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L-Functions

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L-FUNCTIONS

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Abstract. Our aim is to give an introduction to L-functions.

1. Introduction

Our aim in these notes is to give an informal introduction to some of the themes occurring in the theory of L-functions.

1.1. **Euler products, local-global principle.** One of the first aspects about the L-functions we consider is that they have an Euler product. Indeed, Euler gave the factorization

$$\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Here p runs over the set of primes. The above factorization is valid when s is taken to be a positive real greater than 1. Formally the product expresses the fact that every natural number can be factorized uniquely as a product of primes.

Euler observed that the product factorization of the zeta function $\zeta(s)$ gives a proof of Euclid's theorem on the infinitude of prime numbers. Indeed if the set of primes is finite, then the product $\prod_p (1-p^{-s})^{-1}$ has a finite limit as $s \to 1$. But it can be seen that as $s \to 1$, the sum $\sum n^{-s}$ tends to ∞.

The L-functions we consider will be of the form,

$$L(s) = \prod_p L_p(s)$$

where the product is valid in some suitable half plane $\mathrm{Re}(s)$ sufficiently large. More generally we consider completed L-functions of the form

$$L(s) = \prod_p L_v(s),$$

where v runs over all the places of a number field (or a global field). Such L-functions will be associated to arithmetical objects, and the

[1]This is an informal set of notes accompanying the lectures given at ICTP. As such there are not many references; lots of mistakes (maybe even conceptual ones) will be there. Comments are most welcome.

local factors $L_p(s)$ carry local information about the object at the prime p.

Example. Let E be an elliptic curve defined over $\mathbb{Q}$, say given by an equation of the form,

$$y^2 = x^3 + ax + b, \quad a,\ b \in \mathbb{Q}.$$

Let p be a prime not dividing the discriminant $\Delta(E)$ having more or less an expression of the form $\Delta(E) = 4a^3 + 27b^2$ (a prime of good reduction for E). Let $p + 1 - a_p(E)$ be the number of solutions $E(\mathbb{F}_p)$ of the reduction mod p of E with values in the finite field $\mathbb{F}_p$. The local factor $L_p(s, E)$ at such good primes is defined as,

$$L_p(s, E) = \left(1 - \frac{a_p(E)}{p^s} + \frac{p}{p^{2s}}\right)^{-1}.$$

The global partial L-function $L^{\Delta}(s, E)$ associated to E is defined as,

$$L^{\Delta}(s, E) = \prod_{(p,\Delta)=1} L_p(s, E).$$

So the local information at p that is encoded here is related to the number of solutions of the elliptic curve E over the finite field $\mathbb{F}_p$ (also over the finite extensions of $\mathbb{F}_p$).

Example. Let q be a natural number. Consider the quadratic Dirichlet character $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ defined by,

$$\chi(p) = \left(\frac{q}{p}\right), \quad (p, q) = 1,$$

where

$$\left(\frac{q}{p}\right) = \begin{cases} 1 & \text{if } q \text{ has a square root in } \mathbb{F}_p \\ -1 & \text{otherwise} \end{cases}$$

is the Legendre symbol. The local Euler factor at p is defined as,

$$L_p(s, \chi) = \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Here the local L-factor carries information on the splitting behaviour of the prime p in the quadratic extension $\mathbb{Q}(\sqrt{q})$.

The global properties satisfied by the L-functions can be considered as a manifestation of the local-global principle; the local information at the different primes is patched together by the Euler product, and in turn the L-function gives global information about the arithmetic object.

Example. As a manifestation of the local-global principle let us consider elliptic curves again. It is known that the group $E(\mathbb{Q})$ of rational points of an elliptic curve E defined over $\mathbb{Q}$ is a finitely generated abelian group. Heuristically if $E(\mathbb{F}_p)$ is large for various primes p, then the curve E should have many rational points, i.e., the rank of $E(\mathbb{Q})$ should be large. Consider the local factor $L_p(s, E)$ evaluated at $s = 1$:

$$L_p(1, E) = \left(\frac{p + 1 - a_p(E)}{p} \right)^{-1} = \left(\frac{|E(\mathbb{F}_p)|}{p} \right)^{-1}.$$

Thus if $E(\mathbb{F}_p)$ is large then $L_p(1, E)$ is small. With some numerical evidence, the following conjecture was made by Birch and Swinnerton-Dyer:

$$\text{rank}_{\mathbb{Z}}(E(\mathbb{Q})) = \text{ord}_{s=1} L(s, E).$$

Example. The first significant application of L-functions to obtain global arithmetical data was made by Dirichlet to obtain class number formulas for quadratic fields. More generally let K be a number field. The Dedekind zeta function associated to K can be defined as an Euler product,

$$\zeta_K(s) = \prod_{\mathcal{P}} \left(1 - N\mathcal{P}^{-s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Here $\mathcal{P}$ runs over the collection of prime ideals in the ring of integers $\mathcal{O}_K$ of K and

$$N\mathcal{P} = |\mathcal{O}_K/\mathcal{P}|,$$

is the norm of the prime ideal $\mathcal{P}$. It was shown in the lectures on Tate's thesis that $\zeta_K(s)$ has a simple pole at $s = 1$ with residue

$$\text{res}_{s=1}\zeta_K(s) = \text{vol}(J_K^1/K^*) = ch_K R_K.$$

Here J_K^1 is the group of ideles of norm 1 of K, h_K is the class number of K and R_K is the regular of K (c is some explicit constant depending on K).

As a corollary of the class number for quadratic fields Dirichlet proved the infinitude of primes in arithmetic progressions:

Theorem 1 (Dirichlet)**.** *Let a, q be coprime natural numbers. Then there are infinitely many primes p such that $p - a$ is divisible by q.*

To see this, introduce for each character $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$ the Dirichlet L-function

$$L^{(q)}(s, \chi) = \prod_{(p,q)=1} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

(the superscript (q) is to indicate that this is an incomplete l-function, in that we have not defined the local factors at the primes p dividing q). Then,

$$\frac{1}{|(\mathbb{Z}/q\mathbb{Z})^*|}\sum_{\chi}\chi(a)^{-1}\log L(s,\chi) = \sum_{p\equiv a \pmod q}\frac{1}{p} + E(s),$$

where $E(s)$ is a function that is bounded as $s \to 1$. Now if $\chi = \chi_0$ is the trivial character then $L(s,\chi_0)$ is equal to the Riemann zeta function $\zeta(s)$ upto some finite number of local L-factors that are non-vanishing at $s = 1$. On the other hand if χ is not the trivial character, then by the method of partial summation we see that $L(s,\chi)$ can be extended to an analytic function in the region $\mathrm{Re}(s) > 0$. Hence if we know that $L(1,\chi) \neq 0$ for a non-trivial character χ, then Dirichlet's theorem follows.

Consider the cyclotomic field,

$$K = \mathbb{Q}(\mu_q)$$

generated by the q^{th} roots of unity. It can be seen that (in fact as a consequence of abelian reciprocity for the cyclotomic field and the inductivity of L-functions attached to Galois representations) that

$$\zeta_K(s) = \prod_{p|q}(1-p^{-s})^{-1}\prod_{\chi}L(s,\chi).$$

From the class number formula for $\zeta_K(s)$ we know that $\zeta_K(s)$ has a simple pole at $s = 1$. Since $L(s,\chi_0)$ has a simple pole at $s = 1$, we conclude from the regularity of $L(s,\chi)$ for χ non-trivial that they do not vanish at $s = 1$. This proves Dirichlet's theorem.

1.2. **Analytic continuation and Functional Equation.** (ref. any book on analytic number theory, eg. Karastuba).

