The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization

IAEA

1833-32

Workshop on Understanding and Evaluating Radioanalytical Measurement Uncertainty

5 - 16 November 2007

Applied Statistics: Basic statistical terms and concepts

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Workshop on
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Applied Statistics: Basic statistical terms and concepts

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Trieste – 8 november 2007

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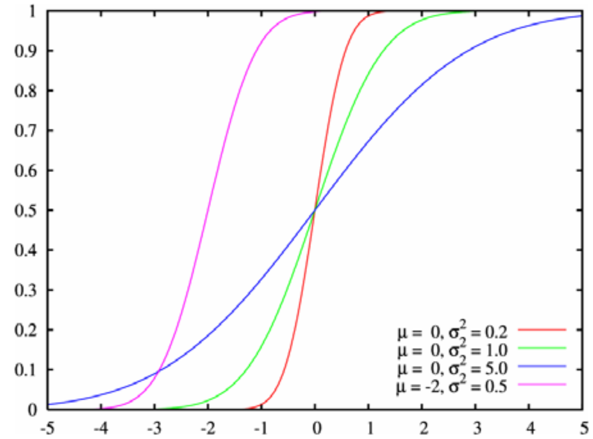
Statistics for evaluation of uncertainty: overwiev

- GUM approach
- Type A evaluation
- Example
- Type B evaluation
- Example
- Law of propagation of uncertainty
- Reporting results



GUM approach

- Define the output quantity, the quantity required to be measured
- Identify the input quantities upon which the output quantity depends
- Develop a model relating the output quantity to these input quantities
- On the basis of available knowledge assign probability density function (normal, uniform, etc.) to the values of the input quantities
- Estimate the uncertainties associated with the input quantities
- Propagate the values of the input quantities and their associated uncertainties throught the model
- Obtain the estimate of the output quantity value and its uncertainty.

GUM approach

A measurement model is expressed by a functional relationship *f*

$$Y = f(X)$$

where Y is a single output quantity and X represents the N input quantities

Quantify uncertainty

Once identified the input quantities, the next step is to quantify the uncertainty arising from these quantities:

- evaluating the uncertainty arising from each individual source and converting them to standard deviation

- determining directly the combined contribution to the uncertainty on the result from some or all of these sources using method performance data.

NOTE:
Not all of the components make a significant contribution to the combined uncertainty.

Convert to standard uncertainty

The uncertainties associated with the input quantities may be grouped into two categories according to the method used to estimate them:

A type A evaluation of standard uncertainty may be based on any valid statistical method for treating data.

A type B evaluation of standard uncertainty is usually based on scientific judgement using all of the relevant information available.

Type A

Examples are:

calculating the standard deviation of the mean of a series of independent observations;

using the method of least squares to fit a curve to data in order to estimate the parameters of the curve and their standard deviations;

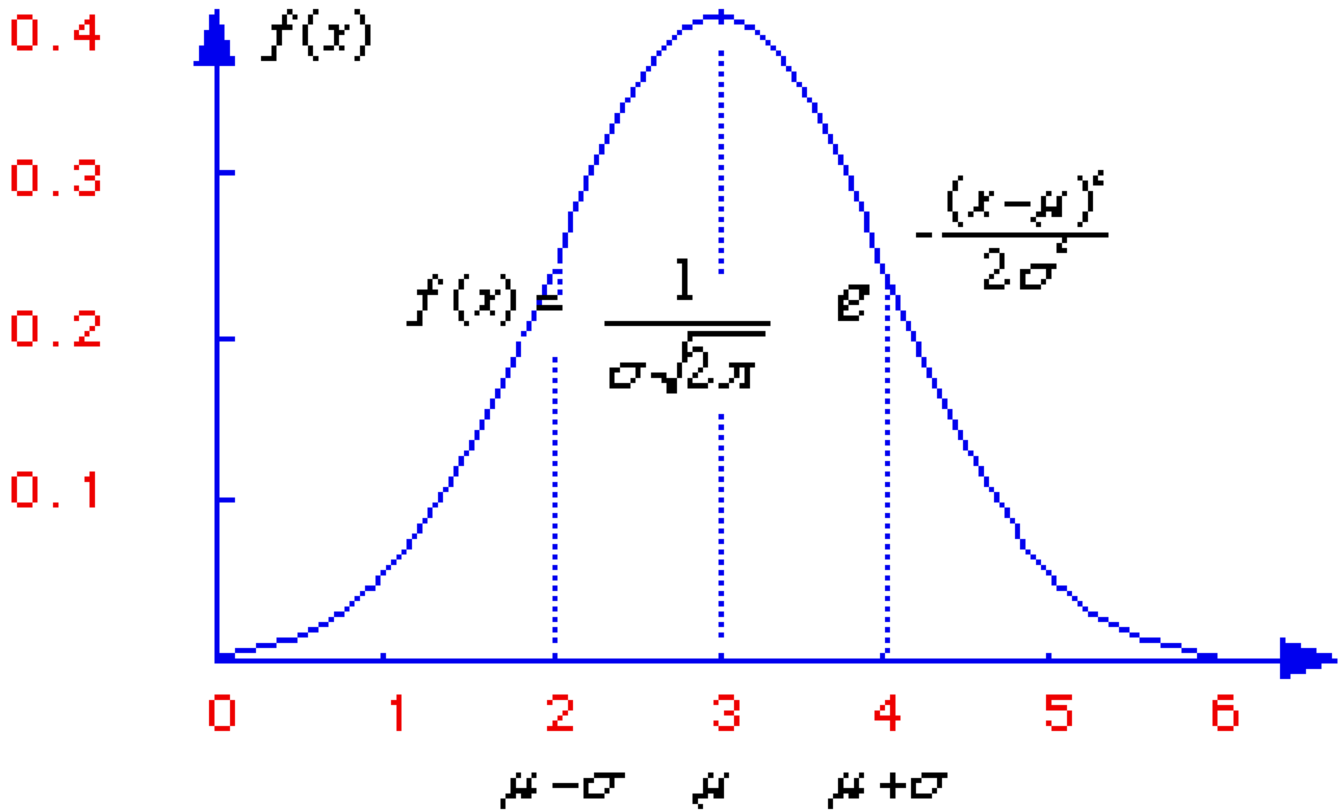
carrying out an analysis of variance.

