The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific
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International Atomic
Energy Agency

SMR 1826 - 8

Preparatory School

to the

Winter College on Fibre Optics, Fibre Lasers and Sensors

5 - 9 February 2007

Introduction to Non Linear Optics

Session 2

M.Centini
Universita' di Roma, "La Sapienza"
Roma, Italy

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Introduction to Non Linear Optics
2nd Session

M.Centini
Universita' di Roma "La Sapienza", Roma, Italy.

Outline

1st session

Introduction:
-The Non linear Optical Susceptibility Tensor;

-On the Physical Origins of the Nonlinear Optical Coefficients;

Second Order Optical Nonlinearities:
-Second Harmonic Generation, Phase matching, examples of phase matching techniques;

-Basic equations of Parametric Amplification, Parametric Oscillation, three-wave mixing, frequency up- and down-conversion.

-**Third Order Optical Nonlinearities:**
-The Nonlinear Constants, Intensity dependence of the refractive index;

-Third harmonic generation; Third order parametric processes, four wave mixing;

-Self Focusing of Optical Beams, Diffraction-less propagation of Optical beams: Spatial Solitons;

2nd session

Third Order Optical Nonlinearities:
-Molecular Raman Scattering,

-Stimulated Raman Scattering;

- Stimulated Brillouin Scattering.

Introduction to Optical Solitons:
- Wave packets, Group velocity and Dispersion;

- Nonlinear wave packets and the nonlinear Schrödinger equation;

- Self phase modulation, Spectral broadening,

- Effects on group velocity dispersion, Pulse Compression;

- Modulation instability;

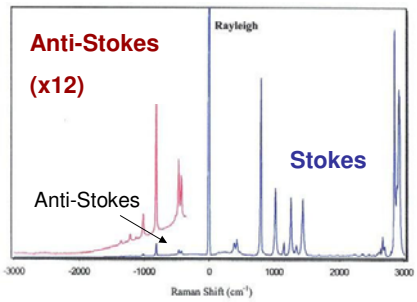
- Fundamental and higher orders Solitons.

Molecular Raman Scattering

In 1921 Prof. C.V. Raman performed experimental studies on light scattering.

His measures shown that the spectrum of scattered light contains weak signals at shifted wavelengths with respect to the pump.

A complete Raman Spectrum of Cyclohexane
(Avalon instruments)

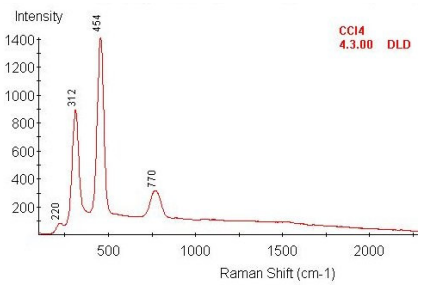


Molecular Raman Scattering

Differences between the spectrum of incident and scattered light correspond to absorption frequencies of the material under investigation.

Raman spectroscopy is widely used to measure vibrational and rotational spectrum of chemical compounds.

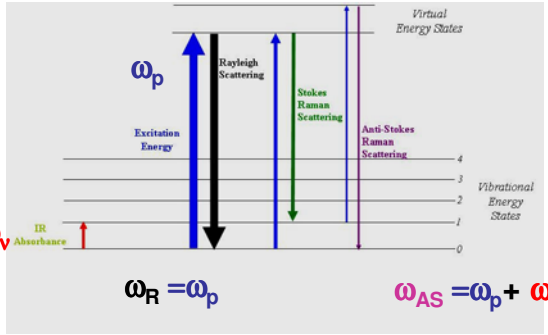
Stokes Raman spectrum for carbon tetrachloride



Molecular Raman Scattering

The Raman effect corresponds, in perturbation theory, to the absorption and subsequent emission of a photon via an intermediate electron state, having a virtual energy level.

A simple physical interpretation is given in terms of scattering between light and a vibrational modes of a molecule.



Molecular Raman Scattering

In other words, Raman scattering is the inelastic scattering between photons and optical phonons.

Anti-Stokes lines are less intense than their respective Stokes line by a factor exp(-hν/KT) due to the probability of finding the molecule on an excited state

Typical Raman shifts are in a range between 500 cm^{-1} and 3000 cm^{-1}

To calculate the value of the scattered wavelength (in cm) we must use the formula:

$$\Delta\nu[cm^{-1}] = \frac{1}{\lambda_{incident}[cm]} - \frac{1}{\lambda_{scattered}[cm]}$$

Exercise 1:Calculate the wavelengths (in μm) corresponding to the Stokes line of the CCl_4 molecule shown in the previous picture for a monochromatic incident light of wavelength 0.632 μm.

Molecular Raman Scattering

Classical interpretation of the Raman effect

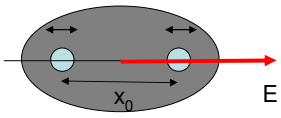
Consider a diatomic molecule irradiated with monochromatic light of frequency ν. If α_0 is the polarizability of the molecule we have an induced dipole momentum (scalar model)

$$M = \alpha_0 E = \alpha_0 E_0 \cos 2\pi\nu t$$

The secondary waves emitted by the molecule at the same frequency as the incident field are part of the Rayleigh scattering.

If the molecule is vibrating along the line that links the nuclei with a frequency ν1. the polarizability, in first approximation has the expression:

$$\alpha \cong \alpha_0 + \alpha_1 x = \alpha_0 + \alpha_1 x_0 \cos 2\pi\nu_1 t$$



Molecular Raman Scattering

Thus

$$M = \alpha_0 E_0 \cos 2\pi\nu t + \frac{1}{2}\alpha_0 E_0 x_0 \{\cos 2\pi(\nu+\nu_1)t + \cos 2\pi(\nu-\nu_1)t\}$$

Rayleigh scattered

Raman shifted frequency (Anti-Stokes)

Raman shifted frequency (Stokes)

NOTE: In order to have Raman shifted frequencies is enough that the polarizability changes with the vibration.

Differences with Standard IR spectroscopy

Absorption of an IR wavelength happens because of the excitation of the vibrational mode if it produces a change in the electric dipole momentum.

Symmetric molecules may have vibrational modes that do not produce a change in the dipole momentum so they are INACTIVE in the infrared. Raman spectroscopy makes it possible to study these modes.

Molecular Raman Scattering

Spontaneous Raman effect is typically a rather weak process.

Stimulated Raman scattering happens under excitation by an intense laser beam.