Riemann showed that the 'completed' L-function

$$\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s),$$

defined originally in the region $\mathrm{Re}(s) > 1$ can be meromorphically continued to the entire plane except for simple poles at $s - 0$, 1 and satisfies the functional equation

$$\Lambda(s) = \Lambda(1-s).$$

Riemann further conjectured deep relationships between the zeros of the zeta function and the distribution of prime numbers. Building on the work of Riemann and others, Hadamard and de Vallee Poussin proved the prime number theorem,

$$\pi(x) := |\{p \leq x\}| = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) \qquad \text{as } x \to \infty.$$

By partial summation, an equivalent version of the prime number theorem

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = x + o(x),$$

where

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for some prime p} \\ 0 & \text{otherwise.} \end{cases}$$

This was obtained as a consequence of a non-vanishing result,

$$\zeta(1 + it) \neq 0.$$

One way of observing that the non-vanishing of the Riemann zeta function $\zeta(s)$ on the line $\mathrm{Re}(s) = 1$ (more precisely a zero free region of the standard form) impies the prime number theorem is via the Explicit formula of Riemann-von Mangoldt-Siegel. The idea of this method is to consider the integral of logarithmic derivative of $\zeta(s)$ against the function x^s/s,

$$\frac{1}{2\pi i} \int_{b-iT}^{b+iT} -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds.$$

The property of integrating against the function of the form x^s/s is that cuts off the infinite sum. The above integral can be estimated as

$$\sum_{n \leq x} \Lambda(n) + ET(b, T, x).$$

We now shiftthe line of integration to the line segment $\sigma_1 - iT$ to $\sigma_1 + iT$ for some $\sigma_1 < 1$ chosen so that there are no zeros of $\zeta(s)$ in the rectangle with vertices

$$\sigma_1 - iT, \ \sigma_1 + iT, \ b + iT, \ b - iT.$$

We pick up a contribution (x) coming from the simple pole of $\zeta(s)$ at $s = 1$. Estimating the contributions we get

$$\psi(x) = x + O(x\exp(-c\sqrt{\log x}).$$

There are a lot of technicalities to be filled in (zero free regions for $\zeta(s)$, the fact that $\Lambda(s)$ is bounded in vertical strips, etc.) in order to rigorously justify the above argument

A different perspective on the Explicit formula was introduced by Weil (ref: Hejhal, Lang's book). Let h and g be sufficiently nice functions which are related mutually by a Fourier-Mellin transform of the form

$$h(r) = \int_{-\infty}^{\infty} g(r)e^{iru}du.$$

Then one form of the explicit formula is as follows:

$$\sum_{\gamma} h(\gamma) - h(i/2) - h(-i/2) - \frac{1}{2\pi}\int_{-\infty}^{infty} h(r)\frac{\Gamma'}{\Gamma}\left(\frac{1}{4} = \frac{ir}{2}\right)dr$$

$$= -g(0)\log(\pi) - 2\sum_{n=1}^{\infty}\frac{\Lambda(n)}{\sqrt{n}}g(\log n).$$

In other words this form of the explicit formula exhibits a duality between the zeros of the Riemann zeta function $\zeta(s)$ and the prime numbers.

2. General L-functions

We now consider some of the defining properties of more general L-functions. Basically these arise from two a priori different contexts. One construction is the L-functions associated to Galois representations arising from a study of the diophantine properties of algebraic varieties over global fields, a study starting with Artin and defined in increasing generality by Hasse, Weil and Grothendieck. The other construction is in the automorphic context initiated by Ramanujan and Hecke, extended by Mass, Siegel and Selberg culminating in the work of Langlands. In the framework of Langlands these two strands have been brought together as a generalization of the abelian reciprocity law.

2.1. **L functions attached to Galois representations; Inductivity.** The model for the definition of L-functions is based on that of L-functions associated to representations of the absolute Galois groups of number fields. Let F be a number field and

$$G_F = \mathrm{Gal}(\bar{F}/F)$$

be the absolute Galois group of an algebraic closure $\bar{F}$ of K over the base field F. Suppose

$$\rho : G_F \to GL(V)(\simeq GL_n(\mathbb{C}))$$

is a continuous n-dimensional linear representation of G_F into $GL(V)$ where V is a n-dimensional vector space over $\mathbb{C}$. Here $GL(V)$ is equipped with the Euclidean topology (more generally one can work with compatible systems of l-adic representations). Since G_F is profinite and ρ is continuous, the representation factors via a finite Galois extension L of F,

$$\rho : G_F \to \mathrm{Gal}(L/F) \to GL(V).$$

For a finite place v of K unramified in L, define the local L-factor

$$L(s, \rho_v) = \det(1 - \rho(\mathrm{Frob}_w) q_v^{-s})^{-1}.$$

Here w is a place of L lying above the place v of K and $Frob_w \in \mathrm{Gal}(L/F)$ is the Frobenius conjugacy class at w. The place v corresponds to a prime ideal $\mathcal{P}_v$ of the ring of integers $\mathcal{O}_F$ of F, and

$$q_v = N\mathcal{P}_v = |\mathcal{O}_F/\mathcal{P}_v|$$

is the norm of the ideal $\mathcal{P}_v$. The local L-factor depends only on the place v, since the various Frobenius conjugacy classes Frob_w for $w|v$ are conjugate inside $\mathrm{Gal}(L/F)$.

At a finite place v of F ramified in L, define the local L-factor as,

$$L(s, \rho_v) = \det(1 - \rho(\mathrm{Frob}_w) \mid_{V^{I_w}} q_v^{-s})^{-1},$$

where $I_w = I_w(L/F)$ is the inertia group at a place $w|v$. Again one can see that this is independent of the choice of w dividing v. By definition, the local factor depends only on the restriction ρ_v of the representation of the Galois group to the decomposition group $D_w(L/F)$ of a place w of L lying over F.

The global L-function is defined as,

$$L(s, \rho) = \prod_{v\Sigma_{F,f}} L(s, \rho_v), \quad \mathrm{Re}(s) > 1,$$

where $\Sigma_{F,f}$ denotes the collection of non-archimedean places of F. It can be seen that that the Euler product converges in the half-plane $\mathrm{Re}(s) > 1$. It is possible to associate at each archimedean place v of F, the local factor $L(\rho_v, s)$ defined in terms of Γ functions, and define the completed L-function

$$\Lambda(s, \rho) = \prod_{v \in \Sigma_F} L(s, \rho_v),$$

where Σ_F runs over the collection of inequivalent places of F.

Example. If $\rho = \rho_0$ is the trivial representation of G_F, then

$$L(s, \rho_0) = \zeta_F(s),$$

the Dedekind zeta function of F.

Example. Let m be a natural number and let

$$\rho : \mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q}) \to \mathbb{C}^*,$$

be a continuous one dimensional character of the Galois group $\mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})$. Now there is a natural identification

$$r : \mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^*,$$

given by the following property

$$\sigma(\mu) = \mu^{r(\sigma)}, \quad \sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q}), \ \mu \in \mu_m.$$

It is a reflection of the abelian reciprocity law for the abelian extension $\mathbb{Q}(\mu_m)$ over $\mathbb{Q}$, that for a Dirichlet character $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$ as above,

$$L^{(q)}(s, \chi) = L^{(q)}(s, \chi \circ R),$$

where the right hand side is defined in the above Galois theoretical context (and the superscript (q) means we have omitted the local factors at the primes dividing q in the definition of $L(\rho, s)$).

The assignment $\rho \mapsto L(\rho, s)$ is additive,

$$L(s, \rho_1 \oplus \rho_2) = L(s, \rho_1)L(s, \rho_2).$$

The defining property of this definition of L-functions is that it is *inductive*: suppose M is a finite Galois extension of F and

$$\eta : G_M \to GL(W)$$

is a continuous linear representation of G_M. Now G_M is a subgroup of finite index in G_K. Suppose

$$\rho := \mathrm{Ind}_{G_M}^{G_F}(\eta),$$

is the representation of G_F induced from η. Then

$$L(s, \rho) = L(s, \eta).$$

The existence of such an inductive definition of L-functions associated to Galois representations and compatible with the known examples of L-functions was discovered by Artin. (ref: Frohlich's volume).

