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Maxwell's Equations & The Electromagnetic Wave Equation

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&
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Maxwell’s Equations

- Introduction
- Historical background
- Electrodynamics before Maxwell
- Maxwell’s correction to Ampere’s law
- General form of Maxwell’s equations
- Maxwell’s equations in vacuum
- Maxwell’s equations inside matter

Introduction

- **In electrodynamics Maxwell’s equations are a set of four equations, that describes the behavior of both the electric and magnetic fields as well as their interaction with matter**
- **Maxwell’s four equations express**
  - How electric charges produce electric field **(Gauss’s law)**
  - The absence of magnetic monopoles
  - How currents and changing electric fields produces magnetic fields **(Ampere’s law)**
  - How changing magnetic fields produces electric fields **(Faraday’s law of induction)**

Historical Background

- **1864 Maxwell in his paper "A Dynamical Theory of the Electromagnetic Field" collected all four equations**

- **1884 Oliver Heaviside and Willard Gibbs gave the modern mathematical formulation using vector calculus.**

- **The change to vector notation produced a symmetric mathematical representation, that reinforced the perception of physical symmetries between the various fields.**

Nomenclature

- E = Electric field
- D = Electric displacement
- B = Magnetic flux density
- H = Auxiliary field
- ρ= Charge density
- j = Current density
- μ_0 (permeability of free space) = $4\pi\times10^{-7}$
- ε_0 (permittivity of free space) = 8.854×10^{-12}
- c (speed of light) = 2.99792458×10^{8} m/s

Electrodynamics Before Maxwell

Gauss's Law $(i)\vec{\nabla}\bullet\vec{E}=\frac{\rho}{\varepsilon_o}$

No name $(ii)\vec{\nabla}\bullet\vec{B}=0$

Faraday's Law $(iii)\vec{\nabla}\times\vec{E}=\frac{\partial\vec{B}}{\partial t}$

Ampere's Law $(iv)\vec{\nabla}\times\vec{B}=\mu_o\vec{J}$

$$\vec{E}=-\vec{\nabla}W-\frac{\partial\vec{A}}{\partial t}$$

$$\vec{B}=\vec{\nabla}\times\vec{A}$$

Electrodynamics Before Maxwell (Cont'd)

Apply divergence to (iii)

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{E}\right) = \vec{\nabla} \bullet \left(-\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\vec{\nabla} \bullet \vec{B}\right)$$

The left hand side is zero, because divergence of a curl is zero.

The right hand side is zero because $\vec{\nabla} \bullet \vec{B} = 0$.

Apply divergence to (iv)

$$\vec{\nabla} \bullet \left(\vec{\nabla} \times \vec{B}\right) = \mu_o \left(\vec{\nabla} \bullet \vec{J}\right)$$

Electrodynamics Before Maxwell (Cont'd)

- The left hand side is zero, because divergence of a curl is zero.
- The right hand side is zero for steady currents i.e.,

$$\vec{\nabla} \bullet \vec{J} = 0$$

- In electrodynamics from conservation of charge

$$\vec{\nabla} \bullet \vec{J} = -\frac{\partial \rho}{\partial t}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = 0$$

ρ is constant at any point in space which is wrong.

Maxwell's Correction to Ampere's Law

Consider Gauss's Law

$$\vec{\nabla} \bullet \varepsilon_o \vec{E} = \rho$$

$$\frac{\partial}{\partial t}\left(\vec{\nabla} \bullet \varepsilon_o \vec{E}\right) = \frac{\partial \rho}{\partial t}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = \vec{\nabla} \bullet \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

$$\frac{\partial \vec{D}}{\partial t} = \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Displacement current

This result along with Ampere's law and the conservation of charge equation suggest that there are actually two sources of magnetic field. The current density and displacement current.

