Preferred Habitat, Time Deformation and the Yield Curve *

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Abstract

The paper approaches the modeling of the yield curve from a stochastic volatility perspective based on time deformation. The way in which we model time deformation is new and differs from alternatives that currently exist in the literature and is based on market microstructure theory of the impact of information flow on a market. We model the stochastic volatility process by modeling the instantaneous volatility as a function of price intensity in the spirit of Cho and Frees (1988), Engle and Russell (1998) and Gerhard and Hautsch (2002). One contribution of the paper therefore lies with the introduction of a new transaction level approach to the econometric modelling of stochastic volatility in a multivariate framework exploiting intensity-based point processes previously used by Bowsher (2003), Hall and Haustch (2003). We find that the individual yields of U.S. treasury notes and bonds appear to be driven by different "operational" clocks as suggested by the market segmentation theory of the Term Structure but these are related to each other through a multivariate Hawkes model which effectively coordinates activity along the yield curve. The results offer some support to the Market Segmentation or Preferred Habitat models as the univariate Hawkes models we have found at each maturity are statistically significantly different from each other and the major impact on each maturity is activity at that maturity. However there are flows between the different maturities that die away relatively quickly which indicates that the markets are not completely segmented. Diagnostic tests show that the point process models are relatively well specified and a robustness comparison with realized volatility indicates the close relationship between the two estimators of integrated volatility but also some differences between the structural intensity model and the model free realized volatility. We have also shown that bond returns standardized by the instantaneous volatility estimated from our Hawkes model are Gaussian which is consistent with the theory of time deformation for security prices quite generally.

JEL: C16, E43, F3, G1, G12

Keywords: Term structure, Interest rates, Multivariate modeling, Hawkes process, Time deformation.

1 Introduction

Financial markets evolve on a time scale that is invariably different from chronological or clock time which is, after all, only determined by the rate by which the earth revolves around the sun. The

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relevance of this time scale, which may be natural for the analysis of physical phenomena, can be questioned when fixed intervals of clock time contain differing numbers of transactions - markets move fast and slow as noted for instance by Hasbrouck (1999). If the underlying stochastic process that we wish to model is indexed to a different time scale than clock time then it becomes difficult, if not impossible, to properly measure the properties of these processes without taking into account of time deformation[1]. This would seem to be crucial for the statistical analysis of security prices quite generally.

This paper considers these issues in the context of modelling the yield curve and asks the following questions; "Do yields on the different maturities operate on the same time scale, and if not how are they coordinated?", "Does this have any implications for modelling the yield curve as a whole?", and "How is information transferred along the yield curve if different information is relevant at different points on the yield curve?"

We attempt to answer these questions by bringing market microstructure based empirical models of information flow to asset pricing in the yield curve. We propose a new approach by specifying a structural stochastic volatility model for bond prices through a price intensity based measure of instantaneous volatility. Our approach recognizes the influence of information flow on the market via trading activities and that this information flow is lumpy and non smooth in clock time. Hence it is natural to consider a new time scale which is based on information flow when modelling the statistical properties of bond prices.

A central question is how to measure information and several alternatives have been suggested in the literature. We exploit what is perhaps the most natural measure of trading activity by using the price intensity to measure market time.[2] We model the conditional price intensity at different points on the yield curve through a dynamic non-homogenous Poisson process that captures the clustering of activity as a function of backward recurrence time and this provides a structural explanation of the transformation between market time and clock time.[3] The estimated price intensity is then used to estimate the instantaneous volatility in the relevant market, in our case, 2, 5 and 30 year US treasury notes and bonds. This provides a new structural approach to stochastic volatility modelling through a point process modelling of activity in the market based on information flow rather than a linear autoregressive model that is frequently used. These instantaneous volatilities may then be used to calibrate a yield curve model such as the Heath, Jarrow and Morton model (1992), to price derivatives and to study how different news events change the shape of the yield curve.[4] In the current paper we focus on analyzing whether there is a single time scale in the yield curve or whether there are different time scales corresponding to different news events and hence information at different points on the yield curve. This has a direct interpretation in terms of the Market Segmentation or Preferred Habitat model of the term structure, see Culbertson (1957) and Modigliani and Sutch (1966) or more recently Vayanos and Vila (2007). We find, using a multivariate Hawkes model to model the multivariate intensity processes corresponding to the different yield-to-maturities, that there are in fact different time scales at each point in the yield curve but these are in turn coordinated by the multivariate Hawkes model into a common time scale that ensures the bond market functions coherently.

In section 2 we briefly describe the Market Segmentation and Preferred Habitat models of the term structure of interest rates. In Sections 3 and 4, we discuss time deformation and our approach to using a multivariate point process model to model stochastic volatility at different maturities on the yield curve. In section 5, we discuss the estimation of our model by Maximum Likelihood Methods and the diagnostic tests we apply. We then describe the data in section 6 and the empirical results in section 7 before drawing some conclusions.

[1] A translation of clock time into a time scale that reflects market activity or information flow.

[2] Price intensity measures the rate of price changing trades.

[3] The backward recurrence time is the time elasped since the last event and is a left-continuous function before t.

[4] We consider this issue in a related paper- Information Flow down the Yield Curve.

2 Market Segmentation and the Preferred Habitat Model

In contrast to modern theories of the Yield Curve which rests on a representative agent who will seek out arbitrage opportunities at any point in the yield curve, the Market Segmentation (Culbertson, 1957) and Preferred Habitat (Modigiani and Sutch, 1966) models both rest on the view that investors differ in the maturites in which they are active given their liability structure. So for instance, Pension Funds may be more interested in long dated bonds, say 30 years, while Life Assurance companies will be active in shorter maturities, perhaps 15 years and asset managers will be interested in shorter bonds. In both these theories, investors at each maturity are different and concentrate their activity on the specific maturity in which they are primarily concerned and so the bond price at each point in the yield curve will reflect the different liquidity structures and the separate forces of demand and supply at each maturity. The shape and movement of the yield curve is then seen to be determined by the different market forces and pressures at each maturity. Modern theories of the Yield Curve downplay the heterogeneity implied by these theories and the different preference and information structure that is implied at each point in the yield curve. In contrast the Market Segmentation Theory assumes that neither investors nor borrowers are willing to move from one maturity to another to take advantage of arbitrage opportunities provided by changing expectations and forward rates. The Preferred Habitat Theory recognizes that the term structure reflects expectations of the future path of interest rates and a risk premium but rejects the uniform rise in the risk premium with maturity. Instead Modigliani and Sutch (1966) argue that there will be some movement in activity between maturities that are in excess supply and those in excess demand at any point but that the compensation required, through the risk premium, will reflect the individual inducement required for the market participant to move out of their preferred habitat. Individuals have different risk preferences and different risk horizons.

Vayanos and Vila (2007) have recently developed a modern Preferred Habitat model in which investors with preferences for specific maturities trade with risk averse arbitrageurs and it is the arbitrageurs who integrate the markets for different maturities by incorporating information about expected short rates into bond prices. The point process models that we develop below coincide almost exactly with this theoretical structure. By focusing on activity at each point in the yield curve separately we can examine if there are critical differences in each market and what the interactions are at the highest frequency of trading. In our model, the multivariate Hawkes model serves as the device that integrates the market as a whole and hence captures the arbitrageurs activity. There are important implications for policy in examining how information flows down the yield curve as it is often seen to be critical to understand how changes in the policy rate which immediately affect the short end of the yield curve are transferred to the long end. We can examine this question precisely using the multivariate Hawkes model developed below.

3 Time Deformation

Most empirical models of security prices have ignored the question of what time scale should be employed to measure the properties of the asset of interest. Common practice uses a fixed interval of clock time, be it one minute or one day, and these do not necessarily reflect the varying amounts of information on the underlying process in these fixed intervals. Although such temporal aggregation superficially facilitates empirical analysis it masks the true stochastic process that drives asset prices and confuses the ability of statistical methods to measure the process. Hasbrouck (1999) notes that the time aggregation of data smears the impact of individual events and aggravates problems of simultaneity. Ait-Sahalia and Mykland (2003) highlights the biases and adverse effects of sampling discreteness and randomness when using a discrete sampling scheme. Information on the underlying process is simply lost on aggregation to fixed intervals of clock time and *ad hoc* assumptions must be used to find representative values for the interval of clock time; such as the choice to use daily closing values to represent the entire day.

Time deformation is a more subtle and deeper issue that is independent of temporal aggregation and rests on the recognition that stochastic processes may evolve on different time scales than natural clock time. For instance consider a sequence of transactions that occur at irregular points in clock time carried out at different prices and different volumes; we can create different time scales defined by volume for instance or price so that the volume time scale increased by one unit each time, say, 10 million dollars had been transacted since the last "time" point or alternatively the price change had increased by 5 ticks[5]. These two time scales would not necessarily coincide of course and represent two very simply possible choices from a large number of different potential time scales that could be considered; see Le Fol and Mercier (1998). Clark (1973) demonstrated that the subordinated stochastic structure implied by time deformation can potentially reconcile the observed non-Gaussianity of returns if a proper measure of information flow can be recovered. The argument being that information flow measures the correct time scale on which the underlying price process should be measured and once returns are conditioned on this measure of information flow (or in this time scale) then the central limit theorem applies and Gaussianity is returned. Market microstructure theory is consistent with this approach. Diamond and Verecchia (1987) and Easley and O'Hara (1992) have noted that when traders are informed, they are more likely to be impatient (short durations) and when traders are informed, they would ideally prefer to trade large volumes or successive waves of smaller trades. The critical question lies in how to measure information.

