The Abdus Salam International Centre for Theoretical Physics

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IAEA International Atomic Energy Agency

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International Workshop on QCD at Cosmic Energies III

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Lecture Notes

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Some physics and mathematics of the parton evolution

Yuri Dokshitzer

Paris–Jussieu & St. Petersburg

Trieste, QCD Cosmic, 1.06 2007

QCD made simple (?)

plan

We are witnessing explosive progress in analytical and numerical methods and techniques for deriving sophisticated high accuracy pQCD predictions, prompted to a large extent by the LHC needs.

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- Ambitious programme

trigger

S. Moch, J.A.M. Vermaseren and A. Vogt
[results March 2004 – 2006 and counting]

The Three-Loop Splitting Functions in QCD:

The Non-Singlet Case	[03.04]
The Singlet Case	[04.04]

...

Higher-Order Corrections in Threshold Resummmation	[06.05]
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Three-Loop Results for Quark and Gluon Form Factors	[08.05]
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A. Mitov, S. Moch, A. Vogt
Next-to-Next-to-Leading Order Evolution of Non-Singlet Fragmentation Functions [04.06]

3rd loop non-singlet a.d.

$$P^{(2)+}_{\mathrm{ns}}(x) = 16 C_A C_F n_f \Bigg(\frac{1}{6} p_{\mathrm{qq}}(x) \Bigg[\frac{10}{3}\zeta_2 - \frac{209}{36} - 9\zeta_3 - \frac{167}{18}\mathrm{H}_0 + 2\mathrm{H}_0\zeta_2 - 7\mathrm{H}_0$$

$$+3\mathrm{H}_{1,0,0} - \mathrm{H}_3 \Bigg] + \frac{1}{3} p_{\mathrm{qq}}(-x) \Bigg[\frac{3}{2}\zeta_3 - \frac{5}{3}\zeta_2 - \mathrm{H}_{-2,0} - 2\mathrm{H}_{-1}\zeta_2 - \frac{10}{3}\mathrm{H}_{-1,0} - \mathrm{H}_{-}$$

$$+2\mathrm{H}_{-1,2} + \frac{1}{2}\mathrm{H}_0\zeta_2 + \frac{5}{3}\mathrm{H}_{0,0} + \mathrm{H}_{0,0,0} - \mathrm{H}_3 \Bigg] + (1-x)\Bigg[\frac{1}{6}\zeta_2 - \frac{257}{54} - \frac{43}{18}\mathrm{H}_0 -$$

$$-(1+x)\Bigg[\frac{2}{3}\mathrm{H}_{-1,0} + \frac{1}{2}\mathrm{H}_2 \Bigg] + \frac{1}{3}\zeta_2 + \mathrm{H}_0 + \frac{1}{6}\mathrm{H}_{0,0} + \delta(1-x)\Bigg[\frac{5}{4} - \frac{167}{54}\zeta_2 + \frac{1}{20}\zeta_2$$

$$+16\, C_A {C_F}^2 \Bigg(p_{\mathrm{qq}}(x) \Bigg[\frac{5}{6}\zeta_3 - \frac{69}{20}\zeta_2{}^2 - \mathrm{H}_{-3,0} - 3\mathrm{H}_{-2}\zeta_2 - 14\mathrm{H}_{-2,-1,0} + 3\mathrm{H}_{-2,0}$$

$$-4\mathrm{H}_{-2,2} - \frac{151}{48}\mathrm{H}_0 + \frac{41}{12}\mathrm{H}_0\zeta_2 - \frac{17}{2}\mathrm{H}_0\zeta_3 - \frac{13}{4}\mathrm{H}_{0,0} - 4\mathrm{H}_{0,0}\zeta_2 - \frac{23}{12}\mathrm{H}_{0,0,0} + 5\mathrm{H}$$

$$-24\mathrm{H}_1\zeta_3 - 16\mathrm{H}_{1,-2,0} + \frac{67}{9}\mathrm{H}_{1,0} - 2\mathrm{H}_{1,0}\zeta_2 + \frac{31}{3}\mathrm{H}_{1,0,0} + 11\mathrm{H}_{1,0,0,0} + 8\mathrm{H}_{1,1,0,0}$$

3rd loop, more

$$+\frac{67}{9}\mathrm{H}_2 - 2\mathrm{H}_2\zeta_2 + \frac{11}{3}\mathrm{H}_{2,0} + 5\mathrm{H}_{2,0,0} + \mathrm{H}_{3,0}\Big] + p_{\mathrm{qq}}(-x)\Big[\frac{1}{4}\zeta_2{}^2 - \frac{67}{9}\zeta_2 + \frac{31}{4}\zeta$$

$$-32\mathrm{H}_{-2}\zeta_2 - 4\mathrm{H}_{-2,-1,0} - \frac{31}{6}\mathrm{H}_{-2,0} + 21\mathrm{H}_{-2,0,0} + 30\mathrm{H}_{-2,2} - \frac{31}{3}\mathrm{H}_{-1}\zeta_2 - 42\mathrm{H}$$

$$-4\mathrm{H}_{-1,-2,0} + 56\mathrm{H}_{-1,-1}\zeta_2 - 36\mathrm{H}_{-1,-1,0,0} - 56\mathrm{H}_{-1,-1,2} - \frac{134}{9}\mathrm{H}_{-1,0} - 42\mathrm{H}_{-1}$$

$$+32\mathrm{H}_{-1,3} - \frac{31}{6}\mathrm{H}_{-1,0,0} + 17\mathrm{H}_{-1,0,0,0} + \frac{31}{3}\mathrm{H}_{-1,2} + 2\mathrm{H}_{-1,2,0} + \frac{13}{12}\mathrm{H}_0\zeta_2 + \frac{29}{2}\mathrm{H}$$

$$+13\mathrm{H}_{0,0}\zeta_2 + \frac{89}{12}\mathrm{H}_{0,0,0} - 5\mathrm{H}_{0,0,0,0} - 7\mathrm{H}_2\zeta_2 - \frac{31}{6}\mathrm{H}_3 - 10\mathrm{H}_4\Big] + (1-x)\Big[\frac{133}{36} +$$

$$-\frac{167}{4}\zeta_3 - 2\mathrm{H}_0\zeta_3 - 2\mathrm{H}_{-3,0} + \mathrm{H}_{-2}\zeta_2 + 2\mathrm{H}_{-2,-1,0} - 3\mathrm{H}_{-2,0,0} + \frac{77}{4}\mathrm{H}_{0,0,0} - \frac{20}{6}$$

$$+4\mathrm{H}_{1,0,0} + \frac{14}{3}\mathrm{H}_{1,0}\Big] + (1+x)\Big[\frac{43}{2}\zeta_2 - 3\zeta_2{}^2 + \frac{25}{2}\mathrm{H}_{-2,0} - 31\mathrm{H}_{-1}\zeta_2 - 14\mathrm{H}_{-1,-}$$

$$+24\mathrm{H}_{-1,2} + 23\mathrm{H}_{-1,0,0} + \frac{55}{2}\mathrm{H}_0\zeta_2 + 5\mathrm{H}_{0,0}\zeta_2 + \frac{1457}{48}\mathrm{H}_0 - \frac{1025}{36}\mathrm{H}_{0,0} - \frac{155}{6}\mathrm{H}_2$$

3rd loop, and more

$$+2\mathrm{H}_{2,0,0} - 3\mathrm{H}_{4}\Big] - 5\zeta_2 - \frac{1}{2}{\zeta_2}^2 + 50\zeta_3 - 2\mathrm{H}_{-3,0} - 7\mathrm{H}_{-2,0} - \mathrm{H}_0\zeta_3 - \frac{37}{2}\mathrm{H}_0\zeta_2$$

$$-2\mathrm{H}_{0,0}\zeta_2 + \frac{185}{6}\mathrm{H}_{0,0} - 22\mathrm{H}_{0,0,0} - 4\mathrm{H}_{0,0,0,0} + \frac{28}{3}\mathrm{H}_2 + 6\mathrm{H}_3 + \delta(1-x)\Big[\frac{151}{64} +$$

$$-\frac{247}{60}{\zeta_2}^2 + \frac{211}{12}\zeta_3 + \frac{15}{2}\zeta_5\Big]\Big) + 16\, {C_A}^2 C_F \Big(p_{\mathrm{qq}}(x)\Big[\frac{245}{48} - \frac{67}{18}\zeta_2 + \frac{12}{5}{\zeta_2}^2 + \frac{1}{2}$$

$$+\mathrm{H}_{-3,0} + 4\mathrm{H}_{-2,-1,0} - \frac{3}{2}\mathrm{H}_{-2,0} - \mathrm{H}_{-2,0,0} + 2\mathrm{H}_{-2,2} - \frac{31}{12}\mathrm{H}_0\zeta_2 + 4\mathrm{H}_0\zeta_3 + \frac{389}{72}$$

$$-\mathrm{H}_{0,0,0,0} + 9\mathrm{H}_1\zeta_3 + 6\mathrm{H}_{1,-2,0} - \mathrm{H}_{1,0}\zeta_2 - \frac{11}{4}\mathrm{H}_{1,0,0} - 3\mathrm{H}_{1,0,0,0} - 4\mathrm{H}_{1,1,0,0} + 4$$

$$+\frac{11}{12}\mathrm{H}_3 + \mathrm{H}_4\Big] + p_{\mathrm{qq}}(-x)\Big[\frac{67}{18}\zeta_2 - {\zeta_2}^2 - \frac{11}{4}\zeta_3 - \mathrm{H}_{-3,0} + 8\mathrm{H}_{-2}\zeta_2 + \frac{11}{6}\mathrm{H}_{-2,0}$$

$$-3\mathrm{H}_{-1,0,0,0} + \frac{11}{3}\mathrm{H}_{-1}\zeta_2 + 12\mathrm{H}_{-1}\zeta_3 - 16\mathrm{H}_{-1,-1}\zeta_2 + 8\mathrm{H}_{-1,-1,0,0} + 16\mathrm{H}_{-1,-1,2}$$

$$-8\mathrm{H}_{-2,2} + 11\mathrm{H}_{-1,0}\zeta_2 + \frac{11}{6}\mathrm{H}_{-1,0,0} - \frac{11}{3}\mathrm{H}_{-1,2} - 8\mathrm{H}_{-1,3} - \frac{3}{4}\mathrm{H}_0 - \frac{1}{6}\mathrm{H}_0\zeta_2 - 4$$

3rd loop, and again

$$-3\mathrm{H}_{0,0}\zeta_2 - \frac{31}{12}\mathrm{H}_{0,0,0} + \mathrm{H}_{0,0,0,0} + 2\mathrm{H}_2\zeta_2 + \frac{11}{6}\mathrm{H}_3 + 2\mathrm{H}_4\Big] + (1-x)\Big[\frac{1883}{108} - \frac{1}{2}$$

$$-\mathrm{H}_{-2,-1,0} + \frac{1}{2}\mathrm{H}_{-3,0} - \frac{1}{2}\mathrm{H}_{-2}\zeta_2 + \frac{1}{2}\mathrm{H}_{-2,0,0} + \frac{523}{36}\mathrm{H}_0 + \mathrm{H}_0\zeta_3 - \frac{13}{3}\mathrm{H}_{0,0} - \frac{5}{2}\mathrm{H}$$

$$-2\mathrm{H}_{1,0,0}\Big] + (1+x)\Big[8\mathrm{H}_{-1}\zeta_2 + 4\mathrm{H}_{-1,-1,0} + \frac{8}{3}\mathrm{H}_{-1,0} - 5\mathrm{H}_{-1,0,0} - 6\mathrm{H}_{-1,2} - \frac{13}{3}$$

$$-\frac{43}{4}\zeta_3 - \frac{5}{2}\mathrm{H}_{-2,0} - \frac{11}{2}\mathrm{H}_0\zeta_2 - \frac{1}{2}\mathrm{H}_2\zeta_2 - \frac{5}{4}\mathrm{H}_{0,0}\zeta_2 + 7\mathrm{H}_2 - \frac{1}{4}\mathrm{H}_{2,0,0} + 3\mathrm{H}_3 + \frac{3}{4}$$

$$+\frac{1}{4}\zeta_2{}^2 - \frac{8}{3}\zeta_2 + \frac{17}{2}\zeta_3 + \mathrm{H}_{-2,0} - \frac{19}{2}\mathrm{H}_0 + \frac{5}{2}\mathrm{H}_0\zeta_2 - \mathrm{H}_0\zeta_3 + \frac{13}{3}\mathrm{H}_{0,0} + \frac{5}{2}\mathrm{H}_{0,0,0}$$

$$-\delta(1-x)\Big[\frac{1657}{576} - \frac{281}{27}\zeta_2 + \frac{1}{8}\zeta_2{}^2 + \frac{97}{9}\zeta_3 - \frac{5}{2}\zeta_5\Big]\Big) + 16\, C_F n_f^2 \Big(\frac{1}{18}p_{\mathrm{qq}}(x)\Big[\mathrm{H}_{0,}$$

$$+(1-x)\Big[\frac{13}{54} + \frac{1}{9}\mathrm{H}_0\Big] - \delta(1-x)\Big[\frac{17}{144} - \frac{5}{27}\zeta_2 + \frac{1}{9}\zeta_3\Big]\Big) + 16\, C_F{}^2 n_f \Big(\frac{1}{3}p_{\mathrm{qq}}(x)\Big[$$

3rd loop, and still some more

$$-\frac{55}{16}+\frac{5}{8}\mathrm{H}_0+\mathrm{H}_0\zeta_2+\frac{3}{2}\mathrm{H}_{0,0}-\mathrm{H}_{0,0,0}-\frac{10}{3}\mathrm{H}_{1,0}-\frac{10}{3}\mathrm{H}_2-2\mathrm{H}_{2,0}-2\mathrm{H}_3\Big]+\frac{2}{3}$$

$$-\frac{3}{2}\zeta_3+\mathrm{H}_{-2,0}+2\mathrm{H}_{-1}\zeta_2+\frac{10}{3}\mathrm{H}_{-1,0}+\mathrm{H}_{-1,0,0}-2\mathrm{H}_{-1,2}-\frac{1}{2}\mathrm{H}_0\zeta_2-\frac{5}{3}\mathrm{H}_{0,0}-$$

$$-(1-x)\Big[\frac{10}{9}+\frac{19}{18}\mathrm{H}_{0,0}-\frac{4}{3}\mathrm{H}_1+\frac{2}{3}\mathrm{H}_{1,0}+\frac{4}{3}\mathrm{H}_2\Big]+(1+x)\Big[\frac{4}{3}\mathrm{H}_{-1,0}-\frac{25}{24}\mathrm{H}_0+$$

$$+\frac{7}{9}\mathrm{H}_{0,0}+\frac{4}{3}\mathrm{H}_2-\delta(1-x)\Big[\frac{23}{16}-\frac{5}{12}\zeta_2-\frac{29}{30}{\zeta_2}^2+\frac{17}{6}\zeta_3\Big]\Big)+16\,{C_F}^3\Big(p_{qq}(x)\Big[$$

$$+6\mathrm{H}_{-2}\zeta_2+12\mathrm{H}_{-2,-1,0}-6\mathrm{H}_{-2,0,0}-\frac{3}{16}\mathrm{H}_0-\frac{3}{2}\mathrm{H}_0\zeta_2+\mathrm{H}_0\zeta_3+\frac{13}{8}\mathrm{H}_{0,0}-2\mathrm{H}_0$$

$$+12\mathrm{H}_1\zeta_3+8\mathrm{H}_{1,-2,0}-6\mathrm{H}_{1,0,0}-4\mathrm{H}_{1,0,0,0}+4\mathrm{H}_{1,2,0}-3\mathrm{H}_{2,0}+2\mathrm{H}_{2,0,0}+4\mathrm{H}_{2,1}$$

$$+4\mathrm{H}_{3,0}+4\mathrm{H}_{3,1}+2\mathrm{H}_4\Big]+p_{qq}(-x)\Big[\frac{7}{2}{\zeta_2}^2-\frac{9}{2}\zeta_3-6\mathrm{H}_{-3,0}+32\mathrm{H}_{-2}\zeta_2+8\mathrm{H}_{-2}$$

$$-26\mathrm{H}_{-2,0,0}-28\mathrm{H}_{-2,2}+6\mathrm{H}_{-1}\zeta_2+36\mathrm{H}_{-1}\zeta_3+8\mathrm{H}_{-1,-2,0}-48\mathrm{H}_{-1,-1}\zeta_2+40\mathrm{I}$$

3rd loop, and UFF

$$+48\mathrm{H}_{-1,-1,2}+40\mathrm{H}_{-1,0}\zeta_2+3\mathrm{H}_{-1,0,0}-22\mathrm{H}_{-1,0,0,0}-6\mathrm{H}_{-1,2}-4\mathrm{H}_{-1,2,0}-32$$

$$-\frac{3}{2}\mathrm{H}_0\zeta_2-13\mathrm{H}_0\zeta_3-14\mathrm{H}_{0,0}\zeta_2-\frac{9}{2}\mathrm{H}_{0,0,0}+6\mathrm{H}_{0,0,0,0}+6\mathrm{H}_2\zeta_2+3\mathrm{H}_3+2\mathrm{H}_{3,0}\text{-}$$

$$+(1-x)\Big[2\mathrm{H}_{-3,0}-\frac{31}{8}+4\mathrm{H}_{-2,0,0}+\mathrm{H}_{0,0}\zeta_2-3\mathrm{H}_{0,0,0,0}+35\mathrm{H}_1+6\mathrm{H}_1\zeta_2-\mathrm{H}_{1,}$$

$$+(1+x)\Big[\frac{37}{10}\zeta_2{}^2-\frac{93}{4}\zeta_2-\frac{81}{2}\zeta_3-15\mathrm{H}_{-2,0}+30\mathrm{H}_{-1}\zeta_2+12\mathrm{H}_{-1,-1,0}-2\mathrm{H}_{-1,0}$$

$$-24\mathrm{H}_{-1,2}-\frac{539}{16}\mathrm{H}_0-28\mathrm{H}_0\zeta_2+\frac{191}{8}\mathrm{H}_{0,0}+20\mathrm{H}_{0,0,0}+\frac{85}{4}\mathrm{H}_2-3\mathrm{H}_{2,0,0}-2\mathrm{H}_3$$

$$-\mathrm{H}_4\Big]+4\zeta_2+33\zeta_3+4\mathrm{H}_{-3,0}+10\mathrm{H}_{-2,0}+\frac{67}{2}\mathrm{H}_0+6\mathrm{H}_0\zeta_3+19\mathrm{H}_0\zeta_2-25\mathrm{H}_{0,0}$$

$$-2\mathrm{H}_2-\mathrm{H}_{2,0}-4\mathrm{H}_3+\delta(1-x)\Big[\frac{29}{32}-2\zeta_2\zeta_3+\frac{9}{8}\zeta_2+\frac{18}{5}\zeta_2{}^2+\frac{17}{4}\zeta_3-15\zeta_5\Big]\Big)$$

music of the spheres

2×2 anomalous dimension matrix occupies

1 st loop: $1/10$ page

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Moch, Vermaseren and Vogt

[waterfall of results launched
March 2004, and counting]

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$$V \sim \begin{cases} 10^{\frac{N(N-1)}{2}-1} \\ 10^{2^{N-1}-2} \end{cases}$$

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not too encouraging a trend . . .

status quo

More importantly, without understanding the essence of the series — the "physics" that underlines the appearance of this or that structure — one may not hope to improve the perturbative expansion.

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More importantly, without understanding the essence of the series — the "physics" that underlines the appearance of this or that structure — one may not hope to improve the perturbative expansion. *What for* ?

Numerically, α_s is not such a magnificent expansion parameter ...
Therefore, it is mandatory to apply as much grey substance as we possibly could to re-arrange the perturbative series to ensure *better convergence*

improvements

Parton splitting functions $P(x,\alpha_s)$ are routinely equated with the (Mellin transformed) anomalous dimensions $\gamma_N(\alpha_s)$.
Scheme dependence enters beyond the LLA (1 loop).

$\overline{MS}$ — a well formulated and convenient renormalization scheme, *BUT...*
Among known troubles:

- $P^{(k)}(x)$ singular at $x\to 1$ [as $P^{(1)}(x)$]
- $\alpha_{\overline{\mathrm{MS}}}$ an unphysical expansion parameter
- no respect to deep symmetries (SUSY)

- Be smart with soft gluons (Low theorem)
- Dimensional regularization → *Dimensional Reduction*

Another [hidden] symmetry —
inter-relation between DIS and annihilation channels.

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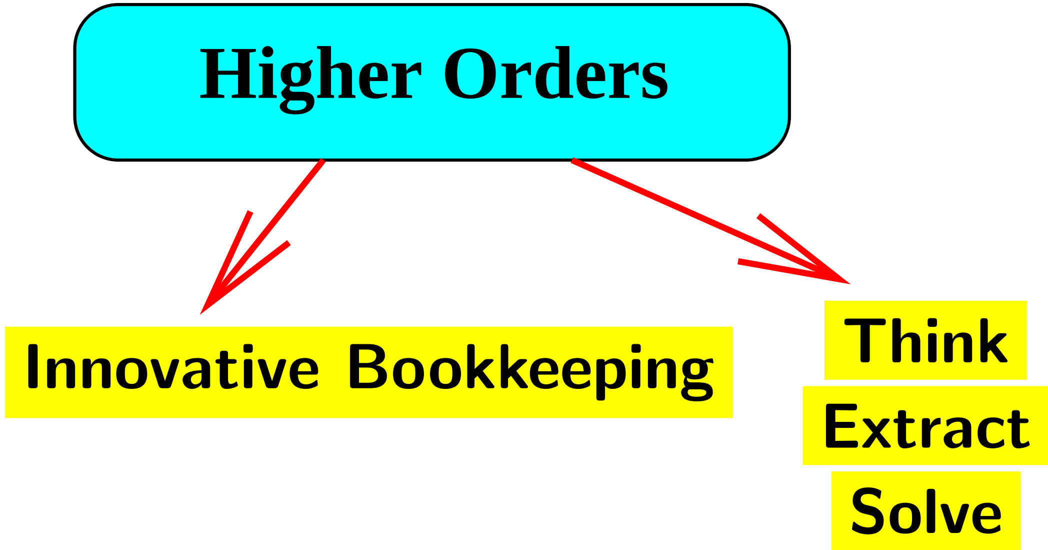

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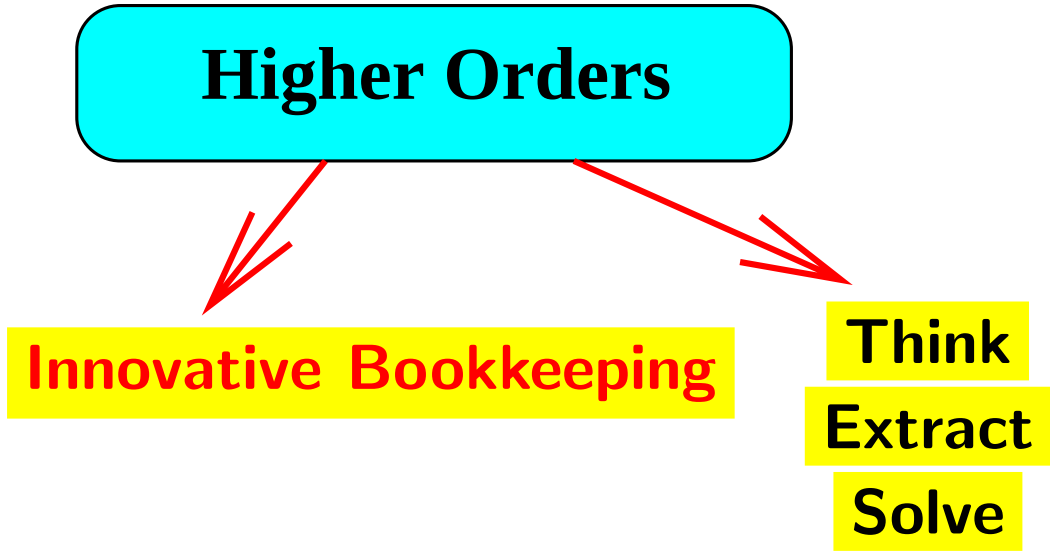

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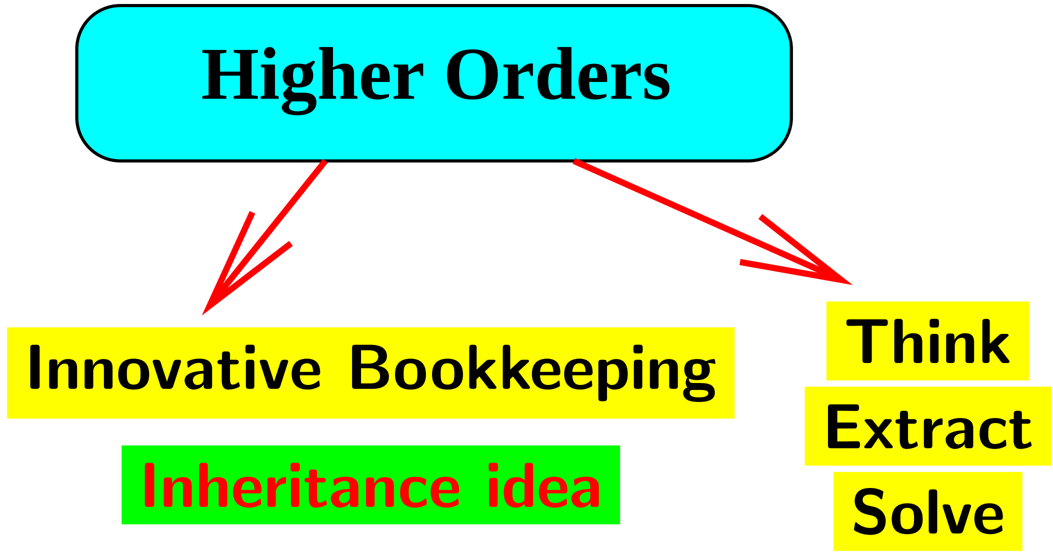

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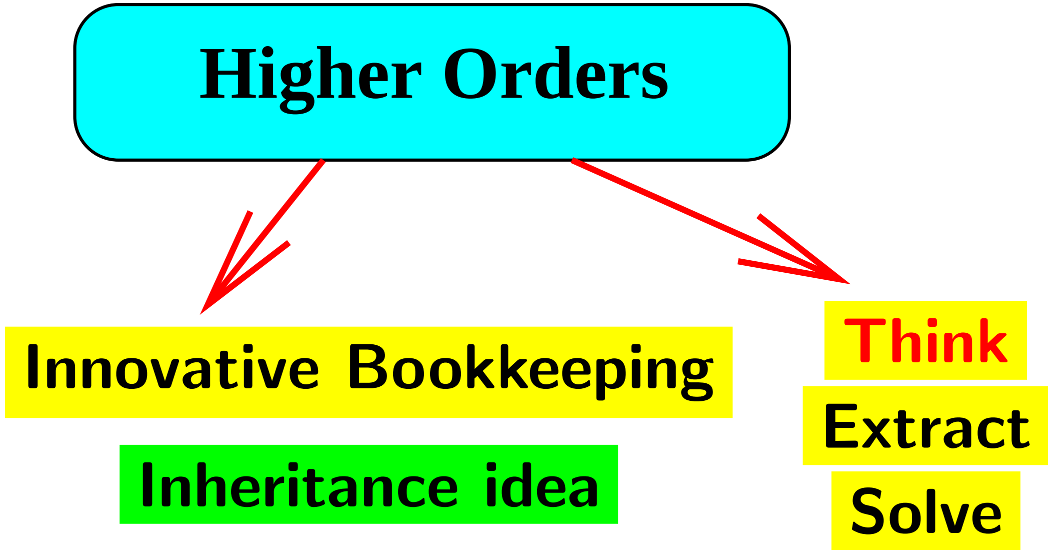

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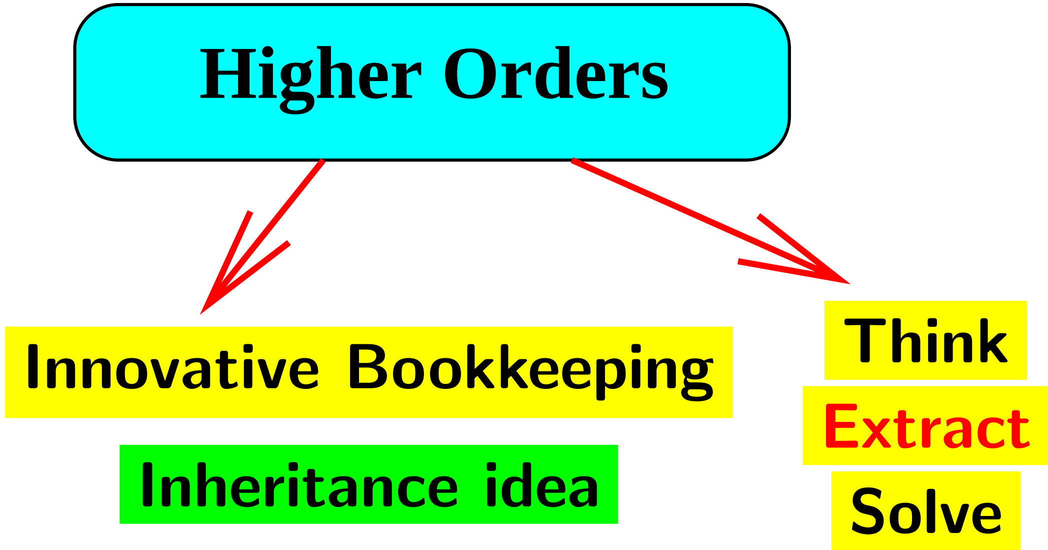

An essential part of gluon dynamics is Classical. (F.Low)

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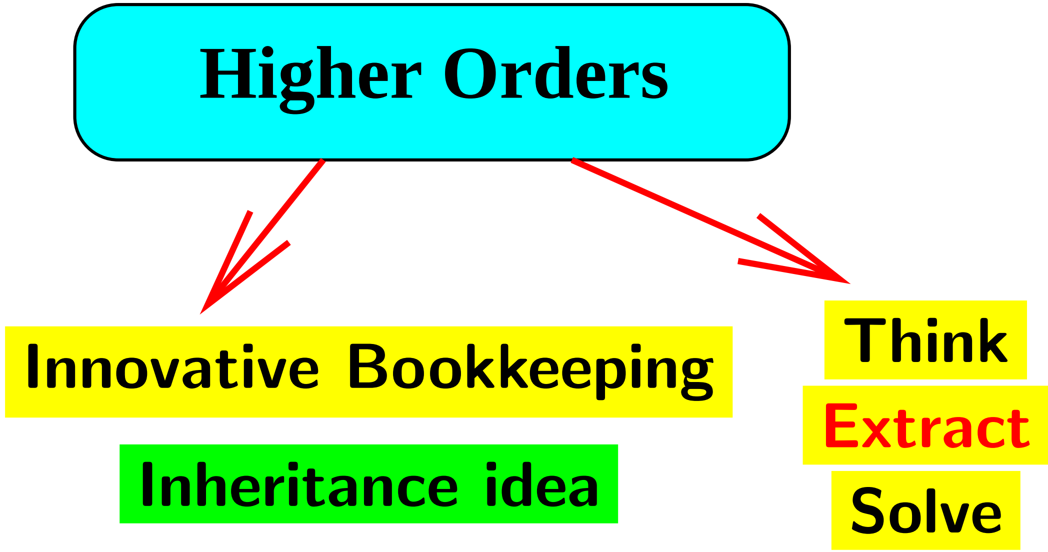