Type A

Depending on the statistical distribution of data, it is possible to estimate the corrisponding standard deviation:

	Normal	Binomial	Poisson
Standard deviation	$s_{\overline{x}} = \sqrt{\dfrac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{n(n-1)}}$	$s = \sqrt{np(1-p)}$	$s = \sqrt{\lambda}$

Normal Distribution



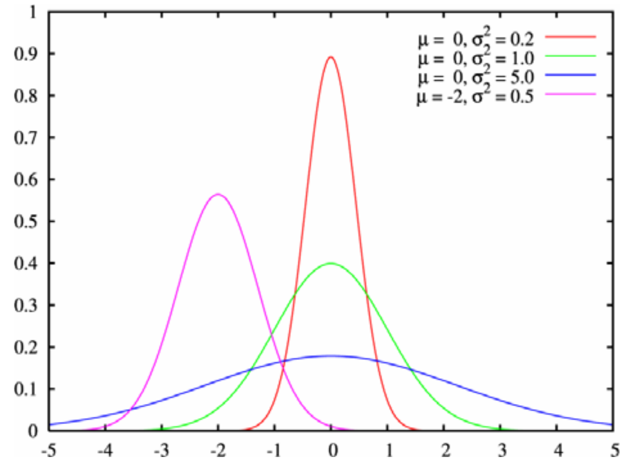
Normal Distribution

For a set of n values x_i

Average (mean value)

Arithmetic mean value of a sample of n results

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n}(x_i)$$

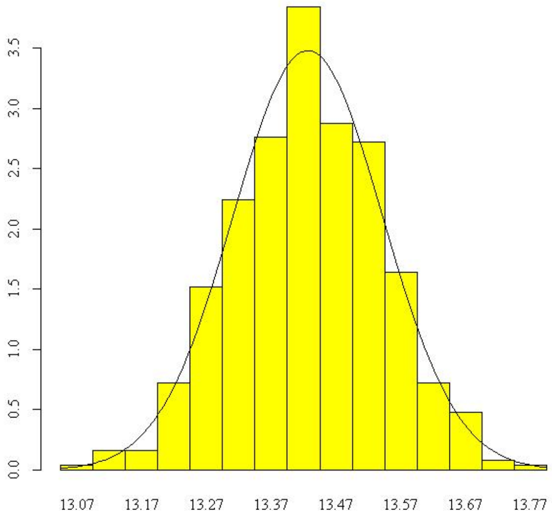


Normal Distribution

For a set of n values x_i

Standard deviation: quantity characterizing the dispersion of the values

$$s(x_i) = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$



Normal Distribution

Variance $$V(x_i) = s(x_i)^2$$

Standard deviation of the mean $$s(\bar{x}) = \frac{s(x_i)}{\sqrt{n}}$$

Relative Standard deviation

$$RSD = \frac{s(x_i)}{\bar{x}}$$ or $$RSD(\%) = CV = \frac{s(x_i)}{\bar{x}} * 100$$

Confidence Interval

A confidence interval for some population parameter is an interval constructed from a sample of n observations so that it will contain the parameter with some specified probability (1-α)100% where α is some fraction between 0 and 1 (usually α is less than 0.5).

Confidence interval of the mean of a normal distribution

$$95\%\ \mathrm{CI} = t(0.05, n-1) * \frac{s}{\sqrt{n}}$$

$$\mu = \bar{x} \pm (1-\alpha)\%\ CI(n)$$

Binomial distribution

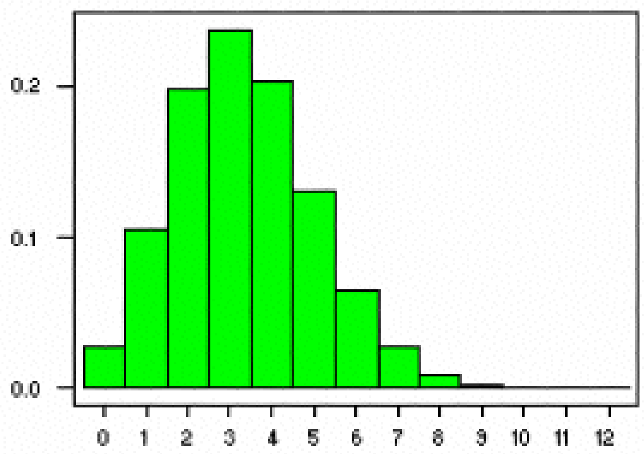
- An elementary example is this: Roll a standard die ten times and count the number of sixes. The distribution of this random number is a binomial distribution with $n = 10$ and $p = 1/6$.

