The Abdus Salam
International Centre for Theoretical Physics

United Nations
Educational, Scientific and
Cultural Organization

IAEA
International Atomic Energy Agency

SMR1852/3

Summer School and Conference on Automorphic Forms and Shimura Varieties

(9 - 27 July 2007)

Automorphic Forms on GL(2)

Arvind Nair
Tata Institute of Fundamental Research
School of Mathematics
Mumbai - 400 005
India

Section 1 recalls some facts about modular forms, Hecke operators, Euler products in the classical setting and gives some concrete examples. In section 2 classical modular forms are related to automorphic forms for $GL(2)$ of the adeles and proofs of the basic analytic facts about automorphic forms (analyticity, finite-dimensionality of spaces of automorphic forms, discreteness of the cuspidal spectrum) are sketched. Section 3 discusses automorphic representations of $GL(2)$, the relation with modular forms, and introduces the L-function of an automorphic representation, relating it to the classical Dirichlet series of a cusp form.

The subject matter of these notes is covered in several places (from which I have borrowed): For classical modular forms see the books [10, 11]. For the analytic properties of automorphic forms see [5, 7, 1]. The modern adelic approach to automorphic forms was introduced by Jacquet-Langlands in [6]. This approach is also covered in [4]; a more recent (and exhaustive) treatment is [1].

1. Classical modular forms

A **holomorphic modular form** of weight k for $SL(2,\mathbb{Z})$ is a holomorphic function f on the upper half-plane $\mathbb{H}$ which satisfies

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

and which is "holomorphic at the cusp". This condition means the following: Since f is invariant under $z \mapsto z+1$ it is a function on the punctured disk $0 < q < 1$ where $q = e^{2\pi i z}$. Then f is holomorphic at $i\infty$ if it admits an expansion (usually referred to as the Fourier expansion)

$$f(q) = \sum_{n\in\mathbb{Z}} a_n\, q^n$$

in which $a_n = 0$ for $n < 0$. If, in addition, the constant term a_0 vanishes then f is called a **holomorphic cusp form**. Note that since $-Id \in SL(2,\mathbb{Z})$ there are only nonzero modular forms for even weights.

More generally, in number theory one considers the Hecke congruence subgroups

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c = 0(N) \right\}$$

and, for a character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$, the functions which satisfy

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

It is sometimes convenient to use the notation

$$f\big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{k/2} (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right);$$

for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})$ with positive determinant. The transformation rule becomes

$$f\big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi(d)f \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

Since $z \mapsto z+1$ belongs to $\Gamma_0(N)$, f is again a function of $q = e^{2\pi i z}$ and we require that f is holomorphic at $q = 0$, i.e. $f = \sum_{n\geq 0} a_n q^n$. There are now other cusps, and one imposes a holomorphy condition at these too, e.g. by conjugating each one to $i\infty$ using an element $\gamma \in SL(2,\mathbb{Z})$ and using a Fourier expansion. (The subgroup of the conjugated subgroup fixing the cusp will be generated by a translation $z \mapsto z+h$ for some $h \geq 1$ (or will contain an index two subgroup generated by such a translation); the Fourier expansion is then in powers of $e^{2\pi i z/h}$.) A function f satisfying the transformation law above and holomorphic at all cusps is called a **holomorphic modular form** of **weight** k**, level** N and **character** (or **Nebentypus**) χ. The space of such forms is denoted

$$M_k(\Gamma_0(N),\chi).$$

(Note that since $Id \in \Gamma_0(N)$, $M_k(\Gamma_0(N),\chi) = \{0\}$ unless $\chi(-1) = (-1)^k$.) If the constant term vanishes at all cusps then f is called a holomorphic cusp form of weight k, level N and character χ. The space of such cusp forms is denoted

$$S_k(\Gamma_0(N),\chi).$$

The spaces $S_k(\Gamma_0(N),\chi)$ and $M_k(\Gamma_0(N),\chi)$ are finite-dimensional and for $k \geq 2$ their dimensions can be calculated (and, in particular, shown to be nonzero). Suppose that χ is trivial. The space $S_k(\Gamma_0(N))$ can be interpreted as the space of sections of a certain line bundle on the curve $X_0(N) = \Gamma_0(N)\backslash\mathbb{H}^*$ and hence its dimension can be computed using Riemann-Roch. (Recall that $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$ and $X_0(N)$ is the cusp compactification of $Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}$. This has the structure of a complete Riemann surface, cf. e.g. [11, §1.5].) For example, if $k = 2$, $f(z)dz$ extends holomorphically to the cusps and we have an isomorphism

$$S_2(\Gamma_0(N)) \cong \mathrm{H}^0(\Gamma_0(N)\backslash\mathbb{H}^*, \Omega^1).$$