A number of different proxies have been put forward, for example; *volume*- Clark (1973), Epps and Epps (1976), Tauchen and Pitts (1983), Karpoff (1987), Gallant, Rossi and Tauchen (1992), Blume, Easley and O'Hara (1994), the *number of trades*- Jones, Kaul and Lipson (1994), Ané and Geman (2000), *trade duration*- Russell and Engle (1998), Engle (2000) and *trade intensity* - Salmon and McCullouch (2005). In this paper, we differ from Salmon and McCullouch (2005) by using the *price* intensity because of its relationship with the volatility of returns. If informed traders trade more frequently after an information event through a series of price changing trades, there is a direct relationship from information to the volatility of asset returns via the price intensity.

Time deformation has been modelled in several different ways in the literature and a natural representation through stochastic subordination is in terms of stochastic volatility models. Stochastic volatility represents a latent process and one approach developed by Stock (1998) assumes that the multivariate latent process $\Im(s)$, the volatility at each maturity in the yield curve, we are interested in modelling, evolves smoothly in operational or market time s. Stock assumes a mapping between market time s and clock time t : $s = g(t)$ which describes the relationship between the two time scales.(Stock, 1988; Ghysels and Jasiak, 1995). This approach to stochastic volatility is structural and rests on the idea that further market or economic variables appear within the function $g(\cdot)$ directing the process by which information flow affects the relationship between clock time and market time. The approach we follow is similar but employs a more natural specification by exploiting an underlying point process model for the stochastic price intensity process at each point in the yield curve. The univariate and multivariate Hawkes processes that we describe below allow self and cross excitation from the activity at different maturities along the yield curve to account for the clustering of activity and information transfer between the different yields. These Hawkes models are then used to model the price intensity which we take as our market time scale and effectively replace Stock's logistic specification for $g(t)$.

4 The Model of Bond Prices and Price Intensity

Time deformation, through a subordinated stochastic process, naturally defines a stochastic volatility model as described originally by Clark (1973). The directing process in our case is the intensity

[5] Another classic example is how to value two identical second hand cars that are of the same age but one has done 100,000 miles and the other 10,000. Clearly their common age measured by clock time would not deliver the correct valuation and some account, a time deformation, would have to be made for milage. Second hand car dealers are arbitrageurs in time deformation!

process which drives the stochastic volatility and hence returns. Consider the following stochastic differential equation for bond returns:

$$\frac{dB(t)}{B(t)} = \mu\ dt + \sigma(t)\, dW_1(t), \tag{1}$$

where $B(t)$ is the bond price and $W_1(t)$ is a standard Wiener process. A standard stochastic volatility model assumes that log volatility is driven by a second Wiener process $W_2(t)$ through an Ornstein-Uhlenbeck type process,

$$d \log \sigma(t) = a((b - \log \sigma(t))dt + cdW_2(t)$$

which assumes that log volatility evolves smoothly in clock time t. Instead we specify the stochastic volatility process by assuming[6]:

$$\frac{dB(t)}{B(t)} = \mu\ dt + \sqrt{\sigma^2(g(t))}dW_1(t) \tag{2}$$

which implicitly implies that log volatility evolves smoothly in market time ($s = g(t)$);

We can estimate the volatility process directly from the price intensity in the following manner;

$$\begin{aligned}
\sigma^2(s) &= \lim_{\Delta \to 0} \frac{1}{\Delta} \left[prob\, |B(t+\Delta) - B(t)| \geq dB \mid \mathcal{F}_t\right] \times \left(\frac{dB(t)}{B(t)}\right)^2 && (3)\\
&= \lim_{\Delta \to 0} \frac{1}{\Delta} \left[prob\left(N^{dB}(t+\Delta) - N^{dB}(t)\right) > 0 \mid \mathcal{F}_t\right] \times \left(\frac{dB(t)}{B(t)}\right)^2 && (4)\\
\sigma^2(s) &= \lambda^{dp}(t; \mathcal{F}_t) \left(\frac{dB(t)}{B(t)}\right)^2, \\
\text{where } s &= g(t) = \lambda^{dp}(t; \mathcal{F}_t) = \mu + \int_{(0,t)} W(t-u, \theta)\, dN(u) && (5)
\end{aligned}$$

where $\lambda^{dp}(t; \mathcal{F}_t)$ is the conditional price intensity at time t, and $\sigma^2\left(\lambda^{dp}(t; \mathcal{F}_t)\right)$ is the directing stochastic volatility. The mapping between market time, $s = \lambda^{dp}(t; \mathcal{F}_t)$, and the calendar time,$t$, given by the stochastic intensity model, specifies a structural stochastic volatility model that depends on how we specify the intensity model and serves as the work horse for our time deformation process. Such a specification for volatility, where the instantaneous volatility is modelled using the price intensity, is not new to financial economics (see for instance,Cho and Frees (1988), Engle and Russell (1998) and Gerhard and Hautsch (2002)). Cho and Frees(1988) argue that measuring volatility by how quickly price changes rather than by how much asymptotically eliminates the resulting measure from micro-structure biases created for instance by bid ask bounce. In addition volatility specified in this form reflects the economic intuition linking the rate of trading activity and the 'stylized fact' of volatility clustering. As new information arrives, if informed traders are impatient and carry out order splitting strategies to exploit their information advantage as quickly as possible, as suggested by the market microstructure literature, there will be clustering of trades and thus volatility clustering. This also suggests that information is incorporated into prices through packets of trades, either fast or slow, motivating the use of price intensity as the directing process for our time deformation model.

Empirically our approach is supported by Garbade and Lieber (1977) who found that a homogenous Poisson process captures the majority of trade activity relatively well, except for the irregular

[6] This is different from Ghysels and Jasiak (1995) which specify the stochastic volatility process as:

$$d \log(\sigma(s)) = a(b - \log \sigma(s))\, ds + cdW_2(s),$$

as an Ornstein-Uhlenbeck type process estimated using Kalman filter and efficient method of moments.

bursts of trade arrivals which invalidate the homogeneous assumption and Engle and Russell (1998) who observe trade clustering in the NYSE.

The model we develop below draws in particular on Bowsher (2005) and Hawkes (1971) and as opposed to Engle (2000) or Bauwens and Hautsch (2003), provides a direct specification for the stochastic intensity process in terms of the backward recurrence time of events rather than by way of duration.

First, assume $N(t)$ to be a simple point process in $[0, \infty)$ on $(\Omega, \mathcal{F}, P)$ that is adapted to some filtration $\{\mathcal{F}_t\}$, and that $\lambda(t \mid \mathcal{F}_t)$ is a positive process with a sample path that is left-continuous with right-hand limits. Then, the conditional trade intensity can be written as:

$$\lambda(t \mid \mathcal{F}_t) = \lim_{\Delta \longrightarrow 0} \frac{1}{\Delta} E\left[N(t+\Delta) - N(t) \mid \mathcal{F}_t\right], \tag{6}$$

where $N(t)$ represents the number of events that have occurred up to and including time t. We refer to $\lambda(t \mid \mathcal{F}_t)$ as $\lambda(t)$, the value of λ at time t, and $\mathcal{F}_t$ is defined as the natural filtration up to and including time t. Within this framework, information flow is captured by trade intensity; the higher trade intensity the shorter trade duration indicating the presence of information.

The time deformation model for the treasury securities is fully specified in a multivariate framework, where each maturity in the yield curve or the number of variates is represented by the index $m \in M$, assuming that $\{t_i\}_{i \in \{1,2,\cdots\}}$ is a simple point process in $[0, \infty)$, defined on $(\Omega, \mathcal{F}, P)$, and where $\{z_i\}_{i \in \{1,2,\cdots\}}$ is a sequence of $\{1, 2, \cdots, M\}$ - valued random variables. Then the double sequence of $\{t_i, z_i\}_{i \in \{1,2,\cdots\}}$ is an M-variate point process on $[0, \infty)$.

Definition 1 *For all m, $1 \leq m \leq M$, and all $t \geq 0$,*

$$N_m(t) = \sum_{i \geq 1} \mathbf{1}(t_i \leq t)\, \mathbf{1}(z_i = m), \tag{7}$$

then the M-vector process $N(t) = (N_1(t), \cdots, N_m(t))$ is an M-variate counting process associated with $\{t_i, z_i\}_{i \in \{1,2,\cdots,m\}}$ reflecting the trading process at each maturity m.

The fully specified intensity model for each maturity is then written as:

$$\lambda_m(t, \theta) = \mu_m + \sum_{m=1}^{M} \int_{(0,t)} W(t-u, \theta)\, dN(u), \tag{8}$$

where $\mu_m > 0$ is the baseline intensity, a deterministic m-vector-valued function on R^+; $W > 0$ is an $m \times m$ matrix-valued function on R^+; θ is a vector of unknown parameters. The functions within W will be specified in our model as nonlinear autoregressive excitation processes, where the pattern of trading in the past increases current intensities and trading activity. In the m-variate case, $\lambda_m(t, \theta)$ is driven by the backward recurrence time of m autoregressive exciting effects; each capturing the past occurrence of type m events. The diagonal entries of W will be called self-exciting effects, while off-diagonal entries are cross-exciting effects. Note that if $\mu_m = 0$ then the self and cross exciting effects in W would never be realized as there would be nothing to start the self exciting process and so $\lambda \equiv 0$. For non-trivial results we therefore require that μ_m is positive. Previous work specifies μ_m using a spline or a fast Fourier transform (i.e. Bowsher, 2005; Hall and Hautsch, 2004) in order to capture the intra-day deterministic effects. Since our objective is to examine time deformation effects and in particular intra-day effects we have chosen instead not to extract any deterministic seasonal pattern and assume μ_m is constant. In part we have also taken this decision to avoid any mis-specification created by inappropriate modelling of the diurnal effect.

