose that u_{λ_n} has $n + k \geq 2n - 1$ nodal domains. Let N represent the cardinality of $\mathcal{I} := \{(i,j) : 1 \leq i \leq m, j > 1\}$. Notice that $\{\Omega_{ij} : (i,j) \in \mathcal{I}\}$ includes all of the nodal domains except the first in each neighborhood. Since each nodal domain neighborhood contains at least two members, we have $N \geq \frac{1}{2}(2n-1)$, so $N \geq n$. Let $u_{ij} := u_{\lambda_n} \cdot \chi_{\Omega_{ij}}$ and define

$$\mathcal{A} := \{\sum_{\mathcal{I}} \gamma_{ij} u_{ij} : \sum_{\mathcal{I}} |\gamma_{ij}|^p \int_{\Omega_{ij}} |u_{ij}|^p = 1\}.$$

Proof of Theorem 10



As in the previous proofs, it is straightforward to check that $\mathcal{A} \in \mathcal{F}_N$ and that $I(u) = \lambda_n$ for $u \in \mathcal{A}$. Suppose that $u^*_{\lambda_n} \in \mathcal{A}$ is a critical point for I, and thus an eigenfunction. Notice that $u^*_{\lambda_n} \equiv 0$ on the nodal domains Ω_{i1} for $1 \leq i \leq m$. By Lemma 6 it follows that $u^*_{\lambda_n} \equiv 0$ on every nodal domain that can be connected to an Ω_{i1} by a finite sequence of neighbors. Therefore $u^*_{\lambda_n} \equiv 0$ in Ω, which is a contradiction because $0 \notin \mathcal{A} \subset \mathcal{S}$. Hence $\mathcal{A}$ contains no critical points. Now the proof can be finished exactly as in the proof of Theorem 8.

In particular, the second eigenfunction has at most 2 nodal domains.

Since it has to change sign, it has exactly 2 nodal domains.

Theorem 11

Let u_{λ_n} be an eigenfunction associated with λ_n such that u_{λ_n} has $n+k$ nodal domains, then there exists another eigenfunction $u^*_{\lambda_n}$ with at most $n-k$ nodal domains.

Divide Ω into nodal domain neighborhoods exactly as in the proof of Theorem 10. Notice that there must be at least $k+1$ neighborhoods, else the cardinality of $\{\Omega_{ij} : 1 \le i \le m, j > 1\}$ will be at least n, and we can apply the proof of Theorem 10 to obtain a contradiction. Now define an index set $\mathcal{I} := \{(i,j) : 1 \le i \le k, j > 1\} \bigcup \{(i,j) : i \ge k+1, j \ge 1\}$, so that $\{\Omega_{ij} : (i,j) \in \mathcal{I}\}$ omits one nodal domain in each of the first k nodal domain neighborhoods, but includes all of the nodal domains from the remaining neighborhoods. Thus $\mathcal{I}$ has cardinality n.