Regarding the analytic properties of the Artin L-functions, it follows from the abelian reciprocity law due to Artin that $L(s, \rho)$ is entire when ρ is an abelian character. Based on this, Artin conjectured the following:

Conjecture 2.1. Let ρ be irreducible (non-trivial). Then $L(s, \rho)$ is entire.

Using some results from representation theory of finite groups and the abelian case, Brauer proved the following:

Theorem 2. *Let F be a number field. For any continuous finite dimensional representation*

$$\rho : G_F \to GL(V)$$

the function $\Lambda(s,\rho)$ admits a meromorphic continuation to the entire plane entire and satsifies a functional equation of the form

$$\Lambda(s,\rho) = \epsilon(s,\rho)\Lambda(1-s,\tilde{\rho}),$$

where $\tilde{\rho}$ is the contragredient representation of ρ, and $\epsilon(s,\rho)$ of the form AB^s for some constants A and B associated to ρ.

Let $R(G)$ denote the Grothendieck ring associated to finite dimensional complex representations of G. Brauer showed that given any finite dimensional representation ρ of G, there is a collection of subgroups $H_i \subset G$ and abelian characters $\eta_i : H_i \to \mathbb{C}^*$ $(1 \leq i \leq k)$ such that in $R(G)$,

$$\rho = \sum_{i=1}^{k} m_i \mathrm{Ind}_{H_i}^{G}(\eta_i),$$

for some integers m_i. The meromorphicity and functional equation follows from the abelian case and the inductivity property of L-functions.

It should be remarked that the later results proving some class of Artin L-functions are entire do not use Brauer's theorem.

2.2. **Automorphic L-functions.** (Ref: Corvallis)

The most general context in which L-functions can be defined having an Euler product, analytic continuation and functional equation lies in the automorphic context. Let G be a reductive algebraic group defined over a number field F, and π be an irreducible representation of the locally compact group of adele points $G(\mathbb{A}_K)$ of G, where $\mathbb{A} = \mathbb{A}_K$ is the adele ring of K. The representation π can be decomposed as a restricted tensor product,

$$\pi = `\otimes'_{v\in\Sigma_K} \pi_v,$$

where π_v is an irreducible representation associated to the local group $G(F_v)$. Here F_v is the completion of K at the place v of F. There exists a finite set S of places of F such that for v not in S the local component π_v is unramified.

Suppose v is a finite place not in S. To the unramified representation π_v there is associated a semisimple conjugacy class

$$t(\pi_v) \in^L G(\mathbb{C}),$$

by the Satake-Langlands parametrization. For examle, if $G = GL_n$, then

$$^{L}GL_n = GL_n(\mathbb{C}) \times G_F,$$

and the Langlands-Satake parametrization gives us a semisimple conjugacy class in $GL(n, \mathbb{C})$. In general consider a representation,

$$r :^{L} G \to GL_n(\mathbb{C}).$$

Define the local L-factor for v not in S as,

$$L(s, \pi_v, r) = \det(1 - r(t_v)q_v^{-s})^{-1},$$

and the partial global L-function as

$$L^S(s, \pi, r) = \prod_{v \notin S} L(s, \pi_v, r).$$

Now if π were to correspond to a Hecke character or to a modular form and r is taken to be the standard representation, then it can be checked that this gives the usual L-factor associated in these cases. For general $\pi,\ r$, using a formula of Macdonald, Langlands showed that the partial global L-function does converge in some half plane.

If we consider the automorphic forms as generalizations of Hecke characters and modular forms, then we expect to be able to prove analytic continuation and functional equation for the L-functions associated to automorhic representations. However in order to obtain functional equations, we need to know the completed global L-function, i.e., we need to know the local L-factors at all the places and not just the unramified ones. So the problem now is to define the local factors at the bad places that is consistent with various other properties that we expect about these L-functions.

As a generalization of the abelian reciprocity law, Langlands conjectured that associated to anirreducible Galois representation

$$\rho : G_F \to GL_n(\mathbb{C}),$$

there should be associated a unitary cuspidal automorphic representation $\pi(\rho)$ of $GL_n(\mathbb{A}_F)$ such that

$$L(s, \rho) = L(s, \pi(\rho)).$$

Thus the definition of $L(s, \pi)$ should be consistent with the above conjectured reciprocity law, since we can define the completed L-function associated to Galois representations.

Langlands also conjectured that the above global reciprocity law should be compatible with a local reciprocity law. Indeed when the local field F is archimedean Langlands proved that isomorphism classes of irreducible admissible representations of $G(F)$ where G is a reductive algebraic group over F are parametrised by conjugacy classes of homomorphisms of the Weil group

$$W_F \to^L G(\mathbb{C}).$$

More generally Langlands expected that if F is non-archimedean, then isomorphism classes of irreducible admissible representations of $G(F)$ are parametrised by conjugacy classes of homomorphisms of the Weil-Deligne group

$$WD_F \to^L G(\mathbb{C}).$$

such that the Frobenius element acts semisimply.

Coming back to the global situation, since we know how to attach L-factors associated to representations of the Weil group, this will allow us to attach the local L-factors $L(s, \pi_v, r)$ at any place v of the global field F provided we know the local Langlands conjecture.

So we are in a peculiar situation: on the one hand we can define the completed L-functions associated to Galois representations but cannot prove the desired analytic properties. On the automorphic side though we cannot define the correct L-functions but we expect to prove the analytic properties and to work with it (thanks to Hecke, Tate, etc.)

In practice what happened was that L-functions associated to automorphic representations have been defined in a wide variety of contexts by different methods and the required analytic properties have been established. Now it is to be expected that the existence of global functional equations should rigidify the definition of the local L-factors. i.e., the local L-factors should be determined uniquely at the finite set of places in S, provided we know that the partial L-function $L^S(s, \pi, r)$ can be completed to a global L-function having the expected functional equation, etc. If this were so, then we expect that the local L-factors constructed by these analytic methods should give us the correct L-factors, i.e., that they should be compatible with the local Langlands correspondence as and when it is established. Again in practice, the analytic theory has helped in clarifying the nature of the local Langlands correspondence for $GL(n)$ even before the correspondence was established (ref: Henniart).

2.3. **ϵ-factors.** The completed L-function should satisfy a functional equation of the form

$$\Lambda(s, \pi, r) = \epsilon(s, \pi, r)\Lambda(1 - s, \tilde{\pi}, r),$$

for some suitable ϵ-factor $\epsilon(s,\pi,r)$. Tate's thesis expects that given an additive character

$$\psi : \mathbb{A}_F/F \to \mathbb{C}^*,$$

there should be a factorization

$$\epsilon(s,\pi,r) = \prod_{v\in\Sigma_F} \epsilon(s,\pi_v,r,\psi_v),$$

where ψ_v is the local component of ψ at v (an additive charcter of $F_v \to \mathbb{C}^*$). Here for almost all places (the places where π and ψ are unramified), the epsilon factor $\epsilon(s,\pi_v,r,\psi_v)$ should be 1.

If the local Langlands conjecture were to be true, then we should expect to define the epsilon factors associated to representations of the Weil-Deligne group associated to local fields. This was established by Langlands and Deligne:

Theorem 3. *Let F be a non-archimedean local field and $\psi : F \to \mathbb{C}^*$ be an additive character. Then there exists a unique prescription,*

$$\phi \mapsto \epsilon(s,\phi,\psi),$$

associated to any irreducible parameter

$$\phi : W_F \to GL_n(\mathbb{C}),$$

satisfying the following properties:

1. *It is compatible with Tate's definition for the epsilon factors associated to $r(\phi) : F^* \to \mathbb{C}^*$, when ϕ is one-dimensional and $r(\phi)$ is the character associated by local class field theory to ϕ.*
2. *It is inductive in degree zero, i.e., if $\sum_i n_i\phi_i$ is an element in the Grothendieck ring of W_F such that $\sum_i n_i = 0$ and $\phi_i : W_F \to GL_{n_i}(\mathbb{C})$ are irreducible parameters, then*

$$\prod_i \epsilon(s,\phi_i,\psi)^{n_i} = 1.$$

(see Tate's article in volume edited by Frohlich for a precise version).