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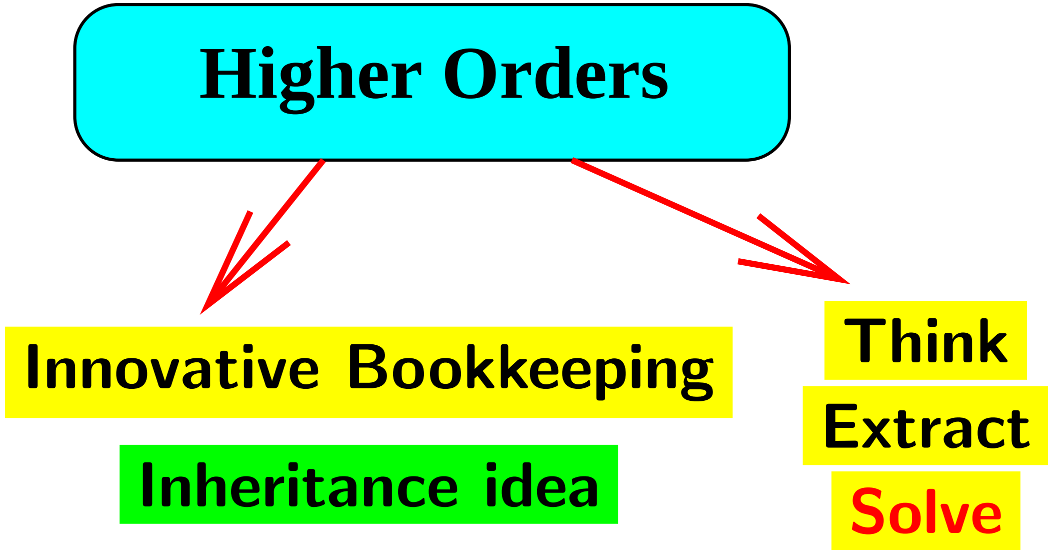

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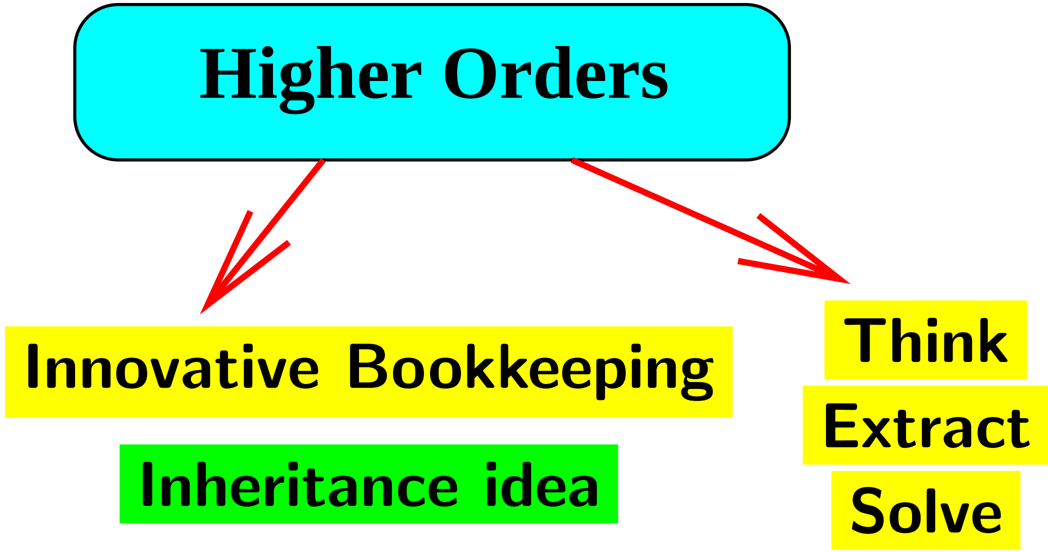

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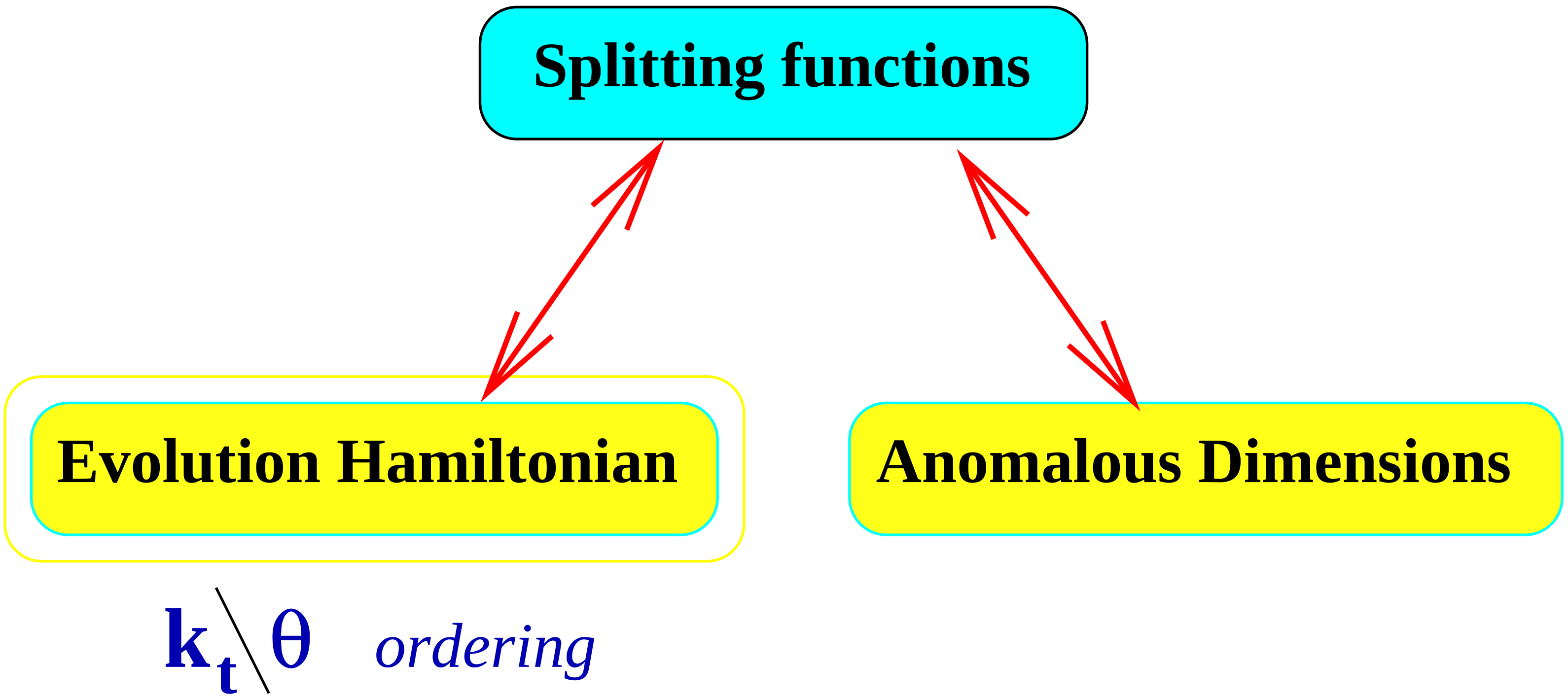
↪ A playing ground for theoretical theory: SUSY, AdS/CFT, . . .

Innovative Bookkeeping

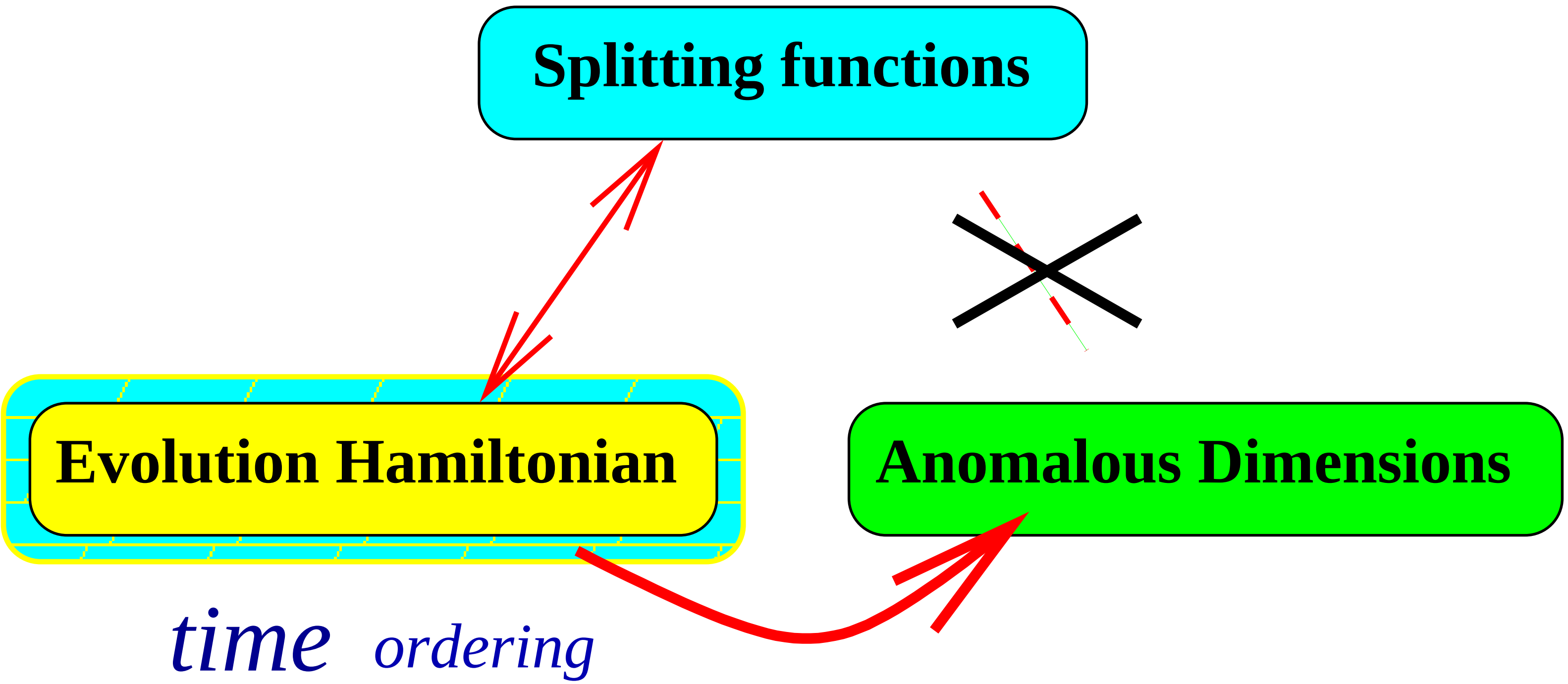
In the standard approach,



- parton splitting functions are equated with anomalous dimensions;
- they are different for DIS and e^+e^- evolution;
- "clever evolution variables" are different too

Innovative Bookkeeping

In the new approach,



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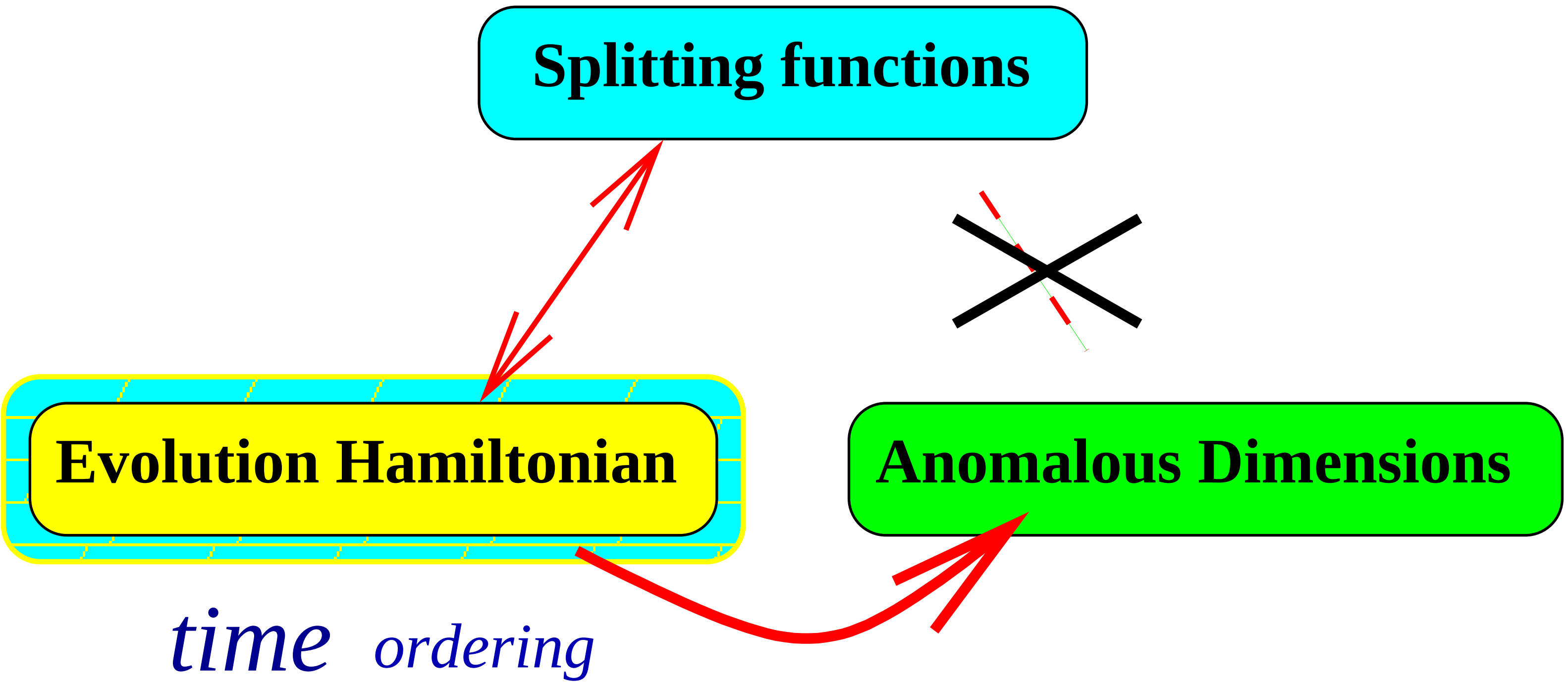

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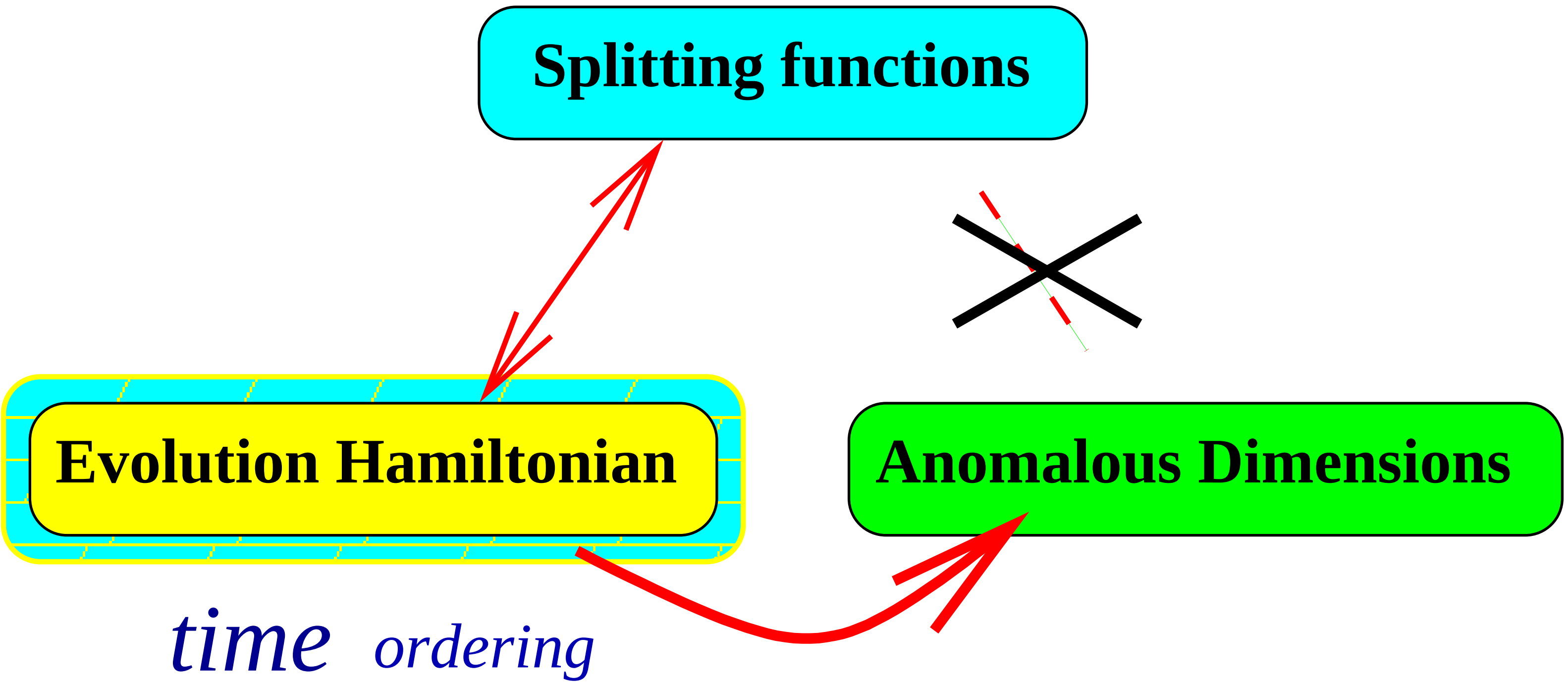

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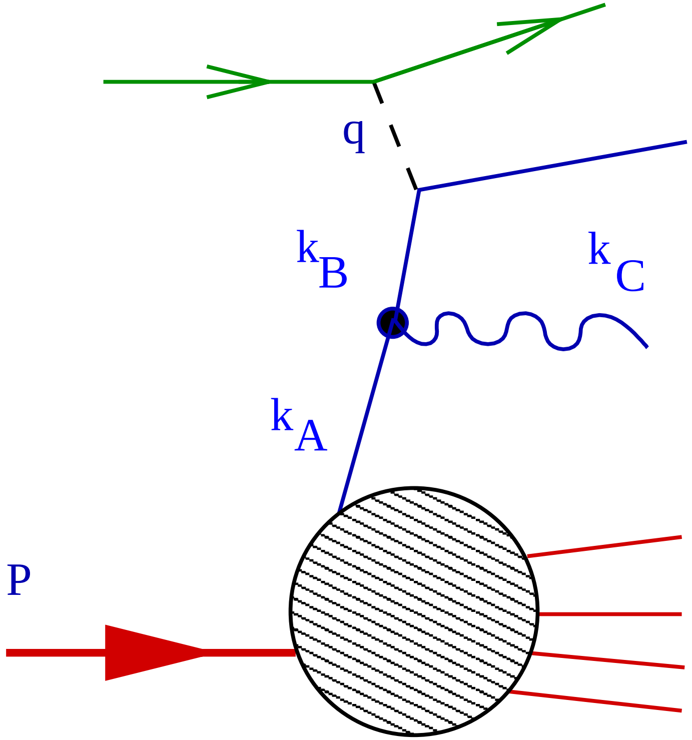
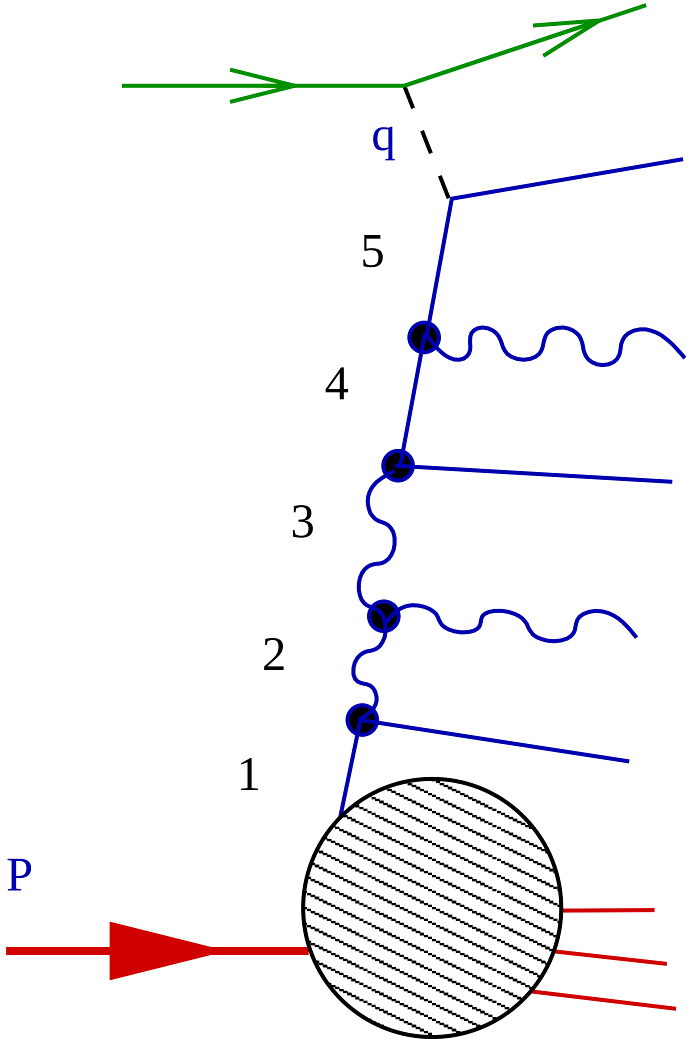
quark–gluon cascades



So long as probability of one extra parton emission is large, one has to consider and treat *arbitrary number* of parton splittings

quark–gluon cascades



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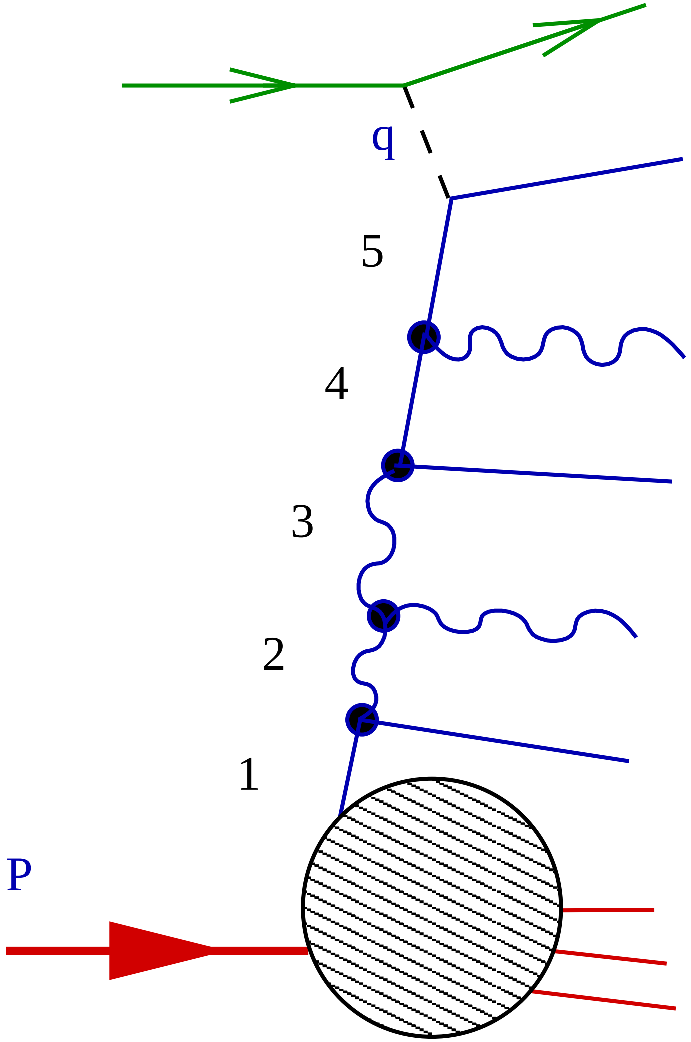

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Four basic splitting processes :

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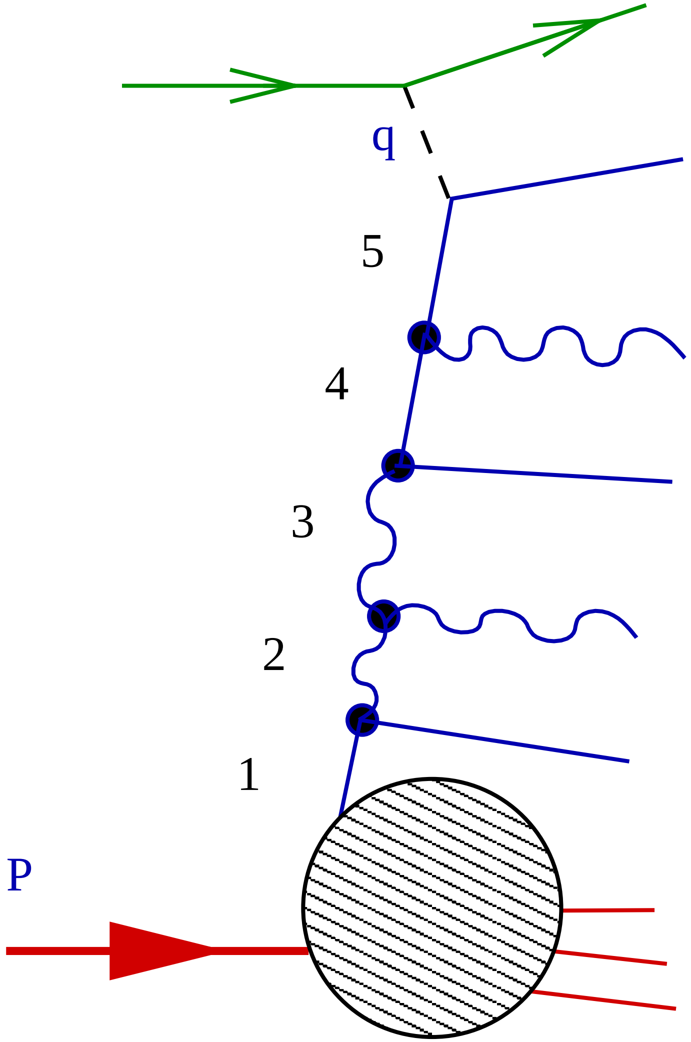

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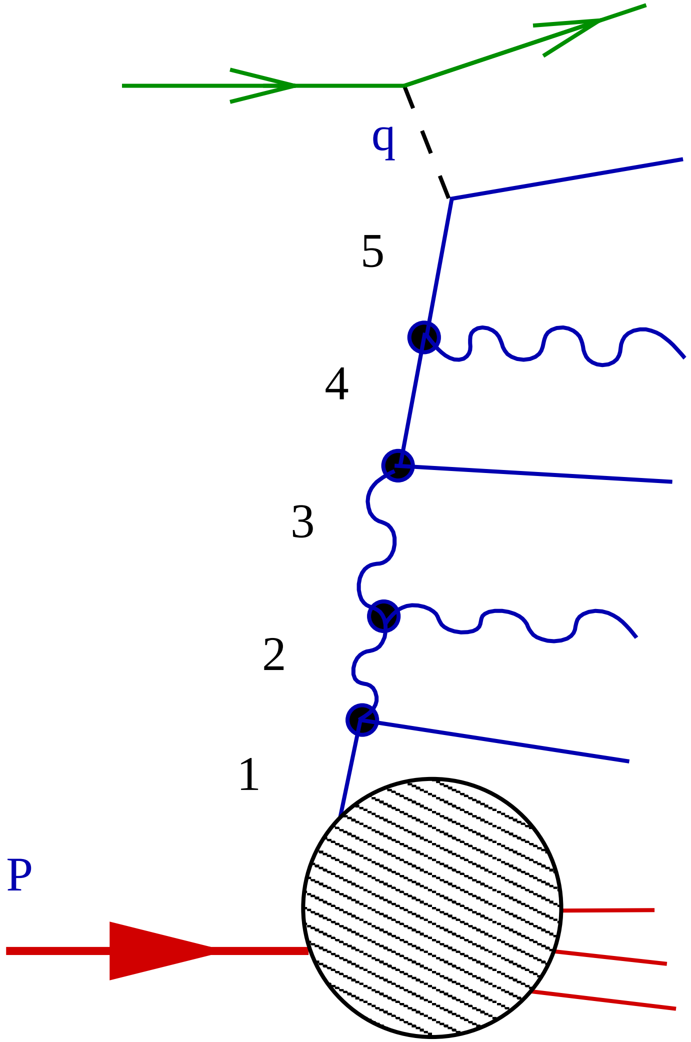

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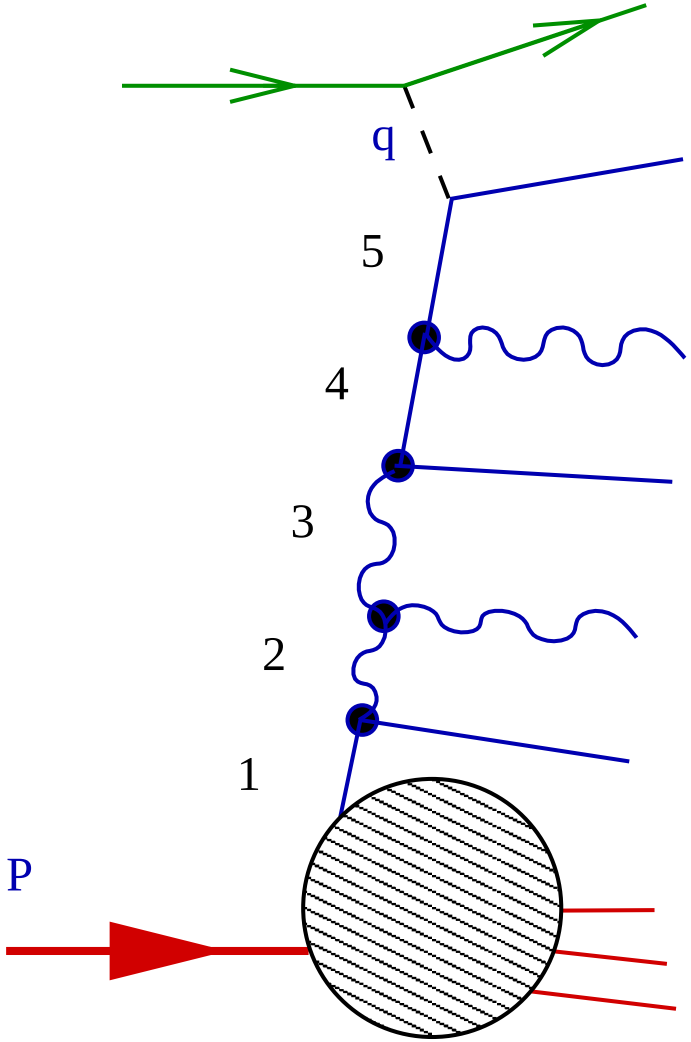

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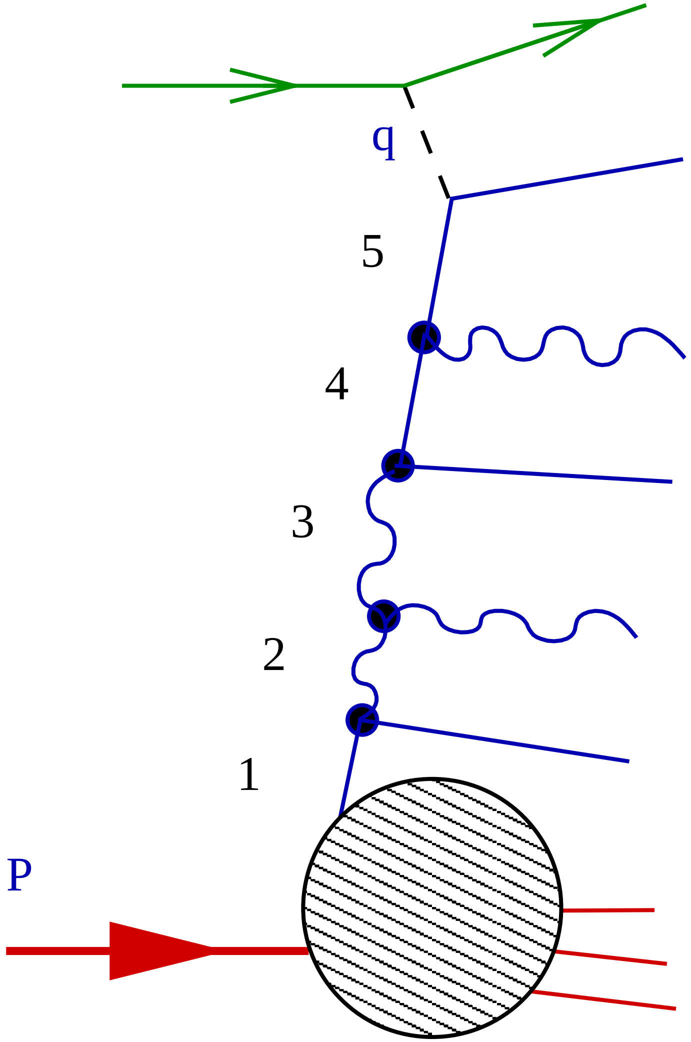

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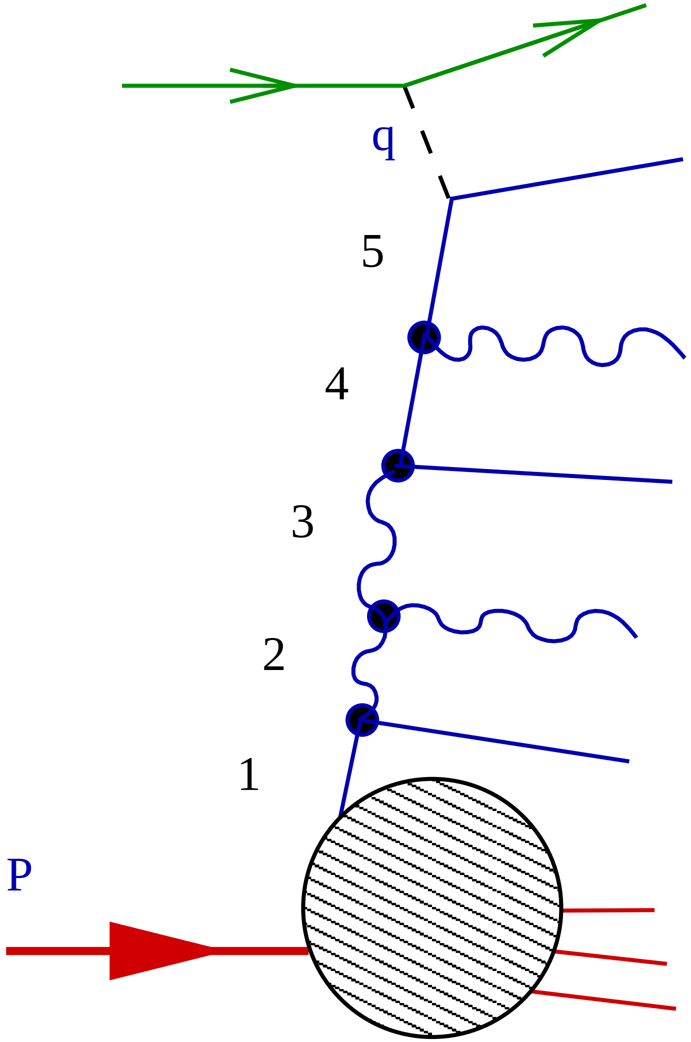

$$\mu^2 \ll k_{1\perp}^2 \ll k_{2\perp}^2 \ll k_{3\perp}^2 \ll k_{4\perp}^2 \ll k_{5\perp}^2 \ll Q^2$$

Four basic splitting processes :

“Hamiltonian” for parton cascades

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Logarithmic “evolution time” $d\xi = \frac{\alpha_s}{2\pi} \frac{dk_\perp^2}{k_\perp^2}$

Relating parton splittings

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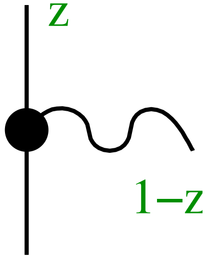
Parton Dynamics turned out to be extremely simple.