A detailed analysis of the Raman effect requires a quantum treatment.

The stimulated process can described through use of the nonlinear polarization.

Typical scattering cross sections per unit volume for Raman Stokes scattering is of the order of 10^{-6} cm^{-1}

Typically, more than 10 percent of the incident laser beam is converted into the Stokes frequency.

Spontaneous process leads to nearly isotropic emission.

Stimulated process leads to a narrow cone emission in forward and backward direction with respect to the pump direction.

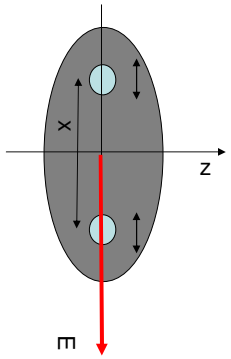
Stimulated Raman Scattering

We assume that the optical field interacts with a vibrational mode of a molecule.

We assume that the vibrational mode can be described as a simple harmonic oscillator of resonance frequency ω_v and damping constant γ.

The equation of motion for a single oscillator is:

$$\frac{d^2X(z,t)}{dt^2}+\gamma\frac{dX(z,t)}{dt}+\omega_v^2X(z,t)=\frac{F(z,t)}{m}$$



The Raman medium is composed of N oscillators per unit of volume, each oscillator is characterized by its position z (1D analysis) and a normal vibrational coordinate X(z,t).

Stimulated Raman Scattering

The driving term can be derived by considering the electrostatic Stored energy density in the presence of the molecule:

$$E = \frac{1}{2}\varepsilon_0 \varepsilon_r \vec{E}\cdot\vec{E}$$

Where:

$$\varepsilon_r = (1+N\alpha) = 1+N\left[\alpha_0 + \left(\frac{\partial\alpha}{\partial X}\right)_{X_0} X\right]$$

Thus, the force per unit volume is:

$$F(z,t) = \frac{1}{N}\frac{\partial}{\partial X} = \frac{1}{2}\varepsilon_0\left(\frac{\partial\alpha}{\partial X}\right)_{X_0}\left\langle \vec{E}\cdot\vec{E}\right\rangle$$

The average is taken over a few optical cycles since the molecules cannot respond to optical frequencies.

Stimulated Raman Scattering

The process can be formulated as follows:

A modulation of ε at ω_v, caused by molecular vibrations, can lead to energy exchange between electromagnetic fields separated in frequency by ω_s such as the laser (ω_l) and the Stoke line at $\omega_s=\omega_l-\omega_v$

We consider the total field as the sum of the Stokes and laser field:

$$\vec{E}(z,t) = \frac{1}{2}\hat{e}_L\xi_L(z)e^{i\omega_L t} + \frac{1}{2}\hat{e}_S\xi_S(z)e^{i\omega_S t} + c.c.$$

So that:

$$\left\langle \vec{E}\cdot\vec{E}\right\rangle = \frac{1}{4}(\hat{e}_S\cdot\hat{e}_L)\xi_S(z)\xi_L^*(z)e^{i(\omega_L-\omega_S)t} + c.c.$$

We look for solutions of the form:

$$X(z,t) = \frac{1}{2}x(z)e^{i(\omega_L-\omega_S)t} + c.c.$$

Stimulated Raman Scattering

Substituting the expression for the fields and the solution, with some algebra we derive the expression for the polarization:

$$\vec{P}(z,t) = \varepsilon_0 N \alpha_0 \vec{E}(z,t) + \varepsilon_0 N \left(\frac{\partial \alpha}{\partial X}\right)_{X_0} X(z,t)\vec{E}(z,t)$$

P_{NL}

Exercise: derive the expression for the nonlinear polarization term oscillating at frequency ω_S.

With some algebra:

$$\vec{P}_{NL}^{(\omega_S)}(z) = \frac{\varepsilon_0^2 N \left(\frac{\partial \alpha}{\partial X}\right)_{X_0} |\xi_L|^2 \hat{e}_L \cdot \hat{e}_S}{8m\left[\omega_v^2 - (\omega_L - \omega_S)^2 - i(\omega_L - \omega_S)\gamma\right]} \xi_S(z)\hat{e}_S$$

NOTE: The nonlinear polarization term can be written as:

$$\vec{P}_{NL}^{(\omega_S)}(z) = \varepsilon_0 \hat{\chi}_{Raman}(\omega_S) \vdots \xi_L \xi_L^* \xi_S \hat{e}_S$$

Stimulated Raman scattering is described as a third order effect.
The term we derived is a special case of a four wave mixing interaction.

Stimulated Raman Scattering

If we consider the wave equation as described in the previous session and we write the equation for the Slowly varying envelope of the Stoke field under the approximation of undepleted laser pump:

$$\frac{\partial}{\partial z}\xi_S(z) = -i\frac{\omega_S}{2cn_S}\chi_{Raman}(\omega_S)|\xi_L(0)|^2\xi_S(z)$$
$$\xi_L(z) = \xi_L(0)$$

Exercise: derive the equations for Raman amplification

Note: Raman Stokes amplification is a process for which Phase matching condition is automatically satisfied. It is a pure gain process.

Indeed the gain for the Stokes field is:

$$g(\omega_S) = -\frac{\omega_S}{2cn_S}\chi''_{Raman}(\omega_S)|\xi_L|^2;$$

$$\chi_{Raman}(\omega_S) = \chi'_{Raman}(\omega_S) - i\chi''_{Raman}(\omega_S);$$

The gain is connected to the imaginary part of the Raman susceptibility that can be evaluated by using the equation for the nonlinear Polarization.