Remark: Tate's construction of the epsilon factors in the one dimensional case is on the automorphic side.

Deligne's proof of this theorem uses global methods, and marks the modern beginning of the use of global methods to prove local results (the first proofs of local class field theory used global class field theory).

The epsilon factors are of the form

$$\epsilon(s,\phi,\psi) = N(\phi)^{s-1/2}W(s,\phi,\psi),$$

where $N(\phi)$ is the conductor of ϕ and $W(s, \phi, \psi)$ is expressed in terms of Gauss sums. It is quite subtle to understand the information carried by the local epsilon factors with regard to the nature of ϕ.

3. Rankin-Selberg L-functions

In this section, we give an introduction to the analytic properties satisfied by the Rankin-Selberg convolution L-functions. We first recall the notion of a unitary, cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$. Let ω be a unitary character of $\mathbb{Z}(\mathbb{A})/Z(F)$, where Z is the center of GL_n. A unitary automorphic representation of $GL_n(\mathbb{A})$ is an irreducible 'constituent' of the regular representation R of $GL_n(\mathbb{A})$ on the space $L^2(Z(\mathbb{A})GL_n(F)\backslash GL_n(\mathbb{A}), \omega)$ consisting of measurable functions,

$$f : GL_n(\mathbb{A}) \to \mathbb{C}$$

satisfying

- $$f(z\gamma g) = \omega(z)f(g) \quad z \in Z(\mathbb{A}),\ \gamma \in GL_n(K),\ g \in GL_n(\mathbb{A}).$$
- $$\int_{Z(\mathbb{A})GL_n(F)\backslash GL_n(\mathbb{A})} |f(g)|^2 dg < \infty.$$

The cuspidal spectrum $L_0^2(Z(\mathbb{A})GL_n(F)\backslash GL_n(\mathbb{A}), \omega)$ is defined as the closed $GL_n(\mathbb{A})$-invariant subspace of $L^2(Z(\mathbb{A})GL_n(F)\backslash GL_n(\mathbb{A}), \omega)$ consisting of functions f as above satisfying the additional 'cuspidality' condition:

$$\int_{N(F)\backslash N(\mathbb{A})} f(ng)dn = 0 \quad \text{for almost all } g,$$

for all unipotent radicals N of proper parabaolic subgroups of GL_n.

By a theorem of Gelfand and Piatetskii-Shapiro, it is known that the cuspidal spectrum decomposes discretely as a direct sum of irreducible representations. By a unitary, cuspidal automorphic representation (with central character ω) we mean an irreducible constituent of $L_0^2(Z(\mathbb{A})GL_n(F)\backslash GL_n(\mathbb{A}), \omega)$.

Given an irreducible representation π of $GL_n(\mathbb{A})$, it breaks up as a restricted tensor product

$$\pi = `\otimes' \pi_v,$$

where v runs over the places of K. Here π_v is an irreducible representation of the local group $GL_n(F_v)$. For almost all finite places v of K, π_v is unramified, i.e., π_v has a nonzero invariant vector with respect to the maximal compact subgroup $K_v = GL_n(\mathcal{O}_v)$ of $GL_n(F_V)$. Here $\mathcal{O}_v$ is the ring of integers in the non-archimedean local field F_v.

In particular, if v is non-archimedean and π_v is an unramified unitary representation of $GL_n(F_v)$, the Langlands-Satake parametrization yields a semisimple conjugacy class,

$$t(\pi_v) \in GL_n(\mathbb{C}).$$

3.1. **The Main Theorem.** We now introduce and state the main properties of the convolution L-functions $L(s, \pi \times \pi')$. The convolution L-function is the automorphic analogue of the $L(s, \rho \otimes \rho')$ associated to the tensor product of $\rho \otimes \rho'$ of Galois representations

$$\rho : G_F \to GL_n(\mathbb{C}) \quad \rho' : G_F \to Gl_m(\mathbb{C}),$$

of the absolute Galois group G_F of a number field F.

Theorem 4. *Let π, π' be irreducible, unitary, cuspidal automorphic representations of $GL_n(\mathbb{A}_F)$ and $Gl_m(\mathbb{A}_F)$ respectively. The Rankin-Selberg L-function $\Lambda(s, \pi \times \pi')$ satisfies the following properties:*

Euler Product: *The function $\Lambda(s, \pi \times \pi')$ admits a decomposition,*

$$\Lambda(s, \pi \times \pi') = \Lambda_\infty(s, \pi \times \pi') L(s, \pi \times \pi').$$

The finite part $L(s, \pi \times \pi')$ admits an Euler product

$$L(s, \pi \times \pi') = \prod_{v \in \Sigma_f} L(s, \pi_v \times \pi'_v),$$

absolutely convergent in the half plane $Re(s) > 1$. The local component $L(s, \pi_v \times \pi'_v)$ is of the form,

$$L(s, \pi_v \times \pi'_v) = P_v(q_v^{-s})^{-1},$$

where P_v is a polynomial of degree at most nm. If v is a finite place of F at which the local components π_v and π'_v are unramified, then

$$L(s, \pi_v \times \pi'_v) = det(1 - t_v(\pi) \otimes t_v(\pi') q_v^{-s})^{-1},$$

where $t_v(\pi)$ (resp. $t_v(\pi')$) is the Langlands-Satake parameter associated to the unramifed representation π_v (resp. π'_v).

The archimedean component $\Lambda_\infty(s, \pi \times \pi')$ admits a decomposition

$$\Lambda_\infty(s, \pi \times \pi') = \prod_{v \in \Sigma_\infty} \Lambda_v(s, \pi \times \pi'),$$

where for each $v \in \Sigma_\infty$,

$$\Lambda_v(s, \pi \times \pi') = \prod_{j=1}^{k} \Gamma_v(s - \mu_j(\pi \times \pi'))$$

for some $k \leq mn$. Here

$$\Gamma_v(s) = \begin{cases} \pi^{-s/2}\Gamma(s/2) & \text{if } v \text{ is real,} \\ (2\pi)^{-s}\Gamma(s) & \text{if } v \text{ is complex.} \end{cases}$$

If v is unramified for both π and π', then

$$\Lambda_v(s, \pi \times \pi') = \prod_{j=1}^{m} \prod_{k=1}^{n} \Gamma_v(s - \mu_j(t(\pi_v)) - \mu_k(t(\pi'_v))).$$

Functional Equation: *The L-function $\Lambda(s, \pi \times \pi')$ extends to a meromorphic function of $s \in \mathbb{C}$ with at most finitely many poles. It satisfies a functional equation,*

$$\Lambda(s, \pi \times \pi') = \epsilon(s, \pi \times \pi')\Lambda(1 - s, \pi \times \pi').$$

Given an additive character $\psi : \mathbb{A}_F/F \to \mathbb{C}^*$, the epsilon factor has a decomposition,

$$\epsilon(s, \pi \times \pi') = \prod_{v} \epsilon(s, \pi_v, \pi'_v, \psi_v)$$

where $\epsilon(s, \pi_v, \pi'_v, \psi_v)$ is a monomial in q_v^{-s}. Morever if v is an unramified place of π, π' and ψ then $\epsilon(s, \pi_v, \pi'_v, \psi_v) = 1$.

Location of poles: *The function $\Lambda(s, \pi \times \pi')$ is entire unless $n = m$, and π is equivalent to the contragredient of π'.*

Non-vanishing: *The function $\Lambda(s, \pi \times \pi')$ is non-vanishing on the line $Re(s) = 1$.*

Bounded in vertical strips: *The function $\Lambda(s, \pi \times \pi')$ is bounded in vertical strips away from the poles.*

Remark 1. The Euler product statement and the functional equation have been proved by two different methods: one the explicit integral method due to Jacquet, Piatetskii-Shapiro and Shalika generalizing the Gl_2 work of Jacquet-Langlands and the work of Rankin-Selberg; the other method is to use the analytic properties of Eisenstein series associated to these automorphic representations due to Langlands and Shahidi. It can be checked that the two methods yield the same L and ϵ-factors.