Maxwell’s Correction to Ampere’s Law (Cont’d)

Amperes law with Maxwell’s correction

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

General Form of Maxwell's Equations

Differential Form

$$\vec{\nabla} \bullet \vec{E} = \frac{\rho}{\varepsilon_o}$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \vec{J} + \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Integral Form

$$\oint_S \vec{E} \bullet d\vec{S} = \frac{1}{\varepsilon_o} \int_V \rho dV$$

$$\oint_S \vec{B} \bullet d\vec{S} = 0$$

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{d}{dt} \int_S \vec{B} \bullet d\vec{S}$$

$$\oint_C \vec{B} \bullet d\vec{l} = \mu_o I_{enc} + \mu_o \varepsilon_o \frac{d}{dt} \oint_S \vec{E} \bullet d\vec{S}$$

Maxwell's Equations in vacuum

- The vacuum is a linear, homogeneous, isotropic and dispersion less medium
- Since there is no current or electric charge is present in the vacuum, hence Maxwell's equations reads as
- These equations have a simple solution interms of traveling sinusoidal waves, with the electric and magnetic fields direction orthogonal to each other and the direction of travel

$$\vec{\nabla} \bullet \vec{E} = 0$$

$$\vec{\nabla} \bullet \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_o \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

Maxwell's Equations Inside Matter

Maxwell's equations are modified for polarized and magnetized materials.

For linear materials the polarization ***P*** and magnetization ***M*** is given by

$$\overrightarrow{\quad} = \varepsilon_o \chi_e \overrightarrow{E}$$

$$\overrightarrow{M} = \chi_m \overrightarrow{H}$$

And the ***D*** and ***B*** fields are related to ***E*** and ***H*** by

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P} = (1 + \chi_e)\varepsilon_o \vec{E} = \varepsilon \vec{E}$$

$$\vec{B} = \mu_o\left(\vec{H} + \vec{M}\right) = (1 + \chi_m)\mu_o \vec{H} = \mu \vec{H}$$

Where χ_e **is the electric susceptibility of material,**

χ_m **is the magnetic susceptibility of material and .**

Maxwell's Equations Inside Matter (Cont'd)

- For polarized materials we have bound charges in addition to free charges

$$\sigma_b = \vec{P} \bullet \hat{n}$$

$$\rho_b = -\vec{\nabla} \bullet \vec{P}$$

- For magnetized materials we have bound currents

$$\overrightarrow{K_b} = \overrightarrow{M} \times \hat{n}$$

$$\overrightarrow{J_b} = \vec{\nabla} \times \overrightarrow{M}$$

Maxwell's Equations Inside Matter (Cont'd)

- **In electrodynamics any change in the electric polarization involves a flow of bound charges resulting in polarization current J_P**

$$J_p = \frac{\partial \vec{P}}{\partial t}$$

Polarization current density is due to linear motion of charge when the Electric polarization changes

$$\text{Total charge density}$$

$$\rho_t = \rho_f + \rho_b$$

$$\text{Total current density}$$

$$J_t = J_f + J_b + J_p$$

Maxwell's Equations Inside Matter (Cont'd)

- Maxwell's equations inside matter are written as

$$\vec{\nabla} \bullet \vec{\boldsymbol{E}} = \frac{\rho_t}{\varepsilon_o}$$

$$\vec{\nabla} \bullet \vec{\boldsymbol{B}} = 0$$

$$\vec{\nabla} \times \vec{\boldsymbol{E}} = -\frac{\partial \vec{\boldsymbol{B}}}{\partial t}$$

$$\vec{\nabla} \times \vec{\boldsymbol{B}} = \mu_o \vec{\boldsymbol{J}}_f + \mu_o \vec{\boldsymbol{J}}_p + \mu_o \vec{\boldsymbol{J}}_b + \mu_o \varepsilon_o \frac{\partial \vec{\boldsymbol{E}}}{\partial t}$$

$$\vec{\nabla} \times \frac{\vec{B}}{\mu_o} = \vec{J}_f + \frac{\partial \vec{P}}{\partial t} + \vec{\nabla} \times \vec{M} + \varepsilon_o \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_o} - \vec{M} \right) = \vec{J}_f + \frac{\partial}{\partial t} \left(\varepsilon_o \vec{E} + \vec{P} \right)$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial}{\partial t} \vec{D}$$

Maxwell's Equations Inside Matter (Cont'd)

