Critical phenomena in portfolio selection

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Rational portfolio selection is seeking a tradeoff between risk and expected return, by optimizing the risk functional over the portfolio weights, given the expected return and possibly other constraints. In real life the risk functional is not given in advance, but has to be estimated from observations on the market. Since the size N of banking portfolios (i.e. the dimension of space in which the minimum is sought) is large and the number of observed data (the lengths T of the time series for the various assets) is always bounded, we never have a sufficient amount of information to reliably reconstruct the underlying stochastic process and estimate the risk functional. Our estimates of risk will therefore fluctuate from sample to sample, and the weights of the optimal portfolio will fluctuate along with them. This problem has long been known in finance, and a large number of noise reduction techniques have been developed to deal with it over the past decades, at least in the simple case when the risk functional is chosen to be the variance. Variance is, however, not the only risk measure in use, and in some cases (e.g. for fat tailed distributions) it can even be grossly misleading. Noise reduction techniques for alternative risk measures are either much less developed or nonexisting.

Applying a variety of techniques (simulations, random matrix theory, replica method) we have performed a comparative study of the noise sensitivity of various risk measures (variance, mean absolute deviation, expected shortfall, and maximal loss) recently. We have found that the noise sensitivity strongly depends on the ratio N/T, and that different risk measures exhibit very different sensitivity to the same noise. We have also observed that there exists a feasibility boundary, a critical value of N/T, for all these risk measures, beyond which the risk functional becomes unbounded, hence the optimization meaningless. This critical ratio is 1 for the variance and mean absolute deviation, ½ for maximal loss, and a value, smaller than ½ (and depending on the threshold beyond which the conditional average loss is calculated) for expected shortfall. Upon approaching this critical point the fluctuations in the estimation error of risk increase tremendously: the average error diverges, so does also the variance of its distribution. The weights of the optimal portfolio for a given sample also show strong deviations from their ideal values, with the variance and all the higher moments of their distribution diverging as one approaches the critical point.

The critical indices associated with these divergences seem to be universal, i.e. independent of the structure of the market, the risk measure and the underlying process, whereas the prefactors of the scaling laws do depend on the covariance structure (predominantly positive correlations enhancing, negative ones decreasing the strength of the divergence).

When short selling is excluded (or any other constraint is applied that makes the domain over which the optimum is sought finite) sample to sample fluctuations can obviously not diverge any more. Then the instability of portfolio selection manifests itself through some or most of the weights sticking to the boundaries defined by the constraints. In the case of a ban on short selling this leads to a spontaneous reduction of the portfolio size. Clearly, in these cases the solution is determined more by the constraints than by the objective function.

These observations are evidently relevant for portfolio selection, but their scope is, in fact, much wider than finance. The critical phenomena observed in portfolio selection are related to (and represent a simple class of) the phase transitions discovered in complex optimization problems recently. In view of the close relationship between portfolio optimization and linear regression, similar instabilities will appear whenever one tries to construct a model for a complex system, which, almost by definition, must have a large number of explicatory variables, and for which one often has only a limited amount of empirical data. These kind of problems are ubiquitous in a number of fields, ranging from operations research, to machine learning, bioinformatics, medical science, economics, sociology, and technology.

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