Have a deeper look at parton splitting probabilities
– our evolution Hamiltonian –
to fully appreciate the power of the probabilistic interpretation of parton cascades

Apparent and Hidden symmetries



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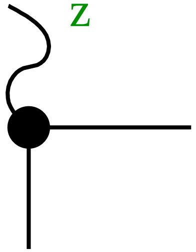

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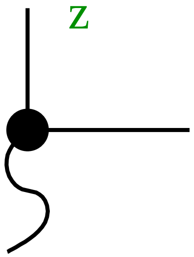

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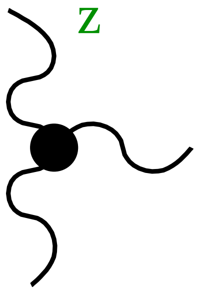

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Four "parton splitting functions"

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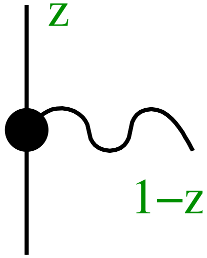

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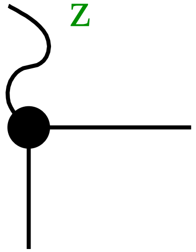

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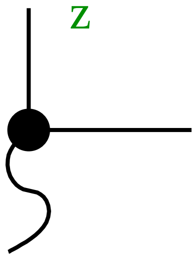

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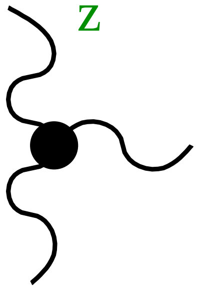

$$= N_c \cdot \frac{1+z^4+(1-z)^4}{z(1-z)}$$

- Exchange the decay products : $z \to 1-z$

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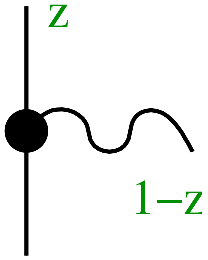
Apparent and Hidden symmetries



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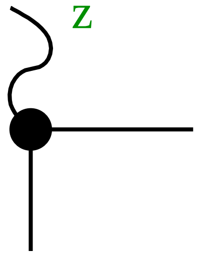

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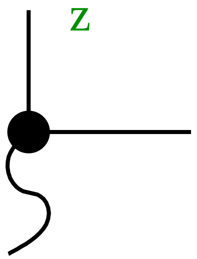

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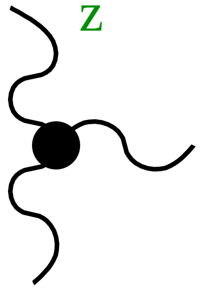

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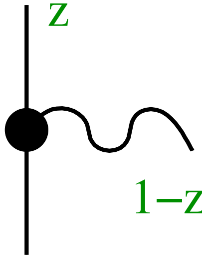

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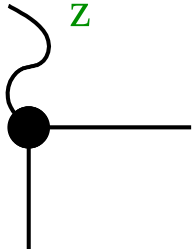

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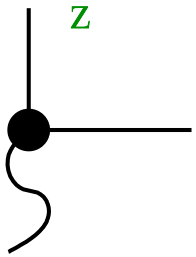

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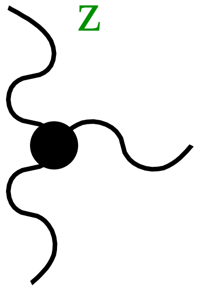

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Three (QED) "kernels" are inter-related; gluon self-interaction stays put :

$$\boxed{ {}_{q}^{q[g]}(z)\,, \quad {}_{q}^{g[q]}(z)\,, \quad {}_{g}^{q[\bar{q}]}(z) } \;;\qquad \boxed{ {}_{g}^{g[g]}(z) }$$

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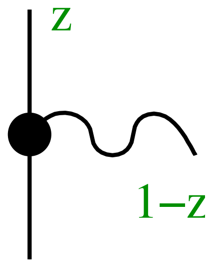

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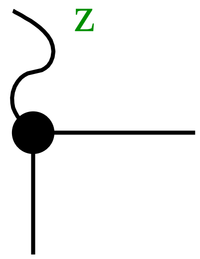

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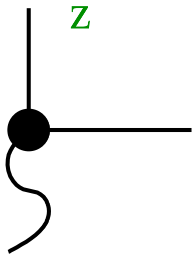

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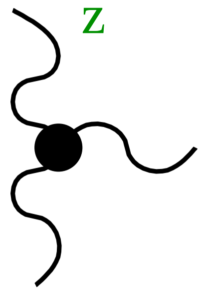

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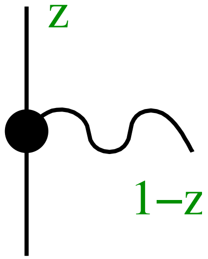

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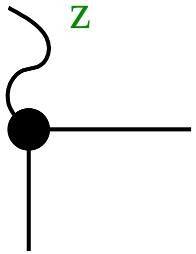

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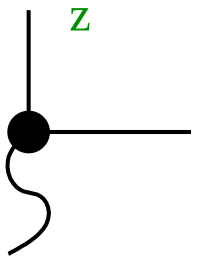

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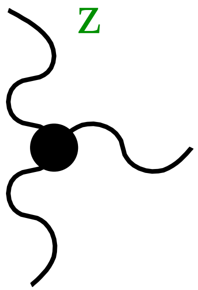

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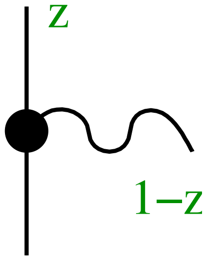

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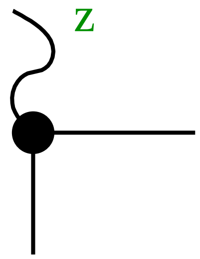

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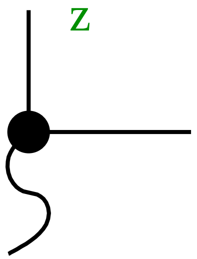

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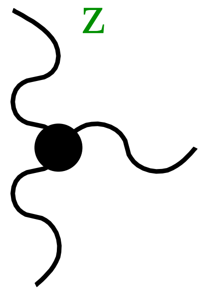

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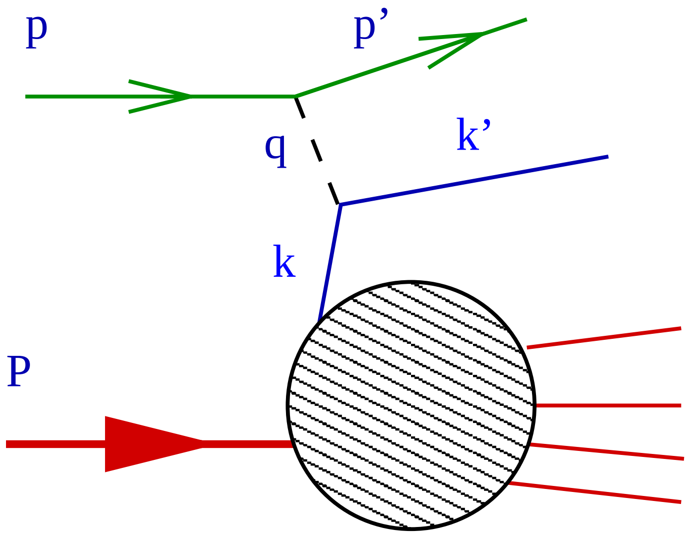
Long-living partons fluctuations



Kinematics of the parton splitting $A \to B + C$

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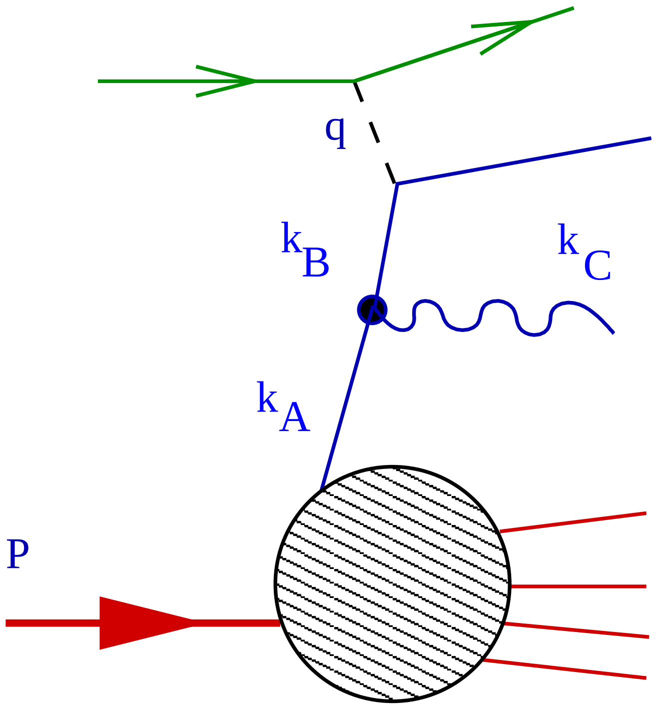

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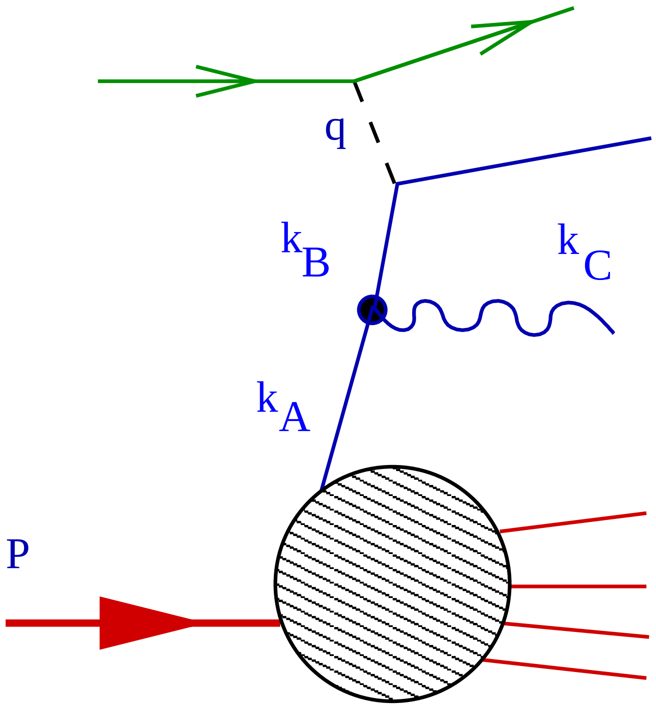

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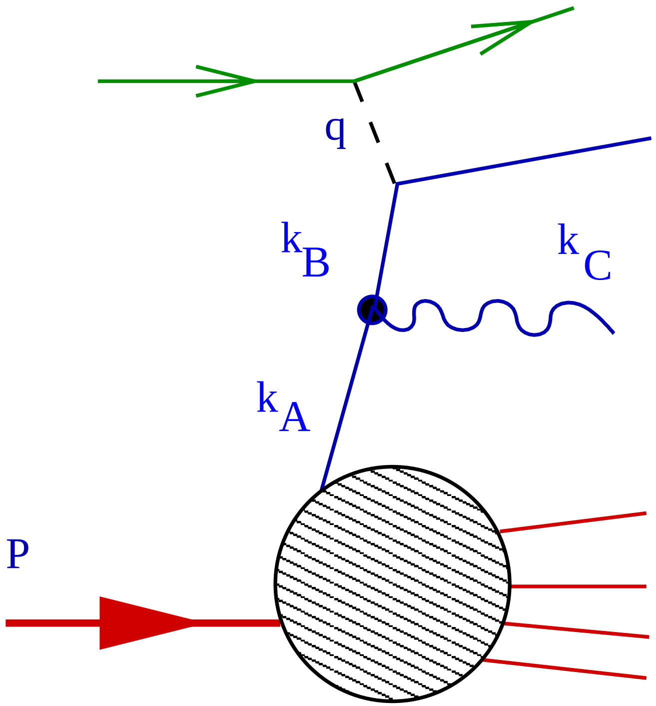

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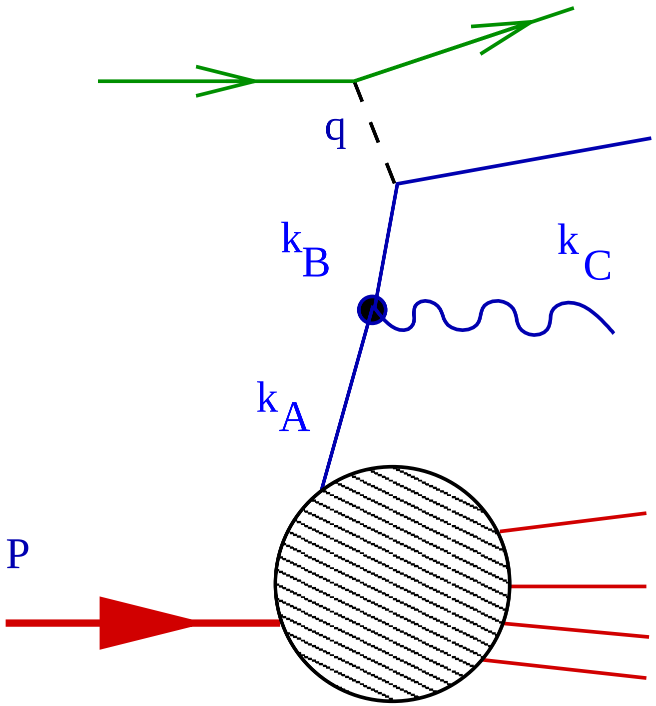

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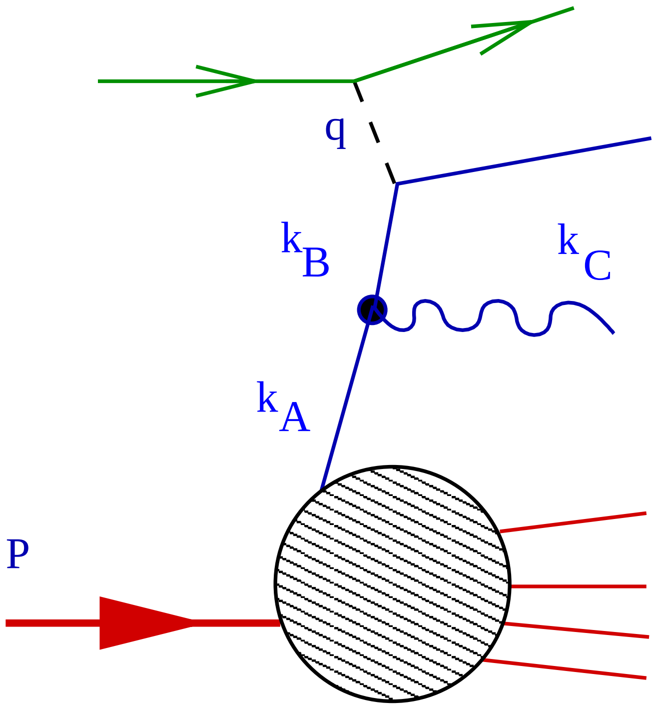

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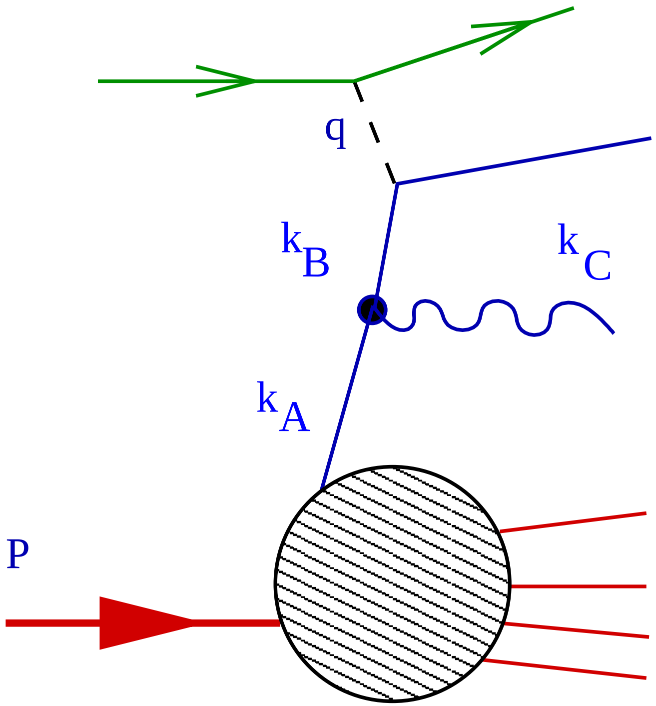

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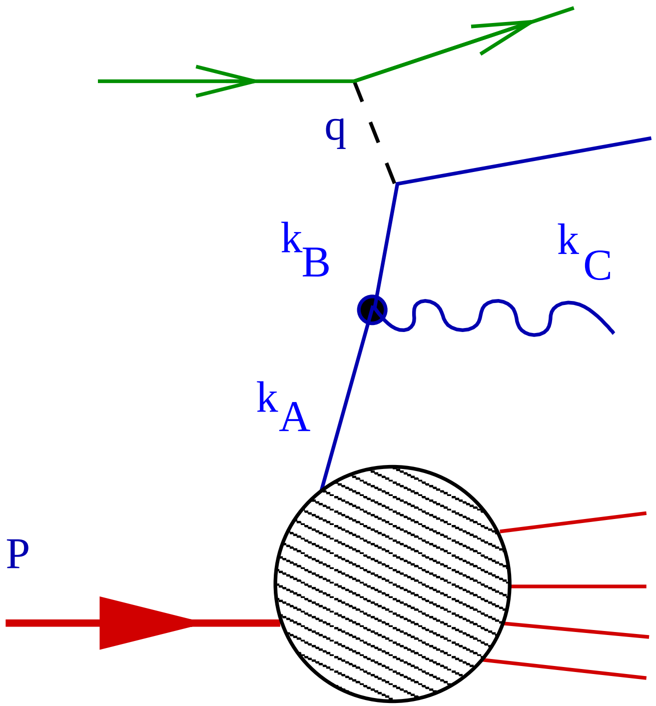

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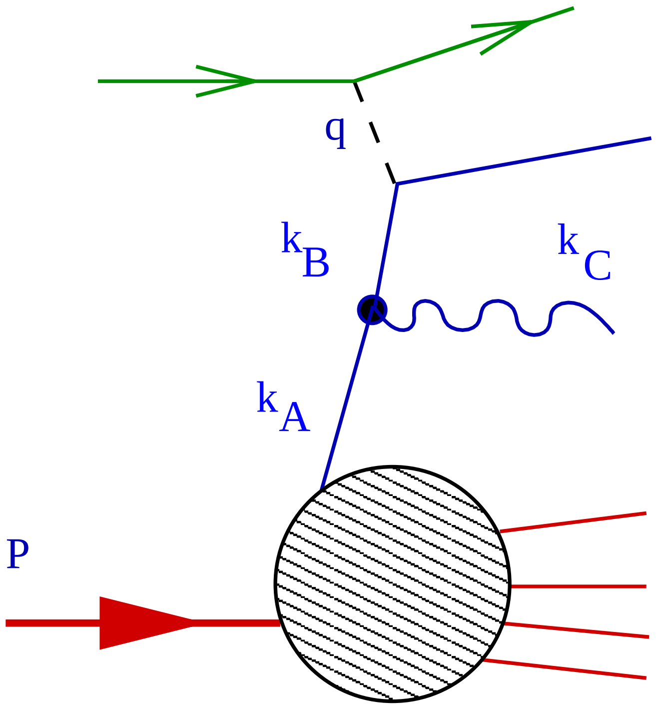

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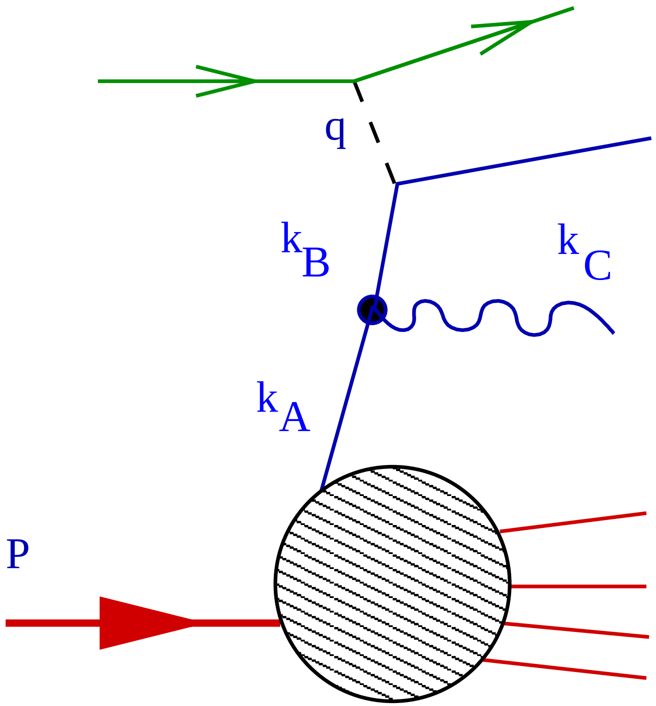

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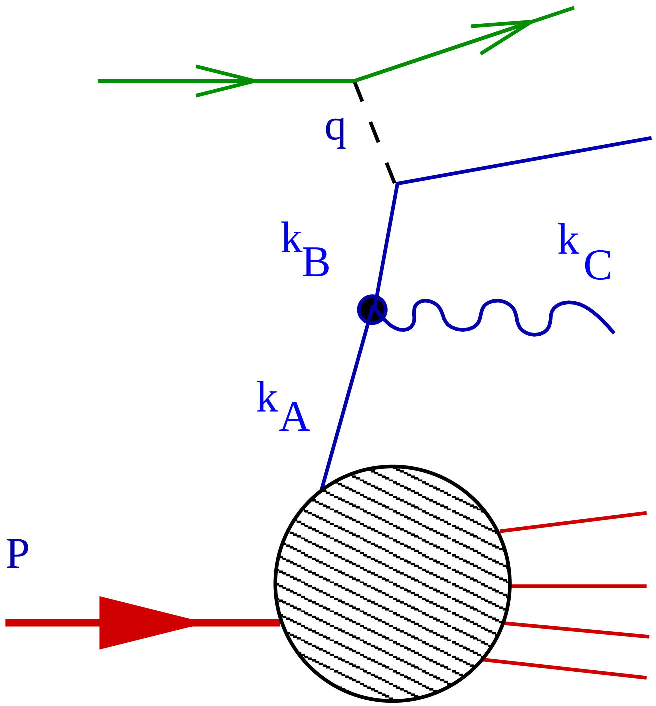

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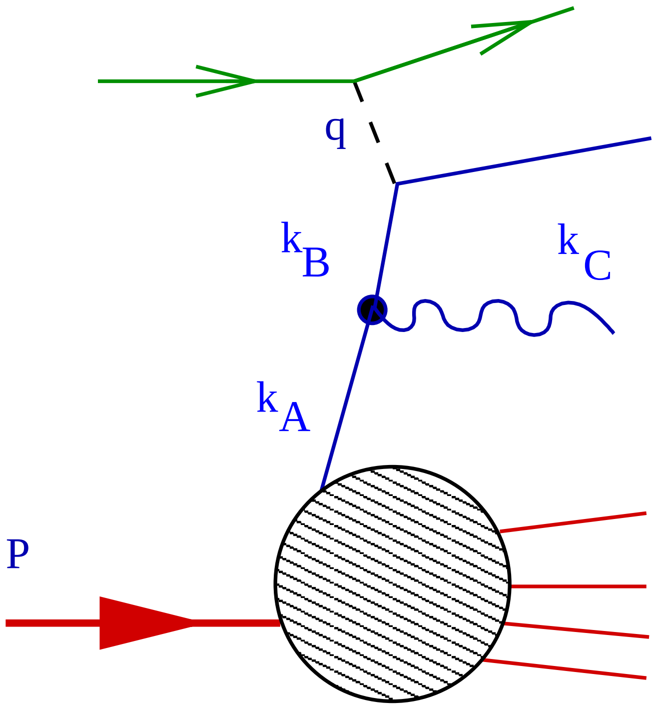

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strongly ordered *lifetimes* of successive parton fluctuations !

How to Order parton splittings?

Beyond the 1st loop, it starts to matter how does one order successive parton splittings that is, what one chooses for "parton evolution time".

The "clever choices" had been established quite some time ago:

$$d\xi = d\ln\frac{k_\perp^2}{1} \quad \text{(space-like)}, \qquad d\xi = d\ln\frac{k_\perp^2}{z^2} \quad \text{(time-like)}.$$

Transverse momentum ordering vs. angular ordering.

Each of these two clever choices — consequence of taking into full consideration soft gluon coherence in order to prevent explosively large terms $(\alpha_s \ln^2 x)^n$ from appearing in higher loop anomalous dimensions.