- In non-dispersive, isotropic media ε and μ are time-independent scalars, and Maxwell's equations reduces to

$$\vec{\nabla} \bullet \varepsilon \vec{E} = \rho$$

$$\vec{\nabla} \bullet \mu \vec{H} = 0$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Maxwell's Equations Inside Matter (Cont'd)

- **In uniform (homogeneous) medium ε and μ are independent of position, hence Maxwell's equations reads as**

$$\vec{\nabla} \bullet \vec{D} = \rho_f \qquad \oint_S \vec{D} \bullet d\vec{S} = Q_{f\,enc}$$

$$\vec{\nabla} \bullet \vec{H} = 0 \qquad \oint_S \vec{H} \bullet d\vec{S} = 0$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \qquad \oint_C \vec{E} \bullet d\vec{l} = -\mu \frac{d}{dt} \int_S \vec{H} \bullet d\vec{S}$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad \oint_C \vec{H} \bullet d\vec{l} = I_{f\,enc} + \frac{d}{dt} \int_S \vec{D} \bullet d\vec{S}$$

Generally, ε and μ can be rank-2 tensor (3X3 matrices) describing birefringent anisotropic materials.

The Electromagnetic Wave Equation (EM Wave)

- The EM wave from Maxwell's Equation
- Solution of EM wave in vacuum
- EM plane wave
- Polarization
- Energy and momentum of EM wave
- Inhomogeneous wave equation

The Electromagnetic Wave from Maxwell's Equations

Take curl of

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla} \times [-\frac{\partial \vec{B}}{\partial t}]$$

Change the order of differentiation on the R.H.S

$$\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}[\vec{\nabla} \times \vec{B}]$$

The Electromagnetic Wave from Maxwell's Equations (cont'd)

As

$$\vec{\nabla}\times\vec{B}=\mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t}$$

Substituting for $\vec{\nabla}\times\vec{B}$ we have

$$\vec{\nabla}\times[\vec{\nabla}\times\vec{E}]=-\frac{\partial\vec{B}}{\partial t}\Rightarrow\vec{\nabla}\times[\vec{\nabla}\times\vec{E}]=-\frac{\partial}{\partial t}[\mu_o\varepsilon_o\frac{\partial\vec{E}}{\partial t}]$$

$$\vec{\nabla}\times[\vec{\nabla}\times\vec{E}]=-\mu_o\varepsilon_o\frac{\partial^2\vec{E}}{\partial t^2}$$

- Assuming that μ_o and ε_o are constant in time

The Electromagnetic Wave from Maxwell's Equations (cont'd)

Using the vector identity $\vec{\nabla} \times \vec{\nabla} \times \vec{E} = -\frac{\partial^2 \vec{E}}{\partial t^2}$

becomes, $\vec{\nabla}(\vec{\nabla} \bullet \vec{E}) - \nabla^2 \vec{E} = -\mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2}$

In free space $\vec{\nabla} \bullet \vec{E} = 0$

And we are left with the wave equation

$$\nabla^2 \vec{E} - \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

The Electromagnetic Wave from Maxwell's Equations (cont'd)

Similarly the wave equation for magnetic field

$$\nabla^2 \vec{B} - \mu_o \varepsilon_o \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

where,

$$c = \frac{1}{\sqrt{\mu_o \varepsilon_o}}$$

Solution of Electromagnetic Waves in Vacuum

The solutions to the wave equations, where there is no source charge is present

$$\nabla^2 \vec{E} - \mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad \nabla^2 \vec{B} - \mu_o \varepsilon_o \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

can be plane waves, obtained by method of separation of variables

Solution of Electromagnetic Waves in Vacuum (Cont'd)

$$\vec{E} = \vec{E}_o e^{i\left(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t\right)}$$

$$\vec{B} = \vec{B}_o e^{i\left(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t\right)}$$

Where E_o and B_o are the complex amplitudes of electric and magnetic fields and related to each other by relation

$$\vec{B}_o = \frac{1}{c}(\hat{k} \times \vec{E}_o)$$

Where $\hat{k}$ is a propagation vector.