Stimulated Raman Scattering

Where:

$$\chi'_{Raman}(\omega_S) = \frac{\varepsilon_0 N\left(\frac{\partial\alpha}{\partial X}\right)^2_{X_0}\left[\omega_v-(\omega_L-\omega_S)\right]}{16m\omega_v\left\{\left[\omega_v^2-(\omega_L-\omega_S)^2\right]+\gamma^2/4\right\}}$$

$$\chi''_{Raman}(\omega_S) = \frac{-\varepsilon_0 N\frac{\gamma}{2}\left(\frac{\partial\alpha}{\partial X}\right)^2_{X_0}}{16m\omega_v\left\{\left[\omega_v^2-(\omega_L-\omega_S)^2\right]+\gamma^2/4\right\}}$$

NOTE: Anti-Stokes radiation amplification can be described as a four wave mixing process where the nonlinear polarization for the AS field at ω_{AS} is given by:

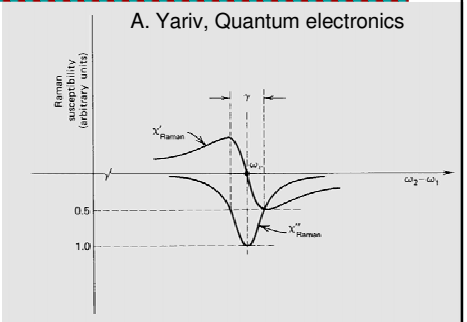
$$\mathbf{P(\omega_{AS})= \alpha E_p E_p E_S^*}$$

This term is subjected to phase matching conditions:

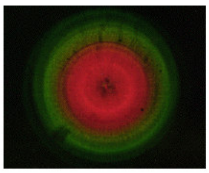
$$2\mathbf{k}_p-\mathbf{k}_S=\mathbf{k}_{AS}$$

They cannot be verified for collinear interaction in isotropic normal dispersive media. Thus AS field is emitted in the form of a conical shell (see figure)

A. Yariv, Quantum electronics



18.7 The in-phase (χ'_{Raman}) and quadrature (χ''_{Raman}) components of the Raman susceptibility as a function of the Stokes frequency ω_2. (ω_1 increases from right



Light created when a ruby laser is focused in a cell of benzene emerges to form a pattern of brightly colored rings. New frequencies come from Raman resonances in the benzene. Courtesy of R.W. Terhune.
From Yariv, Quantum Electronics.

Stimulated Raman Scattering

Since there is no phase matching condition, the interaction between the Stokes wave and the pump under cw conditions in a silica glass fiber can be conveniently described by the following set of equations:

$$\frac{\partial}{\partial z}I_p(z) = -\frac{\omega_p}{\omega_S}g_R I_p(z)I_S(z) - \alpha_p I_p(z)$$

$$\frac{\partial}{\partial z}I_S(z) = g_R I_p(z)I_S(z) - \alpha_S I_S(z)$$

Where **g_R is the Raman gain coefficient**. It can be measured. In silica fibers g_R extend over a large frequency range (up to 40 THz) with a broad dominant peak at 13 THZ. (Typical of amorphous materials).

α_S and α_p account for the fiber loss at the Stokes and pump frequencies.

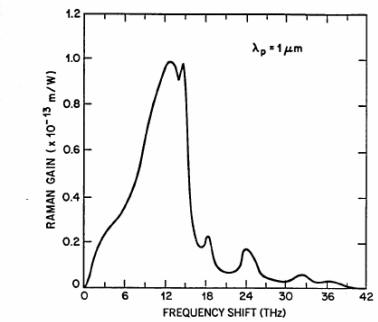


Figure 8.11 Measured Raman gain spectrum for fused silica. Raman gain scales inversely with the pump wavelength λ_p. (After Ref. [29]. ©1972 AIP. Reprinted with permission.)

From Fiber-Optic Communications, 2nd Edition, Govind Agrawal

Stimulated Raman Scattering

When the pump power exceeds a threshold value the Stokes component at the dominant frequency builds up almost exponentially. It is possible to estimate the Raman threshold by neglecting pump depletion due to Raman effect:

$$\frac{\partial}{\partial z} I_S(z) = g_R I_p(z) I_S(z) - \alpha_S I_S(z)$$

$$\frac{\partial}{\partial z} I_p(z) = -\alpha_p I_p(z)$$

neglect

Thus:

$$\frac{\partial}{\partial z} I_S(z) = g_R I_{p0} e^{(-\alpha_p z)} I_S(z) - \alpha_S I_S(z)$$

$$I_p(z) = I_{p0} e^{(-\alpha_p z)}$$

Solution for the Stokes field is:

$$I_S(L) = I_S(0) e^{\left(g_R I_{p0} L_{eff} - \alpha_S L\right)};$$

With:

$$L_{eff} = \frac{1}{\alpha_p}\left(1 - e^{(-\alpha_p L)}\right);$$

Stimulated Raman Scattering

Because of pump absorption due to fiber loss the effective length is L_{eff} rather than L.

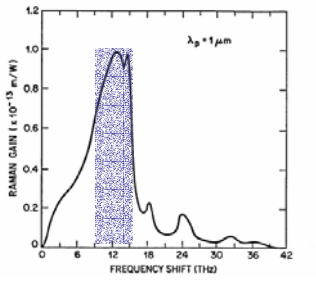
NOTE: a nonzero Stokes input is required. In practice it builds up from spontaneous Raman scattering occurring throughout the fiber length.

It has been shown that it is equivalent to injecting one fictious photon per mode at input. Considering ω_s as the frequency where the Raman gain has a peak and an effective band width of B_{eff}. The Stokes power can be written as:

$$P_S(L) = \hbar \omega_S B_{eff} e^{\left(g_R I_{p0} L_{eff} - \alpha_S L\right)};$$

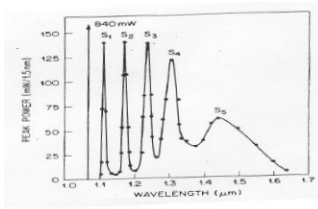
The Raman threshold is defined as the input pump power at which the Stokes Power becomes equal to the pump power at the output. Assuming A_{eff} the effective core area, so that $P_p(0)=I_{p0}A_{eff}$ and assuming $\alpha_s=\alpha_p$ it can be shown that, to a good approximation, the critical pump power is given by:

$$g_R P_0^{cr} L_{eff} / A_{eff} \cong 16;$$



Stimulated Raman Scattering

Exercise: Consider λ_p=1.55 μm,. The silica glass fiber loss is minimum, thus L_{eff} is of the order of 20 km. A typical value for A_{eff} is 50 μm^2. Calculate the predicted Raman threshold. [600 mW]. How does the threshold change if we consider radiation emitted by a Nd:YAG laser with A_{eff} of the order of 15 μm^2 for a 1m long fiber?



If the pump power exceeds the critical power, the Stokes power can become intense enough to pump the next order Stokes line and so on

Applications

Fiber Raman Amplifiers

- **Large bandwidth (more than 5 THz)**
- **20 dB gain demonstrated from 3.7W Q switched YAG laser**
- **Broadband noise from spontaneous Raman scattering. An output filter is needed**

Fiber Raman Lasers

- **Tunability (10 THz or about 10 nm at 500 nm operating wavelength)**
- **Low threshold (about 1 W)**

Drawbacks

Raman induced crosstalk: In multi-channel communications systems, the short wavelength channel is depleted, acting as a pump for longer wavelength channels.