A good *dynamical* move. But a lousy one *kinematically* :

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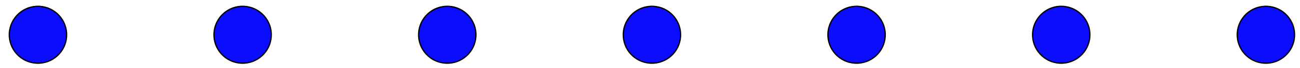
Recall an amazing historical example: Cosmic ray physics (mid 50's); conversion of high energy photons into e^+e^- pairs in the emulsion

Chudakov effect

Charged particle leaves a track of ionized atoms in photo-emulsion.

electron track



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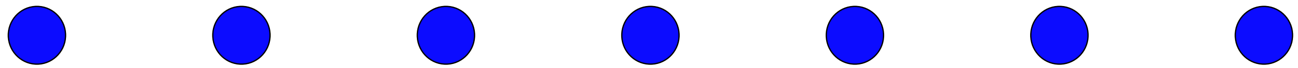
electron track 

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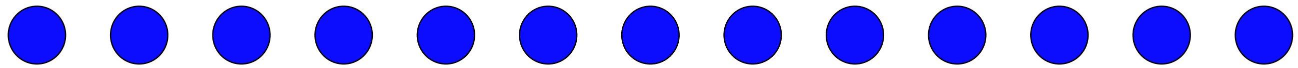
e^+e^- track (expected) 

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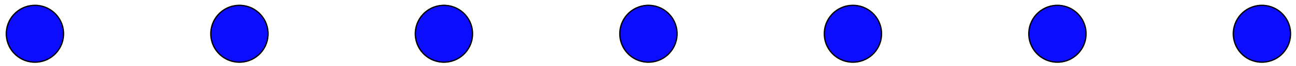
electron track 

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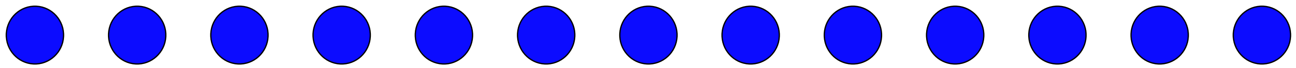
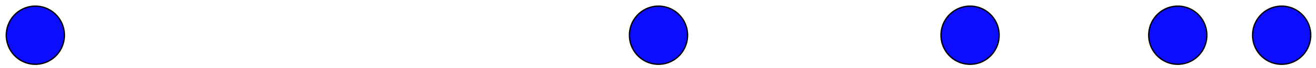
e^+e^- track (expected) 

Why then do we see *this* ?

e^+e^- (observed) 

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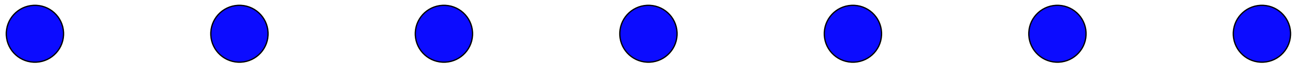
electron track 

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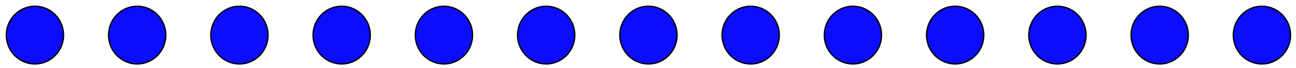
e^+e^- track (expected) 

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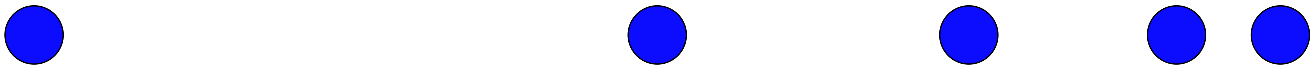
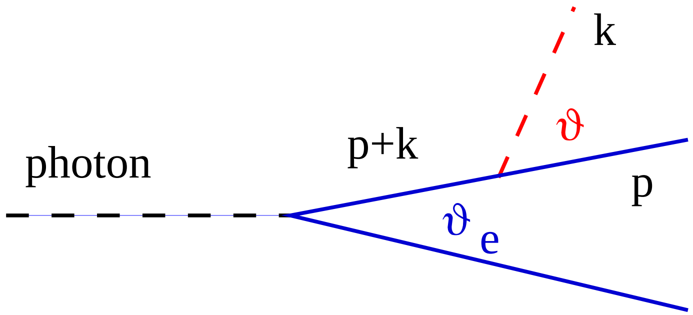
e^+e^- (observed) 

Transverse distance between two charges
(size of the e^+e^- dipole) is

$$\rho_\perp \simeq c\,t \cdot \vartheta_e$$



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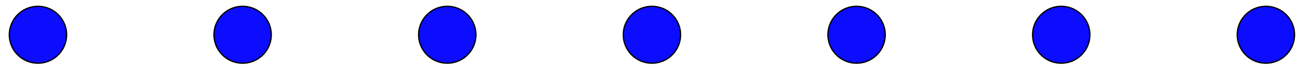
electron track 

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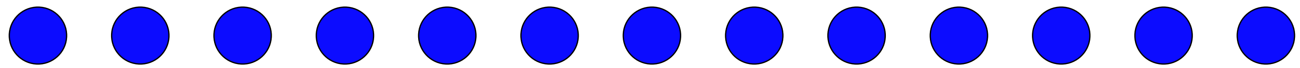
e^+e^- track (expected) 

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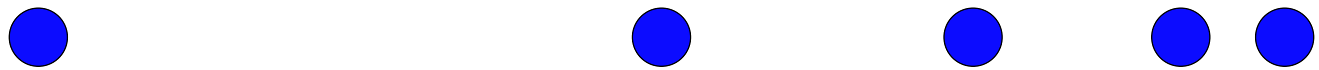
e^+e^- (observed) 

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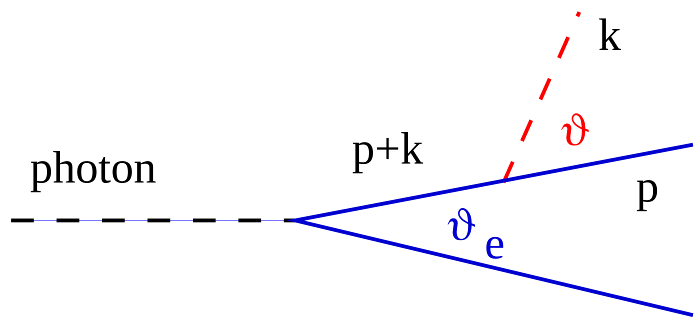

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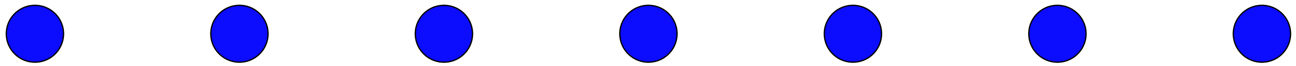
electron track 

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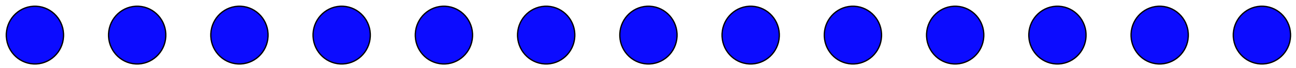
$e^+ e^-$ track (expected) 

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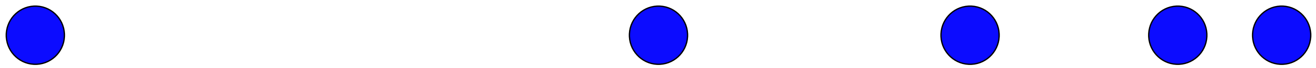
$e^+ e^-$ (observed) 

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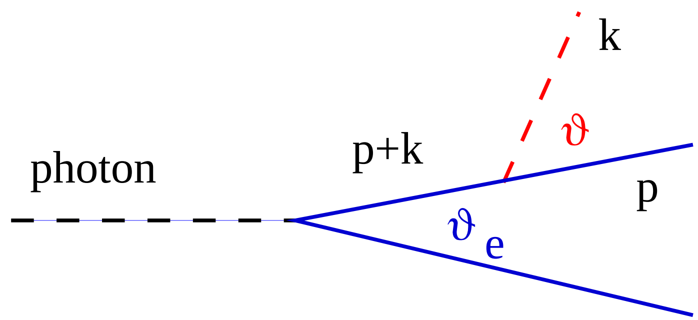
Angular Ordering

$\vartheta < \vartheta_e$ – independent radiation off e^- & e^+



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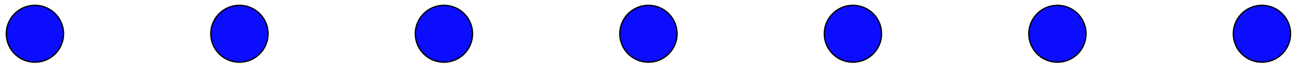
electron track 

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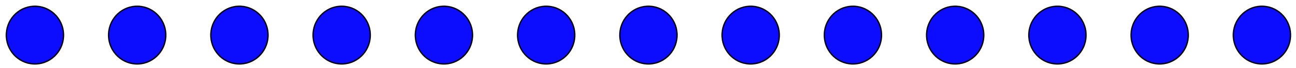
$e^+ e^-$ track (expected) 

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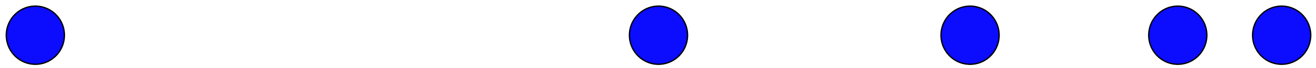
$e^+ e^-$ (observed) 

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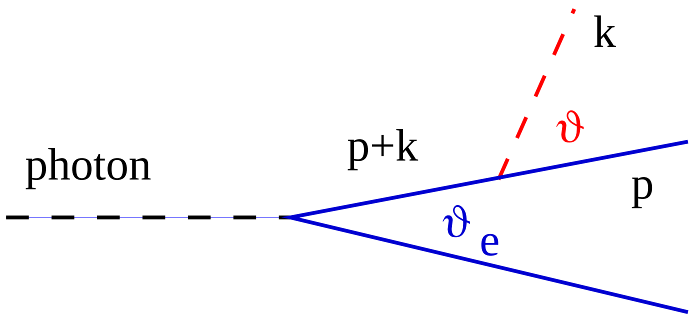

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Angular Ordering is *more restrictive* than the fluctuation time ordering:
$\vartheta \le \vartheta_e$ versus $\vartheta \le \vartheta_e \cdot \sqrt{\frac{p_0}{k_0}}$ that follows from

$$t_\gamma = \frac{p_0}{p_\perp^2} \simeq \frac{1}{p_0 {\vartheta_e}^2} < \frac{1}{k_0 \vartheta^2} \simeq \frac{k_0}{k_\perp^2} = t_e$$

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while the softest particles (that seem to be the easiest to produce) should not multiply at all !

space- and time-like parton multiplication

Space-like parton evolution (S) vs. *time-like* fragmentation (T)

Drell–Levy–Yan relation

$$P_{BA}^{(T)}(x) = \mp x \cdot P_{AB}^{(S)}(x^{-1}).$$

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True in any QFT, it reflects the crossing and allows to link the two channels by analytic continuation, from $x < 1$ to $x > 1$:

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$$P_{BA}^{(T)}(x_{\text{Feynman}}) = P_{BA}^{(S)}(x_{\text{Bjorken}}); \qquad x_B = \frac{-q^2}{2pq}, \quad x_F = \frac{2pq}{q^2}$$

Mark the different meaning of x in the two channels!

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Reciprocity Respecting Evolution

Fluctuation time ordering : D-r (HERA, 1993)

$$\frac{dD^A(x,Q^2)}{d\ln Q^2} = \int_0^1 \frac{dz}{z}\,\mathcal{P}_B^A(z;\alpha_s)\,D^B\left(\frac{x}{z}, z^\sigma Q^2\right)$$

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In the Mellin moment space,

$$P_N \equiv \int_0^1 \frac{dz}{z}\,P(z)\,z^N \qquad \Longrightarrow \qquad \gamma_N \cdot D_N(Q^2) = \mathcal{P}_{N+\sigma d}\cdot D_N(Q^2)$$

the *evolution kernel* $\mathcal{P}$ emerges with the *differential operator* for argument.

Reciprocity Respecting Evolution

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$$\frac{dD^A(x,Q^2)}{d\ln Q^2} = \int_0^1 \frac{dz}{z}\,\mathcal{P}_B^A(z;\alpha_s)\,D^B\left(\frac{x}{z}, z^\sigma Q^2\right), \qquad \sigma = \begin{cases} +1, & (\mathrm{T}) \\ -1, & (\mathrm{S}) \end{cases}$$

which is *non-local* due to the mixing of z and Q^2 in the hardness scale.
This non-locality can be handled using the Taylor series trick:

$$\int_0^1 \frac{dz}{z}\,\mathcal{P}(z,\alpha_s)\,D\left(z^\sigma Q^2\right) = \int_0^1 \frac{dz}{z}\,\mathcal{P}(z)\,z^{\sigma\frac{d}{d\ln Q^2}}\,D(Q^2), \quad d \equiv \frac{d}{d\ln Q^2}.$$

In the Mellin moment space,

$$P_N \equiv \int_0^1 \frac{dz}{z}\,P(z)\,z^N \qquad \Longrightarrow \qquad \gamma_N \cdot D_N(Q^2) = \mathcal{P}_{N+\sigma d} \cdot D_N(Q^2)$$

the *evolution kernel* $\mathcal{P}$ emerges with the *differential operator* for argument.

Expanding, get an equation for the an.dim. γ

$$\gamma[\alpha] = \mathcal{P} + \dot{\mathcal{P}}\cdot(\sigma\gamma + \beta/\alpha) + \tfrac{1}{2}\ddot{\mathcal{P}}\cdot\left[\gamma^2 + \sigma(2\beta/\alpha\,\gamma + \beta\partial_\alpha\gamma) + \beta/\alpha\,\partial_\alpha\beta\right] + \mathcal{O}(\alpha^4).$$

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GLR beyond the 1st loop

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More generally, a *renormalization scheme transformation* as a cure for/against GLR violation was proposed by Stratmann & Vogelsang (1996)

large x

Another important aspect of the RREE is the "double nature" of the perturbative expansion — in $\alpha_{\rm phys}$ and, at the same time, in $(1-x)$:

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A gap between *classical radiation* (Low–Burnett–Kroll wisdom)

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and *quantum fluctuations*

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small x : 2 puzzles

RREE relates two long-standing puzzles :

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$1 \to 1 + 2 \implies$ Exact Angular Ordering still intact !

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so-called "Malaza puzzle"

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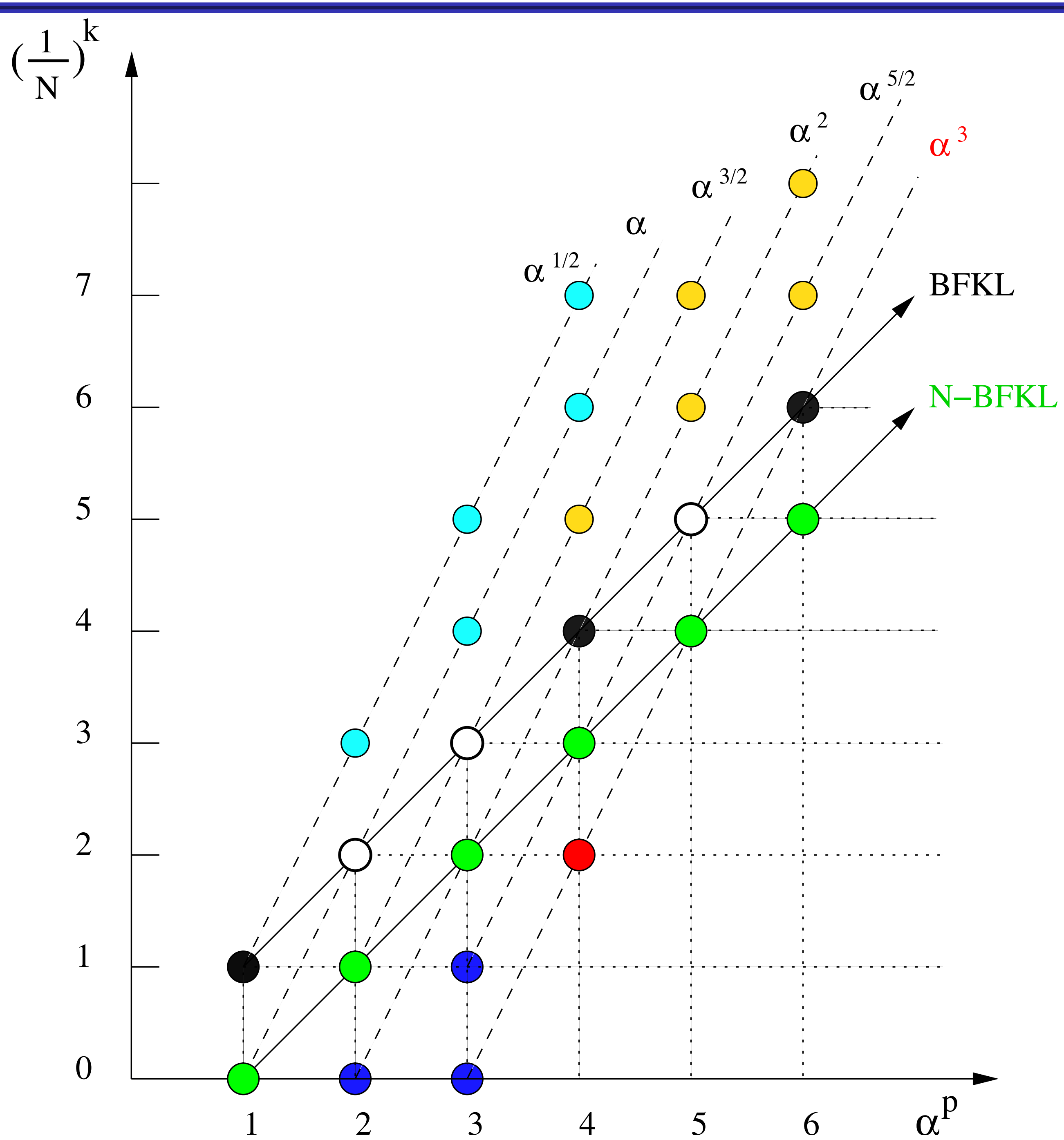
small x chart



Solid – BFKL (black) and N-BFKL (green) known in all orders.

Dashed blue – γ_+ terms generated by α/N and α.

Yellow – unknown.

Space-Time bookkeeping

The origin of the GL reciprocity violation is essentially kinematical :
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Moreover, the most resent result, still smoking : in $\mathcal{N}=4$

✗ GLR holds for twist 3, in $3+4$ loops Matteo Beccaria et. al (2007)

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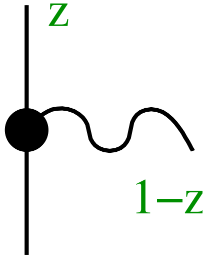
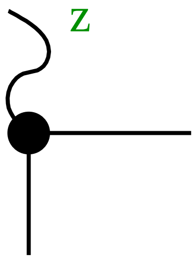
Recall an old hint from QCD ...

Relating parton splittings



$$= C_F \cdot \frac{1+z^2}{1-z}$$



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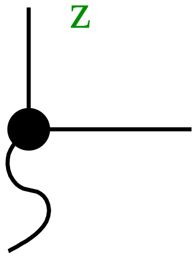

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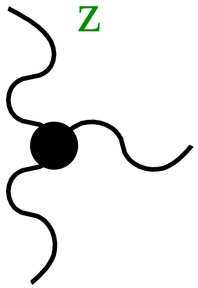

$$= N_c \cdot \frac{1+z^4+(1-z)^4}{z(1-z)}$$

Four “parton splitting functions”

$$ {}_{q}^{q[g]}(z)\,, \qquad {}_{q}^{g[q]}(z)\,, \qquad {}_{g}^{q[\bar{q}]}(z)\,, \qquad {}_{g}^{g[g]}(z)$$

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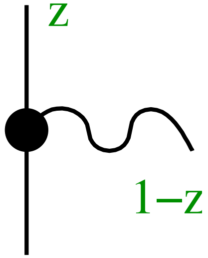

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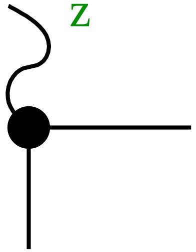

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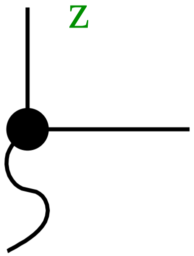

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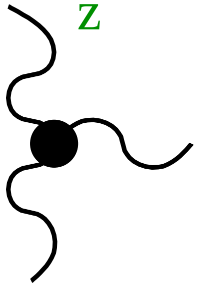

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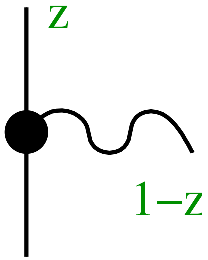

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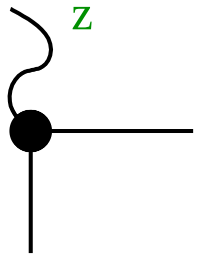

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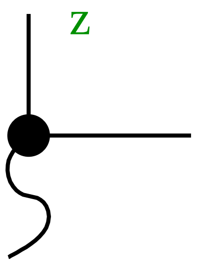

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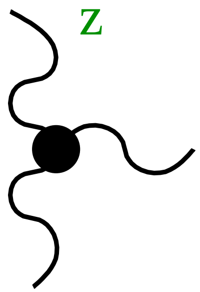

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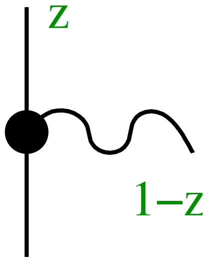

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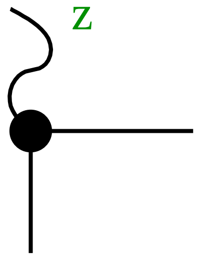

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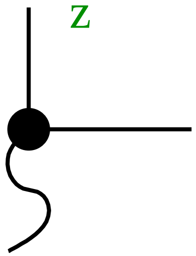

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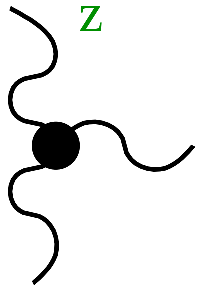

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Three (QED) "kernels" are inter-related; gluon self-interaction stays put :

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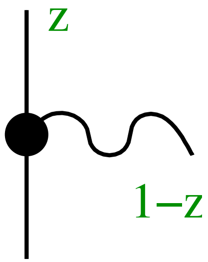

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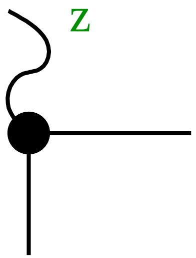

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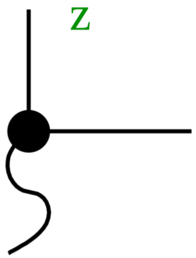

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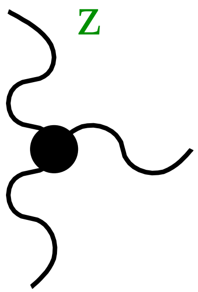

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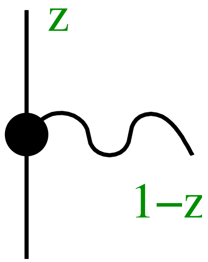

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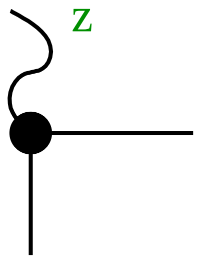

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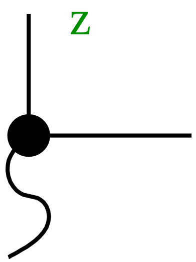

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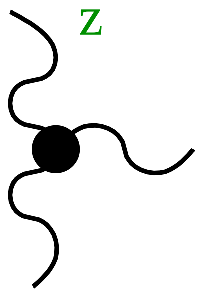

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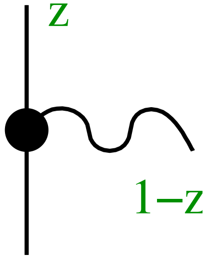

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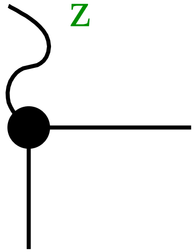

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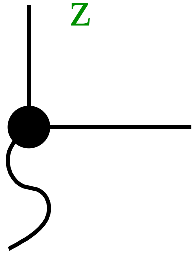

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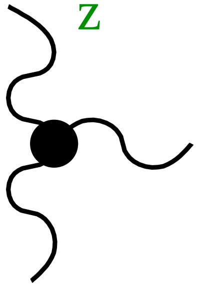

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from Bookkeeping to Solving

The integrability feature manifests itself already in *certain sectors* of QCD, in specific problems where one can *identify* QCD with SUSY-QCD :

✓ the Regge behaviour (large N_c)	Lipatov Faddeev & Korchemsky (1994)
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WHY and **WHAT FOR** ?

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And here we arrive at the second — Divide and Conquer — issue

LBK wisdom

Recall the diagonal first loop anomalous dimensions:

$$\tilde{\gamma}_{q\to q(x)+g} = \frac{C_F\alpha_s}{\pi}\left[\frac{x}{1-x} + (1-x)\cdot\frac{1}{2}\right],$$

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Let us look at the rôles these animals play on the QCD stage

Gluenatomy

Clagons :

- ✗ Classical Field
- ✓ infrared singular, $d\omega/\omega$
- ✓ define the physical coupling
- ✓ responsible for
  - ↪ DL radiative effects,
  - ↪ reggeization,
  - ↪ QCD/Lund string (gluers)
- ✓ play the major rôle in evolution

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- ✗ Quantum d.o.f.s (constituents)
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In addition,

- ✗ Tree multi-clagon (Parke–Taylor) amplitudes are *known exactly*
- ✗ It is clagons which dominate in all the *integrability cases*

N=4 SUSY Yang–Mills

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Matter content = 4 Majorana fermions, 6 scalars;
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Now, $\mathcal{N}=4$ SUSY :

$$\frac{{C_A}^{-1}\,d}{d\ln\mu^2}\left(\frac{\alpha(\mu^2)}{4\pi}\right)^{-1} = -\frac{11}{3} + \frac{4}{2}\cdot\int_0^1 dx\, 2[x^2+(1-x)^2] + \frac{6}{2!}\cdot\int_0^1 dx\, 2x(1-x)$$

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Euler–Zagier harmonic sums

In spite of having many states ($s = 0, \frac{1}{2}, 1$), the SYM-4 parton dynamics is built of a single "universal" anomalous dimension:

$$\gamma_+(N+2) = \tilde{\gamma}_+(N+1) = \gamma_0(N) = \tilde{\gamma}_-(N-1) = \gamma_-(N-2) \equiv \gamma_{\text{uni}}(N)$$

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$$S_1(N) = \sum_{k=1}^{N} \frac{1}{k} \;=\; \psi(N+1) - \psi(1).$$

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as we as multiple indices — *nested sums*

$$S_{m,\vec{\rho}}(N) = \sum_{k=1}^N \frac{S_{\vec{\rho}}(k)}{k^m} \qquad (\vec{\rho} = (m_1, m_2, \ldots, m_i)),$$

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generates positives and simplifies negatives.

RR evolution kernel

In terms of the perturbative expansion in the physical coupling,

$$a_{\mathrm{ph}} = a\left(1 - \frac{1}{2}\zeta_2 a + \frac{11}{20}\zeta_2^2 a^2 + \ldots\right),$$

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Twist-3

The $\mathfrak{sl}(2)$ sector of planar $\mathcal{N}=4$ SYM contains single trace states which are linear combinations of the basic operators

$$\mathrm{Tr}\left\{(\mathcal{D}^{s_1} Z) \cdots (\mathcal{D}^{s_L} Z)\right\}, \quad s_1 + \cdots + s_L = N,$$

where Z is one of the three complex scalar fields and $\mathcal{D}$ is a light-cone covariant derivative. The numbers $\{s_i\}$ are non-negative integers and N is the total spin. The number L of Z fields is the twist of the operator, *i.e.* the classical dimension minus spin.

The anomalous dimensions of these states are the eigenvalues $\gamma_L(N; g)$ of the dilatation operator — integrable Hamiltonian.

These values were obtained by solving numerically the Bethe Ansatz equations (BAE), order by order in g^2, and guessing the answer in terms of harmonic sums of transcedentality $\tau = 2n-1$, at n loops.

Since *wrapping problems*, delayed by supersymmetry, appear at $L+2$ loop order for twist-L operators, the BAE for twist-3 are reliable up to *four loops* (including, at the fourth loop, the dressing factor).

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The anomalous dimensions of these states are the eigenvalues $\gamma_L(N; g)$ of the dilatation operator — integrable Hamiltonian.

These values were obtained by solving numerically the Bethe Ansatz equations (BAE), order by order in g^2, and guessing the answer in terms of harmonic sums of transcedentality $\tau = 2n-1$, at n loops.

Since *wrapping problems*, delayed by supersymmetry, appear at $L+2$ loop order for twist-L operators, the BAE for twist-3 are reliable up to *four loops* (including, at the fourth loop, the dressing factor).

Twist-3

The $\mathfrak{sl}(2)$ sector of planar $\mathcal{N} = 4$ SYM contains single trace states which are linear combinations of the basic operators

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Twist-3 : Answer

$$
\begin{aligned}
\gamma_3^{(1)} &= 4\,S_1\,, \\
\gamma_3^{(2)} &= -2\,(S_3 + 2\,S_1 S_2) \\
\gamma_3^{(3)} &= 5\,S_5 + 6\,S_2\,S_3 - 8\,S_{3,1,1} + 4\,S_{4,1} - 4\,S_{2,3} + S_1(4\,S_2^2 + 2\,S_4 + 8\,S_{3,1}), \\
\gamma_3^{(4)} &= \tfrac{1}{2}\,S_7 + 7\,S_{1,6} + 15\,S_{2,5} - 5\,S_{3,4} - 29\,S_{4,3} - 21\,S_{5,2} - 5\,S_{6,1} \\
&\quad -40\,S_{1,1,5} - 32\,S_{1,2,4} + 24\,S_{1,3,3} + 32\,S_{1,4,2} - 32\,S_{2,1,4} + 20\,S_{2,2,3} \\
&\quad +40\,S_{2,3,2} + 4\,S_{2,4,1} + 24\,S_{3,1,3} + 44\,S_{3,2,2} + 24\,S_{3,3,1} + 36\,S_{4,1,2} \\
&\quad +36\,S_{4,2,1} + 24\,S_{5,1,1} + 80\,S_{1,1,1,4} - 16\,S_{1,1,3,2} + 32\,S_{1,1,4,1} \\
&\quad -24\,S_{1,2,2,2} + 16\,S_{1,2,3,1} - 24\,S_{1,3,1,2} - 24\,S_{1,3,2,1} - 24\,S_{1,4,1,1} \\
&\quad -24\,S_{2,1,2,2} + 16\,S_{2,1,3,1} - 24\,S_{2,2,1,2} - 24\,S_{2,2,2,1} - 24\,S_{2,3,1,1} \\
&\quad -24\,S_{3,1,1,2} - 24\,S_{3,1,2,1} - 24\,S_{3,2,1,1} - 24\,S_{4,1,1,1} - 64\,S_{1,1,1,3,1} \\
&\quad -8\,\beta\,S_1\,S_3.
\end{aligned}
$$

The last term, with $\beta = \zeta_3$, is the contribution from the dressing factor that appears in the BAE at the fourth loop.

Twist-3 : features

The twist-3 anomalous dimension has two characteristic features:

1. All harmonic functions $S_{\vec{a}}$ are evaluated at half the spin, $S_{\mathbf{a}} \equiv S_{\mathbf{a}}(N/2)$. On the integrability side, this does not look unwarranted, since only *even* N belong to the non-degenerate ground state of the magnet.
2. No negative indices appear at twist-3, while in the case of twist-2 negative index sums were present starting from the second loop.