We specify W using Hawkes models for the stochastic intensity with a non-negative exponentially decaying function, which capture the clustering of trading. This allows us to measure the impact of periods with different information flow and their rates of decay. Note that μ_m must be

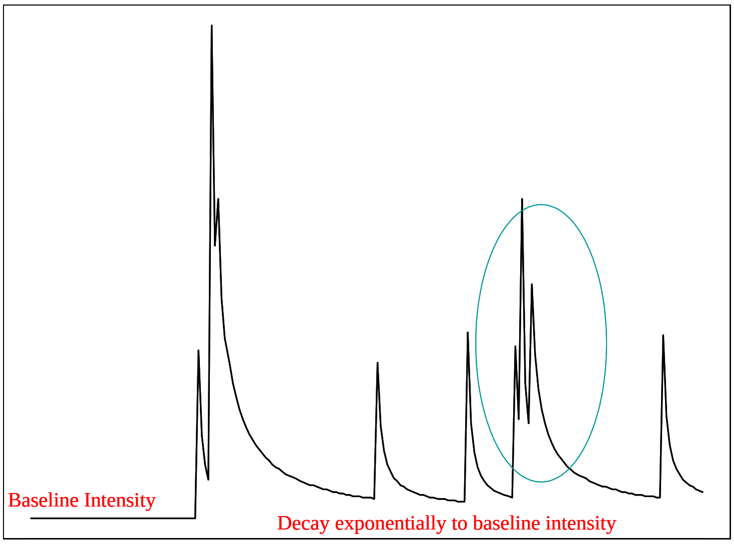

Figure 1: Graphical Illustration of a Hawkes Model

non negative and that different forms of the decay function of the backward recurrence time are possible but, in common with earlier work, we will employ an exponential form; $W(t-u,\theta) = \alpha^j_{m,r} \exp\left[-\beta^j_{m,r}(t-\tau_{r,k})\right]$.[7] The final specification of our time deformation model is then given as:

$$\lambda_m(t) = \mu_m + \sum_{r=1}^{M}\sum_{j=1}^{D}\sum_{k=1}^{\mathcal{N}_r(t)} \alpha^j_{m,r} \exp\left[-\beta^j_{m,r}(t-\tau_{r,k})\right], \tag{9}$$

where M is the number of maturities we consider, D is the number of dimensions (exponential functions), $\mathcal{N}_r$ is the number of data points of type r, and $\tau_{r,k}$ is the occurrence time for the $k'th$ data point of process m.

In this model, we, therefore, allow for market time to take multiple scales corresponding to multiple types of information- if there were a single market time scale then a univariate Hawkes model would be common for all maturities or the impact factor $\alpha^j_{m,r}$ will be the same for all α's.[8] A graphical illustration of a Hawkes model is presented in Figure 1 below which shows hte self exciting feature of the model.

5 Maximum Likelihood Estimation

Ogata (1978) established, under certain regularity conditions, that the MLE for a simple, stationary univariate point process model is consistent and asymptotically normal as $T \to \infty$, and the likelihood

[7] For a more detailed analysis of the asymptotic properties and stationarity conditions see Hawkes (1971).

[8] Our model differs from the g-HawkesE model in Bowsher (2005), which uses only the past events of the same trading day and the previous day's intensity. This specification is more flexible, but it is difficult to derive conditions for stationarity, as the additive effect of the previous events is fixed as endpoints of each individual trading day.

ratio test of a simple null hypothesis possesses the standard χ^2 asymptotic null distribution. .

The assumption that the point process is simple implies that the intensity of each type m gives the conditional probability of a trade occurring per unit of time as the time interval tends to zero. If the sample path of the point process is integrable, then there exists an analytic likelihood function for the specified conditional intensity. Given a parameterized Hawkes' model, the unknown parameters, $\theta = (\mu_m, \alpha^j_{m,r}, \beta^j_{m,r})$, can be estimated using MLE with a likelihood function given by:

$$\mathcal{L}\left(\theta; \{N(t)\}_{t\in(0,T]}\right) = \exp \sum_{m=1}^{M} \left[\int_{(0,T]} \log(\lambda_m(s;\theta \mid \mathcal{F}_s))dN_m(s) + \int_0^T (1-\lambda_m(s \mid \mathcal{F}_s))\, ds \right]. \quad (10)$$

The parameters associated with $\lambda_m(t;\theta \mid \mathcal{F}_s)$ in a multivariate Hawkes model are often assumed to be variation-free or separable (Bowsher, 2005; Hall and Hautsch, 2004), allowing for separate maximization of m log likelihood components and avoiding the curse of dimensionality. Separability is important in a point process model because it eases the complexity of estimation as a parameter or a set of parameters may be estimated individually[9]. However, as far as we can see, the full impact of the variation-free assumption or separability has not yet been rigorously examined in this particualr context and the impact on the asymptotic properties of the MLE estimates are therefore still unclear. We have been able to avoid any assumption of separability by rewriting the joint likelihood in a recursive conditional mannerand directly maximizing the joint likelihood following the procedure outlined in Sohrmann and Tham (2006).

5.1 Stationarity Conditions

Assuming μ to be constant rather than a spline allows one to derive the stationarity conditions for the multivariate case. If

$$\lambda_s(t) = \mu_s + \sum_{r=1}^{k} \int_{-\infty}^{t} g_{sr}(t-u)dN_s(u) \quad (11)$$

$$\boldsymbol{\Lambda}(t) = \boldsymbol{\mu} + \int_{-\infty}^{t} \boldsymbol{G}(t-u)d\boldsymbol{N}(u), \quad (12)$$

then, the vector of stationary densities is:

$$\boldsymbol{\Lambda} = (1-\Gamma)^{-1}\boldsymbol{\mu} \quad (13)$$

$$\Gamma = \int_0^{\infty} \boldsymbol{G}(\upsilon)d\upsilon\,, \quad (14)$$

given $\boldsymbol{\Lambda} > \boldsymbol{0}$. For an invertable $(1-\Gamma)$, this condition will give k rows of stationarity conditions. This integral equation is in general difficult to solve analytically. However, an analytic solution may be obtained when $g(\tau - \upsilon)$ decays exponentially (Hawkes, 1971) and in our case corresponds to the multivariate stationarity condition.

$$0 < \sum_{r=1}^{M}\sum_{j=1}^{D} \frac{\alpha^j_{m,r}}{\beta^j_{m,r}} < 1$$

In each case reported below in our empirical results we find this conditon is statisfied and therefore the models we have estimated are stationary.

[9] For a more detailed discussion of separability in multi-dimensional point process, refer to Schoenberg (2003) (2004).

5.2 Generalized Residuals and Specification Testing

A common approach to evaluating point process models is to examine discrimination criteria such as the Akaike Information Criterion or the Bayesian Information Criterion (e.g. Ogata, 1988). These provide useful numerical comparisons of the relative goodness of fit of competing models but cannot shed light on the specification of a particular model as provided by formal specification tests. In particular, they cannot identify where a model fits poorly nor where it fails.

Residual analysis in other statistical contexts is a standard and powerful tool for locating defects in the fitted model and for suggesting how the model should be improved. While the same is true in point process modelling, you need to be careful in how you define the residuals in this context as they will be yet another point process, the residual process is therefore similar to the generalized residuals considered in Cox and Snell (1968). Much diagnostic checking in the point process literature is carried out by examining various plots constructed from the residual process, see Baddeley et al (2005) and Zhuang (2006).

If $N(t)$ denotes the count process of trades then residuals can be constructed from the fact that the unobservable error or innovation process;

$$I(t) = N(t) - \int_0^t \lambda(s)ds$$

is a martingale with $E[I(t)] = 0$ when the model is true. When the point process model is fitted to the data and parameters, θ, estimated as,$\hat{\theta}$, then the estimated parameters would be substituted into $\lambda(t) = \lambda_\theta(t)$ to give the estimated conditional intensity $\hat{\lambda}(t) = \lambda_{\hat{\theta}}(t)$ from which we can compute the *raw residual process*

$$R(t) = N(t) - \int_0^t \hat{\lambda}(s)ds$$

Increments in $R(t)$ are equivalent to the residuals (observed minus fitted) in a regression model and the adequacy of the fitted model can be examined by checking if $R(t) \approx 0$.

Another standard approach is to examine if the estimated conditional intensity function $\hat{\lambda}(t)$ delivers a homogeneous Poisson residual process. Suppose we observe a one-dimensional point process $t_1, \cdots, t_n$ with conditional intensity $\lambda(t)$ on an interval $[0, T]$. Papangelou (1974) shows that the integrated conditional intensity of the process forms a homogeneous Poisson process with a unit rate on the interval $[0, n]$. If the estimated intensity $\hat{\lambda}(t)$ is close to the true conditional intensity, then the residual process should resemble a unit rate homogenous Poisson process.

Theorem 2 *Let $N(t)$ be a simple point process on $[0, \infty)$. Suppose that $N(t)$ has the intensity function $\lambda(t \mid \mathcal{F}_t)$ that satisfies:*

$$\int_0^\infty \lambda(t \mid \mathcal{F}_t)\, dt = \infty, \tag{15}$$

define for $\forall t$, the stopping time τ_t as the solution to:

$$\int_0^{\tau_t} \lambda(s \mid \mathcal{F}_s)\, ds = t, \tag{16}$$

then, the point process $\tilde{N}(t) = N(\tau_t)$ is a homogenous Poisson process with intensity $\lambda = 1$. Proof is shown in Bremaud (1981).