  the expected value of Y is

  $$y=np$$

  the standard deviation is

  $$s(y) = \sqrt{np(1-p)}$$



Poisson distribution

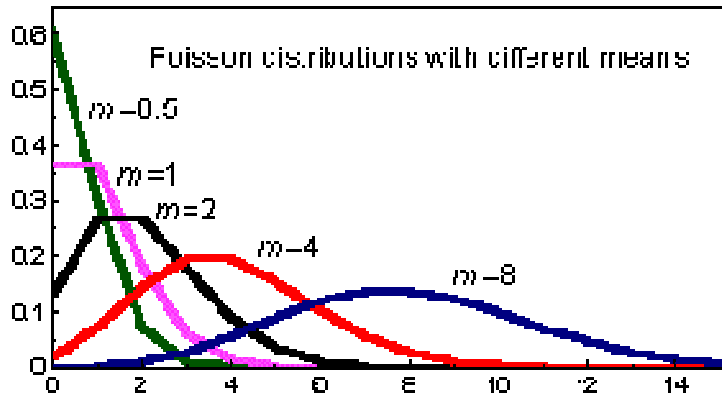
The poisson distribution is generally appropriate for counting experiments where the data represent the number of events observed per unit interval. It is important in the study of random processes such as those associated with the radioactive decay of elementary particles or nuclear states.

the expected value of *Y* is

$$y = \lambda$$

the standard deviation is

$$\boldsymbol{s}(\boldsymbol{y}) = \sqrt{\lambda}$$



λ= average number of events that occur within the given time period (or area)

Type B

Relevant information available:

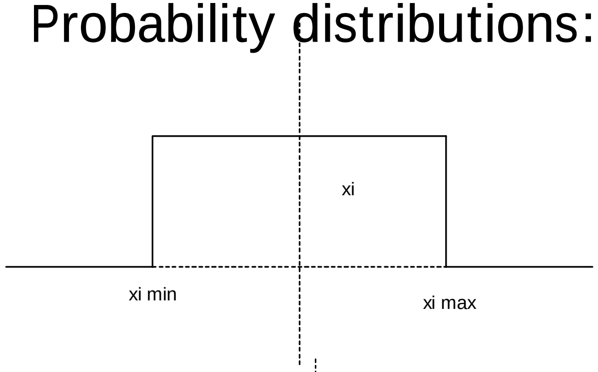
- previous measurement data
- experience with, or general knowledge of, the behavior and property of relevant materials and instruments
- manufacturer's specifications

- data provided in calibration and other reports
- uncertainties assigned to reference data taken from handbooks

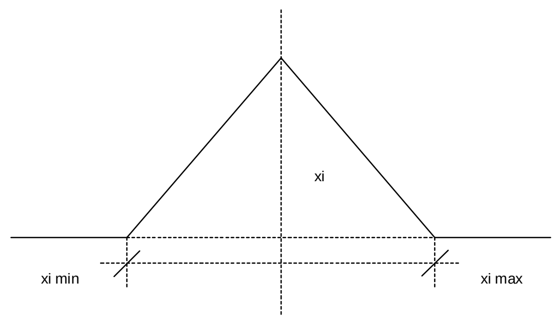
Type B

Depending on the information available, it is possible to estimate the standard uncertainty on the basis of assumed probability distributions which the input quantities approximate.

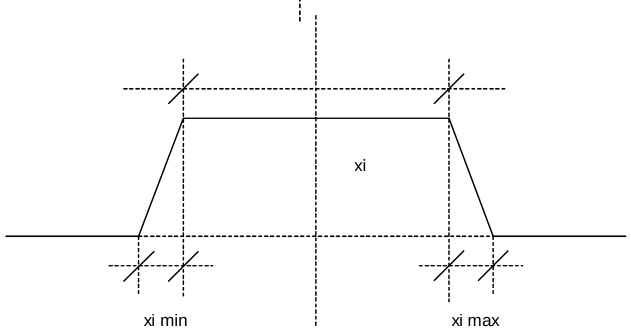
Probability distributions:



$$u(x) = \frac{x_{i\min} - x_{i\max}}{2 \cdot \sqrt{3}}$$



$$u(x) = \frac{(x_{i\min} - x_{i\max})}{2 \cdot \sqrt{6}}$$



$$u(x) = \frac{(x_{i\min} - x_{i\max}) \cdot \sqrt{(1 + \beta^2)}}{2 \cdot \sqrt{6}}$$

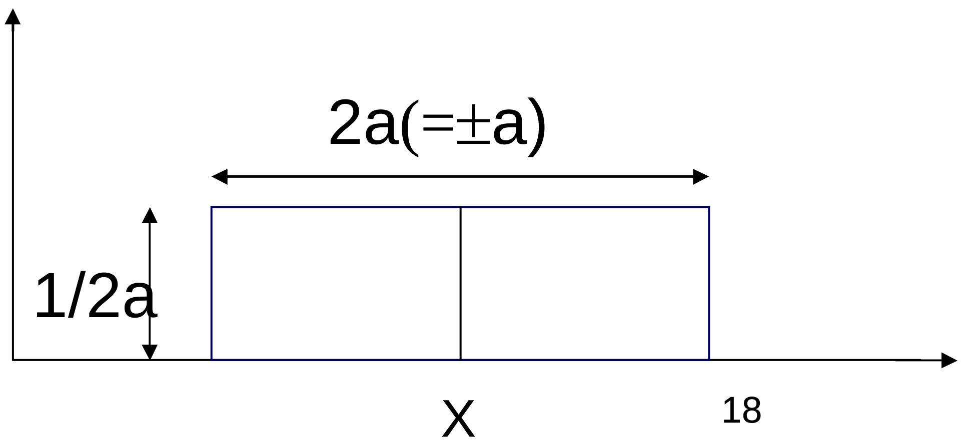
rectangular distribution

the input quantity value is between the limits $a_- \ldots\ldots a_+$

the expectation $y = x \pm a$

estimated standard uncertainty $s = u(x) = a/\sqrt{3}$

Little information are available about the input quantity and all one can do is suppose that the input quantity is described by a uniform distribution with a constant probability for the value to lie anywhere within the interval