The last space has dimension the genus of $\Gamma_0(N)\backslash\mathbb{H}^*$, which can be computed using the ramified covering $X_0(N) \to SL(2,\mathbb{Z})\backslash\mathbb{H}^* = \mathbb{P}^1$. For detailed computations of dimensions for $k \geq 2$ see [11, §2.6]. In the case of $SL(2,\mathbb{Z})$ there is a more elementary computation (see [10, p. 88]) which shows that

$$\dim M_k(SL(2,\mathbb{Z})) = \begin{cases} \lfloor k/12 \rfloor & \text{if } k = 2 \bmod 12, k \geq 0 \\ \lfloor k/12 \rfloor + 1 & \text{for other } k \geq 0. \end{cases}.$$

The space $\dim M_k(\Gamma_0(N))$ is the direct sum of $\dim S_k(\Gamma_0(N))$ and the space of Eisenstein series.

Having seen that modular forms exist (at least for $k \geq 2$), we would like to have some concrete examples. The simplest way to construct holomorphic modular forms for $\Gamma_0(N)$ is to use an averaging procedure: Take a holomorphic function f invariant under the subgroup

Γ_∞ (generated, up to $\pm Id$, by $z \mapsto z+1$) fixing the cusp $i\infty$ and "average" it (with a suitable transformation factor) over the cosets $\Gamma_\infty \backslash \Gamma_0(N)$:

$$\sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \backslash \Gamma_0(N)} f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-k}.$$

Ignoring convergence questions for the moment, this sum has the transformation property of a modular form of weight k (and trivial character) for $\Gamma_0(N)$. (If one wants forms with character χ an extra factor of $\chi(d)^{-1}$ is put in.) Of course, the function f is has a series expansion in $q = e^{2\pi iz}$, so in fact we can look at each $q^m = e^{2\pi imz}$ separately. We will consider the case $m = 0$ first, which leads to Eisenstein series. (The case $m > 0$, which leads to Poincaré series, will come up later.)

Suppose $m = 0$ and $f \equiv 1$ and $N = 1$. The sum above can then be rewritten as

$$G_k(z) := \sum_{(m,n) \neq (0,0)} \frac{1}{(mz+n)^k}$$

which, for $k \geq 2$, converges absolutely and uniformly on compact sets to a holomorphic function on the upper half-plane $\mathbb{H}$, the **holomorphic Eisenstein series** of weight k. (It is zero if k is odd.) The tranformation property under $SL(2,\mathbb{Z})$ is clear, so to check that G_k is a modular form for $SL(2,\mathbb{Z})$ we must show that it is holomorphic at ∞. It is not too difficult to show that

$$G_k(q) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n.$$

(See [10, p. 92].) Here ζ is the Riemann zeta function and $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the divisor function. (In fact, the constant term is easy to compute once we know the function is well-defined at $i\infty$ (and the sum absolutely convergent in neighbourhoods of $i\infty$): The sum and limit can be interchanged to get $\lim_{y\to\infty} G_k(x+iy) = \sum_{(m,n)} \lim_{y\to\infty}(mz+n)^{-k} = 2\sum_{n\geq 1} n^{-k} = 2\zeta(k)$.) Thus $G_k \in M_k(SL(2,\mathbb{Z}))$. Note that G_k is never a cusp form since $\zeta(k) \neq 0$ for even $k \geq 0$. The space of modular forms breaks up as

$$M_k(SL(2,\mathbb{Z})) = S_k(SL(2,\mathbb{Z})) \oplus \mathbb{C}G_k$$

for $k \geq 2$. (This is clear from Fourier expansions at $i\infty$.) In fact, the ring of all modular forms for $SL(2,\mathbb{Z})$ is $\mathbb{C}[E_4, E_6]$ [10, p. 93].

Knowing the constant term at $i\infty$ of E_k allows us to construct some cusp forms using the ring structure. The famous cusp form $\Delta \in S_{12}(SL(2,\mathbb{Z}))$ is defined by

$$\Delta = (60G_4)^3 - 27(140G_6)^2.$$

It is the unique cusp form of weight 12 up to multiples (because $\dim S_{12}(SL(2,\mathbb{Z})) = \dim M_{12}(SL(2,\mathbb{Z})) - 1 = 1$). Another way to define Δ is via Jacobi's product expansion in $q = e^{2\pi iz}$:

$$\Delta = (2\pi)^{12} q \prod_{n\geq 1}(1-q^n)^{24}.$$

(For a proof see [10, p. 95].) Ramanujan's τ-function, defined by the Fourier expansion of $(2\pi)^{-12}\Delta$:

$$(2\pi)^{-12}\Delta(q) = \sum_{n\geq 1} \tau(n) q^n,$$

is multiplicative: $\tau(mn) = \tau(m)\tau(n)$ for $(m,n) = 1$. This property played an important role in the development of the theory; we will see why it holds in a moment.