The only condition for the above relations to hold is the assumption of a simple point process where there is zero probability of the occurrence of more than one event at any single point in time. From the theorem, it can be shown that $\tilde{t}_i - \tilde{t}_{i-1} = \int_{t_{i-1}}^{t_i} \lambda(s \mid \mathcal{F}_s)\, ds = \Lambda(t_{i-1}, t_i)$, where $\{\tilde{t}_i\}_{i \in \{1,2,\ldots\}}$ denotes the time of occurrence for the sequence of points associated with $\tilde{N}(t)$. Then it follows that:

$$\Lambda\left(t_{i-1}, t_i\right) \sim i.i.d.Exp(1). \tag{17}$$

Note that the above transformation is a time-series transformation of a non-homogeneous Poisson process into a homogenous Poisson process. Moreover, $\Lambda\left(t_{i-1}, t_i\right)$ establishes the link between the intensity function and the duration until the next occurrence of an event. $\Lambda\left(t_{i-1}, t_i\right)$ can be seen as the generalized residual and indicates whether the specified intensity function under-predicts ($\Lambda\left(t_{i-1}, t_i\right) < 1$) or over-predicts ($\Lambda\left(t_{i-1}, t_i\right) > 1$) the number of events at any point in time.

6 Data Description

The data is drawn from BrokerTec, an inter-dealer electronic trading platform of secondary wholesale U.S. treasury bonds that currently has a market share of approximately 60-65% of the active issues. It functions as a limit order book and operates from 7:00 to 17:30 Eastern Standard Time (EST) for the 2, 3, 5, 10, and 30-year treasuries. The paper focuses on the on-the-run notes and bonds, namely 2-year, 5-year notes, and 30-year bond, traded between 7:30 and 17:30 EST. Even though the on-the-run securities represent just a small fraction of all the outstanding treasury securities, they account for about 71% of the activity in the interdealer market (Fabozzi and Fleming, 2000). The three-year note is excluded as the issuance of this security was stopped in 1998. Similarly, Treasury Inflation-Indexed Securities are also excluded because of limited level of trading activities. Our sample period is January 18th, 2005 to December 30th, 2005.

Weekends, holidays, and days with unusually low or no trading activity (due to feed failures) are removed. The time stamp in the dataset is accurate to one thousandth of a second. There are records of simultaneous trades and such trades are differentiated using a uniform distribution between zero and one to ensure the assumption of a simple point process holds. The period of estimation is chosen from7:30 to 17:30 EST as this range of time captures more than 90% of all trading activity. Because of the time gap between each trading day, the data is concatenated together so that 17:30 on Monday is followed by 7:30 on Tuesday in a new time line.

Table 1: Preliminary data analysis

Variables	2-year	5-year	30-year
Mean	276.43	69.2	93.26
Standard Deviation	446.18	113.28	138.76
Skewness	4.28	5.61	4.96
Min	0.10	0.10	0.10
Max	8075.01	3510.89	4015.59
Observations	28644	116163	84651

Table 1 shows we have 28644, 116163, and 84651 observations for the 2-year, 5-year notes, and 30-year bond respectively. The average duration between trades with a price change of 1-tick is 276.43 seconds, 69.2 seconds, and 93.26 seconds respectively. The 5-year note is therefore the most traded security among the three considered. The minimum duration for all securities is 0.1 second while the maximum duration varies between 3510.886 seconds to 4015.596 seconds. Figures 2-4 show the histograms of the price durations for each security, which suggests over-dispersion.

The sizable differences between the means and standard deviations also suggest that the durations are not conditionally exponentially distributed and this is confirmed in Table 2 where a Kolmogorov Smirnov test is reported. This also confirms the finding of Garbade and Lieber (1977) that a homogenous Poisson process is not appropriate for modelling trade arrivals.

Since, the Hawkes model is proposed as a time-deformation model for temporally dependent trades, we need to examine the temporal dependence in the duration between price-changing trades

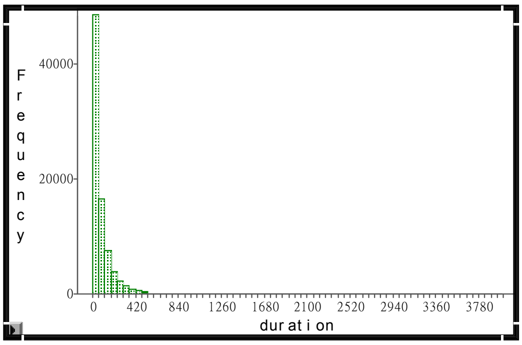

Figure 2: Histogram of the price duration for 2-year notes.

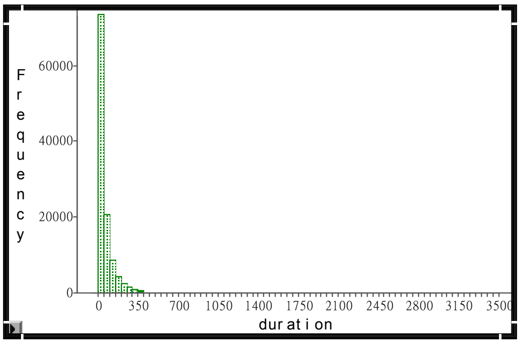

Figure 3: Histogram of the price duration for 5-year notes

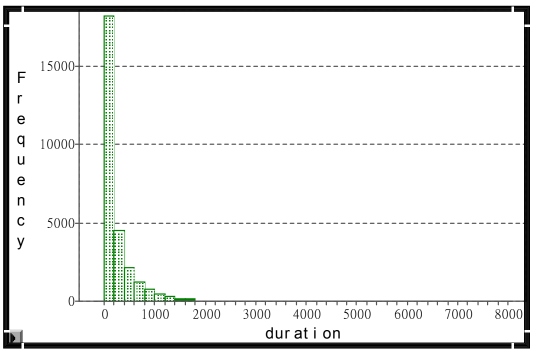

Figure 4: Histogram of the price duration for 30-year notes

Table 2: Kolmogorov-Smirnov test for price duration

Maturity	2-year	5-year	30-year
Kolmogorov-Smirnov test ~Exp			
Asymptotic Statistics	146.86	229.37	211.96
P-value	0.001	0.001	0.001

for potential clustering of trades. The real time autocorrelations in Table 3 are all significantly positive. The Ljung-Box statistic of no autocorrelation is a χ^2_{15} variable with a 5% critical value of 25 and the null of no autocorrelation is therefore easily rejected for all three securities. So, as in Engle and Russell (1997), we find a clear pattern of temporal dependence in price changing trades in the U.S. treasury market.

Table 3: Duration autocorrelation

Duration	2-year	5-year	30-year
Lag 1	0.19	0.26	0.26
Lag 2	0.19	0.25	0.22
Lag 3	0.18	0.24	0.22
Lag 4	0.16	0.23	0.21
Lag 5	0.15	0.22	0.19
Lag 6	0.14	0.20	0.18
Lag 7	0.12	0.18	0.17
Lag 8	0.12	0.19	0.16
Lag 9	0.12	0.18	0.15
Lag 10	0.10	0.18	0.14
Lag 11	0.09	0.17	0.14
Lag 12	0.09	0.17	0.13
Lag 13	0.08	0.16	0.13
Lag 14	0.08	0.14	0.12
Lag 15	0.07	0.13	0.11
L.B (15)	8943.34	9986.76	9910.23

7 Empirical Results

We start our analysis by estimating univariate Hawkes models for each maturity independently as shown in Table 4 , where α is the impact factor of the self-excitation component, β is the decay rate of the self-excitation effect, and μ is the baseline intensity which can be seen as the deterministic component of the non-homogenous Poisson process.

It is not surprising that the 5-year note shows the highest baseline intensity since it is the most heavily traded among the three securities. All three securities exhibit strong autodependence as shown by the relevant estimated α's with half-lives of 5.12, 4.48, and 5.40 minutes for the 2-year, 5-year, and 30-year treasury securities respectively, which confirms the preliminary indication of strong autocorrelation found in Table 3 above. The models for all three securities appear relatively well specified as the residual durations show a mean of 1 and a standard deviation that is close to 1, signalling that all three residuals are Poisson (1) processes with time-transformed durations that are Exponentially (1) distributed. However, the test for independence using Ljung-Box test rejects the null-hypothesis that these residuals are independently distributed suggesting the need for a multi-variate model.

The multivariate Hawkes model is estimated by taking into account the interaction between trading activities at each maturity. Doing so allows us to investigate a much richer picture behind the forces driving the different parts of the yield curve and to investigate the market segmentation hypothesis directly.[10] The model allows us to study the co-dependence or cross excitation of trading

[10]The estimation routine was written in C++ using OPT++ as the optimizing package. The log likelihood function is maximized using quasi-newton non-linear optimization. The C++ algorithm written for the calculation of the log likelihood has been tested against a simulated multivariate Poisson process. Empirical tests of the consistency of estimates have been successfully carried out . All estimated parameters attain convergence against the gradient tolerance of 1E-5 and the standard error is estimated using the inversion of the negative Hessian matrix. All the time

Table 4: Univariate Hawkes estimation

Variables	2-year	5-year	30-year
Panel A. Coefficient Estimates			
α	0.089	0.13	0.10
S.E.	(1.20E-5)	(3.24E-6)	(5.45E-6)
β	0.129	0.154	0.12
S.E.	(2.00E-5)	(4.19E-6)	(7.28E-6)
μ	0.066	0.134	0.104
S.E.	(8.99E-6)	(6.15E-6)	(1.04E-5)
Panel B. Diagnostics			
LogLikelihood	-72408.80	-132731.70	-121999.30
Observations	28644.00	116163.00	84651.00
Mean residual duration	0.9953	1.00	0.997
S.D. residual duration	0.96	0.976	0.956
L.B. test (20 lags)	284.60	106.45	56.14

activity between treasury securities across time alongside the self-excitation that will be reflected to some degree in the autocorrelation patterns reported above. Table 5 shows the estimated parameters for the multivariate Hawkes model as well as the satisfaction of the stationarity condtion.