The location of the poles is due to Jacquet-Shalika and Moeglin-Waldspurger.

The non-vanishing on the line $\text{Re}(s) = 1$ is due to Shahidi.

The bounded in vertical strips was initially proved by Ramakrishnan when $n = m = 2$ and in general by Gelbart and Shahidi.

Remark 2. The basic analytic properties were established by two different methods: one an explicit integral method due to Jacquet, Piatetskii-Shapiro and Shalika. The other method is the Langlands-Shahidi method based on the connection between the constant terms of Eisenstein series and the L-functions associated to the induction data. It is important for application to the converse theorem that the local L and ϵ factors are defined for all irreducible representations of the local groups.

Remark 3. Regarding the location of the pole at $s = 1$, the principle is that a L-function has a pole at $s = 1$, say for example $L(s, \rho)$ for a Galois representation ρ, precisely when ρ contains the trivial representation as a direct summand. This is reflected in the location of poles for the convolution L-functions.

Suppose now that π is a self-dual ($\pi \simeq \tilde{\pi}$) cuspidal automorphic representation. Then it follows from the result concerning the location of poles of $\Lambda(s, \pi \times \tilde{\pi})$ and the following equality

$$\Lambda(s, \pi \times \tilde{\pi}) = \Lambda(s, S^2\pi)\Lambda(s, \Lambda^2\pi)$$

that one of the two L-factors on the right hand side should have a pole. Here $S^2\pi$ and $\Lambda^2\pi$ denote respectively the symmetric and exterior square lifts of π. Based on the above heuristic (say for L-functions associated to Galois representations), it can be expected that if the exterior (resp. symmetric) square L-function has a pole at $s = 1$, then the automorphic representation π should be a lift of an automorphic representation from the symplectic (resp. orthogonal) group.

Thus the presence of poles of L-functions sheds information on the nature of the automorphic representation.

We now consider some applications of this theorem.

3.2. **Ramanujan conjecture.** The first main application is to estimating the size of the Hecke eigenvalues of cuspidal representations of $GL_n(\mathbb{A})$:

Corollary 1. *Let π be an irreducible unitary cuspidal automorphic representation of $GL_n(\mathbb{A})$. Suppose v is a place of F at which the local component is unramified. Then*

$$|log_{Nv}(\mu_j(t(\pi_v))| < \frac{1}{2} \quad \textit{if } v \textit{ is finite},$$

and

$$|Re(\mu_j(t(\pi_v))| < \frac{1}{2} \quad \textit{if } v \textit{ is archimedean.}$$

The Ramanujan and the Selberg conjectures assert that the local component π_v at any place v of F of a cuspidal automorphic representation π of $GL_n(\mathbb{A}_F)$ is tempered. This translates to the estimates that at a place v of F at which π_v is unramified,

$$|\mu_j(t(\pi_v))| = 1 \quad \text{if } v \text{ is finite,}$$

and

$$|\mathrm{Re}(\mu_j(t(\pi_v))| = 0 \quad \text{if } v \text{ is archimedean.}$$

3.3. **Families of L-functions.** (Ref: paper Luo, Rudnick and Sarnak)

With some further input from analytic number theory, the estimates appearing in the Corollary were improved by Luo, Rudnick and Sarnak [?]:

Theorem 5. *Let π be an irreducible unitary cuspidal automorphic representation of $GL_n(\mathbb{A})$, and let v be a place of F at which the local component π_v is unramified. Then*

$$|log_{Nv}(\mu_j(t(\pi_v))| \leq \frac{1}{2} - \frac{1}{n^2+1} \quad \textit{if } v \textit{ is finite,}$$

and

$$|Re(\mu_j(t(\pi_v))| < \frac{1}{2} - \frac{1}{n^2+1} \quad \textit{if } v \textit{ is archimedean.}$$

At the moment these are the best bounds available for general n. The proof of Luo-Rudnick-Sarnak's bound is based on the following observation: suppose there exists an unramified finite place v_0 at which there exists an eigenvalue of the Satake parameter $t(\pi_{v_0})$ of absolute value q_v^α. Then it can be seen that the local L-factor $L(s, \pi_{v_0} \times \tilde{\pi}_{v_0} \times \chi_{v_0})$ has a pole at $s = \alpha$, where χ is any idele class character such that $\chi_{v_0} = 1$. Suppose now that χ is non-trivial. Since the completed L-function $\Lambda(s, \pi \times \tilde{\pi} \times \chi)$ is entire this implies that the partial L-function

$$\Lambda^{v_0}(s, \pi \times \tilde{\pi} \times \chi) = \prod_{v \neq v_0} L(s, \pi_v \times \tilde{\pi}_v \times \chi_v),$$

should vanish at $s = \alpha$. Now consider the average sum over a family of characters,

$$S(\pi, N) = \sum_{\chi} \Lambda^{v_0}(s, \pi \times \tilde{\pi} \times \chi),$$

where the sum is all over characters χ such that $\chi_{v_0} = 1$ and the conductor of χ is at most N. By using methods from analytic number theory (inspired by a result of Rohrlich on the non-vanishing of L-functions), it is shown that for N large the average sum $S(\pi, N)$ is non-zero provided α is as in the theorem and this gives the desired bound.

The method emphasizes the importance of considering families of automorphic L-functions rather than a single L-function. Here the family considered is the family of twists by characters, and it is easier to study the (average) behaviour of the family.

3.4. **Converse theorems.** (Refs: Cogdell's notes, papers of Cogdell and Piatetskii-Shapiro)

One of the most important motivation to consider the convolution L-function is to converse theory. Converse theory is based on the principle that L-functions which having an Euler product, analytic continuation and appropriate functional equation should be L-functions associated to automorphic representations living on Gl_n. In order to carry this out, it is imperative to define the local L and ϵ-factors

$$L(s, \pi \times \tau), \quad \epsilon(s, \pi \times \tau, \psi)$$

where π, τ are irreducible unitary representations of $Gl_n(F)$, and satisfying the appropriate expected properties. Here F is a local field and ψ is an additive character of F. One form of the current state of converse theory is given by the following theorem due to Cogdell and Piatetskii-Shapiro:

Theorem 6. *Let F be a global field and let π be an irreducible unitary representation of $Gl_n(\mathbb{A}_F)$. Suppose for any $m \leq n-2$ if $n \geq 3$ (and $m = 0, 1$ when $n \leq 2$) and any cuspidal automorphic representation τ of $Gl_m(\mathbb{A})$ the completed L-function,*

$$\Lambda(s, \pi \times \tau) = \prod_{v \in \Sigma_F} L(s, \pi_v \times \tau_v)$$

are 'nice' in that they satisfy the following:

(1) *The L-function $\Lambda(s, \pi \times \tau)$ can be analytically continued to the entire plane and satisfies a functional equation of the form*

$$\Lambda(s, \pi \times \tau) = \epsilon(s, \pi \times \tau)\Lambda(1 - s, \tilde{pi} \times \tilde{\tau}),$$

where

$$\epsilon(s, \pi \times \tau) = \prod_v \epsilon(s, \pi_v \times \tau_v, \psi_v)$$

and ψ is an additive character of $\mathbb{A}_F/F$.

(2) *The L-function $\Lambda(s, \pi \times \tau)$ is bounded in vertical strips.*

Then π is an automorphic cuspidal representation of $Gl_n(\mathbb{A}_F)$.