Let

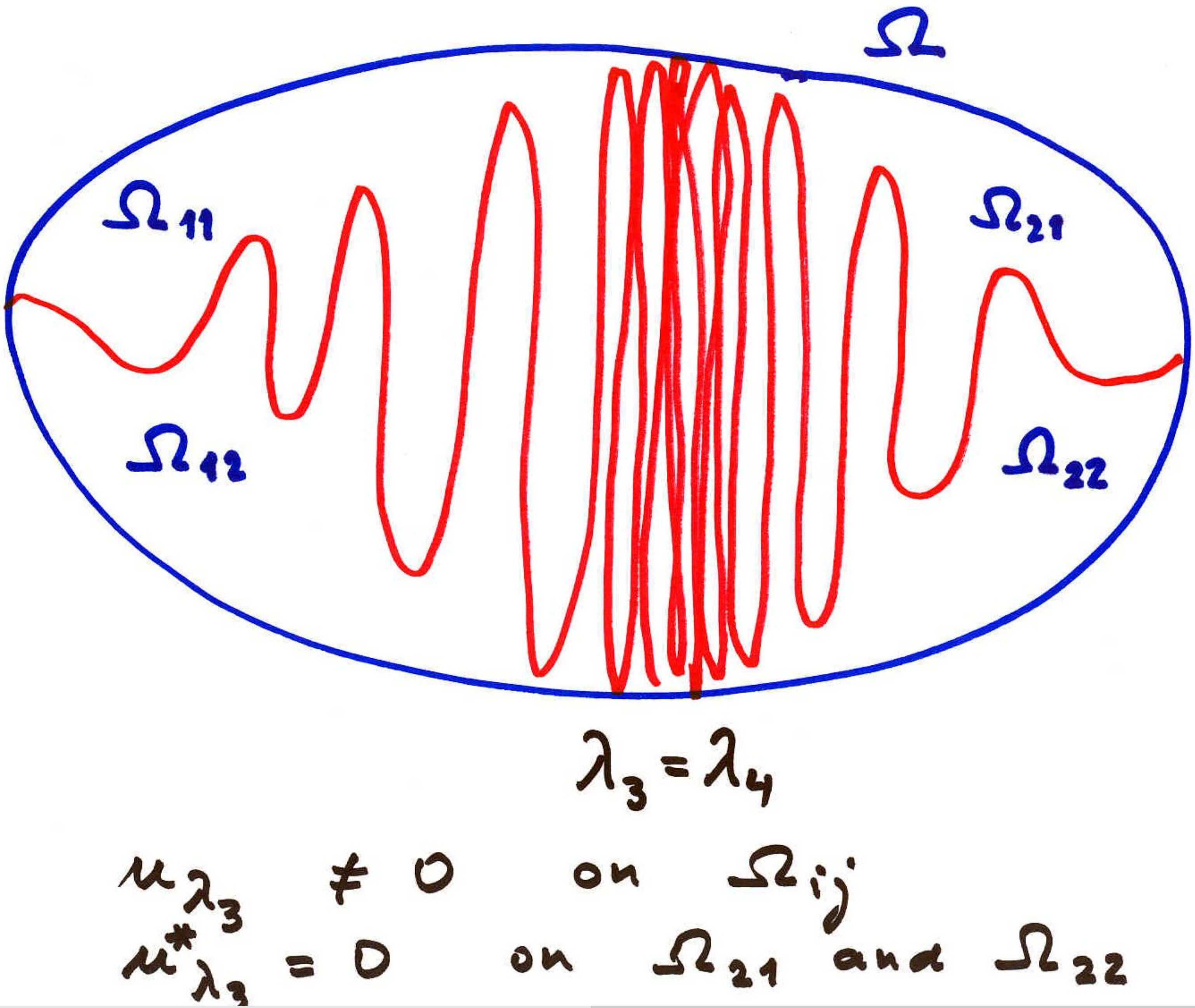
$$\mathcal{A} := \{\sum_{\mathcal{I}} \gamma_{ij} u_{ij} : \sum_{\mathcal{I}} |\gamma_{ij}|^p \int_{\Omega_{ij}} |u_{ij}|^p = 1\}.$$

As in previous proofs we can show that $\mathcal{A} \in \mathcal{F}_n$ with $I \equiv \lambda_n$ on $\mathcal{A}$. $\mathcal{A}$ must contain a critical point of I, else we could derive a contradiction as in previous proofs. Let $u^*_{\lambda_n} \in \mathcal{A}$ be a critical point of I, i.e. another eigenfuction associated with λ_n. Since $u^*_{\lambda_n} \in \mathcal{A}$ we know that $u^*_{\lambda_n} \equiv 0$ in Ω_{i1} for $1 \le i \le k$. As in the proof of Theorem 10, it follows that $u^*_{\lambda_n} \equiv 0$ on each of the first k nodal domain neighborhoods. Notice that the nodal domains for $u^*_{\lambda_n}$ are a subset of the nodal domains in the remaining nodal domain neighborhoods. By removing the first k nodal domain neighborhoods we have removed at least $2k$ nodal domains from consideration. Hence there are at most $n-k$ remaining nodal domains where $u^*_{\lambda_n}$ can be nontrivial.

Corollary 12

For each n there is an eigenfunction, u_{λ_n}, associated with λ_n, such that u_{λ_n} has at most n nodal domains.

Theorem 11


Ω
Ω_{11}
Ω_{12}
Ω_{21}
Ω_{22}
$\lambda_3 = \lambda_4$
$u_{\lambda_3} \neq 0$ on Ω_{ij}
$u^*_{\lambda_3} = 0$ on Ω_{21} and Ω_{22}

Thank you very much for your attention