The columns denote the effect each column security has on the row security and the diagonal shows the self-excitation components. For example, the cross-excitation effect of a 5-year note price changing trade on a 2-year note trade intensity is 0.001746. This suggests that the conditional probability of the trade occurrence of a 2-year note is increased by 0.001746 on top of the baseline 0.0602 and self-excitation 0.08939 effects. This increase in the probability of occurrence of a price changing trade is, however, decaying at the speed of 0.6 units per minute.

The estimated parameters from the multivariate model show self-excitation effects that are very similar to those found in the univariate Hawkes model, but the standard errors of the estimated parameters suggest a significant influence of cross-trade effects in this multivariate framework. However, it is clear that the majority of the trade intensity is *self*-excited for all maturities. For instance, the cross-excitation terms in the 2-year note suggest that price movements are driven by trades occurring in the 5-year and 30-year markets *but* these effects (while significant) are relatively much short lived and the cross impact effects much smaller. In each case, the decay pattern is much slower for the own-market event compared to the cross-market effects. However we can also see that each market is different as the trade intensity is primarily driven by self-excitation effect and the coefficients of all impact and decay rates are significantly different from each-other. This does suggest that the markets are segmented to a degree and hence responding to different information, or there is more than one source of information that is driving the price process for each security. From the time deformation point of view,this also indicates that there are different time scales in each market and that the multivariate Hawkes model serves to coordinate activity between the different maturities. The estimated parameters also highlight that the long end of the yield curve has a significant impact on the shorter end and vice-versa. One can also observe that the all-weather 5-year note is a security that is the least affected by movements in other parts of the yield curve.

In brief, the markets do appear to be segmented to a considerable degree but not totally and there seem to be different 'operational clocks' or time scales at work in the treasury market. We now need to systematically test for the presence of a single market time scale using the likelihood ratio test and carry out various specification tests which we turn to in the next section.

of occurrence of price-changing trades is scaled in terms of minutes for numerical efficiency reasons. The multivariate Hawkes model is also linearised mathematically to improve on computational time. For more details, see Sohrmann and Tham (2006).

Table 5: Multivariate Hawkes estimation

Variables	2-year	5-year	30-year
Panel A. 2-year bond			
α	0.08939	0.001746	0.003673
S.E.	(1.25E-05)	(3.81E-06)	(5.06E-06)
β	0.1296	0.6	0.6397
S.E.	(2.10E-05)	(4.25E-06)	(4.29E-06)
μ	0.0602		
S.E.	(1.02E-05)		
Stationarity Condition $\frac{\alpha}{\beta} < 1$	0.69	0.003	0.006
Panel B. 5-year bond			
α	0.0003386	0.1309	0.0003843
S.E.	(1.08E-05)	(9.47E-06)	(9.61E-06)
β	0.639	0.1549	0.5800
S.E.	(4.21E-06)	(1.23E-05)	(4.22E-06)
μ		0.1341	
S.E.		(1.80E-05)	
Stationarity Condition $\frac{\alpha}{\beta} < 1$	0.003	0.22	0.0006
Panel C. 30-year note			
α	0.007165	0.0007342	0.1003
S.E.	(7.48E-06)	(6.29E-06)	(7.77E-06)
β	0.6392	0.6200	0.1201
S.E.	(4.26E-06)	(4.21E-06)	(9.79E-06)
μ			0.1007
S.E.			(1.67E-05)
Stationarity Condition $\frac{\alpha}{\beta} < 1$	0.011	0.0012	0.84

7.1 Diagnostic Tests and Tests for Time Deformation

As a first diagnostic check we can examine if the estimated generalised residuals defined above which follow from the martingale condition defined above in section (5.2) have a zero mean. A simple plot shows two lines that are indistinquishable given the number of residual observations we have and a formal test of a zero mean yields a p value of one for each residual series, univariate and multivariate at each maturity. The residuals process will show no further structure if they are a homogeneous Poisson processes and we examine this, again as described in Section (5.2) using tests for an Exp(1) distribution in Table 7 below.

A further important issue is to examine the presence of time deformation in the original trading process and the most intuitive way of examining this is simply to plot $\int_0^t \lambda^{dp}\left(t; \mathcal{F}_t^2\right)$ against clock time ,t, and see if it is linear and increases proportionately.

Figures (5), (6), (7), (8), (9) and (10), show price intensities and prices for each maturity across time on 30th June 2005; a day with an FOMC meeting and a 2-year note monthly auction where we can see the Preferred Habitat Theory at work. There was an increase on the base interest rate of 25 basis points announced at 8.30 and a reduction of the size of the auction for 2-year note held between 2.00pm and 4.00pm. The market had expected the base rate to rise by 25 basis points according to a market survey from Informa Global Markets' monthly report and it is interesting to see that prices fell immediately and then rebounded to a higher level after the annoucement of the base rate rise. The inital downward jump is congruent with the increase in the base rate and the upward movement afterwards might suggest that expectation hypothesis is at work reflecting the market sentiment that further interest rate rises were unlikely. However, prices become more volatile during the auction period where the bid-to-cover ratio, a measure of demand, stood at 2.23, above the 2.10 average of the previous ten 2-year note sales. The government awarded $20 billion of two-year notes at a yield of 3.65%, reducing the size from the previous auction of $22 billion. Indirect bidders, who include foreign central banks and large institutional investors, bought only 28%, down from 37% in the previous auction, in May. The price of the 2-year note stablised at a lower level after 4pm, suggesting a possible shift of investor interest to the 5-year note and 30-year bond where prices continued to rise after the 2 year auction supporting the presence of Preferred Habitat behaviour in the treasury market.

The figures also indicate, taking account of the different scales, the relative impact that the two year auction had on trading activity at the two other maturities. There is clearly a much greater time deformation impact on the two year note itself as the steepness of the time deformation functon is much greater during the auction which also suggests that the volatility is driven by the reduction in supply which has spilled over to the 5-year and 30-year securities as investors substitute 2-year notes with the 5-year and 30-year bond. The sudden increase in the slope of the integrated intensity, our measurement of trading activty, from 2-4pm also coincides with the period of high volatility. These figures strengthen our claim that trade intensity provides a suitable operational clock.

In order to test the time-deformation hypothesis more formally we can also carry out explicit parameter restriction tests on the estimated multivariate Hawkes model. Using Likelihood ratio (LR) tests we can test the following four hypotheses

1. All three maturities have the same impact effects through the restriction that $H_0 : \alpha_{11} = \alpha_{22} = \alpha_{33}$, $\alpha_{12} = \alpha_{13} = \alpha_{21} = \alpha_{23} = \alpha_{31} = \alpha_{32}$
2. All three maturities have the same decay characteristics through the restriction that: $H_0 : \beta_{11} = \beta_{22} = \beta_{33}$, $\beta_{12} = \beta_{13} = \beta_{21} = \beta_{23} = \beta_{31} = \beta_{32}$
3. That each process a homogeneous Poisson Process: $H_0 : \alpha_{11} = \alpha_{22} = \alpha_{33} = \alpha_{12} = \alpha_{13} = \alpha_{21} = \alpha_{23} = \alpha_{31} = \alpha_{32} = 0$
4. That the Mulivariate Hawkes model does not dominate the univariate Hawkes models: $H_0 : \alpha_{12} = \alpha_{13} = \alpha_{21} = \alpha_{23} = \alpha_{31} = \alpha_{32} = 0$

Table 6 reports the relevant Likelihood Ratio Statistics for each hypothesis.