At the $N \to \infty$ limit, the *minimal* anomalous dimension γ (corresponding to the ground state) must exhibit the universal (LBK-classical) ln N behaviour which depends neither on the twist, nor on the nature of fields under consideration. Computing analytically the large N asymptotics yields

$$\frac{\gamma_3(N)}{\ln N} = 4\,g^2 - \frac{2\pi^2}{3}\,g^4 + \frac{11\pi^4}{45}\,g^6 - \left(4\zeta_3^2 + \frac{73\pi^6}{630}\right)\,g^8 + \mathcal{O}(g^{10})\,,$$

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Twist-3 : Evolution Kernel (rough)

After processing thru $\gamma = \mathcal{P}(N + \frac{1}{2}\gamma)$, in series in $g^2 = \frac{N_c \alpha}{2\pi}$,

$$\begin{aligned} P^{(1)} &= 4\, S_1, \\ P^{(2)} &= -2\, S_3 - 4\, \zeta_2\, S_1, \\ P^{(3)} &= S_5 + 2\, \zeta_2\, S_3 + 4\, (S_{3,2} + S_{4,1} - 2\, S_{3,1,1}) \\ &\quad + 4\, S_1\, (2\, S_{3,1} - S_4 + 4\, \zeta_4) - 4\, S_1^2\, (S_3 - \zeta_3). \end{aligned}$$

The fourth loop kernel we split into two terms: $P^{(4)} = P_S^{(4)} + P_\zeta^{(4)}$.

$$\begin{aligned} P_S^{(4)} &= -8\left[S_{3,3} + S_{1,5} + 2S_{2,4} - 4(S_{2,1,3} + S_{1,2,3} + S_{1,1,4}) + 8S_{1,1,1,3}\right] S_1 \\ &+ \tfrac{3}{2}\, S_7 - 16\left(S_{1,6} + S_{4,3}\right) - 24\left(S_{2,5} + S_{3,4}\right) \\ &\quad + 48\left(S_{1,1,5} + S_{1,3,3} + S_{3,1,3}\right) + 64\left(S_{2,2,3} + S_{2,1,4} + S_{1,2,4}\right) \\ &\quad - 128\left(S_{1,1,1,4} + S_{2,1,1,3} + S_{1,2,1,3} + S_{1,1,2,3}\right) + 256\, S_{1,1,1,1,3}\,, \\ P_\zeta^{(4)} &= 8\zeta_4\, \mathcal{S}_1^3 - 4\left[\zeta_2\zeta_3 + 8\zeta_5\right] \mathcal{S}_1^2 - \left[4(\zeta_3 + 2\beta)\mathcal{S}_3 + 49\zeta_6\right] \mathcal{S}_1 \\ &\quad + (8\mathcal{S}_{1,1,3} - 4\mathcal{S}_{1,4} - 4\mathcal{S}_{2,3} - \mathcal{S}_5)\,\zeta_2 - 8\mathcal{S}_3\, \zeta_4. \end{aligned}$$

RR harmonic functions

Let $\vec{m} = \{m_1, m_2, \ldots, m_\ell\}$, and examine the recurrence relation

$$\tilde{\Phi}_{b,\vec{m}}(x) = -[\Gamma(b)]^{-1} \frac{x}{x-1} \int_x^1 \frac{dz\,(z+1)}{z^2} \ln^{b-1}\frac{z}{x} \cdot \tilde{\Phi}_{\vec{m}}(z),$$

where the single index function coincides with the image of the standard harmonic sum,

$$\tilde{\Phi}_a(x) = [\Gamma(a)]^{-1} \frac{x}{x-1} \ln^{a-1}\frac{1}{x} = \tilde{\mathcal{S}}_a(x).$$

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At the base of the recursion, we have (the *weight* $w \equiv \tau - \ell$)

$$\tilde{\Phi}_a(x) \;=\; \Big(-x\,\tilde{\Phi}_a(x^{-1})\Big) \cdot (-1)^{a-1} \equiv \Big(-x\,\tilde{\Phi}_a(x^{-1})\Big) \cdot (-1)^{w[a]}.$$

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An iteration increases transcedentality $\tau = \sum_{i=1}^{\ell} |m_i|$ of the function by b, and the length ℓ of the index vector by one, so that

$$w[\vec{m}] + b - 1 = w[b, \vec{m}].$$

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Twist-3 : Evolution Kernel (beautified)

Then, in terms of the physical coupling,

$$\mathbf{g}_{\mathrm{ph}}^2 \equiv \frac{N_c\,\alpha_{\mathrm{ph}}}{2\pi} = g^2 - \zeta_2\, g^4 + \frac{11}{5}\zeta_2^2\, g^6 - \left(\frac{73}{10}\zeta_2^3 + \zeta_3^2\right) g^8 + \ldots,$$

the perturbative series for the kernel, $\mathcal{P} = \sum_{n=1} \mathbf{g}_{\mathrm{ph}}^{2n}\, \mathcal{P}_{\mathrm{ph}}^{(n)}$, becomes

$$\begin{aligned}
\mathcal{P}_{\mathrm{ph}}^{(1)} &= 4\,\mathcal{S}_1, \\
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\mathcal{P}_{\mathrm{ph}}^{(3)} &= 3\,\mathcal{S}_5 - 2\,\Phi_{1,1,3} + \zeta_2 \cdot (-2\,\mathcal{S}_3), \\
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where

$$\begin{aligned}
\widehat{\mathcal{A}}_4 &= 2\,\widehat{\Phi}_{1,1,1,3} - \left(\widehat{\Phi}_{1,5} + \widehat{\Phi}_{3,3}\right) - \zeta_3\,\widehat{\mathcal{S}}_3, \\
\mathcal{B}_4 &= 16\,\Phi_{1,1,1,1,3} - 4\left(\Phi_{3,1,3} + \Phi_{1,3,3} + \Phi_{1,1,5}\right) - \frac{5}{2}\,\mathcal{S}_7.
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Since all harmonic functions involved have *even* weights *w*,
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Twist-3 vis a vis Twist-2

This result can be compared with the evolution kernel that generates the twist-2 universal anomalous dimension :

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\mathcal{P}^{(1)}_{\text{ph}} &= 4\,\mathcal{S}_1; \\
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\mathcal{P}^{(3)}_{\text{ph}} &= 8\,\mathcal{S}_5 - 24\,\Phi_{1,1,1,-2} - 8\,\zeta_2\,\mathcal{S}_3 \\
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Remark : in general, the GL parity is

$$\tilde{\Phi}_{\vec{m}}(x) = \left(-\,x\,\tilde{\Phi}_{\vec{m}}(x^{-1})\right)\cdot(-1)^{w[\vec{m}]}\cdot(-1)^{\#\text{ of negative indices}}$$

since

$$\frac{x}{x-1} \Longrightarrow \frac{x}{x+1}$$

Logs in γ and $\mathcal{P}$

General structure of the RR Evolution Kernel

$$\mathcal{P}(N) = \mathcal{S}_1 \cdot \left(\alpha_{\rm ph} + \widehat{\mathcal{A}}\right) + \mathcal{B}, \qquad \widehat{\mathcal{A}} = \mathcal{O}\left(1/N^2\right).$$

This feature is in a marked contrast with the anomalous dimension *per se*, whose large N expansion includes growing powers of $\log N$:

$$\gamma(N) = a \ln N + \sum_{k=0}^{\infty} \frac{1}{N^k} \sum_{m=0}^{k} a_{k,m} \ln^m N.$$

Easy to see from

$$\gamma_\sigma = \mathcal{P}(N + \sigma\gamma) \quad \Longrightarrow \quad \gamma_\sigma(N) = \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sigma \frac{d}{dN}\right)^{k-1} \left[\mathcal{P}(N)\right]^k,$$

Physically, the reduction of singularity of the large N expansion shows that the tower of subleading logarithmic singularities in the anomalous dimension is actually *inherited* from the first loop — the LBK-classical $\gamma^{(1)} = \mathcal{P}^{(1)} \propto S_1$, and the RREE generates them automatically !

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Easy to see from

$$\gamma_\sigma = \mathcal{P}(N + \sigma\gamma) \quad \Longrightarrow \quad \gamma_\sigma(N) = \sum_{k=1}^{\infty} \frac{1}{k!} \left(\sigma \frac{d}{dN}\right)^{k-1} \left[\mathcal{P}(N)\right]^k,$$

Physically, the reduction of singularity of the large N expansion shows that the tower of subleading logarithmic singularities in the anomalous dimension is actually *inherited* from the first loop — the LBK-classical $\gamma^{(1)} = \mathcal{P}^{(1)} \propto S_1$, and the RREE generates them automatically !

Fresh results

- RRE as a natural consequence of the conformal invariance
  "Anomalous dimensions of high-spin operators beyond the leading order"
  Benjamin Basso & Gregory Korchemsky *hep-th/0612247*
- *"N=4 SUSY Yang-Mills: three loops made simple(r)"*
  D-r & Pino Marchesini *hep-th/0612248*
- *"Anomalous dimensions at twist-3 in the sl(2) sector of N=4 SYM"*
  Matteo Beccaria *0704.3570 [hep-th]*
- Bethe Ansatz fails ("maximally") at 4 loops for twist-2
  "Dressing and Wrapping"
  Kotikov, Lipatov, Rej, Staudacher & Velizhanin *0704.3586 [hep-th]*
- twist-3 gaugino = twist-2 "universal"
  "Universality of three gaugino anomalous dimensions in N=4 SYM"
  Beccaria *0705.0663 [hep-th]*
- *"Twist 3 of the sl(2) sector of N=4 SYM and reciprocity respecting evolution"*
  Beccaria, D-r & Marchesini

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$$\frac{\text{clever 2nd loop}}{\text{clever 1st loop}} < 2\% \qquad \left(\begin{array}{c} \text{Heavy quark fragmentation} \\ \text{D-r, Khoze \& Troyan , PRD 1996} \end{array} \right)$$

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Employ $\mathcal{N}=4$ SYM to simplify the essential part of the QCD dynamics

Conclusions

- A steady progress in high order perturbative QCD calculations is worth accompanying by reflections upon the origin and the structure of higher loop correction effects
- Reformulation of parton cascades in terms of Gribov–Lipatov reciprocity respecting evolution equations (RREE)
  - reduces complexity by (at leat) an order of magnitude
  - improves perturbative series (less singular, better "converging")
  - links interesting phenomena in the DIS and e^+e^- annihilation channels
- The Low theorem should be part of theor.phys. curriculum, worldwide
- Complete solution of the $\mathcal{N}=4$ SYM QFT should provide us with a *one-line-all-orders* description of the major part of QCD parton dynamics
- Long live QFT, and perturbative QCD !

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Back to Hadrons at high energies

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- Colour in quark scattering

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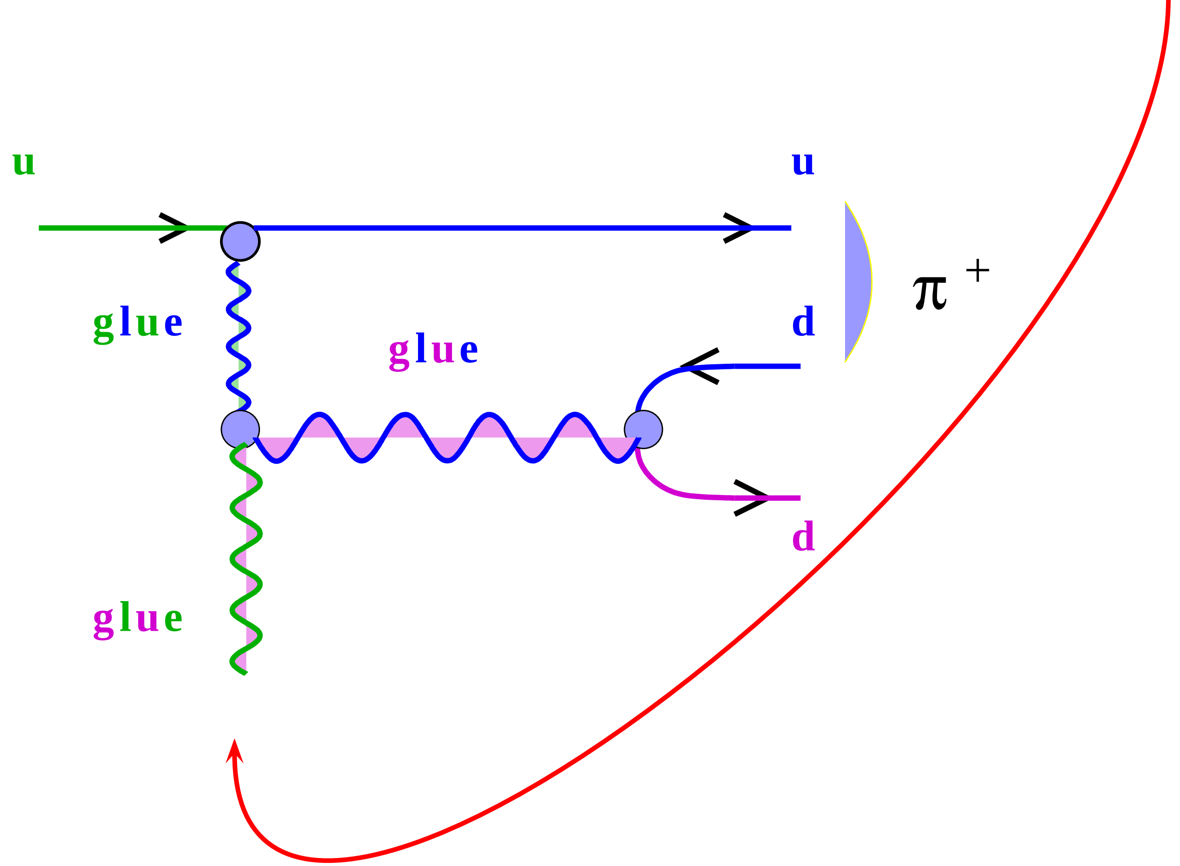
- Confinement in strong Colour field

Colour in Quark scattering

Quark inelastic scattering scenario:

Colour in Quark scattering

Quark inelastic scattering scenario: gluon exchange



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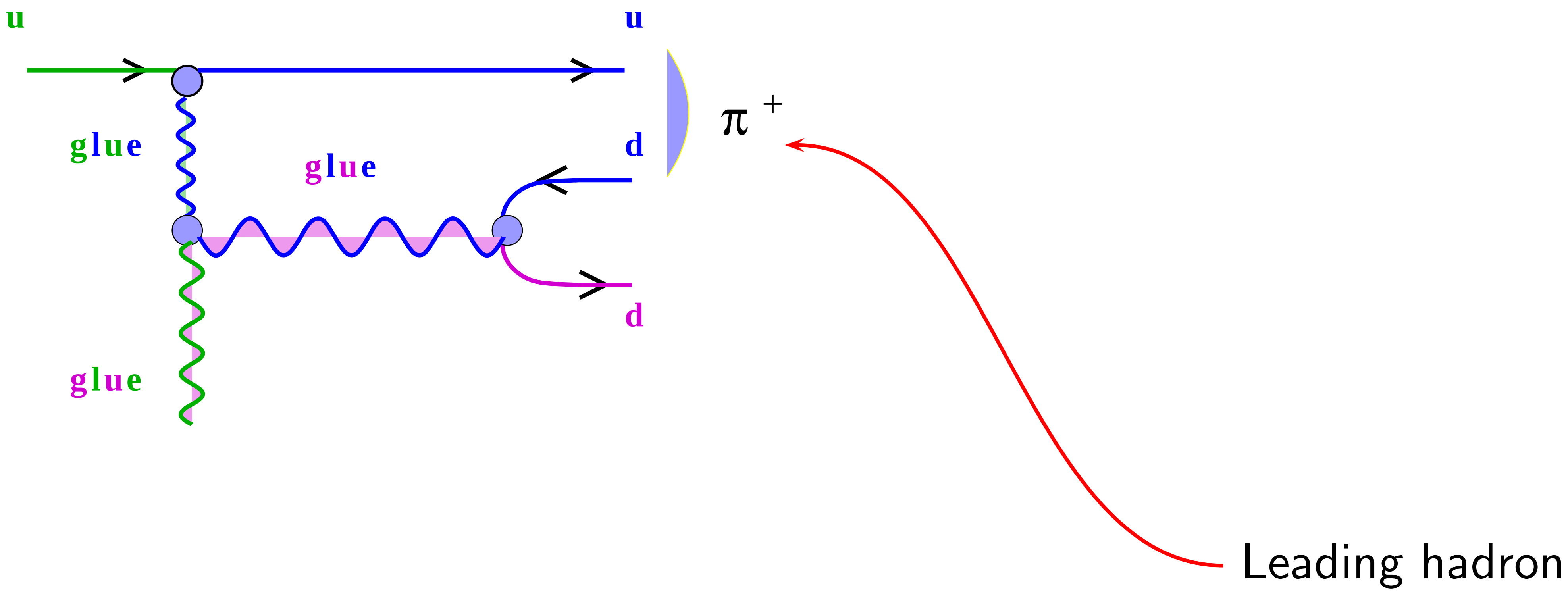

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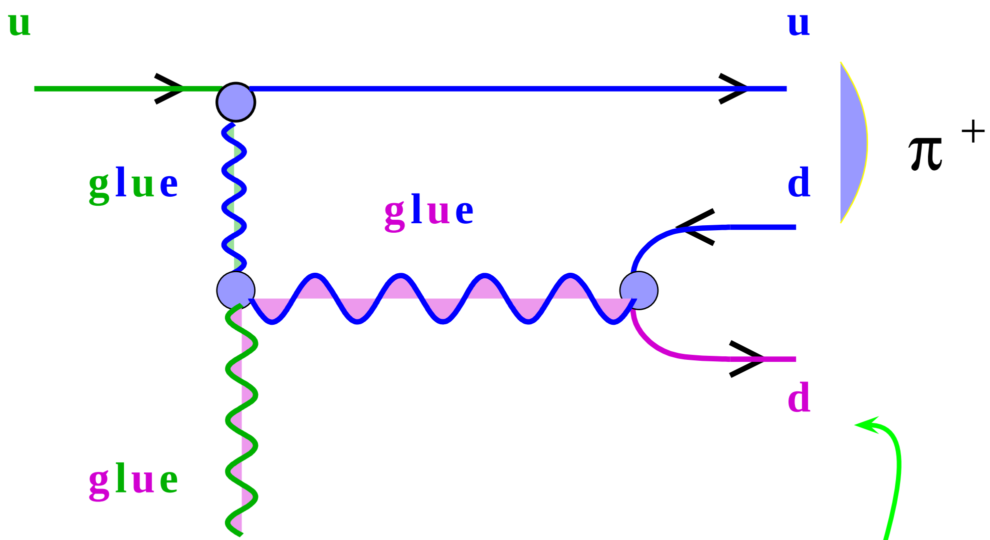
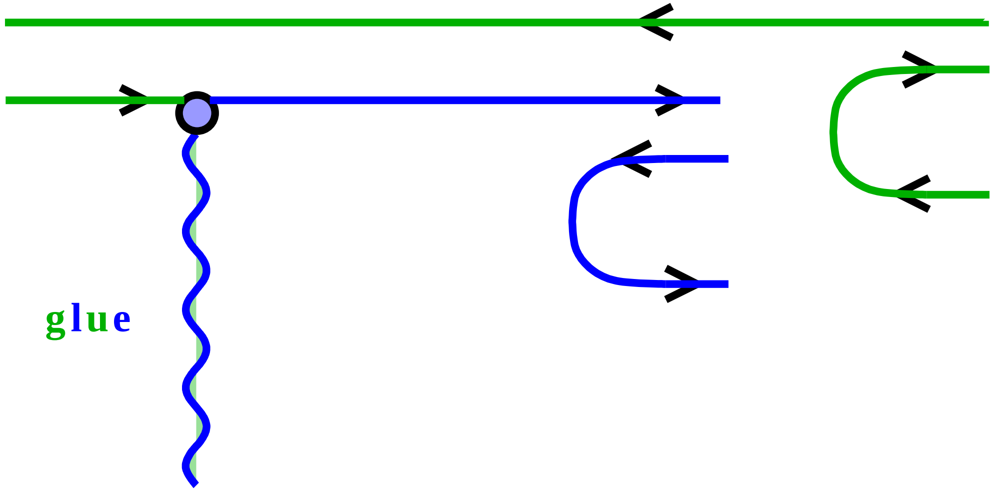


Feynman plateau

Colour in Quark scattering

Meson inelastic scattering scenario: gluon exchange



= two "quark chains"

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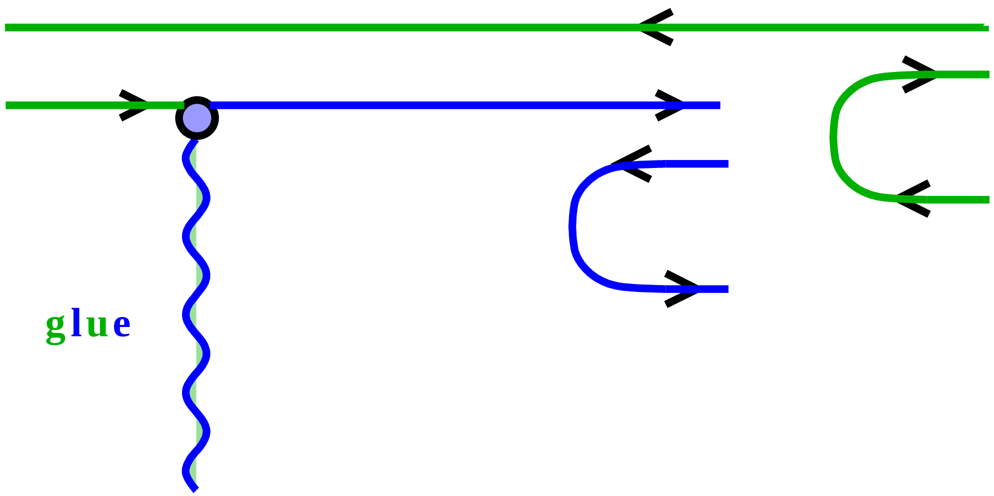

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Look now at the *proton* projectile:

Painting the proton

Single scattering scenario:

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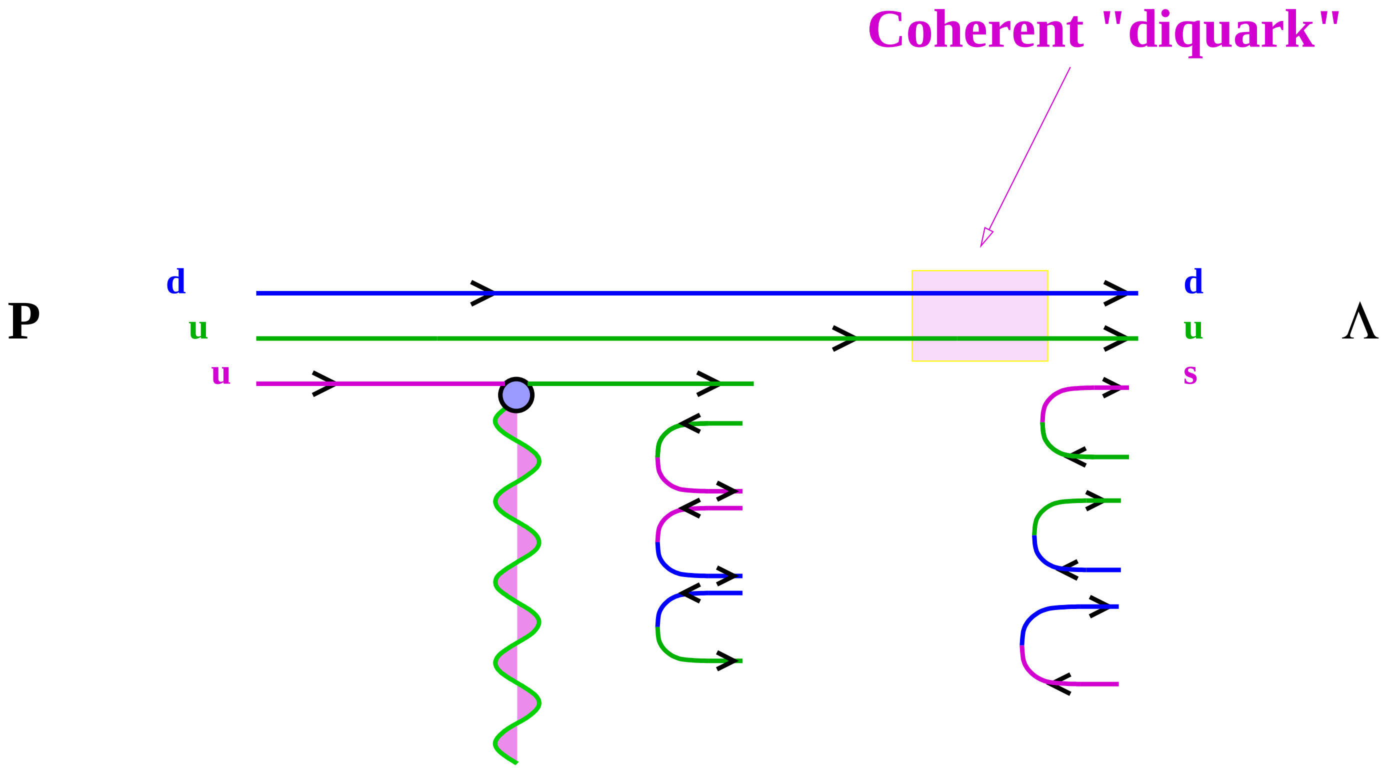

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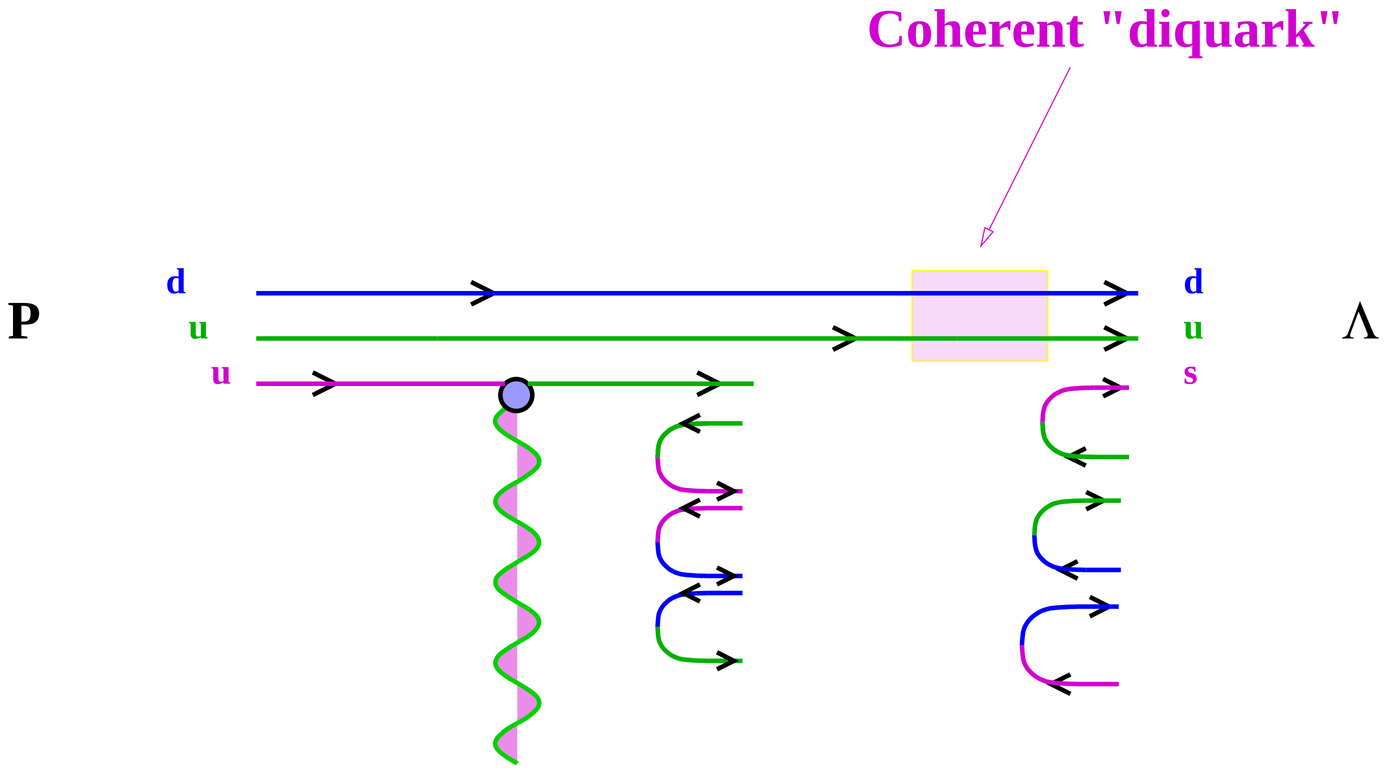

Coherence of the *diquark* ain't broken:

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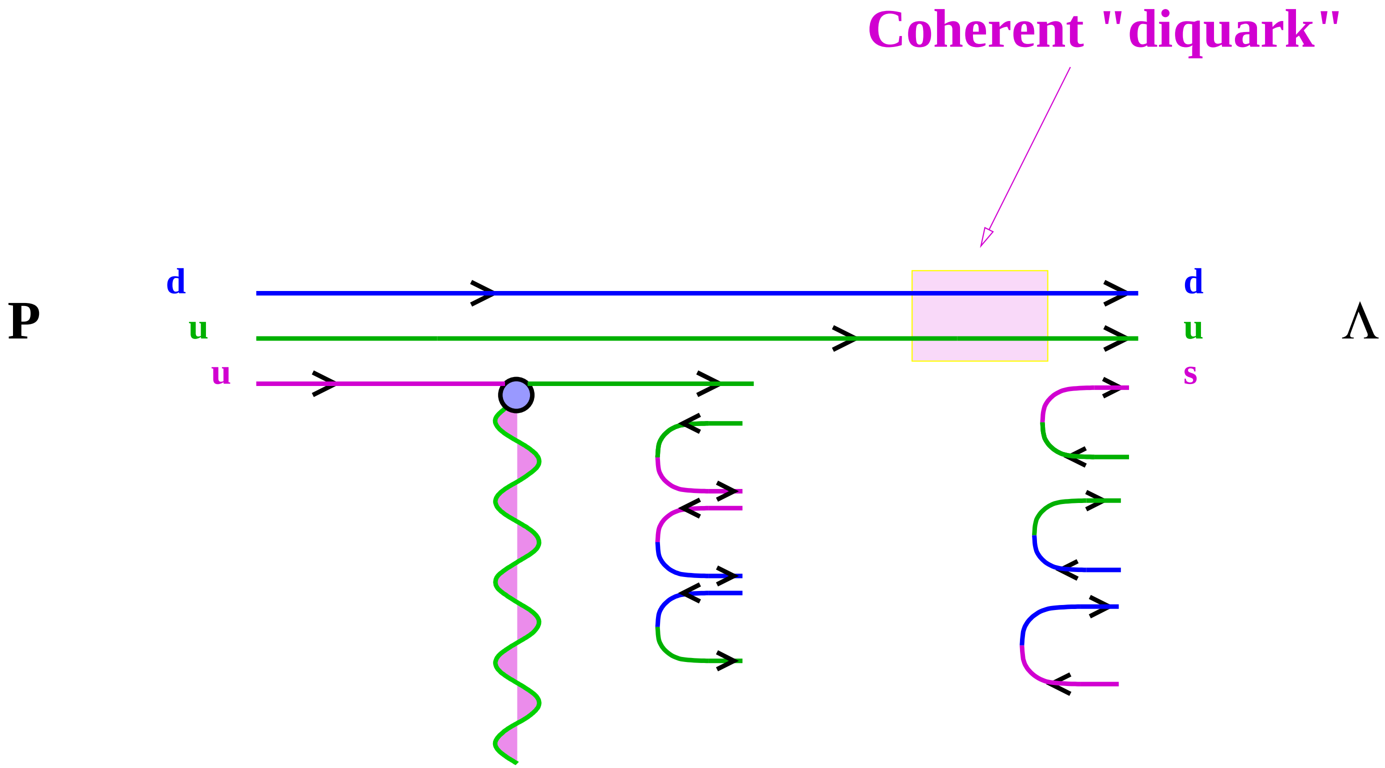

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$\Longrightarrow$ a Leading Baryon: $B(1) \rightarrow B(2/3) + M(1/3) + \ldots$

Repainting the Proton

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Kick it *twice* to break the Colour Coherence of the Valence Quarks:

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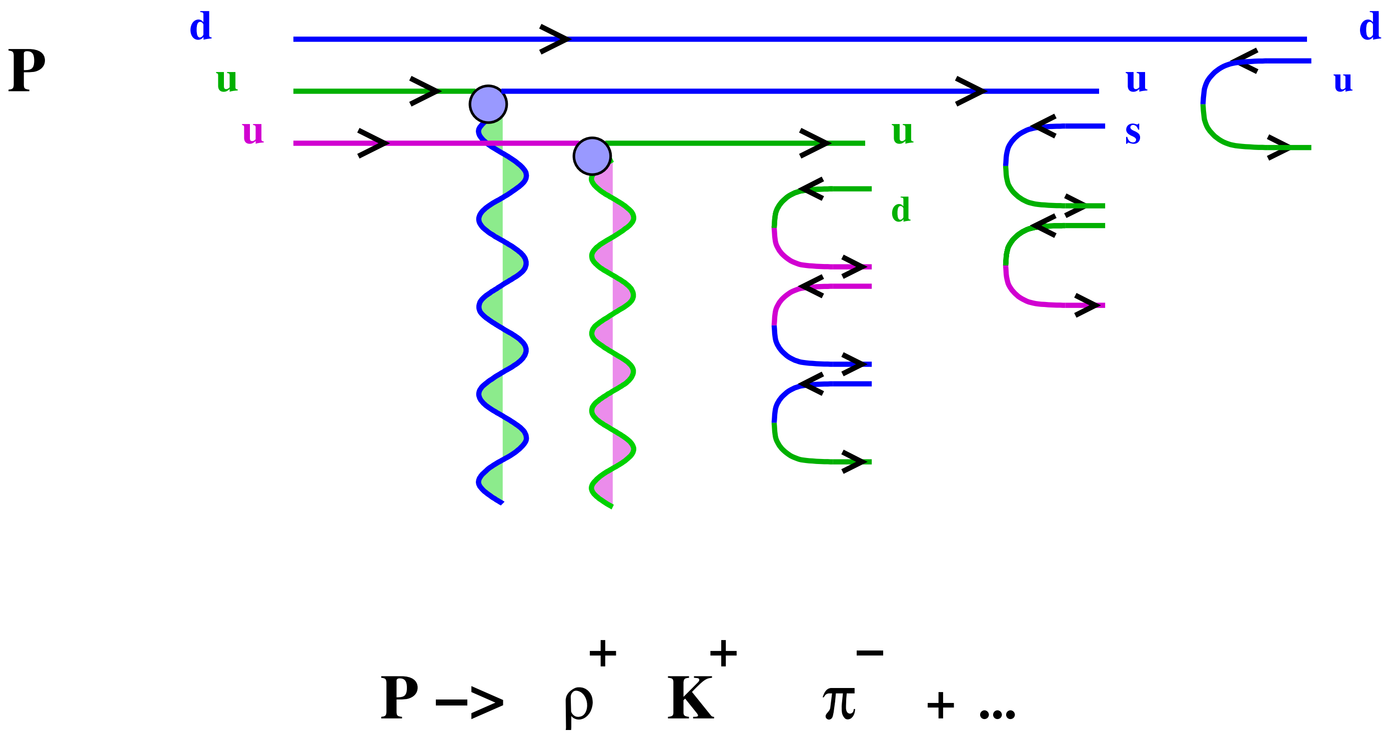

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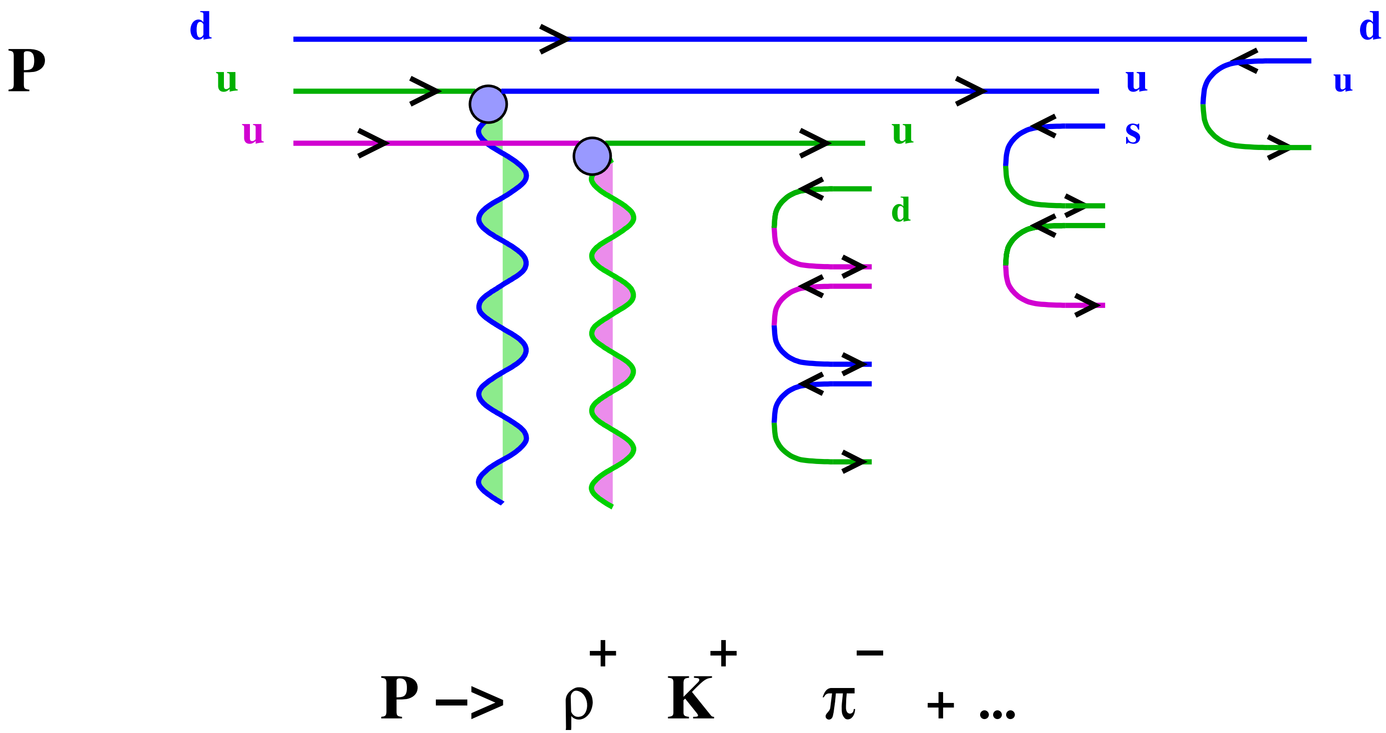

Proton is "*fragile*"

Expect the baryon quantum number *to sink* into the sea :

$$B(1) \to M(1/3) + M(1/3) + M(1/3) + \ldots + B(0)$$

Multiple Proton Scattering: pA, AB

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CERN $\sqrt{s} = 17$ GeV (NA49)

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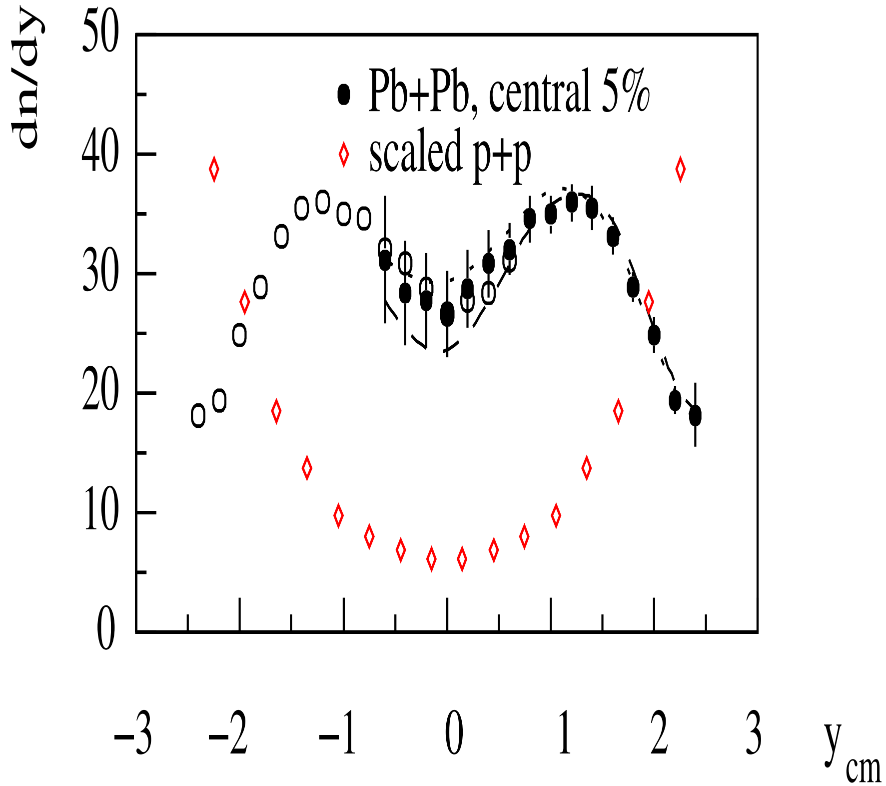

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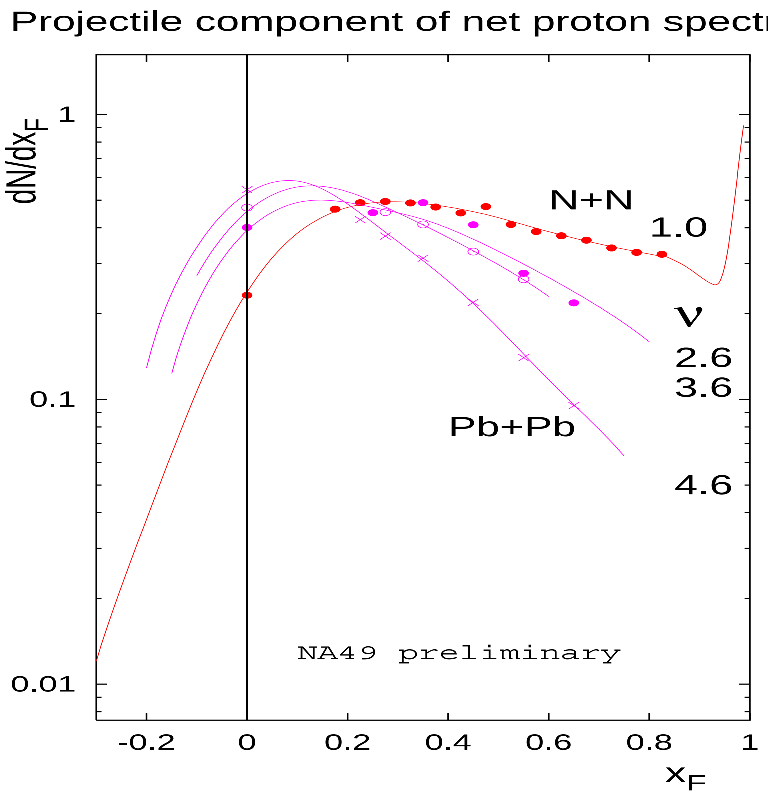

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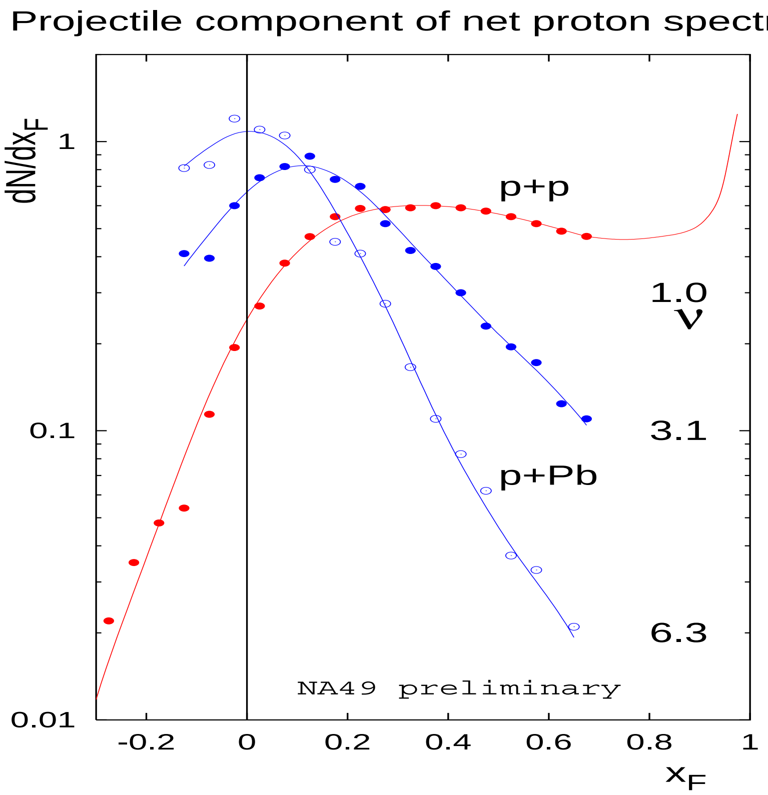

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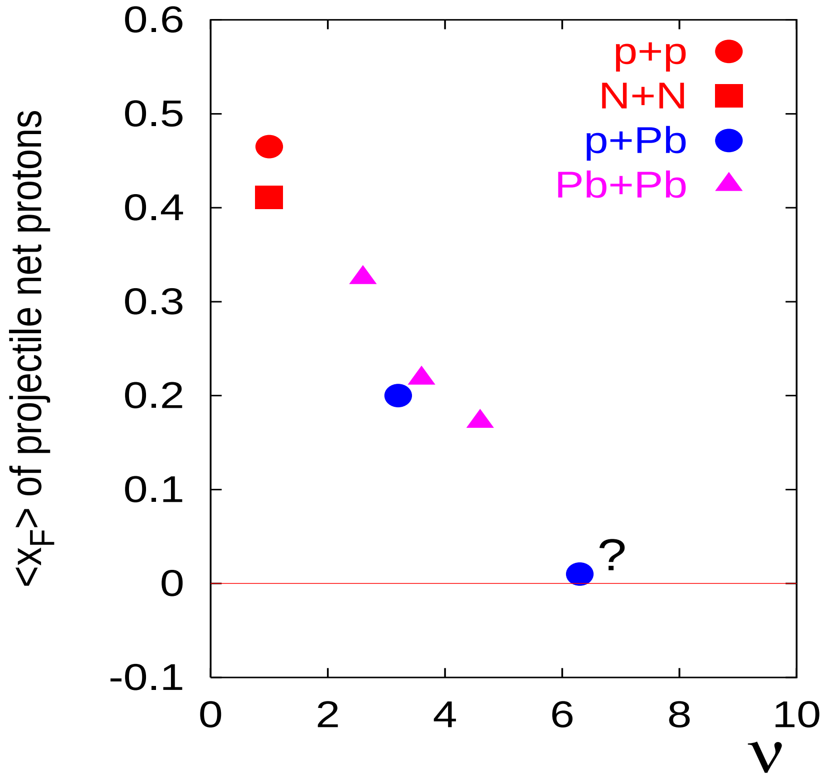

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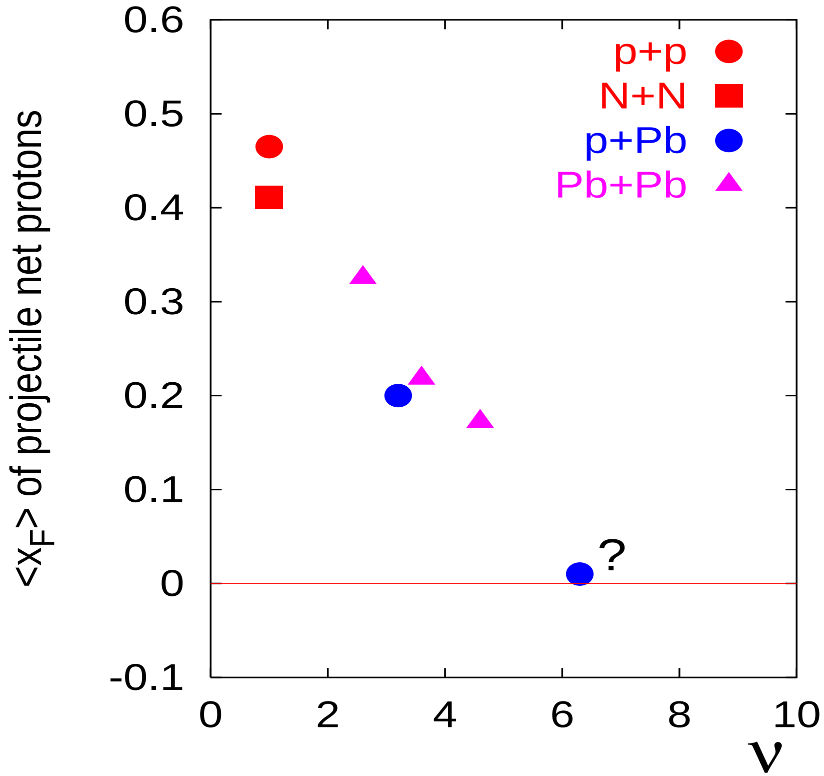

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Known as Proton Stopping.

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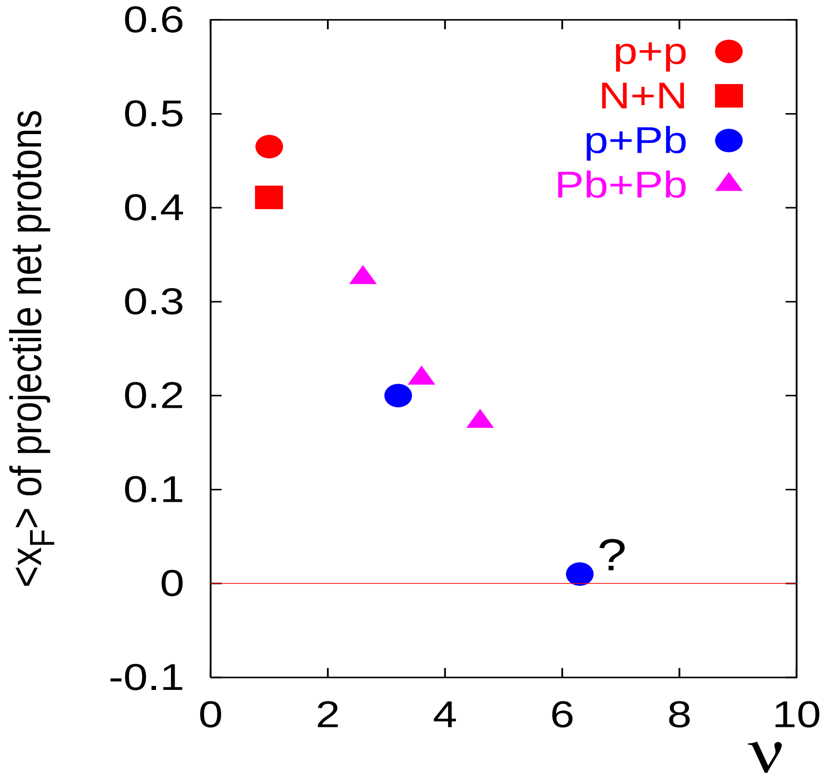

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Known as Proton Stopping. Better be known as Proton Decay

Multiple scattering and *strangeness*

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NA49: strangeness yield vs. target "*thickness*"

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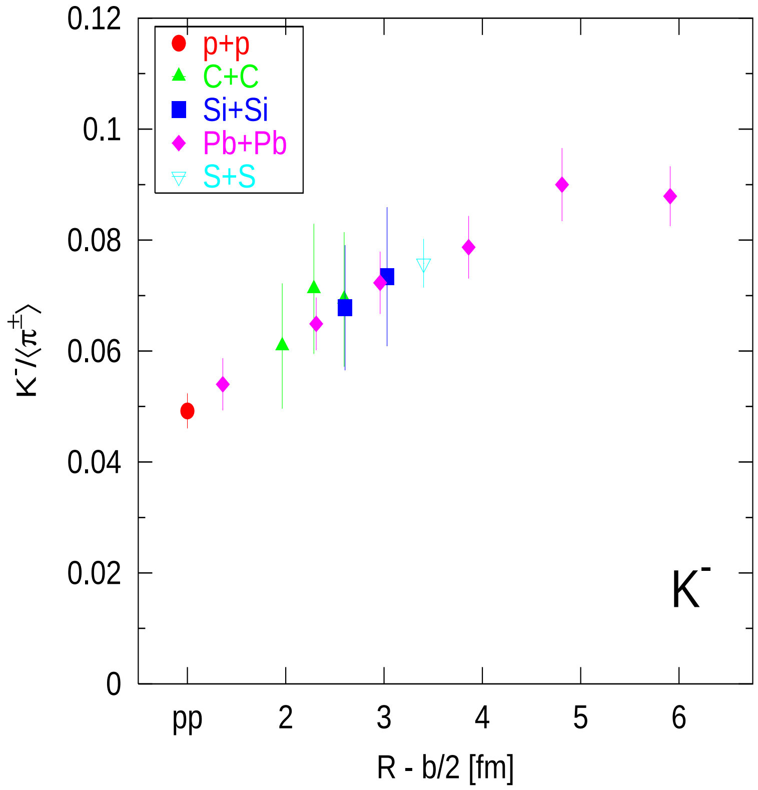
- Negative K to π yield



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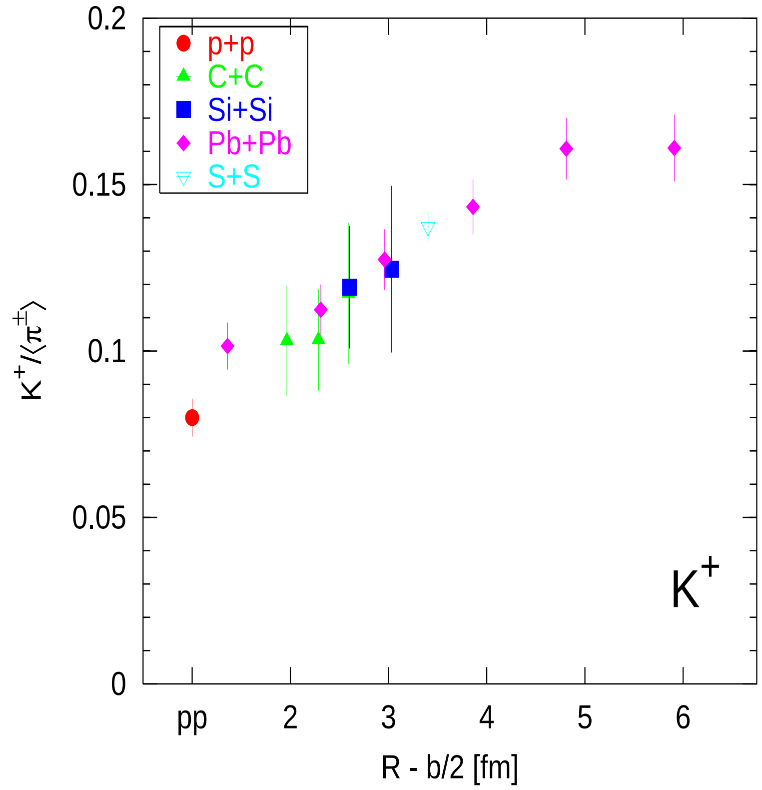

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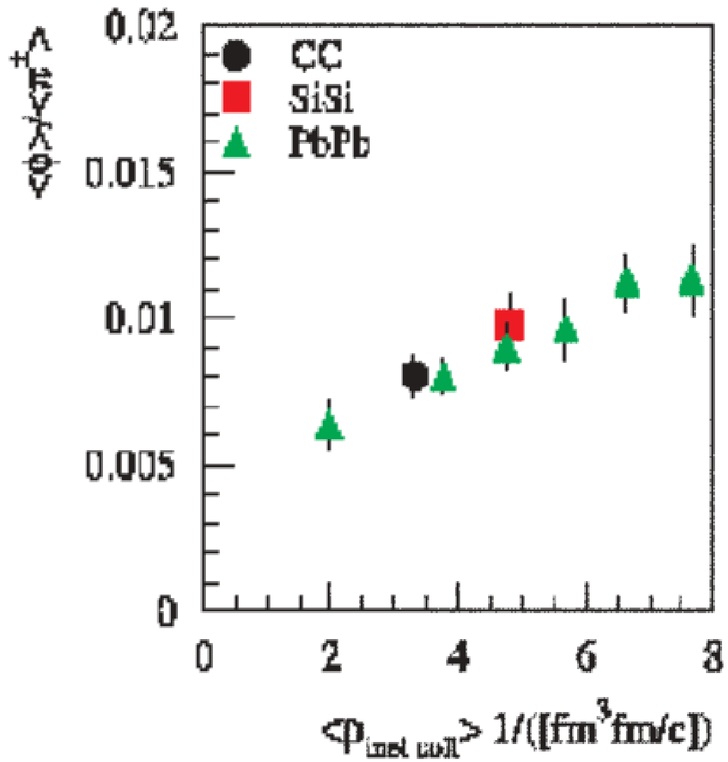

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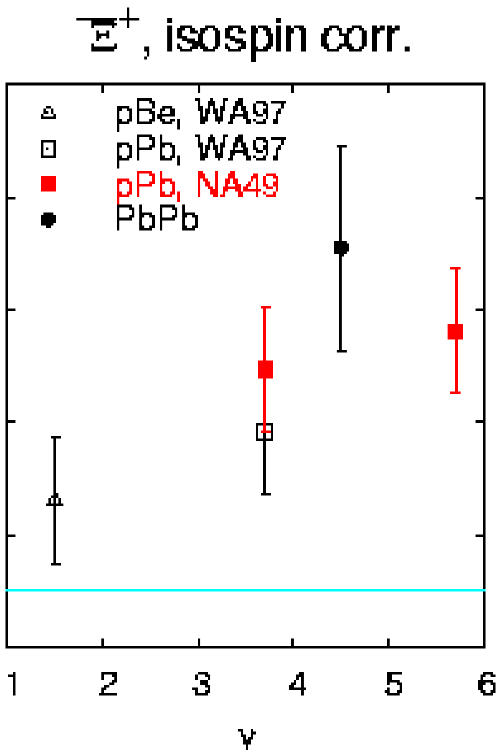

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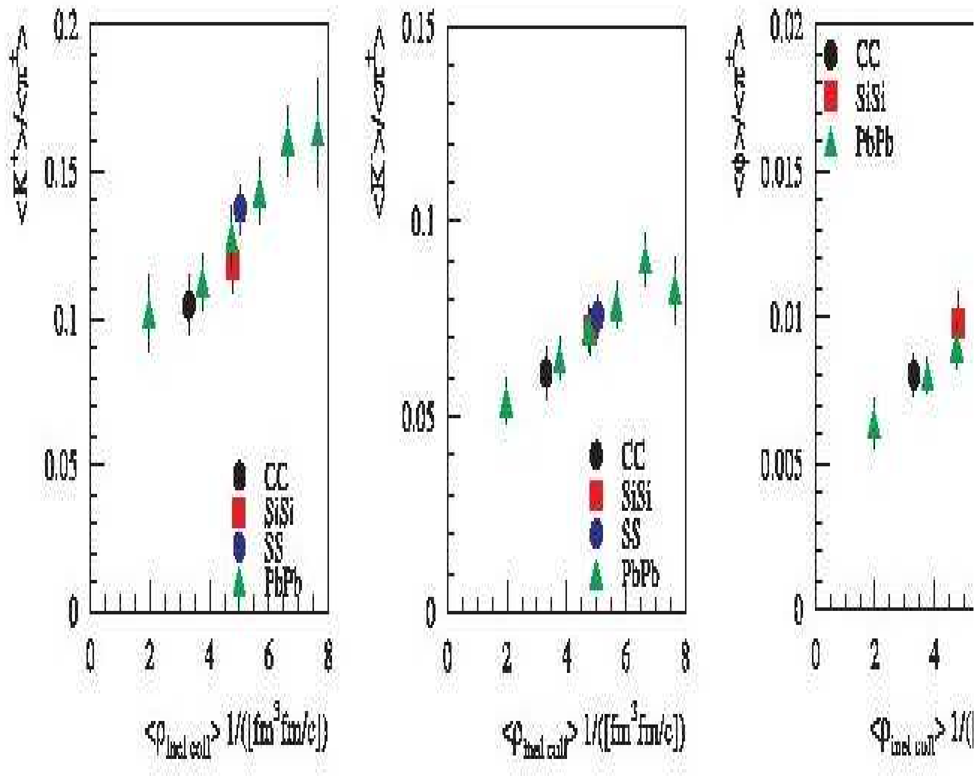

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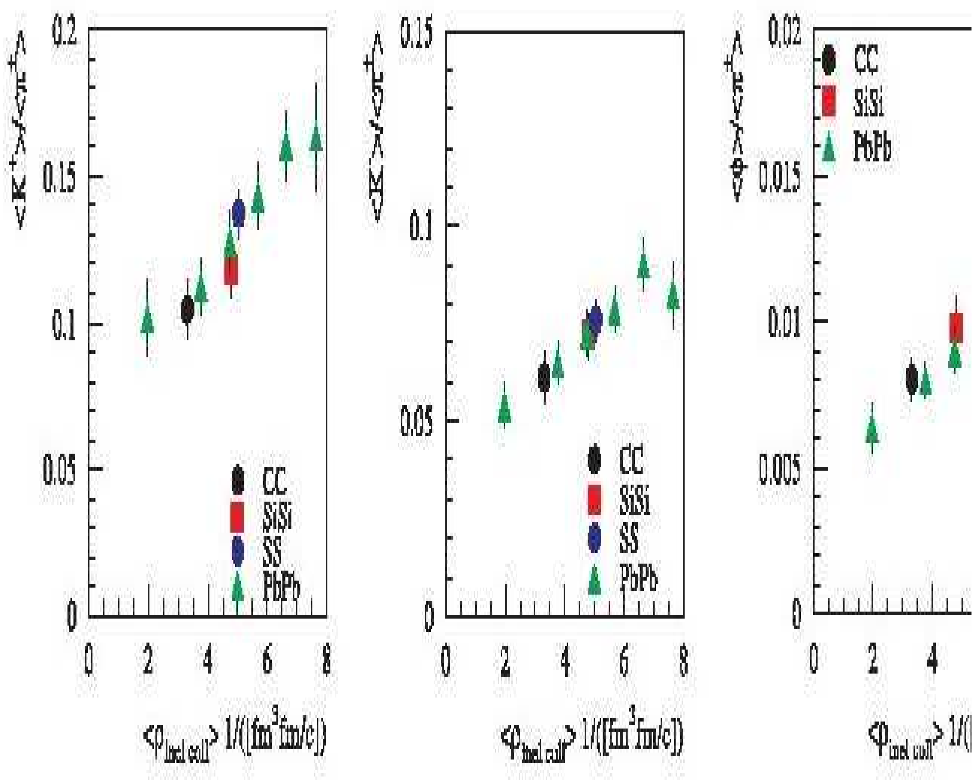

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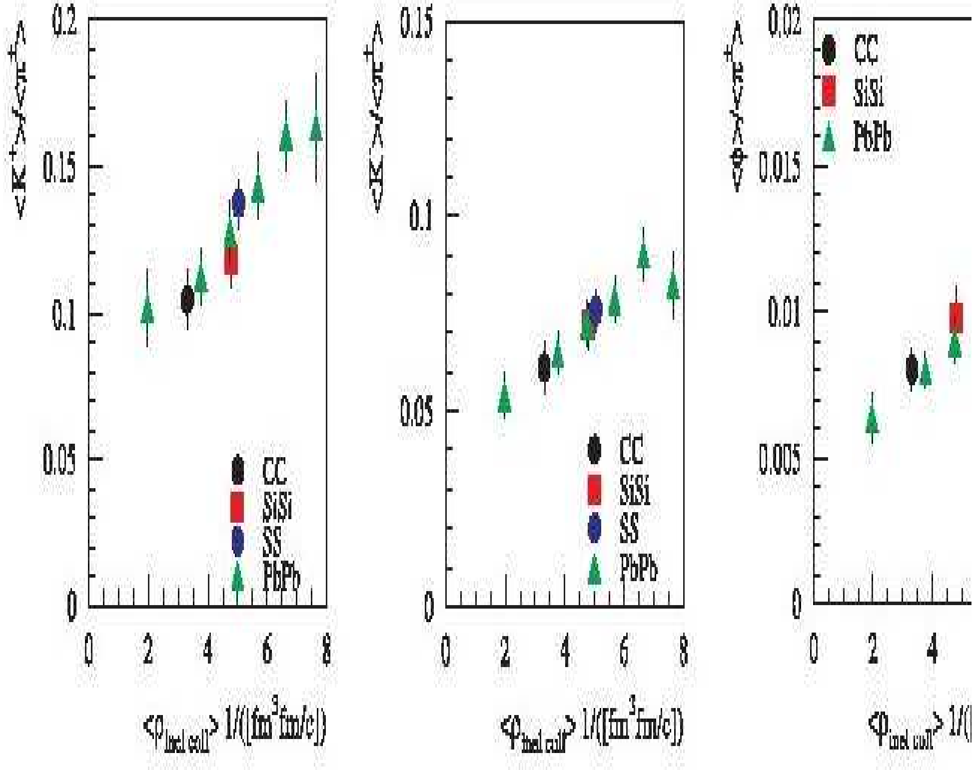