The usefulness of the converse theorem is as follows: suppose G and H are two reductive groups defined over a global field F and suppose there exists a homomorphism

$$\Phi :^L G \to^L H$$

of the associated L-groups. Composing with Φ gives a lifting of the local parameters $W_{F_v} \to^L G$ associated to G to local parameters $W_{F_v} \to^L H$ associated to H associated to H. It follows from the local Langlands correspondence that given an irreducible unitary representation π of $G(\mathbb{A})$ there is associated an irreducible unitary representation $\Phi(\pi)$ of $H(\mathbb{A})$. The coarse expectation of Langlands functoriality is that $\Phi(\pi)$ should be automorphic whenever π is automorphic. This gives a lifting of automorphic representations from G to that of H.

In some cases of Φ it is possible to relate the two groups $G(\mathbb{A})$ and $H(\mathbb{A})$ analytically and to compare the harmonic analysis on the two groups. This should enable one to prove that the L-function associated to $\Phi(\pi)$ should be nice knowing that π is automorphic. Converse theory together with information about the location of poles, etc. now chip in to conclude that $\Phi(\pi)$ is automorphic on $H(\mathbb{A})$.

3.5. **Non-vanishing and Equidistribution.** (Refs: Serre's book 'Abelian l-adic representations'; Langlands article in Corvallis).

Just as for the application of the non-vanishing of the Riemann zeta function on the line $Re(s) = 1$ implies the prime number theorem, the non-vanishing of more general L-functions on the line $Re(s) = 1$ leads to equidistribution results. Given a L-function of the form $L(s) = L(s, \pi, r)$ as considered above, the logarithmic derivative

$$-\frac{L'(s)}{L(s)} = \sum_{v \notin S} \frac{\mathrm{Tr}(r(t(\pi_v))}{q_v^s} + T(s),$$

where $T(s)$ is a term that is under control as $s \to 1$. By a Tauberian theorem argument (or as indicated by an explicit formula type argument if we have a slightly better non-vanishing result), we obtain an estimate of the form

$$\sum_{v \notin S,\ Nv<x} \mathrm{Tr}(r(t(\pi_v)) = o(x) \quad \text{as } x \to \infty,$$

where we have assumed that $L(s)$ is holomorphic and non-vanishing on the line $\mathrm{Re} = 1$.

Let M be a compact group, and x_n be a sequence of conjugacy classes in M. We say that the sequence x_n is equidistributed with respect to the projection of the normalized Haar measure dm onto the space of conjugacy classes in M, if for any class function f on M, the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_n)$$

exists and is equal to

$$\int f(m)dm.$$

By the denseness of characters of irreducible representations in the space of class functions, it is enough to check the above condition for the class functions χ which are characters of irreducible representations ρ of M (this is known as the Kronecker-Weyl criterion).

We now consider a heuristic generalization of the classical Chebotarev density theorem. Consider a cuspidal automorphic representation π of $G(\mathbb{A})$. Assume the following:

(1) For any representation $r :^L G \to^L Gl_n$, the L-functions $L(s,\pi,r)$ can be analytically continued to an entire function on the plane and is non-vanishing on the line $Re(s) = 1$.
(2) The automorphic representation π satisfies the Ramanujan conjecture, i.e., for v an unramified place of π, the Langlands-Satake conjugacy class $t(\pi_v)$ can be conjugated to a maximal compact subgroup K_G^1 of ${}^LG^1$, where ${}^LG^1$ is the kernel of a certain homomorphism (the absolute value of the weight homomorphism?) from ${}^LG \to Gl_1$.

We conclude from the analytic machinery that the conjugacy classes $t(\pi_v) \cap K_G^1$ are equidistributed with respect to the projection of the normalized Haar measure on K_G^1 onto the space of conjugacy classes in K_G^1. This is the generalized Chebotarev density statement.

Example. Let θ be a unitary idele class character

$$J_F/F^* \to S^1,$$

such that θ restricted to J_F^1 is of infinite order. Then the hypothesis of the theorem is satisfied and we get the equidistribution of the Hecke eigenvalues $\theta(\pi_v)$ on the circle.

There are some surprising instances however where we can prove equidistribution, although a priori it may not seem possible. One such example is given by the Chebotarev density theorem:

Theorem 7. *Let E/F be a Galois extension with Galois group $G(E/F)$. Then the Frobenius conjugacy classes in $G(E/F)$ are equidistributed* *with respect to the projection of the Haar measure on the space of conjugacy classes in $G(E/F)$.*

We give a brief outline of the proof. By the above criterion it is required to show that $L(s,\rho)$ does not vanish on the line $Re(s) = 1$ for any irreducible non-trivial representation ρ of $G(E/F)$. If Langlands reciprocity were to hold then this will follow from the non-vanishing result for automorphic L-functions. But this is not possible to prove at this moment, and we argue a bit differently. By Brauer's theorem given any irreducible non-trivial ρ there exist finitely many subgroups $H_i \subset G(E/F)$ and characters χ_i such that

$$\rho = \sum_i n_i \mathrm{Ind}_{H_i}^G(\chi_i)$$

in the Grothendieck ring $R(G(E/F))$ for some integers n_i. By the inductivity property of L-functions,

$$L(s,\rho) = \prod_i L(s,\chi_i)^{n_i}.$$

By Artin reciprocity, if χ_i is non-trivial then $L(s,\chi_i)$ admits an analytic contiuation to the entire plane and is non-vanishing on the line $Re(s) = 1$. The only problem occurs when χ_i is the trivial character, in which case $L(s,\rho)$ can possibly have a zero or a pole at $s = 1$. But the intertwining number

$$0 = (\rho, 1_G) = \sum_j n_i(\mathrm{Ind}_{H_i}^G(\chi_i), 1_G).$$

By Frobenius reciprocity the trivial representation is contained in $\mathrm{Ind}_{H_i}^G(\chi_i)$ if and only if χ_i is trivial. Hence we obtain

$$\sum_{\chi_i = 1} n_i = 0.$$

But this implies that $L(1,\rho)$ does not vanish and proves the theorem.

Remark 4. Chebotarev proved this theorem without using abelian reciprocity. In fact the ideas behind his proof were utilised by Artin to prove the abelian reciprocity law.

4. The methods

We now briefly indicate the various methods that have been used to establish the analytic properties of automorphic L-functions.

4.1. **The method of Tate, Godement and Jacquet.** (Refs: Tate's thesis, Jacquet's article in Corvallis)

This is the generalization of Tate's method of proving the analytic properties of the L-functions attached to Hecke characters to that of L-functions of the form $L(s,\pi,r)$ where (π,V) is an irreducible, unitary cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$ and r is the standard representation of Gl_n. The matrix coefficients for π are functions on $Gl_n(\mathbb{A})$ of the form,

$$\omega(g)=(\pi(g)\phi,\phi') \quad \phi,\ \phi'\in V,$$

where $(.,.)$ denotes the inner product on V. The global integrals considered are of the form,

$$Z(\Phi,\omega,s)=\int_{Gl_n(\mathbb{A})}\Phi(g)\omega(g)|\det(g)|^{s-1/2}dg,$$

where Φ is a Schwarz-Bruhat function on the space $M(n\times n,\mathbb{A})$ of adele valued $n\times n$ matrices. The integral converges in some half plane, the analytic continuation and the functional equation

$$Z(\Phi,\omega,s)=Z(\hat{\Phi},\tilde{\omega},1-s)$$

for such integrals comes by applying the Poisson summation formula. Here

$$\hat{\Phi}=\int_{M(n\times n,\mathbb{A})}\Phi(y)\psi(\mathrm{Tr}(xy))dy,$$

is the Fourier transform of Φ with respect to the additive character $\psi:\mathbb{A}/F\to\mathbb{C}^*$, $\tilde{\omega}$ and

$$\tilde{\omega}(g)=\omega(g^{-1}).$$

To get at the L-functions, the global integral is broken up as a product of local integrals by choosing functions $\Phi=\prod_v\Phi_v$ where Φ_v is the characteristic function of $M(n\times n,\mathcal{O}_v)$ at almost all finite places v of F (and similarly for the matrix coefficients ω written as a product of local data). The local L-factor appears as the gcd of the local analogues of the above global integrals,

$$Z(\Phi_v,\omega,s)=\int_{Gl_n(F_v)}\Phi_v(g)\omega_v(g)|\det(g)|^{s-1)/2}dg,$$

and satisfy the local functional equation,

$$\epsilon(s,\pi,\psi)\frac{Z(\Phi_v,\omega,s)}{L(s,\pi)}=\frac{Z(\hat{\Phi}_v,\tilde{\omega},1-s)}{L(1-s,\tilde{\pi}}.$$

This allows the definition of the local ϵ-factors and also to obtain the analytic continuation and functional equation of the principal L-functions. So the general outline of the method is quite similar to the Gl_1-case. But the disadvantage of this method is that it does not seem to be capable of being generalized to handle the convolution L-functions.