Electromagnetic Plane waves

- Plane electromagnetic waves can be expressed as

$$\vec{E} = \vec{E}_o e^{i\left(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t\right)}\hat{n}$$

$$\vec{B} = \frac{1}{c}\vec{E}_o e^{i\left(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}} - \omega t\right)}\left(\hat{\mathbf{k}} \times \hat{n}\right) = \frac{1}{c}\left(\hat{\mathbf{k}} \times \vec{E}\right)$$

Where $\hat{n}$ is the polarization vector.

Electromagnetic Plane waves

The real electric and magnetic fields in a monochromatic plane wave with propagation vector k̂ and polarization n̂ are therefore

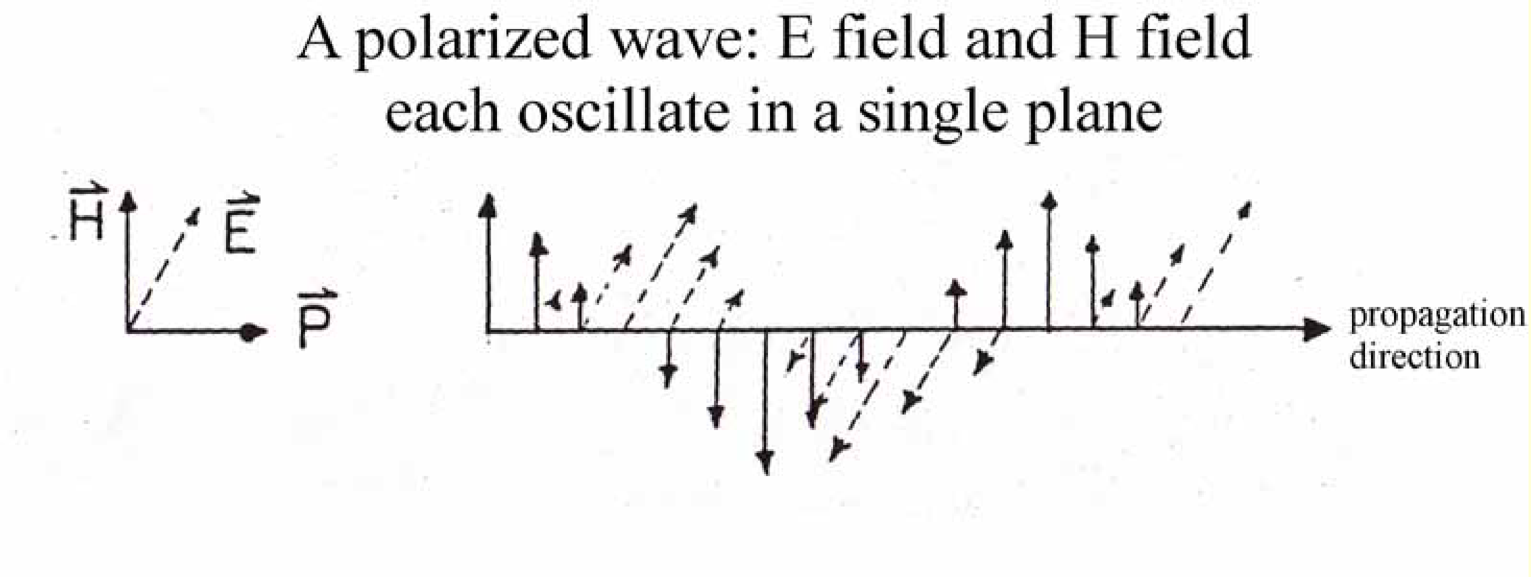
$$\vec{E}\left(\vec{r},t\right)=E_o\cos(\vec{k}\bullet\vec{r}-\omega t)\hat{n}$$

$$\vec{B}\left(\vec{r},t\right)=\frac{1}{c}E_o\cos(\vec{k}\bullet\vec{r}-\omega t)\left(\vec{k}\times\hat{n}\right)$$

Polarization

•The polarization is specified by the orientation of the electromagnetic field.

