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Overview of the Proof

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The Proof of the Bloch-Kato Conjecture

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Let k be a field and ℓ a prime number different from char(k). The *Milnor K-theory* of k is defined to be the quotient $K_*^M(k)$ of the tensor algebra of the abelian group $k^\times$ by the ideal generated by elements of the form $\{a, 1-a\}$, $a \in k - \{0, 1\}$. The Kummer isomorphism $k^\times/\ell \xrightarrow{\sim} H^1_{\text{ét}}(k, \mu_\ell)$ exends to a (graded) *norm residue homomorphism*

$$k_*^M(k)/\ell \longrightarrow \oplus H^n_{\text{ét}}(k, \mu_\ell^{\otimes n}), \tag{0.1}$$

and Milnor asked in [5] whether this map is an isomorphism for $\ell = 2$. This is true, as was proven by Voevodsky in [MC/2].

The same question for ℓ odd was first formulated by Kazuya Kato in [2, p.608] and is known as the *Bloch-Kato Conjecture*. A proof of this was announced in 1998 by Voevodsky, assuming the existence of what we now call a *Rost variety* (see Lecture 1). Rost produced such a variety in 1998 [R-CL], and the proof that (0.1) is an isomorphism appeared in the 2003 preprint [MC/l] — modulo three assertions. One of these, that the Rost variety has certain properties, was established in [9]. The other two assertions, concerning the motivic cohomology groups $H^{**}(X, \mathbb{Z}/\ell)$, are still unknown. In these lectures we shall prove the Bloch-Kato conjecture by establishing parallel assertions concerning the motivic cohomology groups $H^{**}(X, \mathbb{Z})$.

1 Lecture 1: Overview of the Proof

We begin with a series of reductions.

Lemma 1.1. (Voevodsky [12, 5.2]). *If $K_n^M(k)/\ell \to H^n_{\acute{e}t}(k, \mu_\ell^{\otimes n})$ is an isomorphism for all fields of characteristic* 0*, then it is an isomorphism for all fields of characteristic* $\neq \ell$.

Proof. By a standard transfer argument, we may assume that k is perfect. Let R be the ring of Witt vectors over k and K its field of fractions. Then the specialization maps are compatible with the norm residue maps in the sense that

$$\begin{array}{ccc} K_n^M(K)/\ell & \xrightarrow{\simeq} & H^n_{\text{ét}}(K, \mu_\ell^{\otimes n}) \\ \downarrow & & \downarrow \\ K_n^M(k)/\ell & \longrightarrow & H^n_{\text{ét}}(k, \mu_\ell^{\otimes n}) \end{array}$$

commutes. Both specialization maps are known to be split surjections; the result follows. $\square$

Now there is a chain complex $\mathbb{Z}(i)$ of étale sheaves and an isomorphism $H^n_{\text{ét}}(X, \mathbb{Z}/\ell(i)) \cong H^n_{\text{ét}}(X, \mu_\ell^{\otimes i})$ for all $n, i \geq 0$; see [4, 10.2] or [MC/2, 6.1]. We have a diagram

$$\begin{array}{ccccccc} K_n^M(k) & \xrightarrow{\ell} & K_n^M(k) & \longrightarrow & K_n^M(k)/\ell & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^n_{\text{ét}}(k, \mathbb{Z}(n)) & \xrightarrow{\ell} & H^n_{\text{ét}}(k, \mathbb{Z}(n)) & \longrightarrow & H^n_{\text{ét}}(k, \mu_\ell^{\otimes n}) & \longrightarrow & H^{n+1}_{\text{ét}}(k, \mathbb{Z}(n)) \end{array}$$

This motivates the following result, whose proof we omit; compare with [11, 7.1].