Stimulated Brillouin Scattering

Scattering of light from thermally excited acoustic waves (1922-Spontaneous).

The presence of an acoustic wave, modulates the dielectric constant. Exchange of energy between electromagnetic waves whose frequencies differ by an amount equal to the acoustical frequency can happen.

It is analogous to the Raman effect, but light scattering with acoustic waves instead of molecular vibration is considered.

In the Stimulated Brillouin Scattering, the acoustic wave that scatters the optical beam is produced by the optical beam itself (1964).

Main differences with Raman effect:

Typical frequency shifts are smaller (10 GHz, not 12 THz)

Linewidth is narrower (100 MHz, not 5 THz)

Stokes field is backward emitted

Stimulated Brillouin Scattering

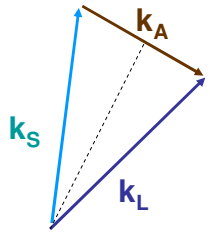
If the optical beam at frequency ω_L is sufficiently intense, there is a simultaneous generation of an optical beam at frequency ω_S and an acoustic wave at $\omega_A=\omega_L-\omega_S$.

Also, momentum must be conserved during each scattering event thus:

$$\mathbf{k_A}=\mathbf{k_L}-\mathbf{k_S}$$

Considering that ω_L is very close to ω_S we can assume $|\mathbf{k_L}|=|\mathbf{k_S}|$ thus:

$$|\mathbf{k_A}|=2|\mathbf{k_L}|\sin(\theta/2)$$



And:

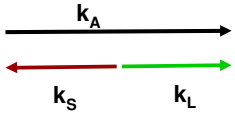
$$\omega_A = V_A\,|\mathbf{k_A}|$$

V_A is the propagation velocity of the acoustic wave in the medium

Stimulated Brillouin Scattering

NOTE: The frequency shift of the Stokes wave depends on the scattering angle. It reaches its maximum value for $\theta=\pi$ (in the backward direction with respect to the laser pump).

$$\Delta\nu_A = \frac{\omega_A}{2\pi} = \frac{2n_L V_A}{\lambda_L}$$



With good approximation we can say that SBS in optical fibers occurs only in the backward direction. If V_A=5.96 km/s and n=1.45 we have for silica fibers n_A=11.1 GHz at λ=1.55 μm.

Measurements of the Brillouin gain in bulk silica were performed in 1950.

Recent measures shown that n_a=34.7 GHz and Δv_a=54 MHz at 486 nm.

Line width is related to the phonon lifetime.

The measure of the gain obtained for CW or quasi CW pump is of the order of 5×10^{-11}m/W.

Stimulated Brillouin Scattering

NOTE: Stimulated Brillouin gain is nearly three orders of magnitude higher than the Raman Gain.

Nevertheless the measured Brillouin gain for pulses of time duration T_0 shorter than the phonon lifetime is much smaller. In particular, for pulses shorter than 1 ns the Brillouin gain is reduced below the Raman gain.

Indeed for a pump spectrum $\Delta\nu_p >> \Delta\nu_b$, the Brillouin gain is reduced by a factor $\Delta\nu_p/\Delta\nu_b$

By following a method similar to that used for the SRS the Brillouin threshold is found to occur at a critical cw pump power given by the relation:

$$g_B P_0^{cr} L_{eff} / A_{eff} \cong 21;$$

Stimulated Brillouin Scattering

Exercise: Consider λ_p=1.55 μm, The silica glass fiber loss is minimum, thus L_{eff} is of the order of 20 km. A typical value for A_{eff} is 50 μm². Calculate the predicted Raman threshold. [600 mW] and brillouin threshold [1 mW].

SBS is a dominant nonlinear process in optical fibers.

At low input powers the reflected light was about 4% due to air-fibre interface.

The Brillouin threshold is reached at an input power of about 5 mW.

Above the threshold, the backward Stokes field dominates.

Transmitted power decreases as a result of pump depletion.

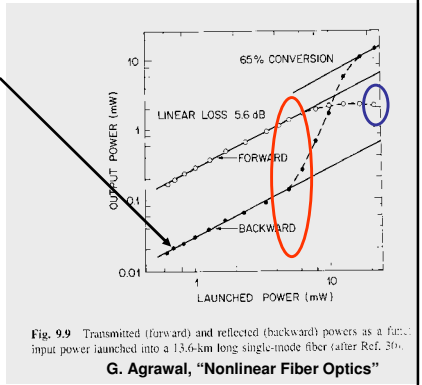


Fig. 9.9 Transmitted (forward) and reflected (backward) powers as a fun... input power launched into a 13.6-km long single-mode fiber (after Ref. 39).

G. Agrawal, "Nonlinear Fiber Optics"

Stimulated Brillouin Scattering

Fiber Brillouin Lasers: Very low threshold (100 mW) and sharp bandwidth.
It is possible to use multiple Stokes and antistokes lines for high pump powers and to obtain short pulse by mode locking. Nevertheless the process is not stable.

Fiber Brillouin Amplifiers: Are capable of providing 20-40 dB gain at a pump power of a few mW. They have a very narrow bandwidth (<100 MHz). Raman amplifiers have about 5 THz bandwidth.

Beneficial effects of SBS:

Amplification can occur with very low pump powers (1-5 mW).

The narrow line width can be used to amplify only a small portion of the signal spectrum.

Stimulated Brillouin Scattering

Undesirable effects of SBS:

Stokes radiation propagates in the opposite direction with respect to the pump.

Signal power at the receiver becomes much lower.

Feedback to the transmitter may destabilize laser operation.

Brillouin threshold increases for short pulses.(Also useful to avoid SBS)

In case of multichannel systems, power in each channel should be kept below the Brillouin threshold. (The narrow frequency range of Brillouin gain make this effect easly avoided).

Exercise: give an estimation of the order of magnitude of pump pulse duration compatible with the results obtained for CW pump. Calculate the time duration of pump pulse in order to get the Raman threshold below the Brillouin threshold.