Example of rectangular distribution

" It is likely that the value is somewhere in that range"

Rectangular distribution is usually described in terms of: the average value and the range (±a)

certificates or other specification give limits where the value could be, without specifying a level of confidence (or degree of freedom).

Examples:

concentration of calibration standard is quoted as (1000 ±2)mg/L assuming rectangular distribution the standard uncertainty is:

$$u(x) = a/\sqrt{3} = 2/\sqrt{3} = 1.16 \text{ mg/L}$$

the purity of the cadmium is given on the certificate as (99.99 ±0.01)% assuming rectangular distribution the standard uncertainty is:

$$u(x) = a/\sqrt{3} = 0.01/\sqrt{3} = 0.0058\ \%$$

triangular distribution

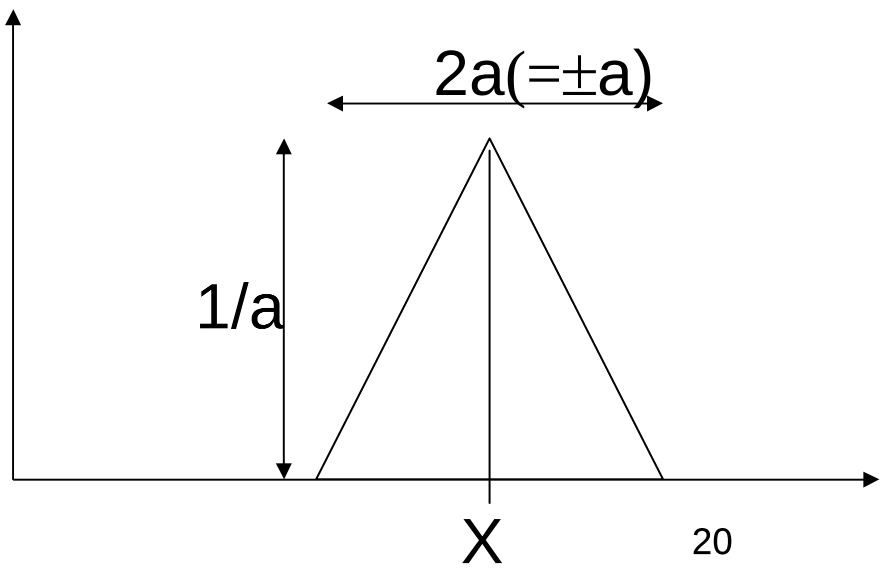
It is used when the available information about the input quantity is less limited than those for the rectangular distribution and it is suggested that values near the centre of the range are more likely than near the extreme

the expectation

$$y = x \pm a$$

estimated standard uncertainty

$$s = u(x) = a / \sqrt{6}$$



Example of triangular distribution

“ values close to x are more likely than near the boundaries”

Examples (volumetric glassware):
The manifacturer quotes a volume for the flask of (100 ±0.1)mL at a T=20°C
Nominal value most probable!
Assuming triangular distribution the standard uncertainty is:

$$u(x) = a / \sqrt{6} = 0.1 / \sqrt{6} = 0.04 \text{ mL}$$

In case of doubt, use the rectangular distribution

trapezoidal distribution

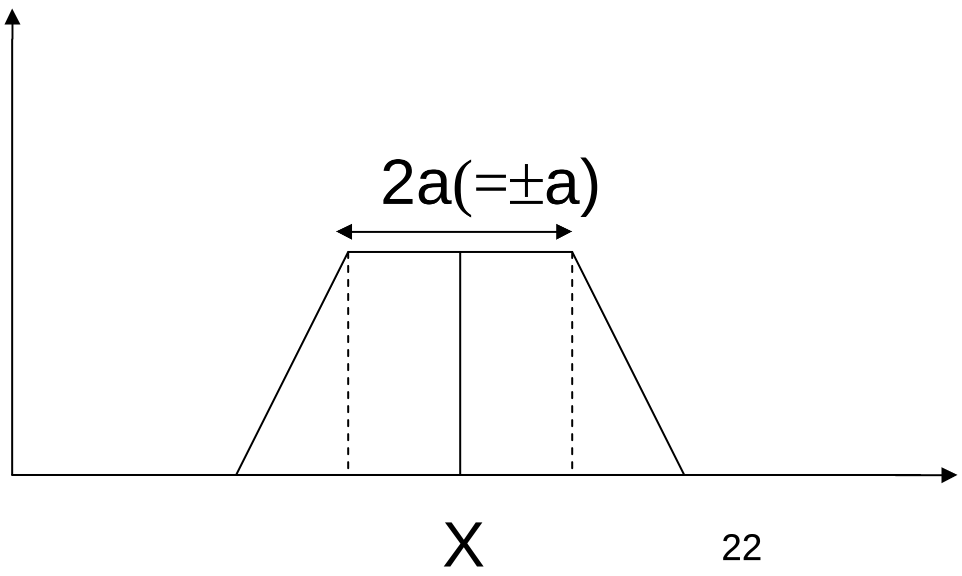
Distribution used when it is suggested that values near the extremes of the range are more likely than near the centre