Theorem 1.2. (Voevodsky [MC/2, 6.10]). *Suppose that $H^{n+1,n}_{\acute{e}t}(k, \mathbb{Z}(n)) = 0$ for every field k of characteristic* 0*. Then $K_n^M(k)/\ell \cong H^n_{\acute{e}t}(k, \mu_\ell^{\otimes n})$.*

We proceed by induction on n, assuming $K_{n-1}^M(k)/\ell \cong H^{n-1}_{\text{ét}}(k, \mu_\ell^{\otimes n})$ for all k.

Proposition 1.3. (Voevodsky [MC/2, p.97]) *Suppose that for every field k and every symbol $\underline{a} = \{a_1, \dots, a_n\}$ in $K_n^M(k)/\ell$ there is a field extension K so that $\underline{a}$ vanishes in $K_n^M(K)/\ell$ and the map $H^{n+1}_{\acute{e}t}(k, \mathbb{Z}(n)) \to H^{n+1}_{\acute{e}t}(K, \mathbb{Z}(n))$ is an injection. Then $H^{n+1}_{\acute{e}t}(k, \mathbb{Z}(n)) = 0$, and hence $K_n^M(k)/\ell \cong H^n_{\acute{e}t}(k, \mu_\ell^{\otimes n})$, for all k.*

Proof. Fix k. By a transfinite process, we can find an extension field L which has $K_n^M(L)/\ell = 0$, L has no prime-to-ℓ extensions, and such that $H^{n+1}_{\text{ét}}(k,\mathbb{Z}(n))$ embeds in $H^{n+1}_{\text{ét}}(L,\mathbb{Z}(n))$. But for such an L we have $H^{n+1}_{\text{ét}}(L,\mathbb{Z}_{(\ell)}(n)) = 0$ by [MC/2, 5.9 and 6.8]. $\square$

Lemma 1.4. (Voevodsky [MC/l, 6.4]) *If $\{a_1,\dots,a_n\}$ is a nonzero symbol in $K_n^M(k)/\ell$, its image is nonzero in $H^n_{\acute{e}t}(k,\mu_\ell^{\otimes n})$.*

Proof. By a standard transfer argument, we may assume k has no prime-to-ℓ extensions. For $E = k(\gamma)$, $\gamma = \sqrt[\ell]{a_n}$, we have a diagram

$$\begin{array}{ccccccc}
K^M_{n-1}(E)/\ell & \xrightarrow{\text{norm}} & K^M_{n-1}(k)/\ell & \xrightarrow{\cup a_n} & K^M_n(k)/\ell & & \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow & & \\
H^{n-1}_{\text{ét}}(E,\mathbb{Z}/\ell) & \xrightarrow{\text{norm}} & H^{n-1}_{\text{ét}}(k,\mathbb{Z}/\ell) & \xrightarrow{\cup[a_n]} & H^n_{\text{ét}}(k,\mathbb{Z}/\ell) & \longrightarrow & H^n_{\text{ét}}(E,\mathbb{Z}/\ell)
\end{array}$$

in which the vertical maps are isomorphisms by induction and the bottom row is exact by [MC/2, 5.2]. Since $\{a_1,\dots,a_n\} \neq 0$, if $\{a_1,\dots,a_n\}$ vanishes in $H^n_{\text{ét}}(k,\mathbb{Z}/\ell)$ then $\{a_1,\dots,a_{n-1}\}$ is the norm of some $s \in K^M_{n-1}(E)/\ell$. But then $\{a_1,\dots,a_n\}$ is the norm of $\{s,a_n\} = \{s,\gamma\}^\ell = 0$ and hence is zero.