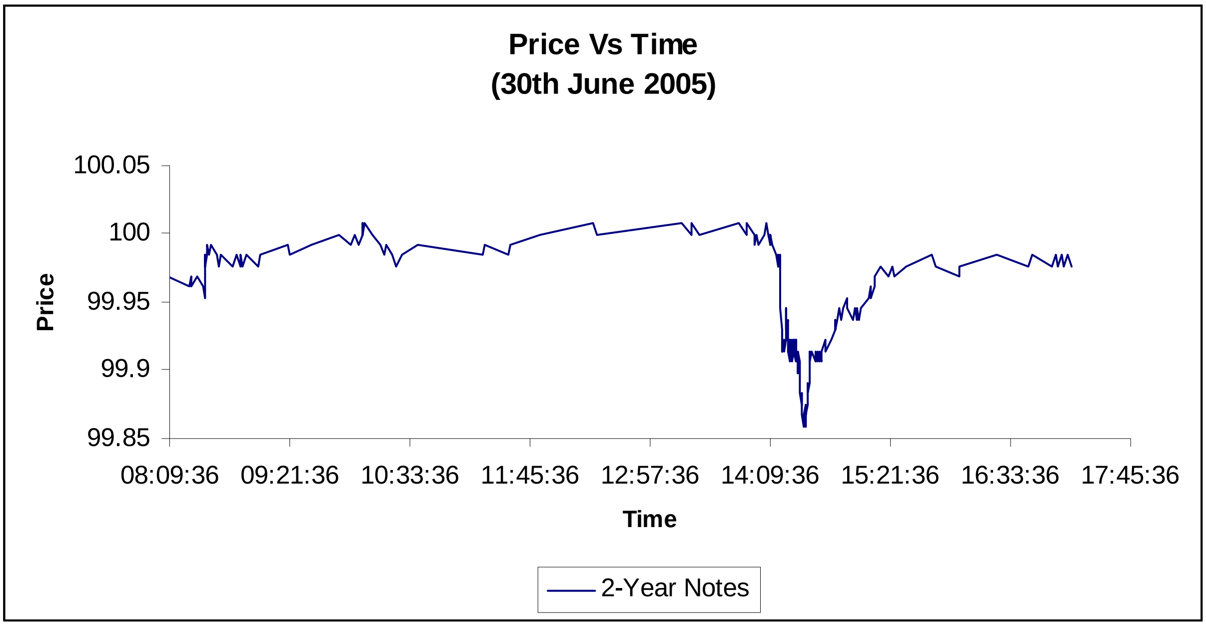

Figure 5: Price of 2-year note vs clock time on 30th June 2005

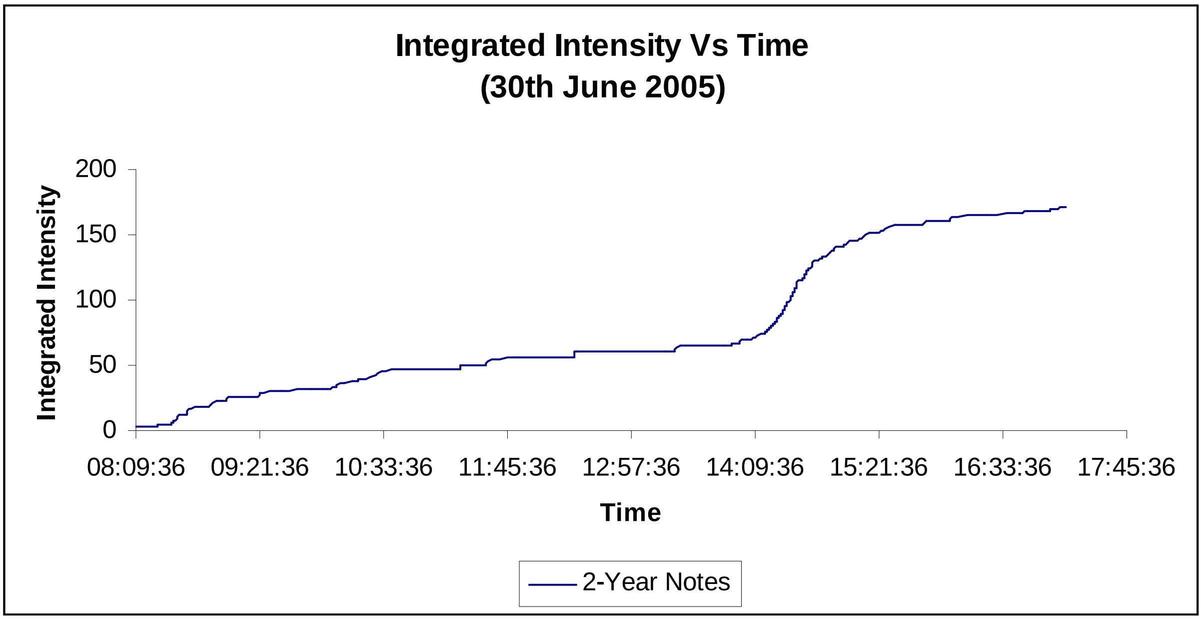

Figure 6: Integrated Intensity of 2-year Notes vs time on 30th June 2005

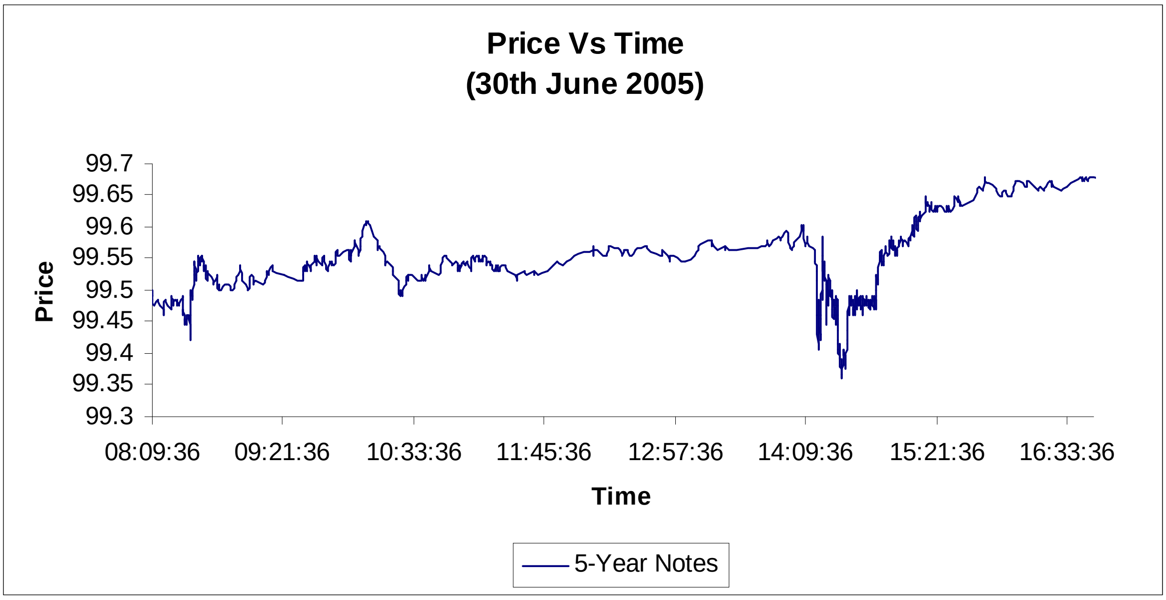

Figure 7: Price of 5-year Notes vs time on 30th June 2005

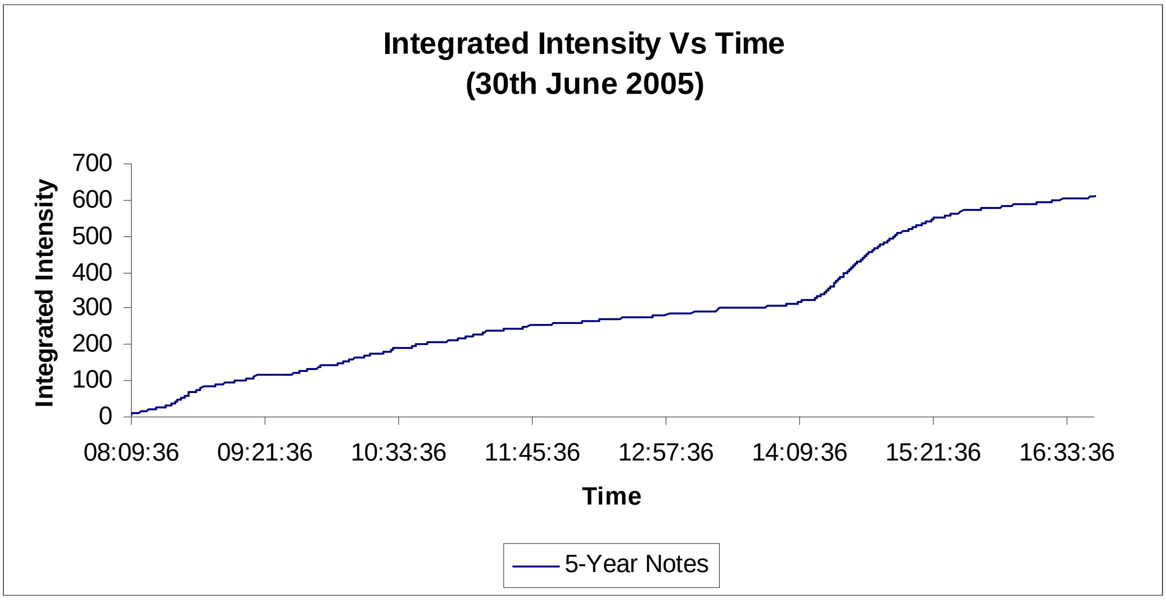

Figure 8: Integrated Intensity of 5-year Notes Vs time on 30th June 2005

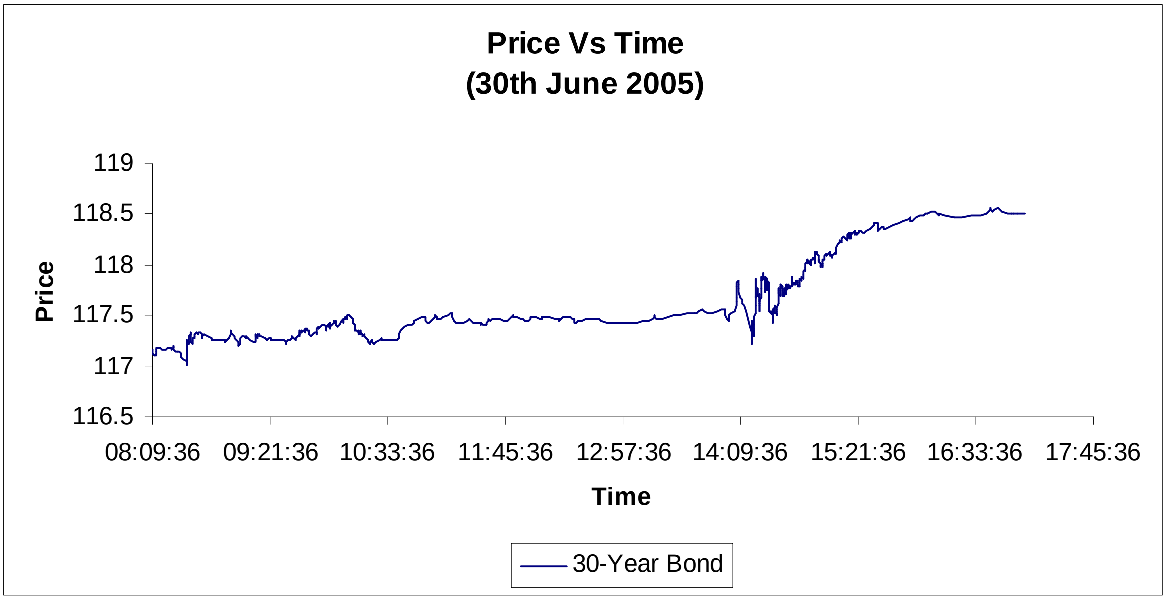

Figure 9: Price of 30-year Bond Vs time on 30th June 2005

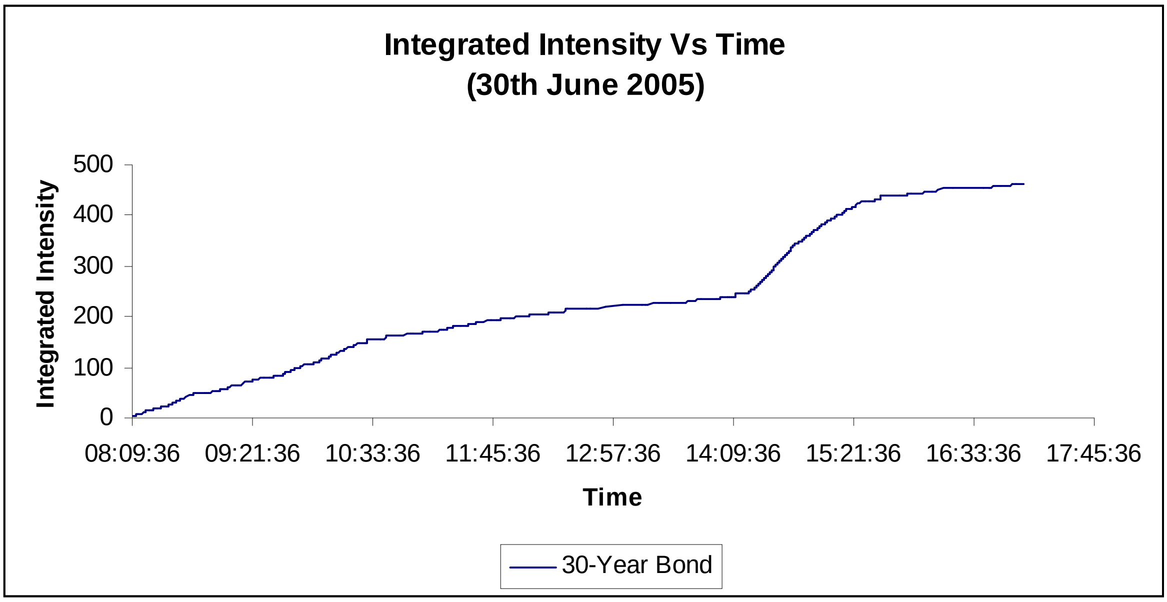

Figure 10: Integrated Intensity of 30-year Bond Vs time on 30th June 2005

Table 6: Diagnostic and Likelihood ratio tests

Variables	2-year	5-year	30-year
Panel A. Diagnostics			
Observations	28644	116163	84651
Mean residual duration	1.00	1.00	1.00
S.D. residuals duration	0.97	0.98	0.96
L.B. test	201.06	78.55	28.87
Panel B. Likelihood Ratio Tests			
Total Log Likelihood of Multivariate Hawkes			-282717
$H_0 : \alpha_{11} = \alpha_{22} = \alpha_{33}, \alpha_{12} = \alpha_{13} = \alpha_{21} = \alpha_{23} = \alpha_{31} = \alpha_{32}$			6868
$H_0 : \beta_{11} = \beta_{22} = \beta_{33}, \beta_{12} = \beta_{13} = \beta_{21} = \beta_{23} = \beta_{31} = \beta_{32}$			112
H_0 : Joint Homogeneous Poisson Process			92728
H_0 : Independent Univariate Hawkes Models			22

All the Likelihood Ratio statistics shown on the right hand side of Panel B reject the associated null hypothesis, in favour of our time deformation model. The second row of Panel B shows that the null of the same impact effects for each maturity is strongly rejected. Similarly the third row indicates that the null hypothesis that the decay rate are the same is also rejected. The null of a homogenous Poisson process is also strongly rejected with a Likelihood Ratio of 92728 indicating the importance of the clustering effects of trade arrival and information flow. Finally when the restrictions are imposed to allow the multivariate Hawkes model to become three independent univariate Hawkes models at each maturity the null is again rejected indicating the importance of the cross market effects but only with a much weaker significance level on this occasion. However, Panel A also shows that again we have residual autocorrelation indicating that a richer dynamic model is called for than we have been able to capture in our Multivariate Hawkes model.