the expectation

$$y = x \pm a$$

estimated standard uncertainty

$$s = u(x) = \frac{a}{\sqrt{6}}\sqrt{(1+\beta^2)}$$



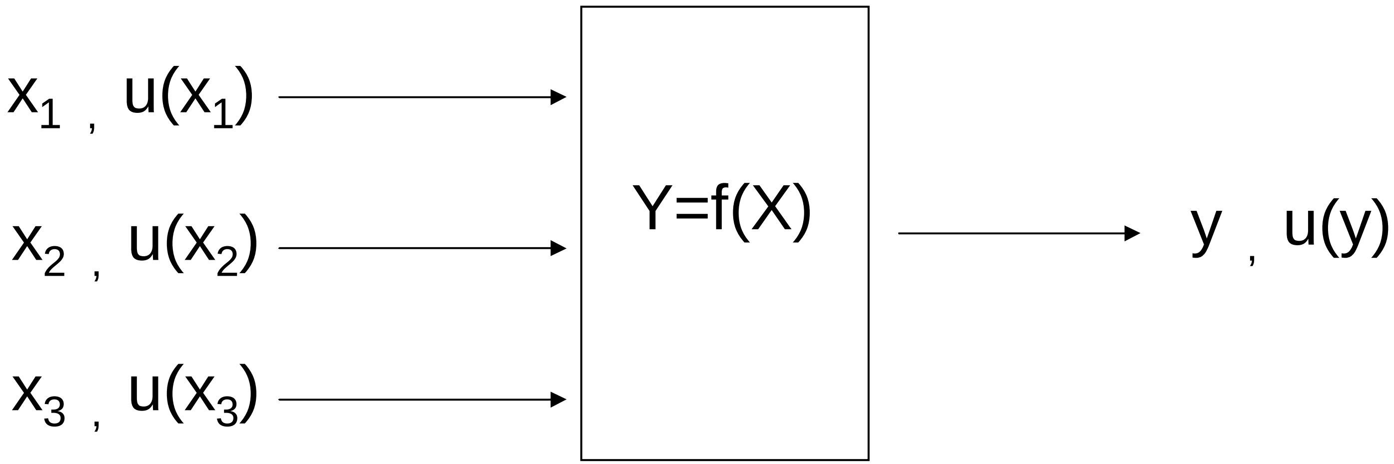
Law of propagation of uncertainty

For the calculation phase of uncertainty evaluation,

GUM

applies the law of propagation of uncertainty

The law of propagation of uncertainty



Combined uncertainty

Following the law of propagation of uncertainty, the combined uncertainty can be calculated combining all the components expressed as standard deviations:

$$u_c(y) = \sqrt{\sum_{i=1}^{n}\left(\frac{\partial y}{\partial x_i}\cdot u(x_i)\right)^2 + 2\sum_{i=1}^{n-1}\sum_{j=1+1}^{n}\left(\frac{\partial y}{\partial x_i}\frac{\partial y}{\partial x_j}\cdot \mathrm{cov}(x_{ij})\right)}$$

where x_i are the input components, while $\mathrm{cov}(x_{ij})$ is the covariance between x_i and x_j.

Combined uncertainty

When there is no correlation between input quantities the covariance is zero and the combined standard uncertainty is evaluated as the square root of the combined variance simplified:

$$u_c(y) = \sqrt{\sum_{i=1}^{n} \left(\frac{dy}{dx_i} \cdot u(x_i) \right)^2}$$

Law of uncertainty propagation without correlation

$$Y = f(X_1, X_2, X_3 \ldots\ldots X_n)$$

$$u_c(y) = \sqrt{\left(\frac{\partial Y}{\partial X_1} u(x_1)\right)^2 + \left(\frac{\partial Y}{\partial X_2} u(x_2)\right)^2 + \ldots. + \left(\frac{\partial Y}{\partial X_n} u(x_n)\right)^2}$$

$$Y = (X_1 + X_2)$$

$$Y = (X_1 - X_2)$$

$$\Rightarrow \quad u_c(Y) = \sqrt{u(X_1)^2 + u(X_2)^2}$$

$$Y = (X_1 \bullet X_2)$$

$$Y = (X_1 / X_2)$$

$$\Rightarrow \quad \frac{u_c(Y)}{y} = \sqrt{\left(\frac{u(X_1)}{X_1}\right)^2 + \left(\frac{u(X_2)}{X_2}\right)^2}$$