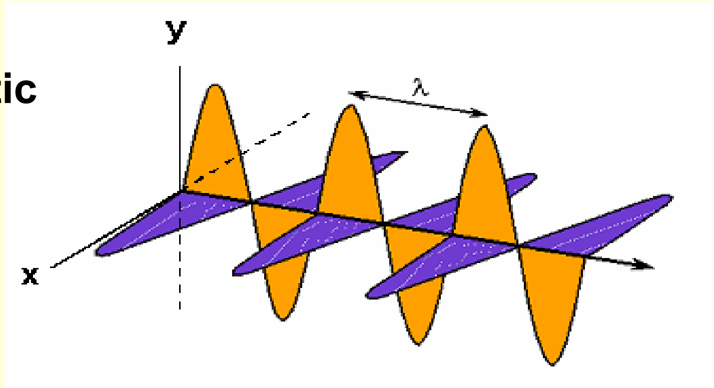
•The plane containing the electric field is called the plane of polarization.



Polarization (Cont'd)

- Can be horizontal, vertical, circular, or elliptical

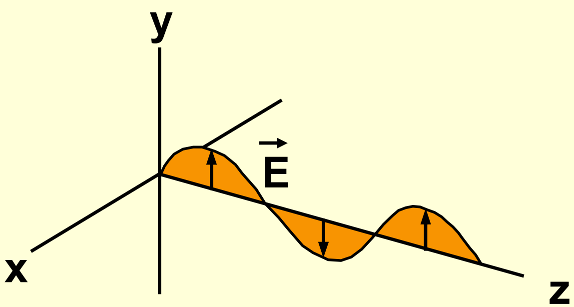
Electromagnetic Wave



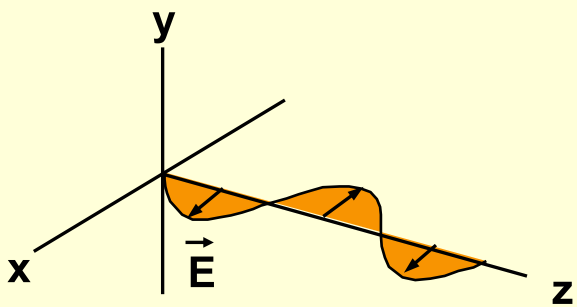
Electric Field

Magnetic Field

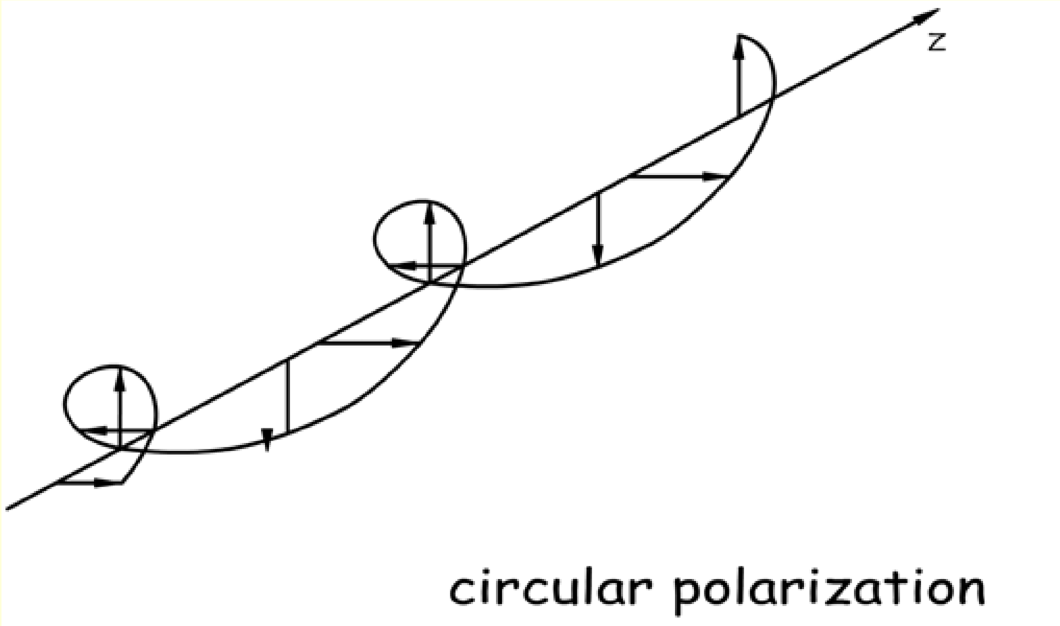
Vertical Polarization



Horizontal Polarization



Polarization (Cont'd)