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develop with the number of inelastic collisions;
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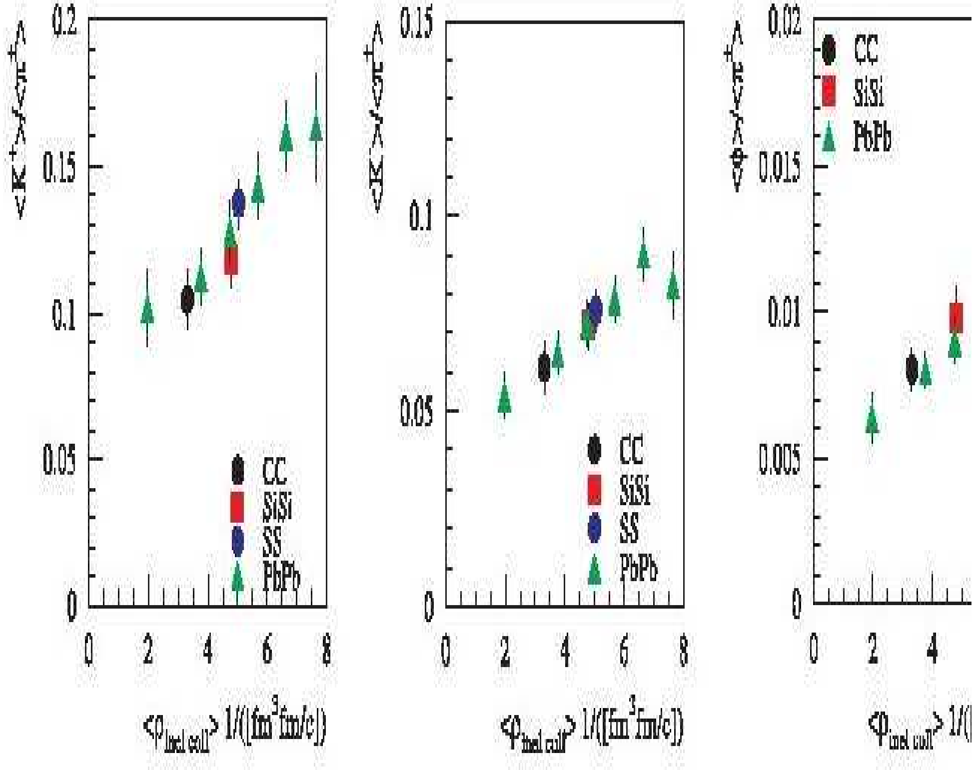

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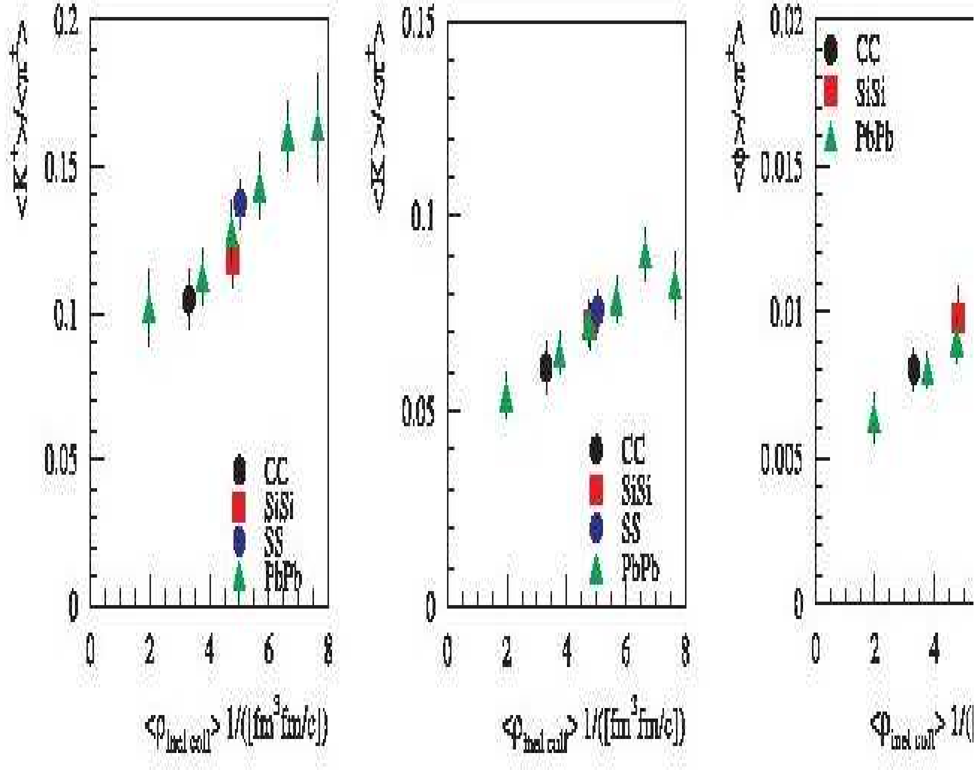

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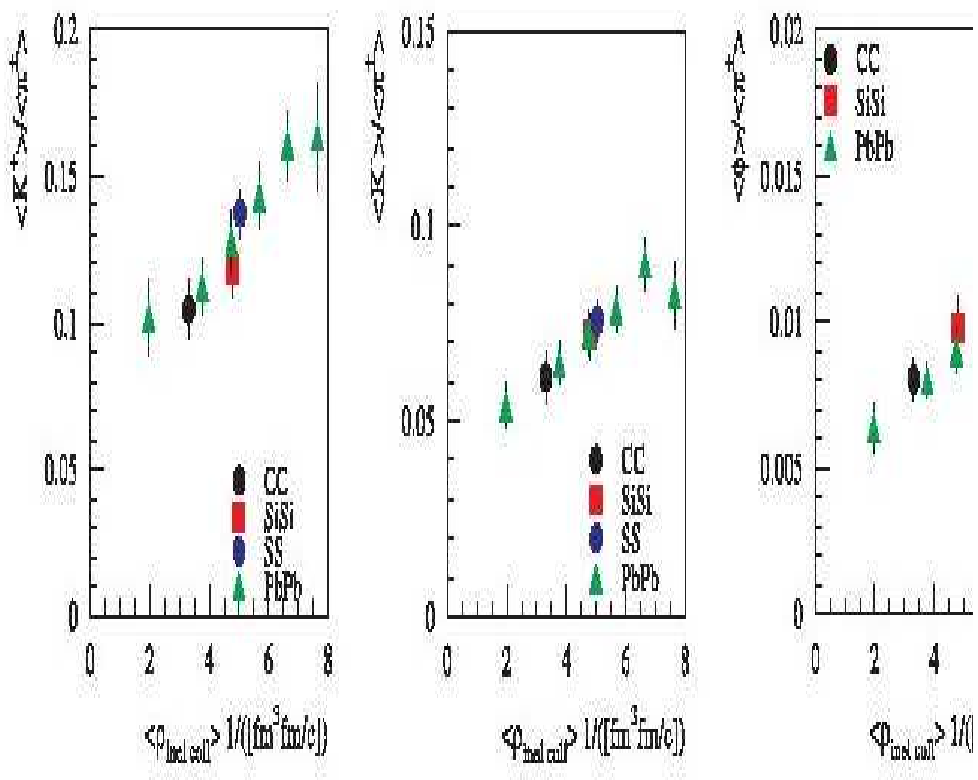

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Multiple collisions in pp

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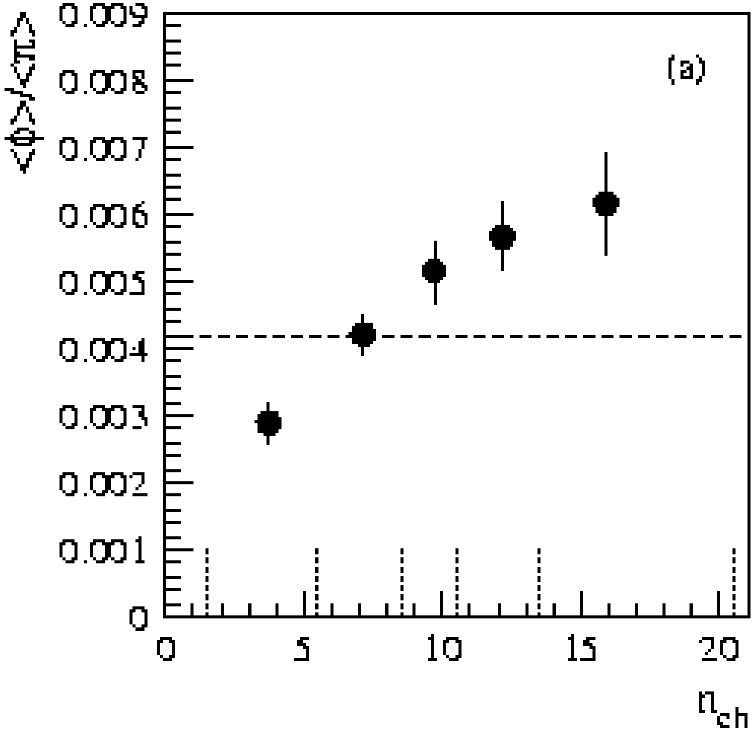
NA–49

ϕ to π

ratio in pp collisions

as a function of event multiplicity

Multiple collisions in pp



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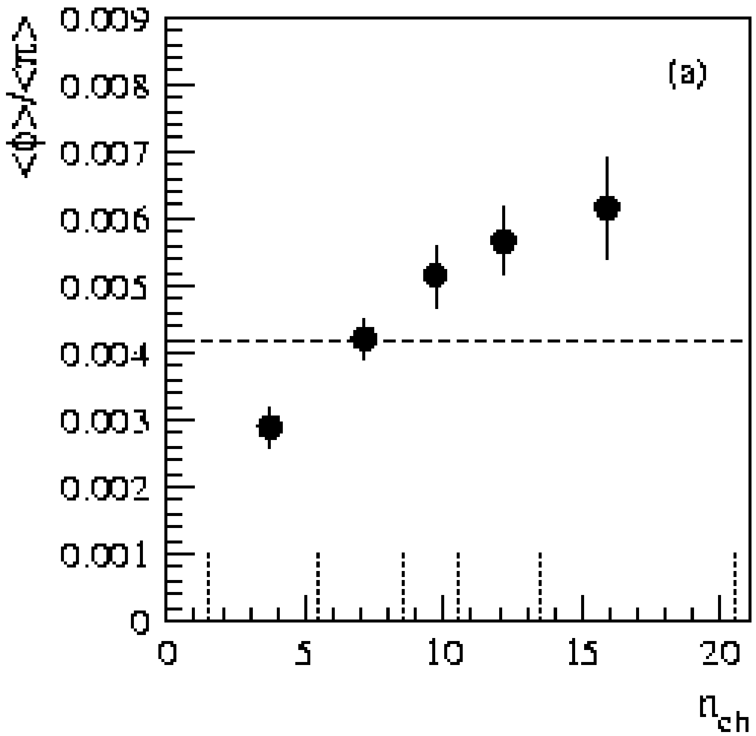

NA–49

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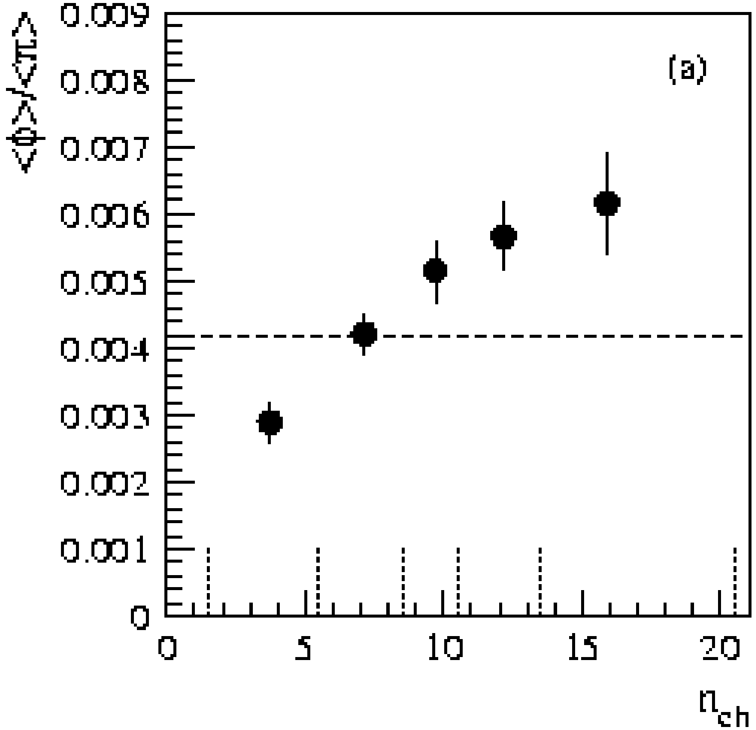
A way to trigger on multiple collisions

Multiple collisions in pp



NA–49

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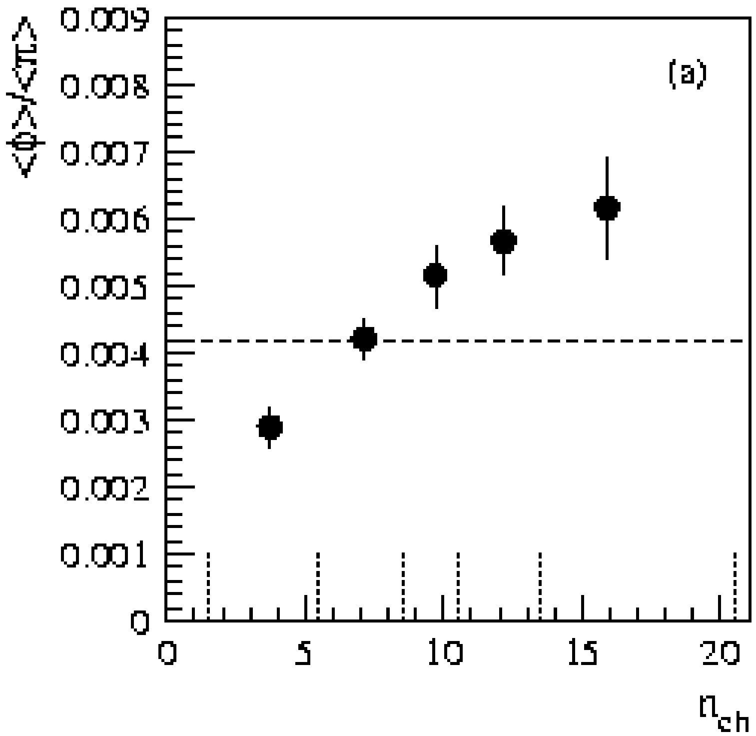

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(or to select *protons–perpetrators*, if you wish)

Would have been extremely interesting to correlate enhanced strangeness yield with *stopping* . . .

Multiple gluon exchange and Hadron Multiplicity

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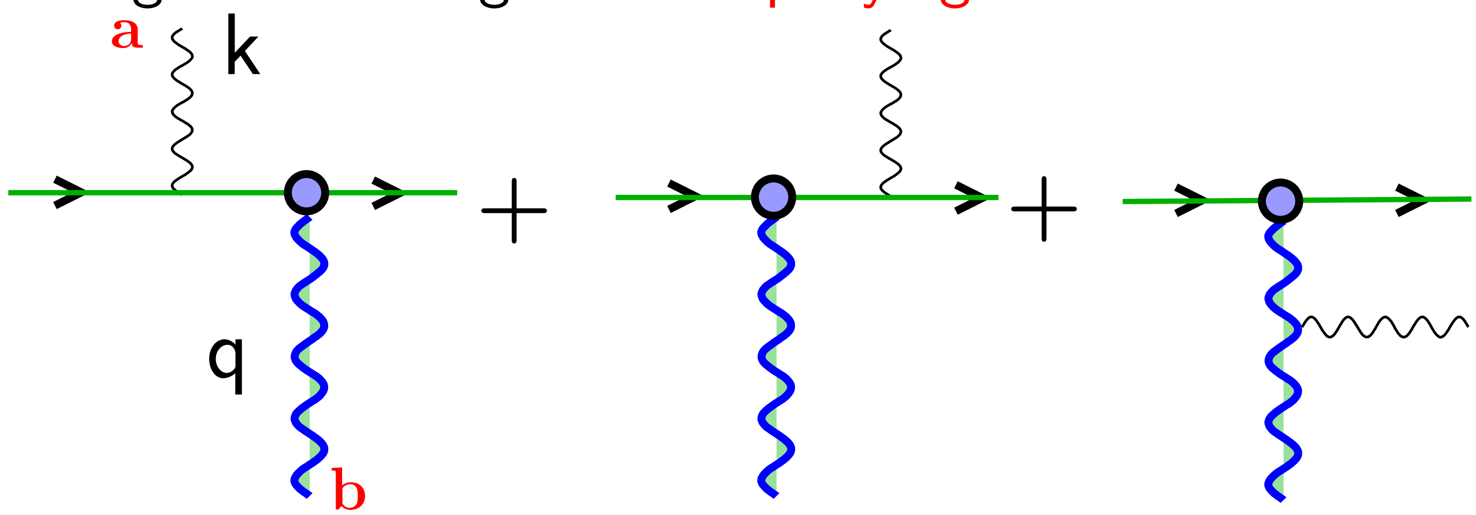
One gluon exchange: Accompanying radiation



$$-\frac{\mathbf{k}_\perp}{\mathbf{k}_\perp^2} \qquad +\frac{\mathbf{k}_\perp}{\mathbf{k}_\perp^2} \qquad +\frac{\mathbf{q}_\perp - \mathbf{k}_\perp}{(\mathbf{q}_\perp - \mathbf{k}_\perp)^2}$$

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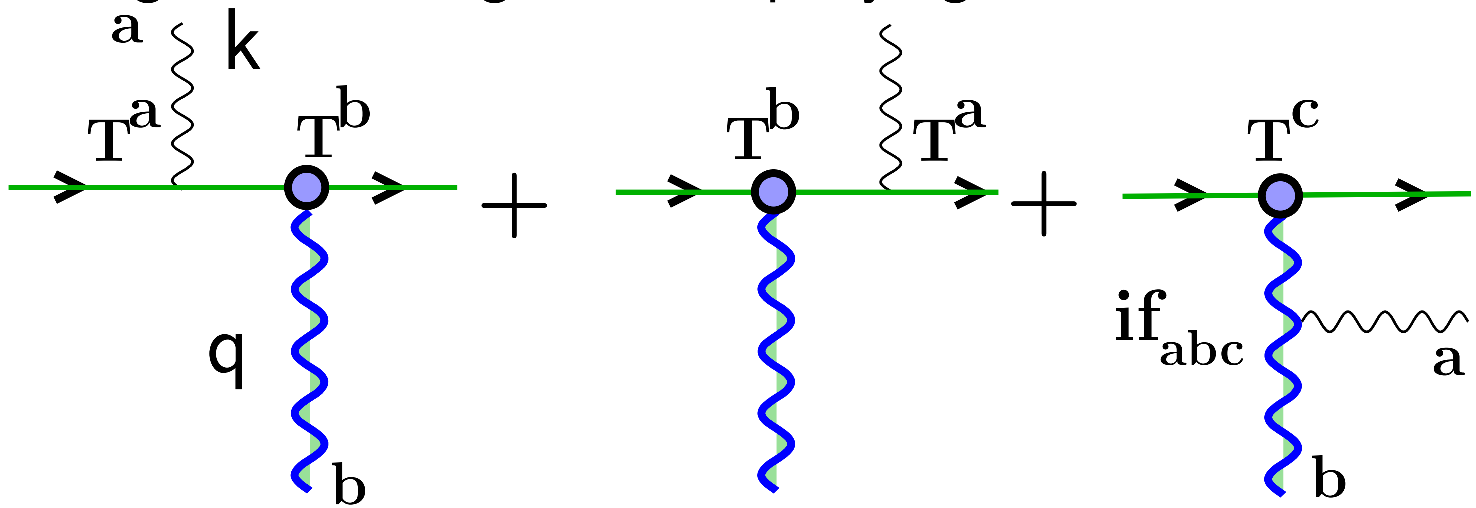

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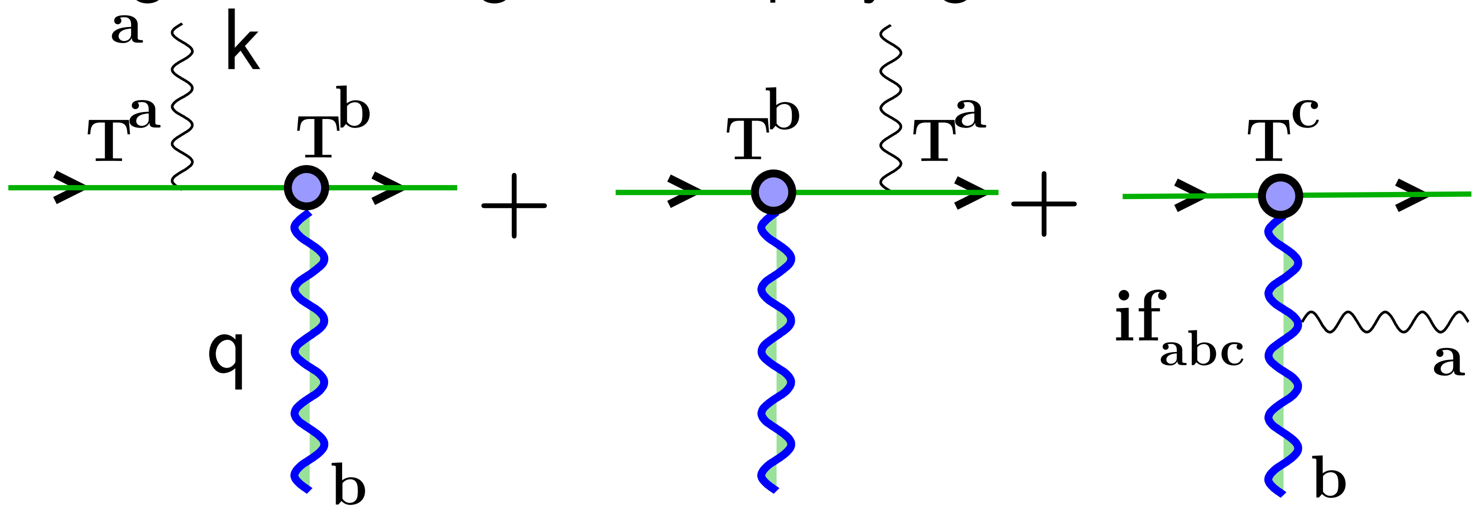

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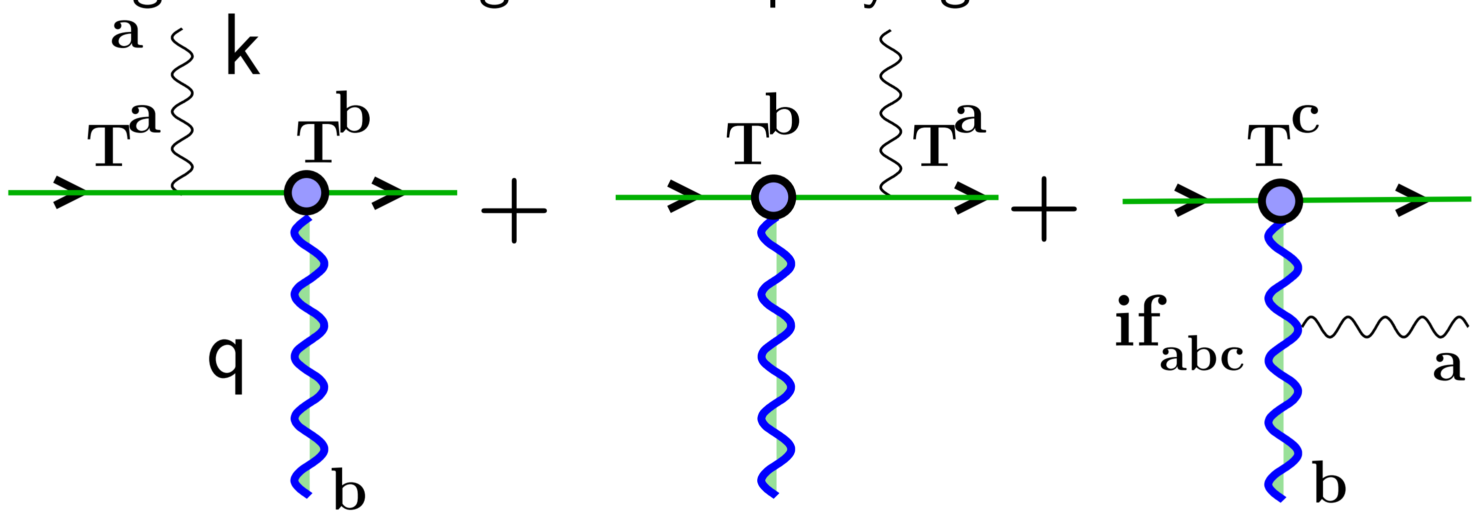

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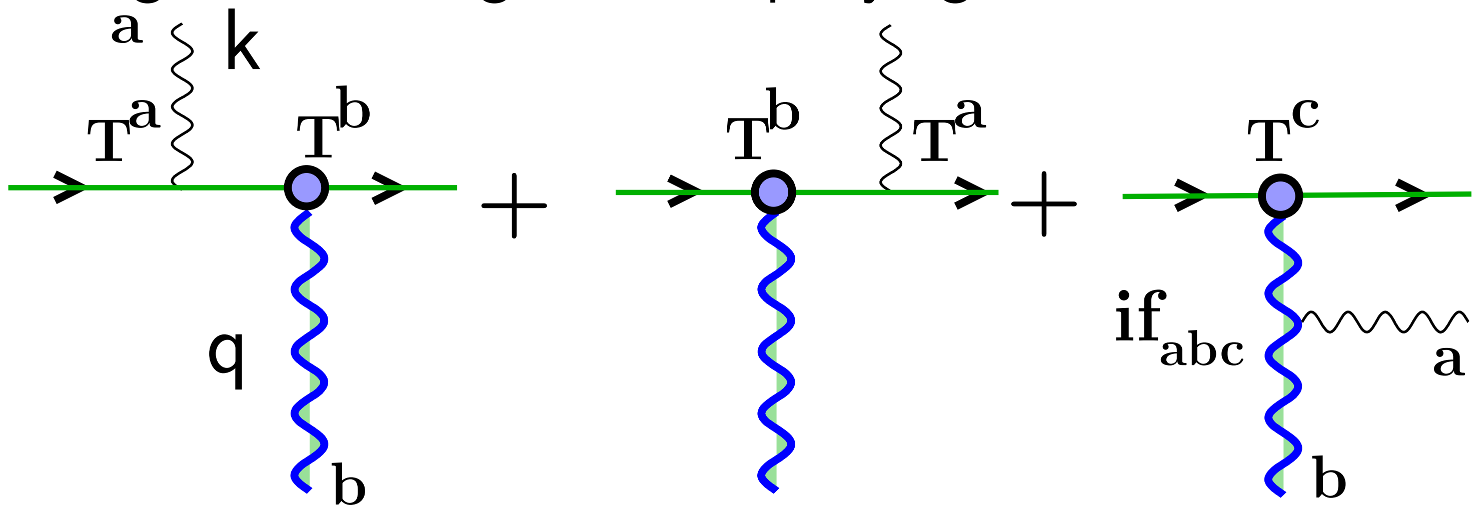

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- Secondary Gluon spectrum
  - $k_\perp < q_\perp \Longrightarrow$ finite transverse momenta;
  - $d\omega/\omega \Longrightarrow$ rapidity plateau

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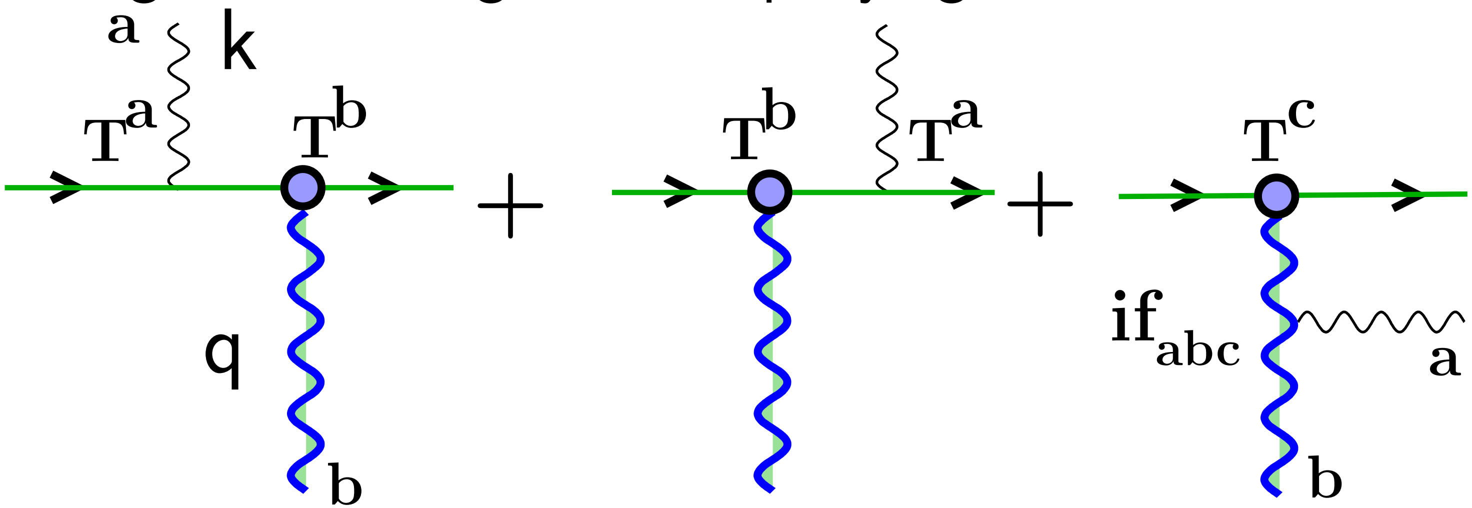