Wavepackets, Group velocity and dispersion

We consider one dimensional wave packets described by the expression:

$$E(z,t)=\frac{1}{2}\left[\psi(z,t)e^{i(k_0 z-\omega_0 t)}+c.c.\right]$$

The equation that governs the propagation of the wave-packet is the wave equation:

$$\frac{\partial^2 E(z,t)}{\partial z^2}-\mu_0\frac{\partial^2 D(z,t)}{\partial t^2}=0$$

If we consider the Fourier transforms of the field and the displacement:

$$E(z,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{E}(z,\omega)e^{-i\omega t}d\omega; \qquad D(z,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{D}(z,\omega)e^{-i\omega t}d\omega;$$

The Fourier amplitudes of the fields are related by:

$$\tilde{D}(z,\omega)=\varepsilon_0\varepsilon(\omega)\tilde{E}(z,\omega);$$

Wavepackets, Group velocity and dispersion

The wave equation, for the Fourier amplitudes becomes:

$$\frac{\partial^2 \tilde{E}(z,\omega)}{\partial z^2}+\frac{\varepsilon(\omega)\omega^2}{c^2}\tilde{E}(z,\omega)=0 \qquad (*)$$

Then we calculate E(z,ω):

$$\tilde{E}(z,\omega)=\int_{-\infty}^{+\infty}E(z,t)e^{i\omega t}dt=\frac{1}{2}\int_{-\infty}^{+\infty}\psi(z,t)e^{ik_0 z}e^{i(\omega-\omega_0)t}dt+\frac{1}{2}\int_{-\infty}^{+\infty}\psi^*(z,t)e^{-ik_0 z}e^{i(\omega+\omega_0)t}dt;$$

Thus:

ψ is a slowly varying envelope so it does not contains Components oscillating at fast angular frequencies

$$\tilde{E}(z,\omega)=\tilde{\psi}(z,\omega-\omega_0)e^{ik_0 z}+\tilde{\psi}^*(z,\omega+\omega_0)e^{-ik_0 z}\cong\tilde{\psi}(z,\omega-\omega_0)e^{ik_0 z} \qquad (+)$$

Substituting (+) in (*) and neglecting second order derivative on the envelope (SVEA) we obtain:

Wavepackets, Group velocity and dispersion

$$2ik_0\frac{\partial\tilde{\psi}}{\partial z}+\left(k^2-k_0^2\right)\tilde{\psi}=0;$$

where: $$k^2=\frac{\varepsilon(\omega)\omega^2}{c^2}$$

In practice k typically differs from k_0 by only a small fractional amount, thus we can make the following approximation

$$\boxed{k^2-k_0^2\cong 2k_0\left(k-k_0\right)}$$

Our equation becomes:

$$\frac{\partial\tilde{\psi}(z,\omega-\omega_0)}{\partial z}-i\left(k-k_0\right)\tilde{\psi}(z,\omega-\omega_0)=0;$$

Wavepackets, Group velocity and dispersion

K depends both on the frequency and on the function ε(ω), if we perform a power expansion up to the second order:

$$k\cong k_0+k_1(\omega-\omega_0)+\frac{1}{2}k_2(\omega-\omega_0)^2;$$

Where:

$$k_1=\left.\frac{dk}{d\omega}\right|_{(\omega=\omega_0)}=\frac{1}{v_g(\omega_0)};\qquad k_2=\left.\frac{d^2k}{d\omega^2}\right|_{(\omega=\omega_0)}=\frac{d}{d\omega}\left[\frac{1}{v_g(\omega_0)}\right]_{(\omega=\omega_0)}=\left(-\frac{1}{v_g^2}\frac{dv_g}{d\omega}\right)_{(\omega=\omega_0)};$$

Group velocity

Group velocity dispersion (GVD)

And the equation becomes:

$$\frac{\partial\tilde{\psi}}{\partial z}-ik_1(\omega-\omega_0)\tilde{\psi}-\frac{1}{2}ik_2(\omega-\omega_0)^2\tilde{\psi}=0;$$

Wavepackets, Group velocity and dispersion

The equation is now transformed from the frequency domain to the time domain, we multiply each term for the factor exp(-i(w-w_0)t) and integrate over all the frequencies. Reminding that:

$$\psi(z,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{\psi}(z,\omega-\omega_0)e^{-i(\omega-\omega_0)t}d(\omega-\omega_0);$$

We finally have:

$$\frac{\partial\psi}{\partial z}+k_1\frac{\partial\psi}{\partial t}+\frac{1}{2}ik_2\frac{\partial^2\psi}{\partial t^2}=0;$$

This equation can be simplified by means of a coordinate transformation:

$$\tau=t-\frac{z}{v_g}=t-k_1z;$$

And the equation becomes:

$$\frac{\partial\psi_s(z,\tau)}{\partial z}+\frac{1}{2}ik_2\frac{\partial^2\psi_s(z,\tau)}{\partial\tau^2}=0; \qquad (@)$$

Exercise: derive eq. (@) applying the coordinate transformation.

Wavepackets, Group velocity and dispersion

In order to make calculation easier we define a variable:

$$A_S=\sqrt{\frac{A_{eff}\varepsilon_0cn_0}{2}}\psi_S$$

So that |A|^2 is the optical power. A_{eff} is the effective core area, it can be evaluated using the modal distribution for the fundamental fibre mode. Depends on the fiber parameter such as core radius, core-cladding index difference….

Typically it takes values in a range of 10-20 μm² in the visible and 50-80 μm² in the near infrared.

And the equation becomes:

$$\frac{\partial A_s(z,\tau)}{\partial z}+\frac{1}{2}ik_2\frac{\partial^2A_s(z,\tau)}{\partial\tau^2}=0;$$

Wavepackets, Group velocity and dispersion

What is the effect of the GVD?

Being v_g function of the frequency, different components of an optical pulse would propagate at different group velocities. Nevertheless, the peak of the pulse moves with the group velocity calculate at ω_0.

k_2>0 (normal dispersion): Red components travel faster than blu components.
k_2<0 (anomalous dispersion): Blu components travel faster than Red components

If we consider a Gaussian pulse at the input (z=0):

$$A_S(0,\tau) = A_0 e^{-\tau^2/2T_0}; \quad \Longrightarrow \quad A(z,\tau) = A_0\left(\frac{T_0^2}{T_0^2 - ik_2 z}\right)^{1/2} e^{-\frac{\tau^2}{2(T_0^2 - ik_2 z)}};$$

$$T_1 = T_0\left(1 + (z/L_D)^2\right)^{1/2}; \qquad L_D = T_0^2/|k_2|;$$

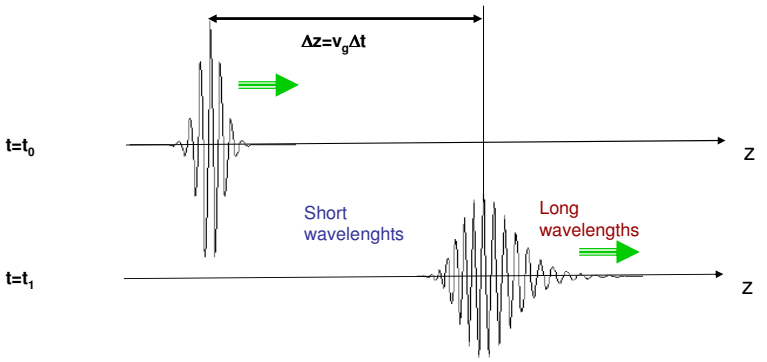
Dispersion length. Pulse-width broadens by a factor of sqrt(2)

Thus a Gaussian unchirped pulse maintains its shape on propagation but its width increases, the peak intensity decreases and becomes linearly chirped.