Energy and Momentum of Electromagnetic Waves

The energy per unit volume stored in electromagnetic field is

$$U = \frac{1}{2}\left(\varepsilon_o E^2 + \frac{1}{\mu_o} B^2 \right)$$

In the case of monochromatic plane wave

$$B^2 = \frac{1}{c^2} E^2 = \mu_o \varepsilon_o E^2$$

$$\Rightarrow U = \varepsilon_o E^2 = \varepsilon_o E_o{}^2 \cos^2(kx - \omega t)$$

Energy and Momentum of Electromagnetic Waves (Cont'd)

- As the wave propagates, it carries this energy along with it. The energy flux density (energy per unit area per unit time) transported by the field is given by the poynting vector

$$\vec{S} = \frac{1}{\mu_o}\left(\vec{E} \times \vec{B}\right)$$

For monochromatic plane waves

$$\vec{S} = c\varepsilon_o E_o^{\,2} \cos^2\left(kx - \omega t\right)\hat{i} = cU\hat{i}$$

Homogenous Wave Equations Inside Matter

Vacuum

$$\frac{1}{\mu_o \varepsilon_o}\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\frac{1}{\mu_o \varepsilon_o}\nabla^2 \vec{B} = \frac{\partial^2 \vec{B}}{\partial t^2}$$

Matter

$$\frac{1}{\mu \varepsilon}\nabla^2 \vec{E} = \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\frac{1}{\mu \varepsilon}\nabla^2 \vec{B} = \frac{\partial^2 \vec{B}}{\partial t^2}$$

Homogenous Wave Equations Inside Matter (cont..)

Permittivity: $\varepsilon=\varepsilon_r\varepsilon_o$ (ε_r is dielectric constant)

Permeability: $\mu=\mu_r\mu_o$ (μ_r is relative permeability ≈1

$$v=\frac{1}{\sqrt{\mu\varepsilon}}=\frac{1}{\sqrt{\mu_r\mu_0\,\varepsilon_r\varepsilon_0}}=\frac{1}{\sqrt{\mu_0\varepsilon_0}}\frac{1}{\sqrt{\mu_r\varepsilon_r}}$$

$=c$ $=n$

$$v=\frac{c}{n}$$

n=Refractive Index

Inhomogeneous Electromagnetic Wave Equation

Inside linear dielectric medium with no free charge present, Maxwell's equations reads as

$$\vec{\nabla} \bullet \vec{D} = 0 \qquad \text{(i)}$$

$$\vec{\nabla} \bullet \vec{B} = 0 \qquad \text{(ii)}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{(iii)}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} \qquad \text{(iv)}$$

Where, $\vec{D} = \varepsilon_o \vec{E} + \vec{P}$ and $\vec{B} = \mu_o \vec{H}$

Inhomogeneous Electromagnetic Wave Equation (Cont'd)

Taking curl of (iii)

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{E}] = -[\vec{\nabla} \times \frac{\partial}{\partial t}\vec{B}] = -\mu_o[\vec{\nabla} \times \frac{\partial}{\partial t}\vec{H}]$$

Using (iv) $\vec{\nabla}[\vec{\nabla} \bullet \vec{E}] - \nabla^2 \vec{E} = -\mu_o \frac{\partial^2 \vec{D}}{\partial t^2}$

$$-\nabla^2 \vec{E} = -\mu_o \varepsilon_o \frac{\partial^2 \vec{E}}{\partial t^2} - \mu_o \frac{\partial^2 \vec{P}}{\partial t^2}$$

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} + \frac{1}{c^2 \varepsilon_o}\frac{\partial^2 \vec{P}}{\partial t^2}$$

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{c^2 \varepsilon_o}\frac{\partial^2 \vec{P}}{\partial t^2}$$

Source term

Solution of Inhomogeneous Electromagnetic Wave Equation

$$\nabla^2 \vec{E} - \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \frac{1}{c^2 \varepsilon_o}\frac{\partial^2 \vec{P}}{\partial t^2}$$

Inhomogeneous wave equation can be solved with the help of Green's Theorem

THANK YOU