This impression is also supported with the results presented in Table 7 where we report formal tests of the model specification by applying the Engle Russell over-dispersion test for the generalized residuals which has a limiting Normal distribution along with a simple χ^2 test for mean and standard deviation equivalence. The 5% critical value for the Engle Russell test is 1.645 and the null of an $Exp(1)$ distribution appears to be rejected for all maturities for both the univariate and multivariate generalized residual durations. The simple χ^2 test indicates acceptance however and so we report in Panel C a specific Information Matrix based test for the $Exp(1)$ distribution given by Acosta and Rojas (2007) which clearly indicates acceptance of the Null of correct specification for both univariate and multivariate Hawkes models .

Table 7: Overdispersion tests

Maturity	2-year	5-year	30-year
Panel A.	*Engle and Russell excess dispersion test*		
	Univariate Hawkes		
Statistics	-2.37	-4.77	-8.06
	Multivariate Hawkes		
Statistics	-3.54	-5.76	-8.72
	Panel B. χ^2 *test*		
D	0.94	0.95	0.92
Statistics	27033.32	110612.80	77694.36
P-value	1.00	1.00	1.00
Panel C.	*Information Matrix test*		
		Univariate Hawkes	
Statistics	3938.81	14164.28	10726.34
P-value	1.00	1.00	1.00
		Multivariate Hawkes	
Statistics	3951.78	14167.1	10715.41
P-value	1.00	1.00	1.00

In the next section we turn to consider two robustness checks of the time deformation model by comparing the structural intensity-based volatility with the model-free realized volatility and then examining the distribution of bond returns standardized by the estimated instantaneous volatility.

7.2 Robustness Checks

7.2.1 Realized Vs Intensity-based Volatility

The daily realized volatility is computed simply as the sum of the intra-day 30-minute returns. The intensity-based volatility is calculated using equations (3)-(5) above so that the daily intensity based volatility is the integrated instantaneous volatility across everyday, using the integrated intensity multiplied by the price changes at different points in time. Figures (11), (12) and (13) show the high co-dependence between the two volatilities supporting the hypothesis of using trade-intensity as a directing process.

Table 8: Realized Vs Intensity-based Volatility

	2year	5year	30year
Correlation	0.8569	0.7954	0.4943

Table 8 indicates a relatively high degree of linear dependence between the two series at least for the 2 year and 5 year bonds. The realized volatility estimates are based on 20 daily observations whereas the intensity based measure is based on approximately 100. The difference between the structural Hawkes-based and the model free realized volatility estimates are interesting and potentially important. The Hawkes model allows for further structural factors to be included in the model but as it stand it can be used to forecast without any assumption of a linear autoregressive information arrival rate which is implicit in time series models frequently used to predict realized volatility.