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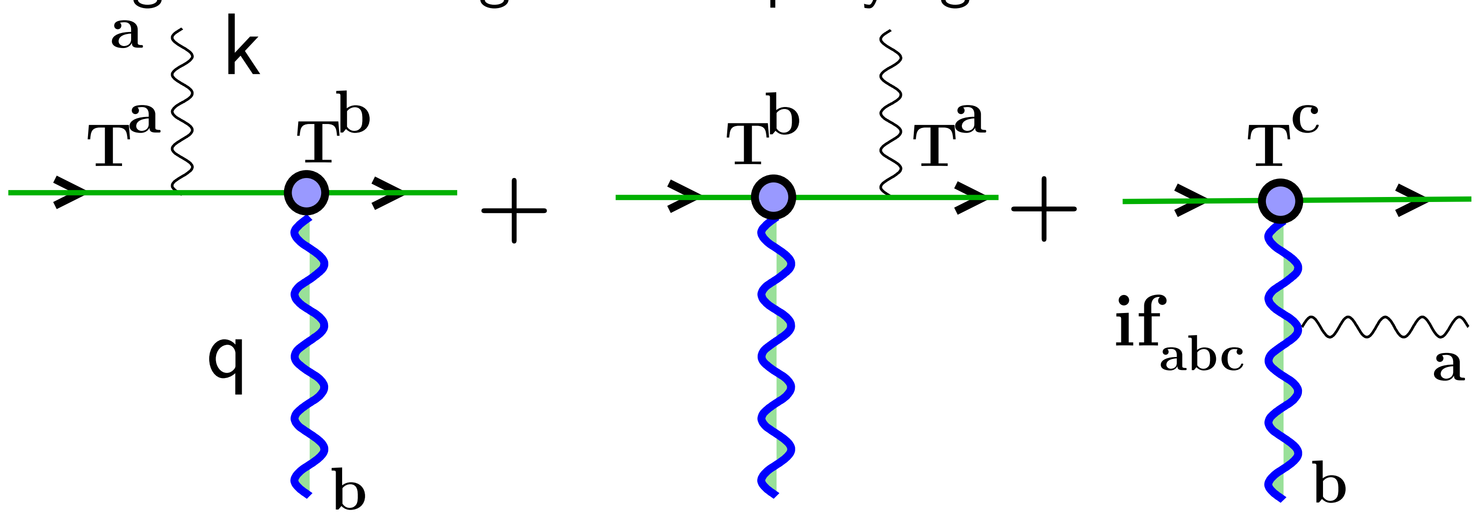

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- Multiple scattering of a quark (or a $q\bar{q}$ meson)

$\Longrightarrow$ NParticipant *scaling*

LPM effect in *hA* scattering

Inclusive spectrum of medium-induced gluon radiation:

$$\frac{\omega\, dn}{d\omega} \simeq \frac{\alpha_s}{\pi} \cdot \left[\frac{L}{\lambda}\right] \cdot \sqrt{\frac{\mu^2 \lambda}{\omega}}, \qquad \mu^2 \lambda < \omega < \mu^2 \lambda \left[\frac{L}{\lambda}\right]^2$$

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Bethe-Heitler spectrum (independent radiation off each scattering centre)

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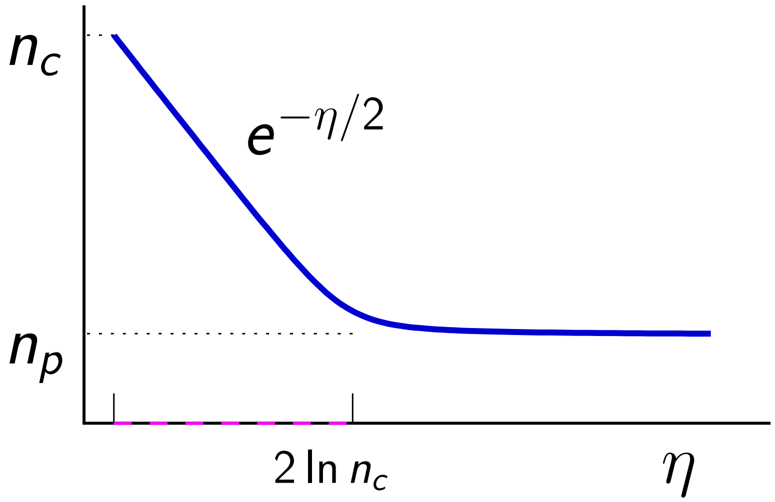

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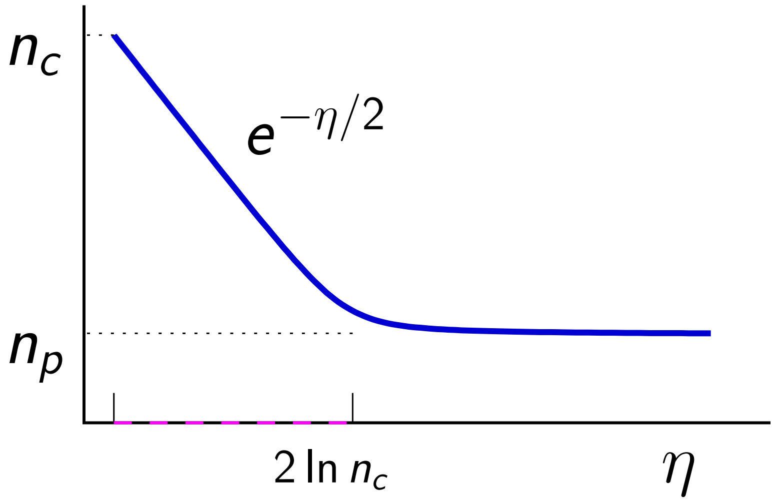

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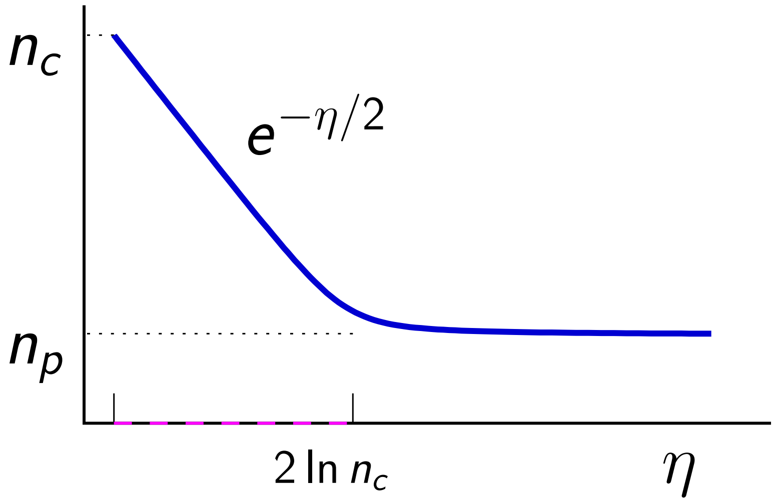
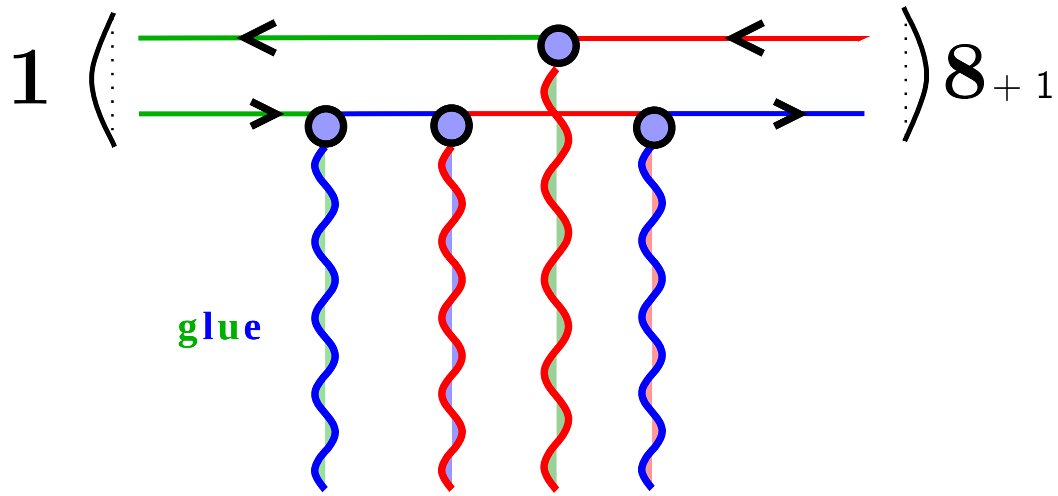


Coherent radiation = "Participant" scaling

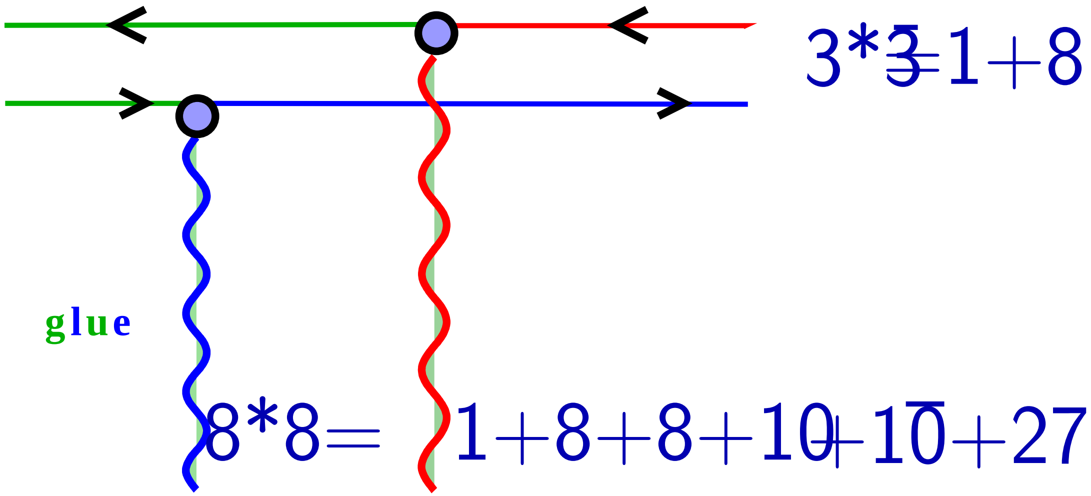
Transition region, down to "Collision" scaling;
occupies finite rapidity range (fragmentation of the nucleus)

Colour capacity



Multiple collisions of a (2-quark) pion

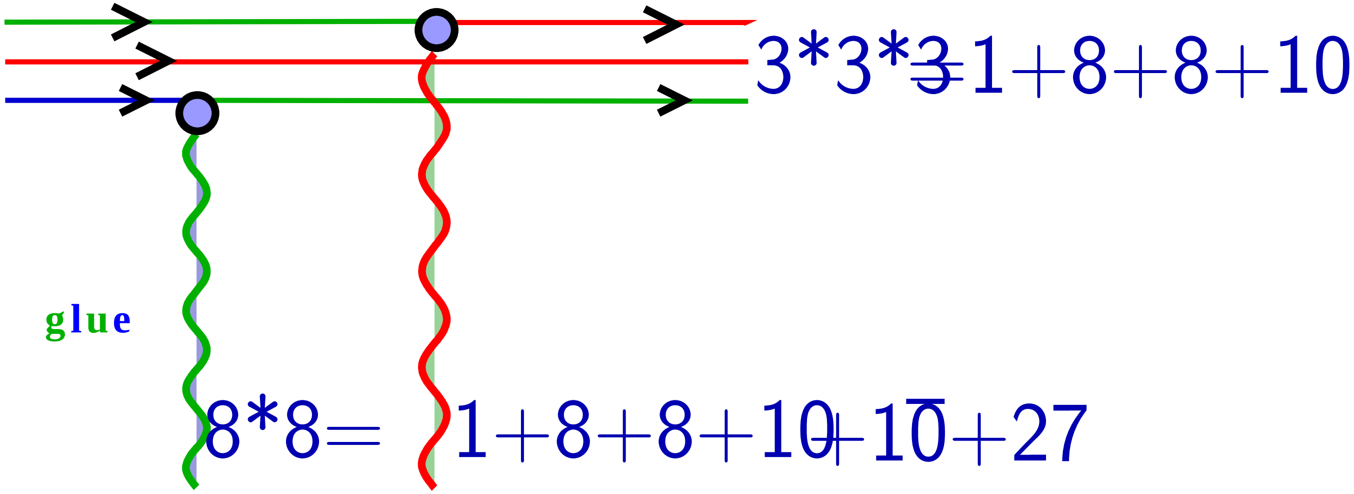
Colour capacity



Consider double scattering (two gluon exchange)

In meson scattering only two colour representations can be realized

Colour capacity



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The (3-quark) proton is more *capacious*, but still . . .

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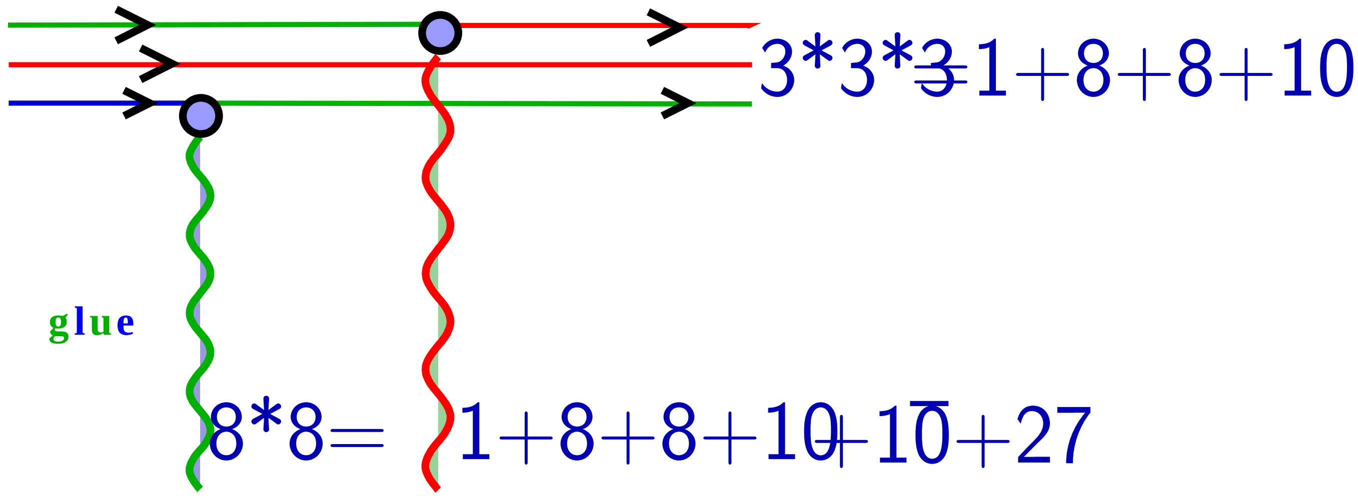

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Calculate the average colour charge of the two-gluon system:

$$\frac{1}{64}\cdot 0+\frac{8+8}{64}\cdot 3+\frac{10+\overline{10}}{64}\cdot 6+\frac{27}{64}\cdot 8=6=2\cdot 3 \Longrightarrow$$ double density of hadrons =2 Pomerons

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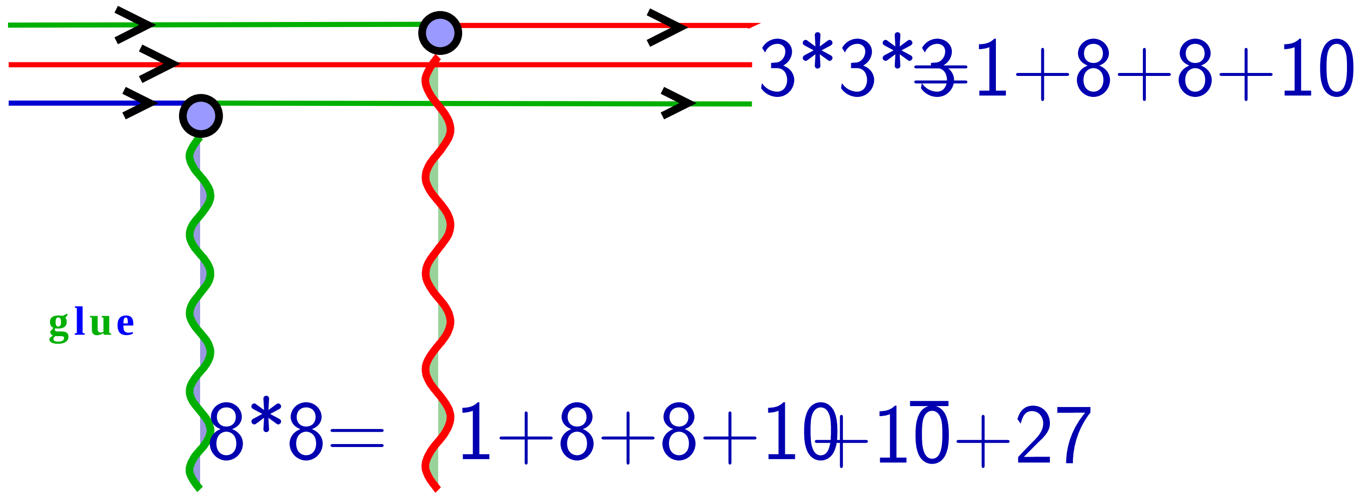

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Cannot be realized on the *valence-built* proton:

$$\frac{1}{27}\cdot 0+\frac{8+8}{27}\cdot 3+\frac{10}{27}\cdot 6=4$$

Colour coherence and breathing projectiles

Coherent picture of hadron accompaniment applies to the bulk of multiplicity (small transverse momentum hadrons) and implies relatively "compact" projectiles (on the *penetrator* side).

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This destructive coherence invalidates the multi-Pomeron exchange picture !

Colour coherence and breathing projectiles

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Recall the good old Amati-Fubini-Stanghellini puzzle.

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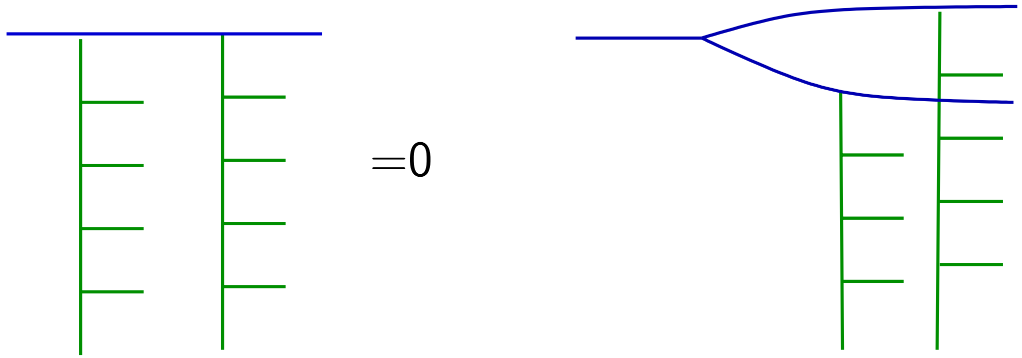

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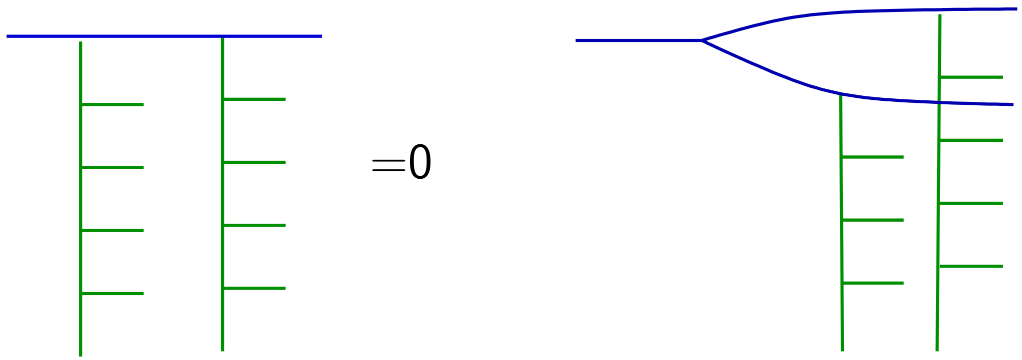

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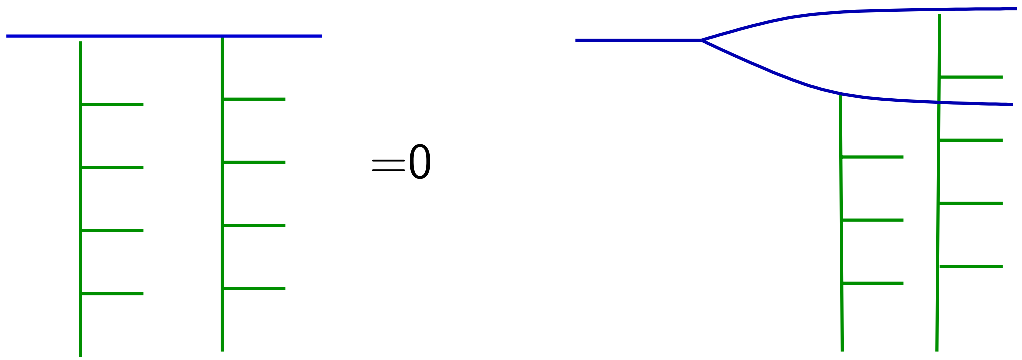

Two ways to break colour coherence:

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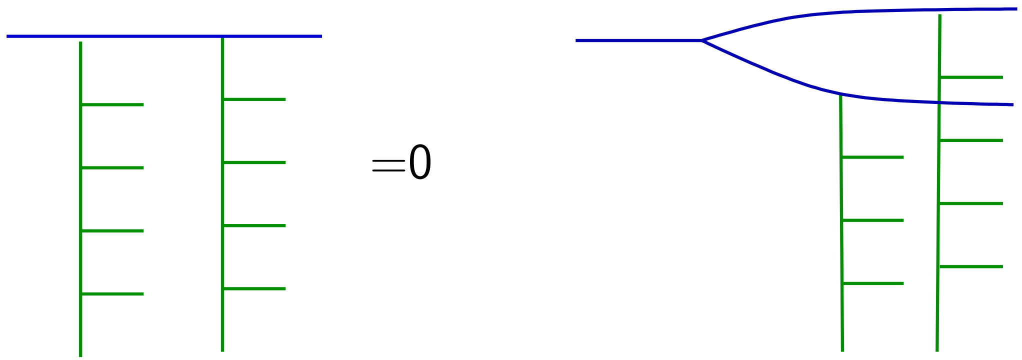

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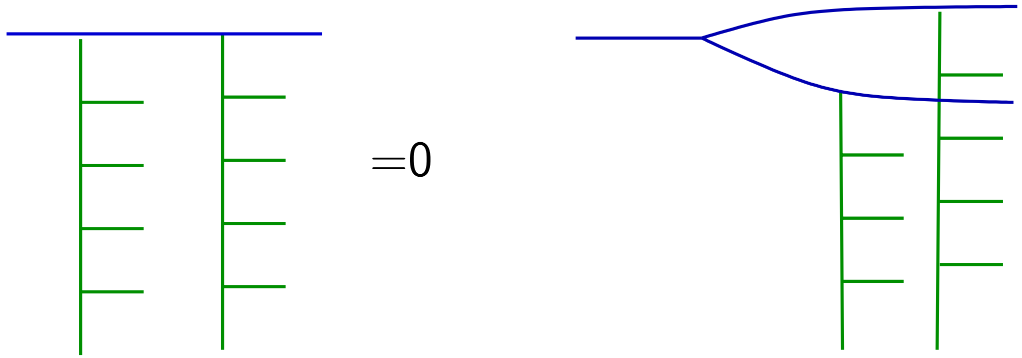

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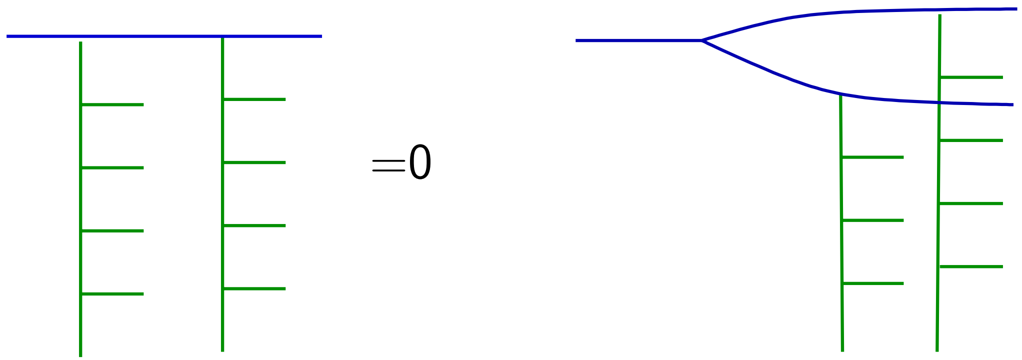

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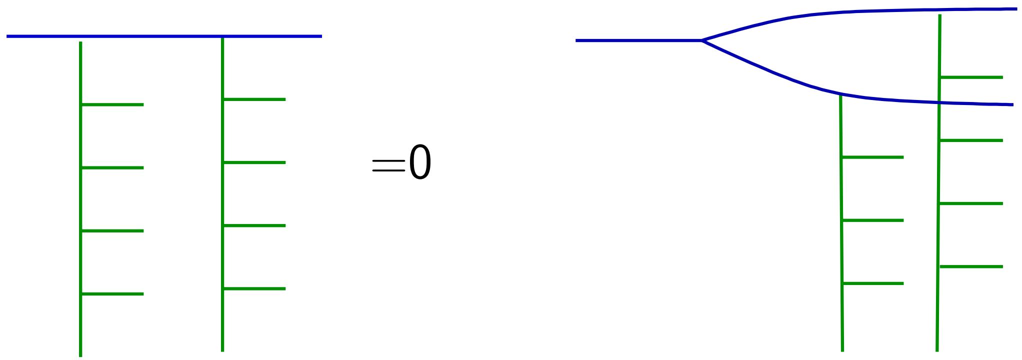

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C increases fast with Q_{res} (hadron transverse momenta),
drops in the fragmentation region, etc

Confinement in Multiple Collisions

In the framework of the standard hadron (multi-Pomeron) picture (e.g., in the successful Dual Parton Model of Capella & Kaidalov et al.) one includes final state interactions to explain spectacular heavy ion phenomena like J/ψ suppression, enhancement of strangeness and alike.

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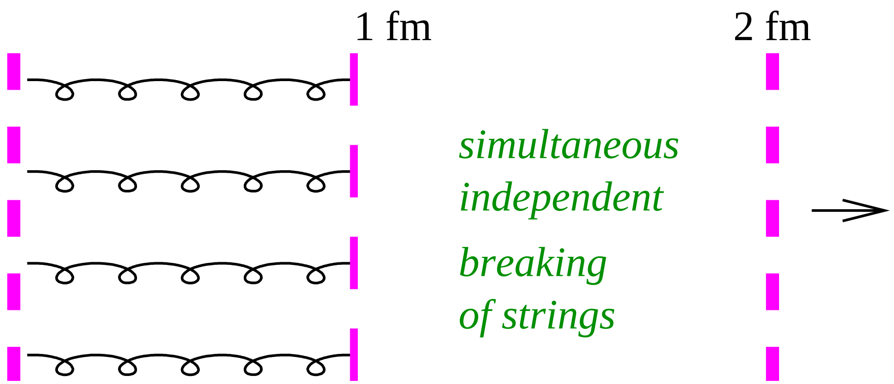

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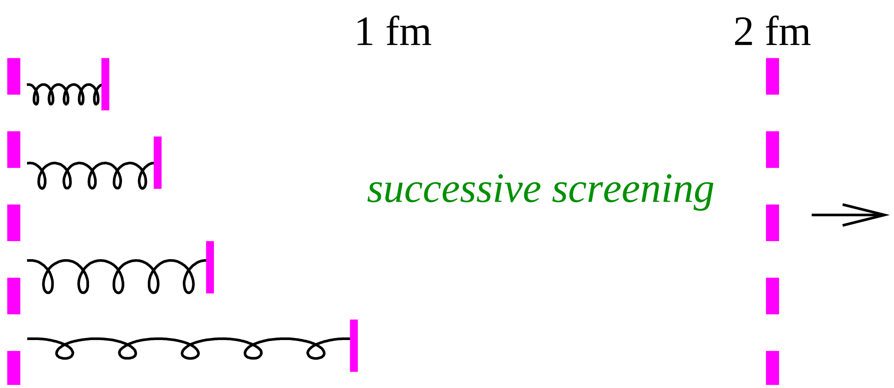

The question is, Does it go

- like BOOOOM (4 Pomerons)
- or rather like TA-TA–TA—TA?
  (new hadron abundances)

Overview

QCD at Terrestrial and Cosmic Energies

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QCD is far from over

Overview

QCD at Terrestrial
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important news for terrestrial/cosmic experimenters :

M.Cacciari and G.Salam, hep-ph/0512210
http://www.lpthe.jussieu.fr/~salam/fastjet/

Extras

non-diagonal transitions

Second loop $G \to G$ [quark box] $(n_f T_R C_F)$

$$P_G^{(S)} = 8x - 16 + \frac{20}{3}x^2 + \frac{4}{3}x^{-1} - (6 + 10x)\ln x - 2(1+x)\ln^2 x,$$

$$P_G^{(T)} = 12x - 4 - \frac{164}{9}x^2 + \frac{92}{9}x^{-1} + (10 + 14x + \frac{16}{3}[x^2 + x^{-1}])\ln x + 2(1+x)\ln^2 x;$$

Non-singlet $F \to F$ [via 2 gluons] $(n_f T_R C_F)$

$$P_F^{(S)} = 12x - 4 - \frac{112}{9}x^2 + \frac{40}{9}x^{-1} + (2 + 10x + \frac{16}{3}x^2)\ln x - 2(1+x)\ln^2 x,$$

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non-diagonal transitions

Second loop $G \to G$ [quark box] $(n_f\, T_R\, C_F)$

$$P_G^{(S)} = 8x - 16 + \frac{20}{3}x^2 + \frac{4}{3}x^{-1} - (6+10x)\ln x - 2(1+x)\ln^2 x,$$

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Integrability

1. anomalous dimensions $\Rightarrow$ eigenvalues of the dilatation operator
2. subset of composite operators su(2) = trace(XXXYYXYXXXYYY) can be mapped onto a spin 1/2 system (X = spin up, Y = spin down)
3. At one loop, it is the Hamiltonian of the integrable XXX spin 1/2 chain
4. At higher loops, a more complicated spin chain, but with spins interacting at neighbouring sites (up to a certain distance)
5. At all loops, there are conjectures for the all loop spin Hamiltonian, exploiting the string results, assuming AdS/CFT duality.
6. Integrability = an infinite number of invariants (conserved quantities).

gluons in-between-jets

2- and 3-prong colour antennae are sort of "trivial": coherence being taken care of, the answers turned out to be essentially additive

The case of $2 \to 2$ hard parton scattering is more involved (4 emitters), especially so for gluon–gluon scattering.
Here one encounters 6 (5 for $SU(3)$) colour channels that mix with each other under soft gluon radiation

The difficult quest of sorting out large angle gluon radiation in all orders in $(\alpha_s \log Q)^n$ was set up and solved by George Sterman and collaborators.

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Puzzle of large angle Soft Gluon radiation

Soft anomalous dimension ,

$$\frac{\partial}{\partial \ln Q} M \propto \left\{ -N_c \ln \left(\frac{t\, u}{s^2} \right) \cdot \hat{\Gamma} \right\} \cdot M, \qquad \hat{\Gamma} V_i = E_i V_i .$$

6=3+3. Three eigenvalues are "simple".

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Three "ain't-so-simple" ones were found to satisfy the cubic equation:

$$\left[E_i - \frac{4}{3}\right]^3 - \frac{(1+3b^2)(1+3x^2)}{3}\left[E_i - \frac{4}{3}\right] - \frac{2(1-9b^2)(1-9x^2)}{27} = 0,$$

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Mark the *mysterious symmetry* w.r.t. to $x \to b$: interchanging internal (group rank) and external (scattering angle) variables of the problem ...