4.2. **Whittaker models, multiplicity one.** (Refs: Bump's book, Cogdell's notes, Bernstein-Zelevinsky)

We now digress a bit to consider Fourier-Whittaker expansions of cusp forms on $GL_n(\mathbb{A})$, since it will be needed to discuss the other methods of establishing the analytic properties of the convolution L-functions.

Given a holomorphic cusp form $f(z)$ on $\Gamma_0(N)$, there is a Fourier expansion

$$f(z) = \sum_{n \geq 1} a_n(f) e^{2\pi i n z}.$$

The theory of Whittaker models allows one to carry over such Fourier expansions in an adelic context. Let $\psi : \mathbb{A}/F \to \mathbb{C}^*$ be a non-trivial additive character of $\mathbb{A}_F$ trivial on F.

Let (π, V) be an irreducible, unitary cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$. Given a cusp form $\phi \in V$, define

$$W_\phi(g) = \int_{N(F)\backslash N(\mathbb{A})} \phi(ng)\psi(n)^{-1} dn \quad g \in GL_2(\mathbb{A})$$

where we consider ψ as a character on the group $N(F)\backslash N(\mathbb{A}) \simeq \mathbb{A}/F$. Since the dual of $\mathbb{A}/F$ is precisely F with the discrete topology, any other character $\eta : \mathbb{A}/F \to \mathbb{C}^*$ is of the form $\eta(x) = \psi(ax),\ x \in \mathbb{A}$ for some $a \in \mathbb{A}$. Using this, Fourier inversion and the fact that ϕ is a cusp form, we get the Fourier-Whittaker expansion,

$$\phi(g) = \sum_{\gamma \in F^*} W_\phi \left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g \right).$$

More generally let N be the unipotent subgroup of GL_n consisting of strictly upper triangular matrices (the diagonal entries are all one). The character ψ can be considered as a character on $N(F)\backslash N(\mathbb{A})$ by the following formula,

$$\psi(n) = \sum_{k=1}^{n-1} \psi(n_{k,(k+1)}).$$

For any cuspidal automorphic representation (π, V) of $GL_n(\mathbb{A}_F)$, and a cusp form $\phi \in V$, define W_ϕ exactly as above. The map

$$\phi \mapsto W_\phi$$

gives a $Gl_n(\mathbb{A})$-equivariant embedding of π into the space $Ind_{N(\mathbb{A})}^{Gl_n(\mathbb{A})}(\psi)$. It follows by an inductive argument that there is a Fourier-Whittaker expansion (or inversion formula),

$$\phi(g) = \sum_{\gamma \in Gl_{n-1}(F)} W_\phi\left(\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

Let F be a local field and

$$\psi : F \to \mathbb{C}^*$$

be an additive character. More generally if G is a quasi-split group over F, and N is the unipotent radical of a Borel subgroup defined by a base Δ of the positive roots, then define ψ on $N(F)$ by the formula,

$$\psi(n) = \sum_{\alpha \in \Delta} \psi(n_\alpha),$$

where n_α is the α-component of n.

Definition 4.1. An irreducible admissible representation (π, V) of $G(F)$ is said to be generic if it admits a non-zero homomorphism,

$$\mathrm{Hom}_N(\pi, \psi) \neq 0.$$

A non-zero homomorphism $\lambda \in \mathrm{Hom}_N(\pi, \psi)$ will be called a Whittaker functional. Equivalently there is an embedding

$$W : (\pi, V) \to \mathrm{Ind}_{N(F)}^{G(F)}(\psi).$$

Giving such an embedding is known as a Whittaker model for the representation (π, V) of $G(F)$.

The main result about the Whittaker models is the uniqueness of Whittaker models due to Gelfand, Kazhdan and Shalika:

Theorem 8. *Let F be a local field and N be the unipotent subgroup of GL_n consisting of strictly upper triangular matrices. Let π be an irreducible admissible representation of $Gl_n(F)$ and ψ be a character of $N(F)$ as above. Then*

$$dim\ Hom_N(F)(\pi, \psi) \leq 1.$$

Thus a Whittaker model if it exists is unique.

Globally an irreducible representation (π, V) of reductive quasi-split group G over a global field F has a Whittaker model (also called generic) if there exists a $G(\mathbb{A})$-equivariant map,

$$W_\pi : (\pi, V) \to \mathrm{Ind}_{N(\mathbb{A})}^{G(\mathbb{A})}(\psi),$$

where ψ is a non-degenerate character of $N(F)\backslash N(\mathbb{A})$ such that there exists some $\phi \in V$ satisfyin

$$\int_{N(F)\backslash N(\mathbb{A})} W_\pi(\phi)(ng)\psi(n)^{-1}dn \neq 0.$$

It follows from the Fourier-Whittaker inversion formula that an irreducible, unitary cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$ is generic.

It follows from the local uniqueness of Whittaker models and the Fourier-Whittaker inversion formula given above, that a multiplicity one theorem holds:

Theorem 9. *Let (π, V) be an irreducible, unitary cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$. Then the mulitplicity of pi in the space of cusp forms of $GL_n(\mathbb{A}_F)$ is precisely one.*

With some further input from analytic number theory and the Jacquet-Shalika theorem the following strong multiplicity one theorem can be established:

Theorem 10. *Let π_1, π_2 be irreducible, unitary cuspidal automorphic representations of $GL_n(\mathbb{A}_F)$. Suppose there exists a finite set of places $S of F$ such that for $v \notin S$, the local components*

$$\pi_{1,v} \simeq \pi_{2,v}.$$

Then $\pi_1 \simeq \pi_2$.

4.3. **Explicit integral method.** (Ref: Cogdell)

Let $f(z) = \sum_{n\geq 1} a_n(f)e^{2\pi inz}$ be a holomorphic cusp form of weight $2k$ for $Sl_2(\mathbb{Z})$. The L-function associated to f is,

$$L(s,f) = \sum_{n\geq 1} a_n(f)n^{-s} \quad \text{Re(s) large.}$$

The completed L-function can be written as a Mellin transform,

$$\Lambda(s,f) = (2\pi)^{-s}\Gamma(s)L(s,f) = \int_0^\infty f(iy)y^s d^*y.$$

The integral converges for all s and defines an entire function. Using the symmetry relation given by the action of the Weyl element $z \to -1/z$, we get the functional equation,

$$\Lambda(s,f) = \Lambda(2k-s,f).$$

Let ϕ_f be the adelic function of $Gl_2(\mathbb{A})$ associated to f in the lectures of A. Nair. Then it can be seen

$$\Lambda(s,f) = \int_{\mathbb{A}^*/\mathbb{Q}^*} \phi_f \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} d^*a.$$

Using Whittaker functions, this can be recast as

$$\Lambda(s,f) = \int_{\mathbb{A}^*} W_{\phi_f}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} |a|^{s-1/2} d^*a.$$

Suppose π is an irreducible, unitary cuspidal automorphic representations of $GL_2(\mathbb{A}_F)$, and χ be a Hecke character. Jacquet and Langlands considered integrals of the form,

$$I(s,\phi,\chi) = \int_{\mathbb{A}^*} W_{\phi}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \chi(a)|a|^{s-1/2} d^*a,$$

where ϕ is a cusp form in the space of ϕ.