NOTE: A chirped, broadened Gaussian pulse, coming from other devices can be recompressed to its minimum time width by propagation trough a dispersive medium with opposite sign.

Wavepackets, Group velocity and dispersion

If we include higher order dispersion terms, the pulse undergoes broadening, Chirping and distortion of the edges. K_2>0



NOTE: Dispersion terms are NOT responsible for spectrum changes. The power spectrum of the input pulse is constant (neglecting losses) during propagation.

Minimum pulse width is obtained when all the spectral components arrive at the same time

Nonlinear Wavepackets, nonlinear Schrödinger equation

If we are interested to high peak intensity pulse propagation, we must take into account that the refractive index of many materials depends on the intensity of the incident optical field. As previously shown, assuming the nonlinear response as instantaneous:

$$\tilde{n}(\omega, I) = n_0(\omega) + n_{2I} I \qquad n_{2I}(\omega) = \frac{3}{4c\varepsilon_0 n_0^2(\omega)} \chi^{(3)}_{xxxx};$$

The wave vector can be expanded as:

$$k \cong k_0 + k_1(\omega - \omega_0) + \frac{1}{2} k_2 (\omega - \omega_0)^2 + \Delta k_{NL};$$

$$\Delta k_{NL} = n_{2I} I \frac{\omega_0}{c} = n_2 \frac{\omega_0}{c A_{eff}} |A|^2 = \gamma |A|^2;$$

Note: Adding nonlinear effects at this stage implies that we are not considering that the group velocity might be intensity dependent. (Self steepening is neglected) Ultra short pulses (T_0<100 fs) needs a more detailed method.

The nonlinear Schrödinger equation is (using retarded coordinates):

$$\frac{\partial A_s(z,\tau)}{\partial z} + \frac{1}{2} i k_2 \frac{\partial^2 A_s(z,\tau)}{\partial \tau^2} = i\gamma |A_s(z,\tau)|^2 A_s(z,\tau);$$

[γ]=[W]$^{-1}$[km]$^{-1}$

NOTE: The non linear term is responsible for a phase modulation of the envelope, molded on the intensity of the same envelope. The effect is called: **SELF PHASE MODULATION**

Nonlinear Wavepackets, nonlinear Schrödinger equation

A more accurate description of the pulse propagation should take into account losses, especially if we are considering optical fibres. Thus the equation is:

$$\frac{\partial A_s(z,\tau)}{\partial z} + \frac{1}{2} i k_2 \frac{\partial^2 A_s(z,\tau)}{\partial \tau^2} = -\frac{\alpha}{2} A_s(z,\tau) + i\gamma |A_s(z,\tau)|^2 A_s(z,\tau);$$

If we define a normalized amplitude U by using:

$$A_S(z,\tau) = \sqrt{P_0} e^{-\alpha z/2} U(z,\tau);$$

The normalized amplitude satisfies the propagation equation given by:

$$\frac{\partial U(z,\tau)}{\partial z} + \frac{1}{2} i k_2 \frac{\partial^2 U(z,\tau)}{\partial \tau^2} = i\gamma P_0 e^{-\alpha z} |U(z,\tau)|^2 U(z,\tau);$$

NOTE: the quantity $\mathbf{L_{NL} = 1/\gamma P_0}$ is called nonlinear length and provides the length scale over which the nonlinear effects become important. This is to be compared to the dispersion length L_D which determines the length over which dispersion effects become important.

SELF PHASE MODULATION

At first we analyze the effect of SPM when the GVD can be neglected. The equation for the normalized amplitude becomes:

$$\frac{\partial U(z,\tau)}{\partial z} = i\gamma P_0 e^{-\alpha z}\left|U(z,\tau)\right|^2 U(z,\tau);$$

Note that the dynamics only affects the phase of U(z,T), while the modulus is a constant. Thus the solution can be written as:

$$U(z,\tau) = U(0,\tau)e^{i\phi_{NL}(z,\tau)}; \qquad \phi_{NL}(z,\tau) = \left|U(0,\tau)\right|^2\left(\gamma P_0 z_{eff}\right);$$

$$with \qquad z_{eff} = \frac{1}{\alpha}\left(1-e^{-\alpha z}\right);$$

SPM induces spectral broadening, indeed temporal varying phase implies that instantaneous optical frequency differs across the pulse from its central value ω_0.

$$\delta\omega(\tau) = -\frac{\partial\phi_{NL}(z,\tau)}{\partial\tau} = -\frac{\partial\left|U(z,\tau)\right|^2}{\partial\tau}\gamma P_0 z_{eff};$$

SELF PHASE MODULATION

A frequency chirp is induced by SPM and it increases in magnitude with the propagated distance. New frequency components are continuously generated as the pulse propagates, corresponding to a spectrum broadening of the pulse.

SPM can broaden the spectrum considerably.
For ultra short pulses the broadened spectrum can extend over 100 THz or more.
This effect is called:
SPERCONTINUUM generation.

Exercise: For a Gaussian pulse, calculate and compare the behavior of the frequency chirp induced by SPM to the frequency chirp induced by GVD.