We say that "X splits $\underline{a}$" if X is a smooth irreducible projective variety over k such that $\underline{a} = 0$ in $K_n^M(k(X))/\ell$. We write $\check{C}(X)$ for the simplicial scheme $\check{C}(X)_n = X^{n+1}$

$$X \leftleftarrows X\times X \mathrel{\substack{\longleftarrow\\[-1ex]\longleftarrow\\[-1ex]\longleftarrow}} X^3 \mathrel{\substack{\longleftarrow\\[-1ex]\longleftarrow\\[-1ex]\longleftarrow\\[-1ex]\longleftarrow}} X^4 \ \cdots\ ,$$

whose face maps are given by projections. By [MC/2, 7.3], $H^p_{\text{ét}}(k,\mathbb{Z}(q)) \xrightarrow{\sim} H^p_{\text{ét}}(\check{C}(X),\mathbb{Z}(q))$ is an isomorphism for all p and q. By [MC/2, 6.9(2)], the motivic $H^p_{\text{nis}}(\check{C}(X),\mathbb{Z}/\ell(q))$ is isomorphic to $H^p_{\text{ét}}(\check{C}(X),\mathbb{Z}/\ell(q))$ for all p, q with $p-1 \le q \le n-1$. $\square$

Lemma 1.5. (Voevodsky [MC/l, 6.5]) *If X splits a nonzero $\underline{a} \in K_n^M(k)/\ell$, then there is a nonzero δ in $H^n(\check{C}(X),\mathbb{Z}/\ell(n-1))$.*

Proof. By induction, the Bloch-Kato conjecture implies (see [11]) that the Leray spectral sequence for $X_{\text{ét}} \to X_{\text{nis}}$ degenerates to yield the exact sequence for $A = \mathbb{Z}/\ell(n-1)$:

$$\begin{array}{ccccccc} & \delta & \mapsto & \underline{a} & \mapsto & 0 \\ 0 \longrightarrow & H^n_{\text{nis}}(\check{C}(X), A) & \longrightarrow & H^n_{\text{ét}}(\check{C}(X), A) & \longrightarrow & H^0_{\text{nis}}(\check{C}(X), \mathcal{H}^n) \\ & & & \cong\uparrow & & \downarrow\text{into} \\ & & & H^n_{\text{ét}}(k, A) & \longrightarrow & H^n_{\text{ét}}(k(X), A). \end{array}$$

Here $\mathcal{H}^n$ is the sheaf associated to $H^n_{\text{ét}}(-, A)$. Now for any simplicial scheme $X_\bullet$, $H^0(X_\bullet, \mathcal{H}^n)$ embeds in $H^0(X_0, \mathcal{H}^n)$. This particular $\mathcal{H}^n$ is a homotopy invariant Nisnevich sheaf with transfers by [4, 6.17 and 22.3], so $H^0(X, \mathcal{H}^n)$ embeds in $H^n_{\text{ét}}(k(X), A)$ by [4, 11.1]. Via a diagram chase, $\underline{a}$ lifts to a nonzero δ in $H^n_{\text{nis}}(\check{C}(X), A)$. □

We will show in Lecture 6 that from the nonzero δ of Lemma 1.5 we can construct something we call a "Rost motive" (this will be defined in 3.4). For this we will need to start with a *Rost variety* X, which is defined in 3.1 and is a variety splitting $\underline{a}$. These varieties were first constructed by Markus Rost in [R-CL]; a proof that they have the defining properties of a Rost variety is published in [9].

When $\ell = 2$, the Rost motive is actually the same as the Rost variety, consided as a motive. Moreover, in that case the Rost variety has a natural interpretation in terms of quadratic forms. Although we do not consider the case $\ell = 2$ in these lectures, Voevodsky's proof in [MC/2] that the norm residue map $K_n^M(k)/2 \to H^n_{\text{ét}}(k, \mathbb{Z}/2)$ is an isomorphism (the "Milnor Conjecture") follows the general lines of our Lectures 1–3.

The goal of Lecture 3 is to use the Rost motive to show that the map $H^{n+1}_{\text{ét}}(k, \mathbb{Z}(n)) \to H^{n+1}_{\text{ét}}(k(X), \mathbb{Z}(n))$ is an injection, which we saw in Proposition 1.3 will imply the Bloch-Kato conjecture. For this, we need several cohomology operations – and that will be the topic of Lecture 2.