More generally if $n' < n$ and $\pi,\ \pi'$ are irreducible, unitary cuspidal automorphic representations of $GL_n(\mathbb{A}_F)$ and $GL_{n'}(\mathbb{A}_F)$ respectively, Jacquet, Piatetskii-Shapiro and Shalika consider global integrals of the form,

$$I(s,\phi,\phi') = \int_{N_{n'}(\mathbb{A})\backslash Gl_{n'}(\mathbb{A})} W_{\phi}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} W_{\phi'}(a)|\det(a)|^{s-(n-n')/2} da,$$

where $\phi,\ \phi'$ are cusp forms in the space of π and π' respectively. The global functional equation for these integrals follow by considering the action of the outer automorphism

$$g \mapsto^t g^{-1}$$

acting on Gl_n.

In analogy with the earlier method it is natural to consider the local theory and to define the local L-and ϵ-factors. The local functional equation is obtained from a uniqueness statement about invariant bilinear forms on appropriate Whittaker spaces.

4.3.1. *Classical Rankin-Selberg method.* (Ref: Bump's book, Garrett's web page)

When $n = n'$, the analytic continuation is obtained in analogy with the method of Rankin-Selberg The Rankin-Selberg method was developed independently by Rankin and Selberg. They showed that if

$$f(z) = \sum_{n\geq 1} a_n e^{2\pi i n z} \quad \text{and} \quad g(z) = \sum_{n\geq 1} b_n e^{2\pi i n z},$$

are holomorphic cusp forms of weight $2k$ of level 1, then the Dirichlet series

$$L(s, f\times g) = \sum_{n\geq 1} a_n b_n n^{-s}$$

has analytic continuation and functional equation of the form,

$$\Lambda(s, f\times g) = \Lambda(4k-1-s, f\times g),$$

where

$$\Lambda(s, f \times g) = (2\pi)^{-2s}\Gamma(s)\Gamma(s-2k+1)\zeta(2s-4k+2)L(s, f \times g).$$

The unfolding trick out here involves the convolution of the invariant function $\phi(z) = f(z)g(\bar{z})y^{2k}$ against the Maass Eisenstein series,

$$\begin{aligned} E(z,s) &= \sum_{N\backslash Sl_2(\mathbb{Z})} \mathrm{Im}(\gamma z)^s \\ &= \frac{1}{2}\sum_{(m,n)=1} \frac{y^s}{(mz+n)^{2s}} \quad \mathrm{Re}(s) > 1. \end{aligned}$$

Here N is the subgroup of $SL(2,\mathbb{Z})$ consisting of the upper triangular integral matrices. By construction for each s in the region of convergence, the Eisenstein series gives a $SL(2,\mathbb{Z})$ invariant function and is an eigenfunction for the hyperbolic Laplacian having moderate growth (hence gives a Maass form). The functional equation is given by,

$$\Lambda(2s)E(z,s) = \Lambda(2-2s)E(z,1-s),$$

where $\Lambda(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function. Further the expressions in the above equation have a simple pole at $s-0,1$.

The Rankin-Selberg method depends on the unfolding of the integral

$$\Lambda(2s)\int_{Sl_2(\mathbb{Z})\backslash\mathbb{H}} E(z,s)\phi(z)\frac{dxdy}{y^2} = \Lambda(2s)\int_{N\backslash\mathbb{H}} y^s\phi(z)\frac{dxdy}{y^2}.$$

A fundamental domain for the action of N on $\mathbb{H}$ is given by the region

$$\{x+iy \mid 0 \leq x \leq 1,\ y > 0\}.$$

Using the Fourier expansions and orthogonality relations the expression on the right hand side yields $\Lambda(s, f \times g)$ (upto some factors?). The analytic continuation and functional equation of the Eisenstein series yields the same for the function $\Lambda(s, f \times g)$.

This is the classical theory. In the higher rank case when $n = n'$, Jacquet, Piatetskii-Shapiro and Shalika consider zeta integrals of the form,

$$I(s,\phi,\phi',\Phi) = \int_{Z(\mathbb{A})Gl_n(F)\backslash Gl_n(\mathbb{A})} \phi(g)\phi'(g)E(g,\Phi,s)dg,$$

where ϕ, ϕ' are cusp forms in the space of π and π' respectively. Here Φ is a Schwarz-Bruhat function on $\mathbb{A}^n$, and $E(g,\Phi,s)$ is a mirabolic Eisenstein series (we have assumed here the central characters of π and

π' to be trivial). Unfolding the Eisenstein series gives an integral of the form,

$$I(s,\phi,\phi',\Phi)=\int_{N(\mathbb{A})\backslash Gl_n(\mathbb{A})} W_\phi(g)W_{\phi'}(g)\Phi(e_ng)|\det(g)|^s dg.$$

where $e_n = (0,\cdots,0,1)$. The analytic continuation and functional equation comes from the functional equation and analytic continuation satisfied by the Eisenstein series. The integral involving the Whittaker functions can be decomposed as a product of local ones, and the local theory gives the desired properties of the global L-function. (Ref: Cogdell).

4.4. **Eisenstein series method.** (Ref. Shahidi's papers).

Classically the constant term of the Eisenstein series $E(z,s)$ considered above is given by

$$a_0(y,s)=y^s+\frac{\Lambda(2-2s)}{\Lambda(2s}y^{1-s}.$$

The analytic continuation of $E(z,s)$ yields the meromorphic continuation for the zeta function $\Lambda(s)$. It remains to control the analytic behaviour of $\Lambda(s)$. This is obtained by computing the Fourier coefficients of $E(z,s)$,

$$a_1(y,s)=\frac{2\sqrt{y}K_{s-1/2}(2\pi y)}{\Lambda(2s)}.$$

The functional equation for the Eisenstein series given above then yields the functional equation for the Riemann zeta function $\Lambda(s)$.

Let $P=GN$ be a parabolic subgroup in a reductive group H with Levi component G and unipotent radical N. Suppose π is a cuspidal representation of $G(\mathbb{A})$. Form the Eisenstein series,

$$E(s,h,\phi)=\sum_{P(F)\backslash H(F)}\phi_s(\gamma h), \quad h\in H(\mathbb{A}).$$

Here ϕ_s is a flat section of induced representation of the form

$$I(s,\pi)=\mathrm{Ind}_{P(\mathbb{A})}^{H(\mathbb{A})}(\pi\otimes\eta_s),$$

where η_s is a character on $G(\mathbb{A})$. Langlands proved the Eisenstein series to have analytic continuation. The constant term of the Eisenstein series along the opposite parabolic $P'=N'M'$ is given by,

$$\int_{N'(F)\backslash N'(\mathbb{A})} E(s,nh,\phi)dn=\phi(h)+M(s,\pi)\phi(h)$$

whee $M(s,\pi)$ intertwines the spaces $I(s,\pi)$ and $I(-s,\tilde{\pi})$. The starting point of this method is the observation that the intertwining operator can be written in terms of ratios of products of certain L-functions of the form $L(s, \pi, r)$ for certain representations r of the dual L-group. From the analytic continuation satisfied by the Eisenstein series, Langlands showed that for the L-functions appearing in constant terms of the Eisenstein series have meromorphic continuation.

Now the problem is twofold: to single out the individual $L(s, \pi, r)$ and show that they are nice. This was carried out by Shahidi by invoking a genericity assumption on π and studying the Fourier coefficients of the Eisenstein series (which involve L-functions again). The poles and zeros of the L-function can be studied in terms of the zeros and poles of the Eisenstein series. But the Eisenstein series take values in the automorphic spectrum of $H(\mathbb{A})$; this allows bringing representation theoretic techniques to control the behaviour of L-functions.

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