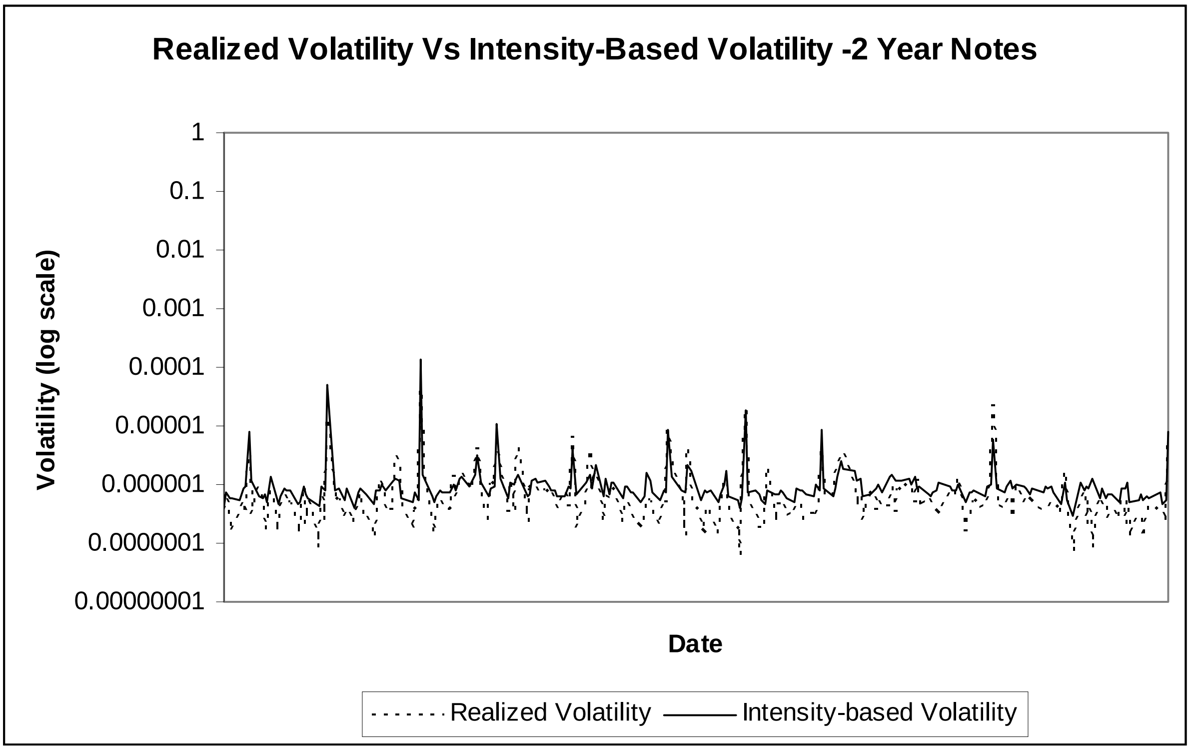

Figure 11: Realized Volatility Vs Intensity-Based Volatility - 2-year Note

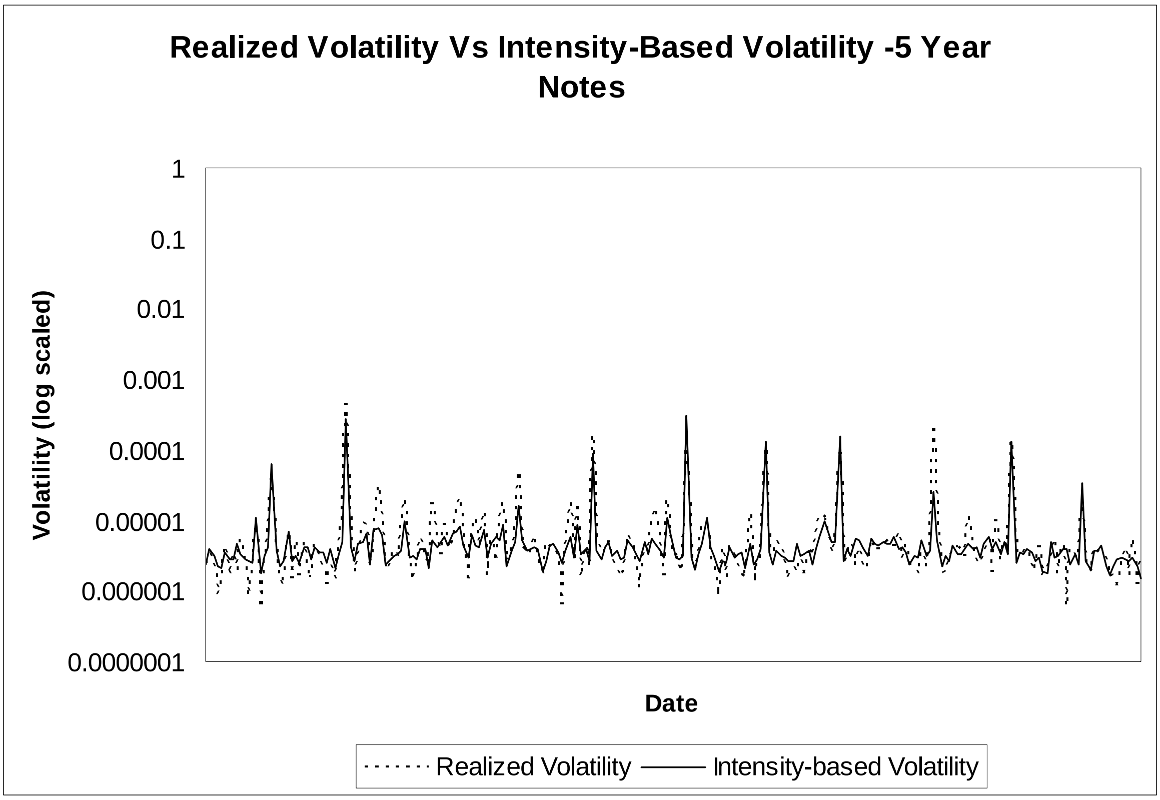

Figure 12: Realized Volatility Vs Intensity-Based Volatility - 5-year Note

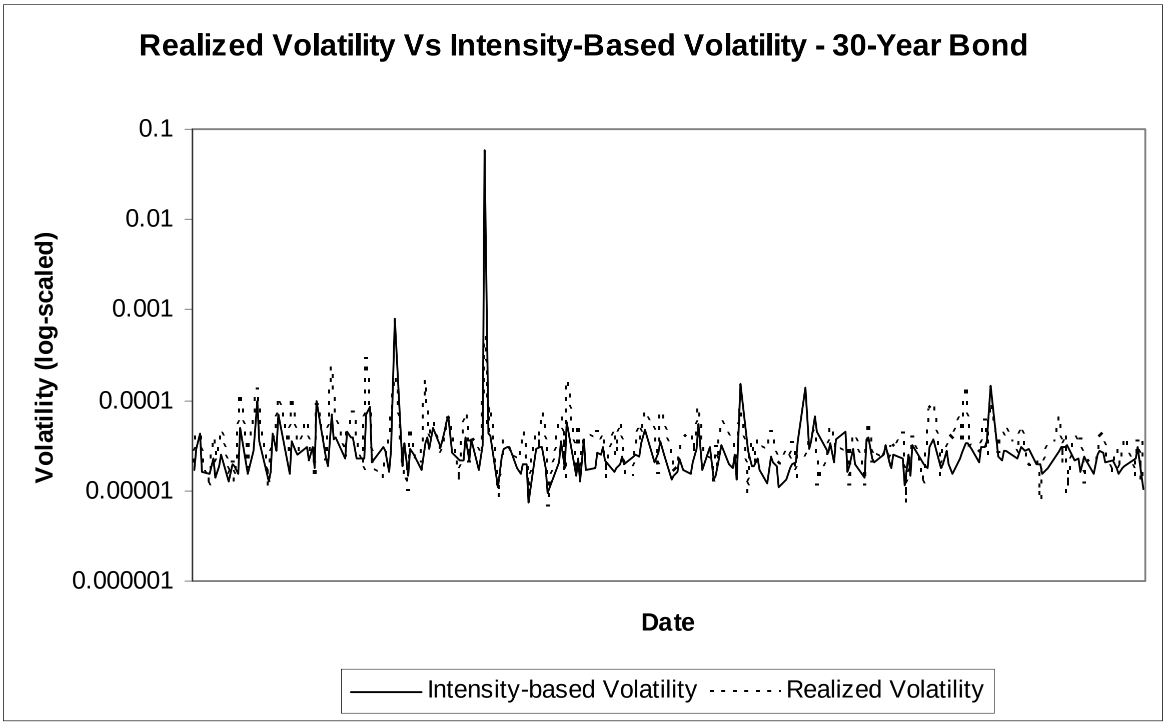

Figure 13: Realized Volatility Vs Intensity-Based Volatility - 30-year Bond

7.2.2 Gaussianity tests

Following Clark(1973), we can also examine the adequacy of our time deformation model by conditioning the observed bond returns on the estimated instantaneous volatility from the Hawkes model. We do this, as does Clark, by standardizing the bond returns with the estimated instantaneous standard deviation. Table 9 provides in Panel A the p values of various normality tests on the normalized bond returns and we can observe clear rejection of Gaussianity with Leptokurtosis strongly particularly in the 2 year and 5 year bonds. Panel B reports the same test statistics for the normalized bond returns and consistent with the theoretical basis the use of our intensity-based measure of volatility through time deformation has returned a conditional Gaussian distribution in each case although there is relatively little effect on the 30 year returns.

8 Conclusion

The paper approaches the modeling of the yield curve from a stochastic volatility perspective based on time deformation. The way in which we model time deformation is new and differs from alternatives that currently exist in the literature and is based on market microstructure theory of the impact of information flow on a market. We model the stochastic volatility process by modeling the instantaneous volatility as a function of price intensity. One contribution of the paper therefore lies with the introduction of a new transaction level approach to the econometric modelling of Yield Curve stochastic volatility in a multivariate framework exploiting intensity-based point processes previously used by Bowsher (2003), Hall and Haustch (2003). We find that the individual yields of U.S. treasury notes and bonds appear to be driven by different "operational" clocks as suggested by the market segmentation theory of the Term Structure but these are related to each other through a multivariate Hawkes model which effectively coordinates activity along the yield curve. The results offer some support to the Market Segmentation or Preferred Habitat models as the univariate Hawkes models we have found at each maturity are statistically significantly different from each other and the major impact on each maturity is activity at that maturity. However there are flows between the different maturities that die away relatively quickly which indicate that the markets are not com-

Table 9: Normality test for daily returns and time-transformed returns

Panel A. Daily Log-Returns			
Variables	2-year	5-year	30-year
Kurtosis	13.19	13.4	**2.45**
Skewness	1.99	1.71	**-0.28**
Shapiro-Wilk	0.00	0.00	0.03
Kolmogorov-Smirnov	0.01	0.01	**0.15**
Cramer-von Mises	0.01	0.01	**0.07**
Anderson-Darling	0.01	0.01	0.05
Bootstrapped Anderson-Darling	0.00	0.00	**0.078**
Bootstrapped Jarque-Bera	0.00	0.00	**0.28**
Panel B. Daily Transformed Returns $\frac{Log-Returns}{Intensity-based\ Volatity}$			
Variables	2-year	5-year	30-year
Kurtosis	**2.77**	**2.47**	**2.23**
Skewness	**-0.02**	**0.03**	**-0.10**
Shapiro-Wilk	**0.65**	**0.37**	0.03
Kolmogorov-Smirnov	**0.15**	**0.15**	**0.13**
Cramer-von Mises	**0.21**	**0.25**	**0.07**
Anderson-Darling	**0.25**	**0.25**	0.05
Bootstrapped Anderson-Darling	**0.14**	**0.18**	**0.079**
Bootstrapped Jarque-Bera	**0.46**	**0.385**	**0.27**

pletely segmented. Diagnostic tests show that the point process models are relatively well specified and a robustness comparison with realized volatility indicates the close relationship between the two estimators of integrated volatility but also some differences between the structural intensity model and the model free realized volatility. We have also shown that bond returns standardized by the instantaneous volatility estimated from our Hawkes model are Gaussian which is consistent with the theory of time deformation for security prices quite generally.

The model can be easily extended by the inclusion of liquidity factors, bid-ask spread and depth as further inputs into the operational or market time scale. The use of a dataset containing the shorter end of the yield curve to study the trade intensity dependence among the different yield-to-maturity would also be valuable as it provides a better understanding to the 'missing' factors or driving force behind the dynamics of the yield curve. Leverage effects can also be included in our model by modelling the conditional intensity of price-changing trade as a function of the backward recurrence time of the buy and sell trade to capture the different self and cross-excitations of trades.

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