Combined uncertainty

The most common quantitative expression relating the value of the measurand to the parameters on which it depends are:

y	$u_c(y)$	$u_r(y)$
$Y = A + B + C$	$\sqrt{u_A^2 + u_B^2 + u_C^2}$	$\sqrt{\frac{A^2 u_{r(A)}^2 + B^2 u_{r(B)}^2 + C^2 u_{r(C)}^2}{(A+B+C)^2}}$
$Y = A \cdot B \cdot C$	$\sqrt{[BC]^2 u_A^2 + [AC]^2 u_B^2 + [AB]^2 u_C^2}$	$\sqrt{u_{r(A)}^2 + u_{r(B)}^2 + u_{r(C)}^2}$
$Y = A \cdot B + hC$	$\sqrt{B^2 u_A^2 + A^2 u_B^2 + h^2 u_C^2}$	$\sqrt{\frac{A^2 B^2 \left(u_{r(A)}^2 u_{r(B)}^2\right) + h^2 C^2 u_{r(C)}^2}{(A \cdot B + h \cdot C)^2}}$
$Y = \frac{A \cdot B \cdot C}{D}$	$\frac{1}{\mathrm{D}}\sqrt{[\mathrm{BC}]^2 \mathrm{u}_\mathrm{A}^2 + [\mathrm{AC}]^2 \mathrm{u}_\mathrm{B}^2 + [\mathrm{AB}]^2 \mathrm{u}_\mathrm{C}^2 + \left(\frac{\mathrm{A \cdot B \cdot C}}{\mathrm{D}}\right)^2 \cdot \mathrm{u}_\mathrm{D}^2}$	$\sqrt{u_{r(A)}^2 + u_{r(B)}^2 + u_{r(C)}^2 + u_{r(D)}^2}$

$u_r(y)$ is the relative combined standard uncertainty = $u_c(y)/|y|$

Reporting results

- A result is given with its uncertainty

$$C_{Cs} = (14.4 \pm 2.2) Bq / kg$$

but...what is 2.2?

- Standard deviation
- Rectangular interval
- Triangular interval
- Confidence interval
- Combined uncertainty
- Expanded uncertainty (with k=?)

THANK YOU

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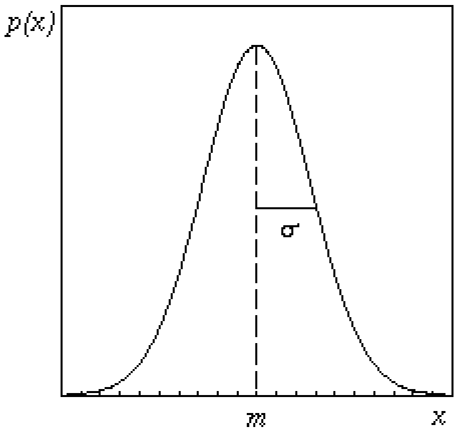
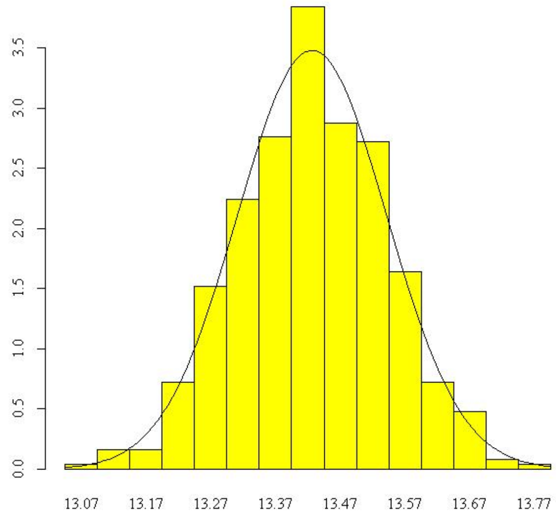
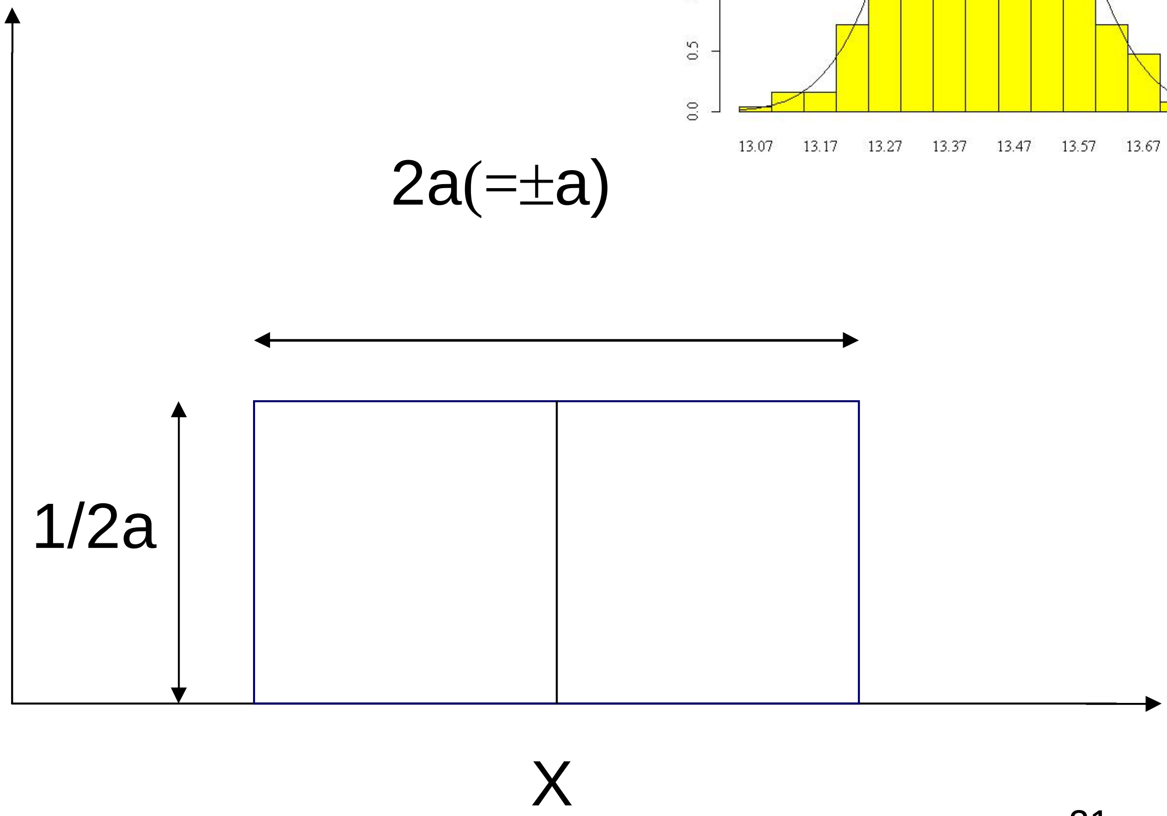
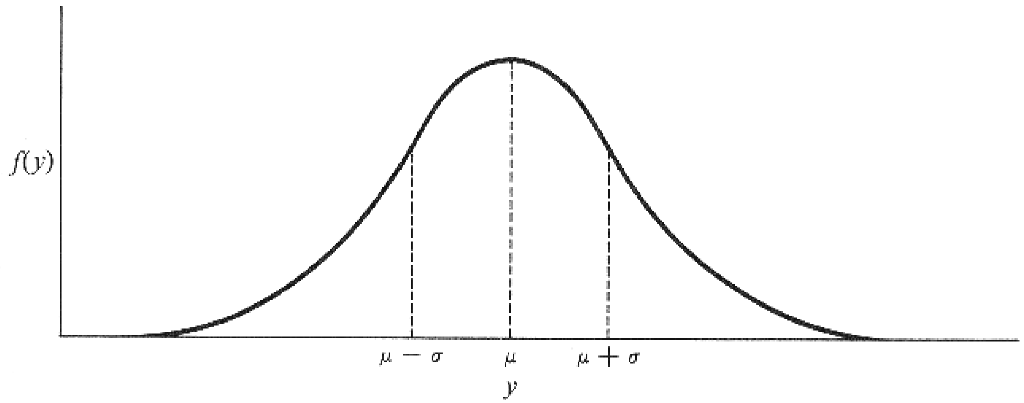
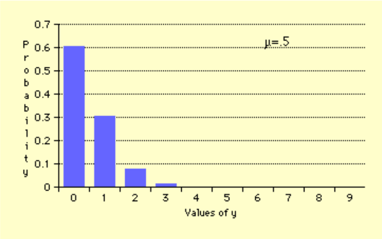
Esempio di funzione di Gauss