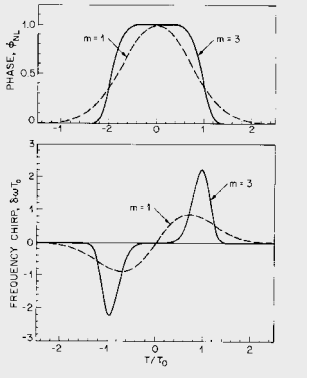
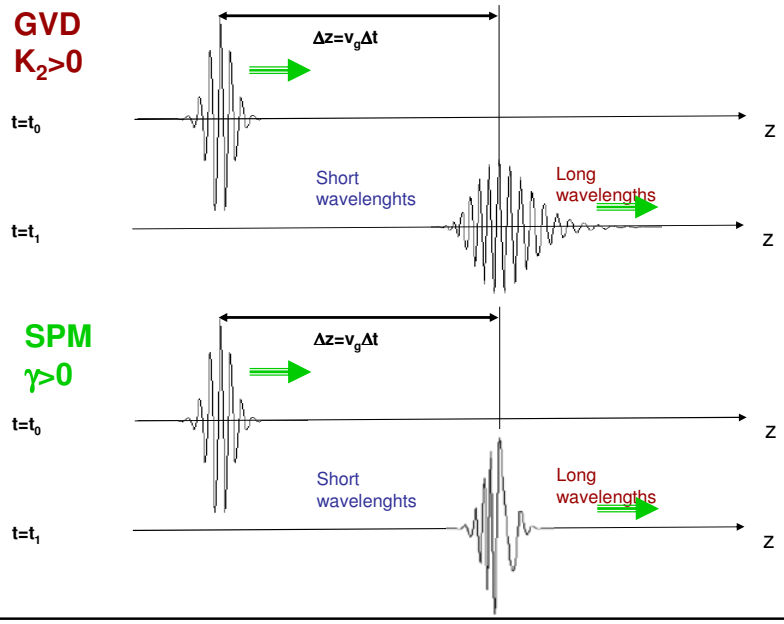


Fig. 4.1 Temporal variation of the phase shift ϕ_{NL} and the frequency induced by SPM for the cases of a Gaussian (dashed curve) and a super (solid curve) pulse.

Agrawal (Nonlinear fiber Optics)

Self Phase Modulation and Group velocity dispersion



Self Phase Modulation and Group velocity dispersion

When L_D and L_{NL} are of the same order of magnitude, it becomes necessary to consider the combined effect of GVD and SPM on pulse evolution along the fiber.

- In the anomalous dispersion regime of the fiber, the two phenomena can cooperate n such a way that the fiber can support optical solitons.

- In the normal dispersion regime combined effects of SPM and GVD are used for pulse compression

Important parameter:

$$N^2 = \frac{L_D}{L_{NL}} = \frac{\gamma P_0 T_0^2}{|k_2|};$$

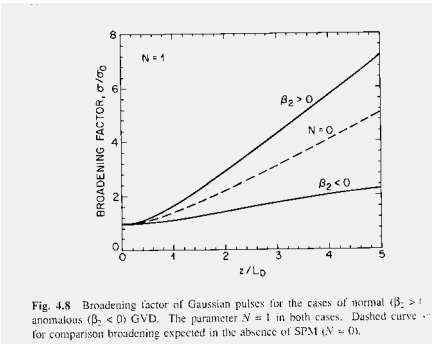


Fig. 4.8 Broadening factor of Gaussian pulses for the cases of normal ($\beta_2 > 0$) and anomalous ($\beta_2 < 0$) GVD. The parameter $N = 1$ in both cases. Dashed curve is for comparison broadening expected in the absence of SPM ($N = 0$).

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Self Phase Modulation and Group velocity dispersion

Typical values for fused silica :

λ_D=1.27 μm

$$L_D = \frac{T_0^2}{|k_2|};$$

If, T_0=1ps, at λ=1.55 mm:
L_{NL}=50 m

Typical value for fibres:

$$\gamma \cong 20W^{-1}km^{-1};$$

$$L_{NL} = \frac{1}{\gamma P_0};$$

If, P_0=1W, at λ=1.55 mm:
L_{NL}=50 m

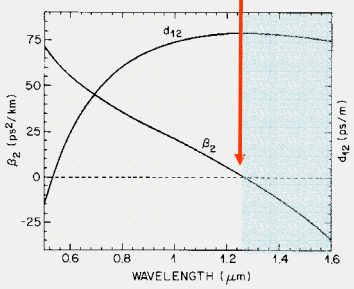

Fig. 1.5 Variation of β_2 and d_{12} with wavelength for fused silica. The dispersi parameter $\beta_2 = 0$ near 1.27 μm. The parameter $d_{12} = \beta_1(\lambda_1) - \beta_1(\lambda_2)$ is plotted as function of λ_2 for $\lambda_1 = 0.532$ μm.

Guided geometry is responsible for modifications of the dispersion relation. So that the zero GVD point is usually shifted at longer wavelengths (around 1.3 μm)

Introduction to Optical Solitons

Modulation Instability

Interplay between nonlinear and dispersive effects may lead to modulation of the steady state. Propagation in the form of continuous wave is instable in the presence of SPM and anomalous GVD.

$$\frac{\partial A_s(z,\tau)}{\partial z} + \frac{1}{2} ik_2 \frac{\partial^2 A_s(z,\tau)}{\partial \tau^2} = i\gamma |A_s(z,\tau)|^2 A_s(z,\tau);$$

For CW, A is independent of τ. Thus the solution is:

$$\overline{A}(z) = \sqrt{P_0} e^{i\phi_{NL}(z)}; \qquad \phi_{NL}(z) = \left(\gamma P_0 z_{eff}\right);$$

Now we introduce a small perturbation:

$$A(z) = \left(\sqrt{P_0} + a\right) e^{i\phi_{NL}(z)};$$

Substituting the perturbed solution in the original equation we find:

$$\frac{\partial a}{\partial z} + \frac{1}{2} ik_2 \frac{\partial^2 a}{\partial \tau^2} = i\gamma P_0 \left(a + a^*\right);$$

Introduction to Optical Solitons

If we assume a general solution of the form:

$$a(z,\tau) = a_1 \cos(\mathrm{K}z - \Omega\tau) + i a_2 \sin(\mathrm{K}z - \Omega\tau);$$

Solutions are possible when K and Ω satisfy the condition:

$$\mathrm{K} = \pm\frac{1}{2}|k_2|\Omega\left(\Omega^2 + \mathrm{sgn}(k_2)\Omega_c^2\right)^{1/2}; \qquad \Omega_c^2 = \frac{4\gamma P_0}{|k_2|}$$

NOTE: For normal GVD ($k_2>0$), K is real for all Ω

For anomalous GVD ($k_2<0$) K becomes imaginary for $\Omega<\Omega_c$ and the perturbation grows exponentially .