Returning to Eisenstein series, the holomorphic Eisenstein series for $\Gamma_0(N)$ (relative to $i\infty$) with character χ is:

$$G_k(z,\chi) = \sum_{(m,n)\neq(0,0)} \frac{\chi(n)^{-1}}{(mNz+n)^k}.$$

This is an element of $M_k(\Gamma_0(N),\chi)$ if $k \geq 2$. (The sum vanishes unless $\chi(-1) = (-1)^k$.) Once again, one can explicitly compute the Fourier expansion at $i\infty$ (and at other cusps) and show that it is holomorphic. The constant term at $i\infty$ is $2L(k,\chi^{-1})$ where $L(s,\chi) = \sum_{n\geq 1} \chi(n) n^{-s}$ is the Dirichlet series associated to χ. (For the Fourier expansion cf. [11].) One can repeat the construction with each cusp of $\Gamma_0(N)$ to get an Eisenstein series for each one; these are linearly independent and span a subspace of $M_k(\Gamma_0(N),\chi)$ complementary to the cusp forms.

A more direct construction of cusp forms is using **Poincaré series**. The mth holomorphic Poincaré series of level N, weight k, and character χ is defined by:

$$P_m(z) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(N)} \chi(d)^{-1}(cz+d)^{-k} e^{2\pi i m\gamma z}$$

where Γ_∞ is the subgroup fixing $i\infty$. When $m = 0$ this is the Eisenstein series we saw before. For $k > 0$ this sum converges absolutely, uniformly on compact subsets of $\mathbb{H}$ to a holomorphic modular form of weight k, level N and character χ. When $m > 0$ this is a cusp form, and in fact the Poincaré series span the space $S_k(\Gamma_0(N),\chi)$. However, this construction is not of much use beyond showing that cusp forms exist as the relations between different Poincaré series are mysterious.

Historically, modular forms first arose in connection with quadratic forms and **theta series**. The sum

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{i\pi n^2 z}$$

converges absolutely for $\mathrm{Im}(z) > 0$ and defines a holomorphic function on the upper half-plane $\mathbb{H}$. This function satisfies the transformation laws

$$\theta(z+2) = \theta(z)$$
$$\theta(-1/z) = (-iz)^{\frac{1}{2}}\theta(z)$$

where the branch of $(-iz)^{\frac{1}{2}}$ is chosen which takes value $\sqrt{y}$ at iy. (The first identity is straightforward; the second can be proved using Poisson summation and the fact that $e^{-\pi x^2}$ is its own Fourier transform.) The transformations $z \mapsto -\frac{1}{z}$ and $z \mapsto z+1$ generate the group $PSL(2,\mathbb{Z})$, so the function $\theta(2z)$ looks like a "modular form of weight 1/2" for $SL(2,\mathbb{Z})$. In fact, $\theta(2z)^k$ is a modular form of weight $k/2$ and

$$\theta(2z)^k \in M_{k/2}(\Gamma_0(4)) \text{ for even } k.$$

If one computes the Fourier expansion one sees that

$$\theta(2z)^k = \sum_{n\geq 0} r_k(n) q^n$$

where $r_k(n)$ is the number of representations of n as a sum of k squares. While $\theta(2z)^k$ is not always cuspidal, it always has a nonzero projection to $S_{k/2}(\Gamma_0(4))$. General bounds on the growth of Fourier coefficients of modular forms imply that $r_k(n) = O(n^{k-1})$ for large (even) k. More generally, if A is an $r \times r$ symmetric integral matrix and $N \geq 1$ is an integer such that NA^{-1} is integral, the sum

$$\theta(z, A) := \sum_{m\in\mathbb{Z}^r} e^{i\pi m^t A m z}$$

defines a holomorphic function on the upper half-plane. If r is even then

$$\theta(z, A) \in M_{r/2}(\Gamma_0(2N)).$$

The Fourier coefficients are related to the number of ways of representing integers using the quadratic form associated to A.