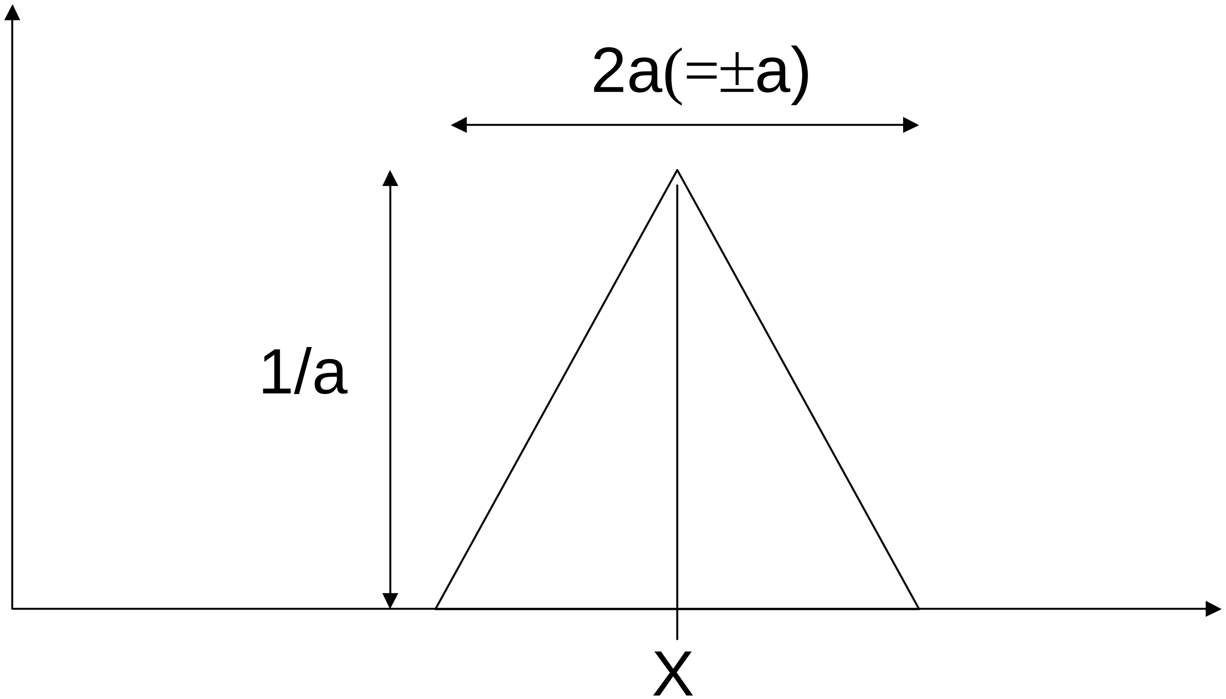


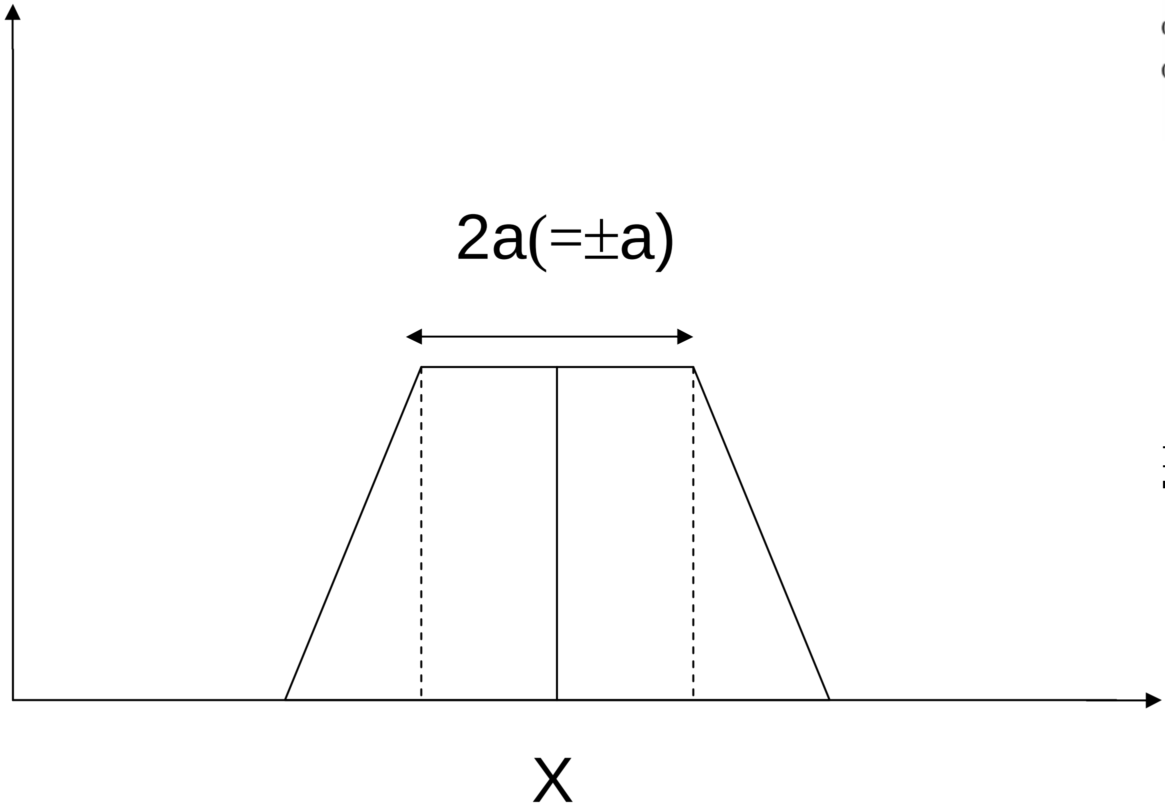


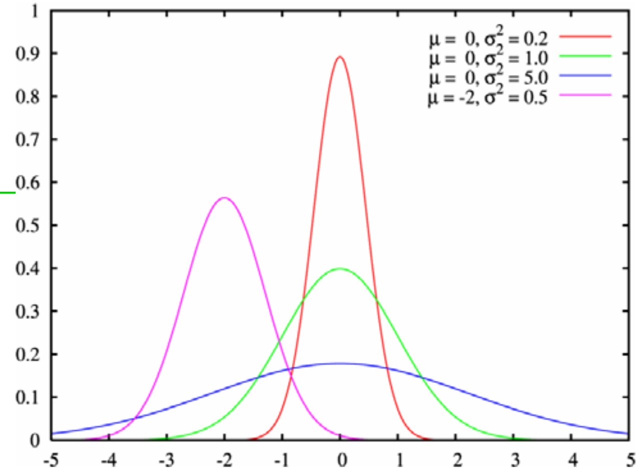


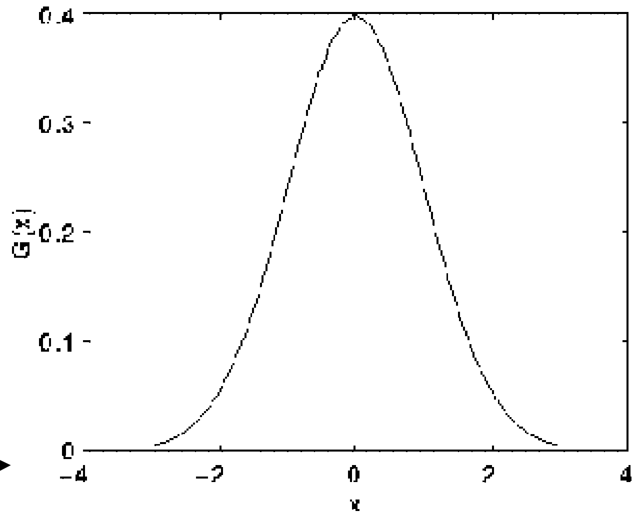

f(y)
μ − σ
μ
μ + σ
y


μ=.5
Probability
Values of y


2a(=±a)
1/a
X


2a(=±a)
X


μ = 0, σ² = 0.2
μ = 0, σ² = 1.0
μ = 0, σ² = 5.0
μ = -2, σ² = 0.5


G(x)
x