Existence of modulation instability for anomalous GVD is an indication of a fundamentally different character of the solution with respect to normal GVD. Indeed the equation has specific solutions which either do not change with z or follow a periodic pattern, called optical solitons

Introduction to Optical Solitons

Fundamental and higher-order solitons.

Neglecting fiber propagation losses and normalizing as follows:

$$U = \frac{A}{\sqrt{P_0}}; \qquad \xi = \frac{z}{L_D}; \qquad \tau = \frac{T}{T_0};$$

Nonlinear Schrödinger equation becomes:

$$i\frac{\partial U}{\partial \xi} = \frac{1}{2}\mathrm{sgn}(k_2)\frac{\partial^2 U}{\partial \tau^2} - N^2|U|^2 U;$$

$$N^2 = \frac{L_D}{L_{NL}} = \frac{\gamma P_0 T_0^2}{|k_2|};$$

If we define *u=NU and we consider the case of anomalous dispersion (sgn(k_2)=-1)* we have the standard form:

$$i\frac{\partial u}{\partial \xi} + \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} + |u|^2 u; \qquad (+)$$

Equation is solved by using the inverse scattering method

Introduction to Optical Solitons

Inverse scattering method: Consists of identifying a suitable scattering problem whose potential is the solution sought. Incident field at z=0 is used to obtain the initial scattering data.

It is possible to show that it exists a solution of the form:

$$u(\xi,\tau)=\operatorname{sech}(\tau)e^{i\xi/2}; \quad (*)$$

$$N^2=\frac{L_D}{L_{NL}}=\frac{\gamma P_0 T_0^2}{|k_2|};$$

If an hyperbolic secant pulse whose width $\mathbf{T_0}$ and peak power $\mathbf{P_0}$ are chosen such that N=1, the pulse will propagate undistorted without change in shape for arbitrary long distances.**[SOLITON]**

Exercise: verify that (*) is solution of (+).

The peak power required to support fundamental soliton is:

$$P_1=\frac{|k_2|}{\gamma T_0^2}=\frac{3.11|k_2|}{\gamma T_{FWHM}^2};$$

Being for the hyperbolic secant $T_{FWHM}=1.76T_0$

Introduction to Optical Solitons

The solution showed belongs to a family of solutions whose initial form at ξ=0 is given by:

$$u(\xi,0)=N\operatorname{sech}(\tau);$$

N is an integer and determines the soliton order. The solution previously described represents the N=1, fundamental order soliton. Dynamics of higher order solitons differs from the one described for fundamental soliton. Indeed it can be shown that the field temporal profile is modified during the propagation along ξ. Nevertheless it follows a periodic pattern.

Power needed to launch a N-th order soliton Is N^2 times higher than the power required to launch fundamental soliton

$$P_N=N^2\frac{|k_2|}{\gamma T_0^2}=N^2\frac{3.11|k_2|}{\gamma T_{FWHM}^2};$$

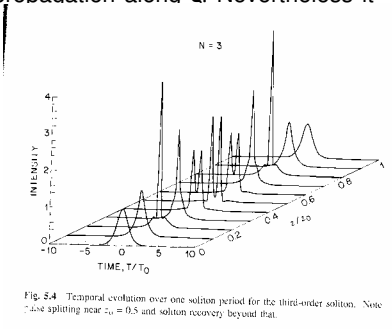

Fig. 5.4 Temporal evolution over one soliton period for the third-order soliton. Note pulse splitting near $z_0 = 0.5$ and soliton recovery beyond that.

Introduction to Optical Solitons

What happens if the initial pulse shape or the peak power are not matched to the requirements?

For example if N' is not an integer
- The pulse adjusts its width as it propagates, to evolve into a soliton. A part of the initial energy is dispersed and the pulse evolves asymptotically into a soliton whose order is an integer N closest to N'.

Similar things happen if the initial shape is not an hyperbolic secant.

Finally, solitons can be excited very easily as long as the peak power of the initial pulse exceeds a threshold value.

Exercise: Consider λ=1.55 μm, n_0=1.45, n_2=3x10^{-20} m^2/W, effective area of 50 μm^2. We assume a transmission rate f_{bit}=10^{10} bit/s using pulses having T_{FWHM}=3x10^{-11}s. Calculate the peak pulse power for the fundamental soliton mode. Compare the results with the Raman and Brillouin thresholds for a 30 km L_{eff} of optical fiber. What happens if we want to communicate at higher bit rates? Can you provide a rough estimation of the shortest soliton pulse duration that can be transmitted without degradation due to stimulated Raman scattering? [175 mW]

Exercise 1 (Molecular Raman scattering): Calculate the wavelengths (in μm) corresponding to the Stokes line of the CCl_4 molecule shown in the previous picture for a monochromatic incident light of wavelength 0.632 μm.

Exercise 2 (Stimulated Raman Scattering): a) derive the expression for the nonlinear polarization term oscillating at frequency ω_s. b)derive the equations for Raman amplification

Exercise 3 (Stimulated Raman Scattering): Consider λ_p=1.55 μm,. The silica glass fiber loss is minimum, thus L_{eff} is of the order of 20 km. A typical value for A_{eff} is 50 μm^2. Calculate the predicted Raman threshold. How does the threshold change if we consider radiation emitted by a Nd:YAG laser with A_{eff} of the order of 15 μm^2 for a 1m long fiber?

Exercise 4 (Stimulated Brillouin Scattering): Consider λ_p=1.55 μm, The silica glass fiber loss is minimum, thus L_{eff} is of the order of 20 km. A typical value for A_{eff} is 50 μm^2. Calculate the predicted Raman threshold and Brillouin threshold.

Exercise 5 (Non linear wavepackets propagation): For a Gaussian pulse, calculate and compare the behavior of the frequency chirp induced by SPM to the frequency chirp induced by GVD.

Exercise 6 (Solitons, Raman and Brillouin thresholds): Consider λ=1.55 μm, n_0=1.45, n_2=3x10^{-20} m^2/W, effective area of 50 μm^2. We assume a transmission rate f_{bit}=10^{10} bit/s using pulses having T_{FWHM}=3x10^{-11}s. Calculate the peak pulse power for the fundamental soliton mode. Compare the results with the Raman and Brillouin thresholds for a 30 km L_{eff} of optical fiber. What happens if we want to communicate at higher bit rates? Can you provide a rough estimation of the shortest soliton pulse duration that can be transmitted without degradation due to stimulated Raman scattering?