Relaxing the holomorphy condition on modular forms leads to **Maass forms** or **nonholomorphic modular forms**. We will not discuss these much here as they will be subsumed in the representation-theoretic discussion later. Suffice it to say that a Maass form of weight k, level N and character χ is a smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying the tranformation rule

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)\left(\frac{c\bar{z}+d}{|cz+d|}\right)^{-k} f(z) \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N),$$

the growth condition

$$|f(x+iy)| \leq y^A \quad \text{for } y > 1$$

for some $A > 0$, the analogous growth condition at other cusps, and is an eigenfunction of the differential operator $\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}$. ($\Delta_0$ is the Laplacian on $\mathbb{H}$.) (By our definition, if $f \in M_k(\Gamma_0(N), \chi)$ then $y^{k/2} f$ is a Maass form as above, with eigenvalue $\lambda = \frac{k}{2}(1 - \frac{k}{2})$ for Δ_k. The growth condition is equivalent to holomorphy of f at infinity.) Since f is invariant under $z \mapsto z+1$ it has a Fourier expansion of the form

$$f(x+iy) = \sum_{n\in\mathbb{Z}} a_n(y) e^{2\pi i n x}$$

where the coefficient $a_n(y)$ depends on y. The growth condition is simply that $|a_n(y)|$ grows polynomially. If $a_0(y) \equiv 0$ and the analogous condition at other cusps holds, then f is called a **Maass cusp form**. The fact that f is an eigenfunction of Δ_k can be used to show that it is real-analytic. The space of Maass forms is finite-dimensional, but no formula for its dimension is known. There are nonholomorphic analogues of Eisenstein series and Poincaré series which can be used to construct Maass forms, which depend on a real parameter (which replaces the weight). Whether Maass forms have a geometric significance is unknown.

In his proof of the multiplicativity of Ramanujan's function τ and the Euler product expansion of $\sum_{n\geq 1}\chi(n)n^{-s}$, Hecke introduced certain operators on modular forms which play a fundamental role. These have several equivalent definitions and here we will give one. (The definitions are for holomorphic modular forms and must be modified for non-holomorphic ones.)

For $g \in GL(2,\mathbb{Q})^+ = \{\gamma \in GL(2,\mathbb{Q}) : \det(\gamma) > 0\}$ and any congruence subgroup Γ, the double coset $\Gamma g \Gamma$ is a finite union of left or right double cosets:

$$\Gamma g \Gamma = \coprod_i \Gamma \alpha_i = \coprod_j \beta_j \Gamma.$$

This makes the formal linear sums $\sum_i a_i \Gamma g \Gamma$ into an algebra in the obvious way. For any γ one defines a right action of this algebra on $M_k(\Gamma)$ by

$$f \cdot \Gamma g \Gamma := \det(g)^{k/2-1} \sum_i f|_k \alpha_i \tag{1}$$

where

$$f\big|_k \gamma = \det(\gamma)^{k/2} f(\gamma z)(cz+d)^{-k} \qquad (\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Q})^+).$$

This preserves the space of cusp forms. When $\Gamma = \Gamma_0(N)$ this defines endomorphisms of $M_k(\Gamma_0(N))$ and $S_k(\Gamma_0(N))$. If a character χ is given we modify (1) as follows:

$$f \cdot \Gamma g \Gamma := \det(g)^{k/2-1} \sum_i \chi(\alpha_i) f|_k \alpha_i \tag{2}$$

Taking $g = \begin{pmatrix} 1 & \\ & p \end{pmatrix}$ defines the Hecke operator $T(p)$. Explicitly, the double coset decomposes as:

$$\Gamma_0(N) \begin{pmatrix} 1 & \\ & p \end{pmatrix} \Gamma_0(N) = \coprod_{\substack{a>0 \\ (a,N)=1}, ad=p} \; \coprod_{0\leq b\leq d-1} \Gamma_0(N) \sigma_a \begin{pmatrix} a & b \\ & d \end{pmatrix} \tag{3}$$

where $\sigma_a \in SL(2,\mathbb{Z})$ is chosen congruent to $\begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$ mod N. The operator $T(p)$ on $M_k(\Gamma_0(N),\chi)$ is therefore given explicitly by:

$$T(p)f = p^{k-1} \sum_{a>0, ad=p, (a,N)=1} \; \sum_{b=0}^{d-1} \chi(a) f\left(\frac{az+b}{d}\right) d^{-k}.$$

(The operators $T(p)$ commute among themselves, so we write their action on the left henceforth.) The operator defined by $\begin{pmatrix} p & \\ & p \end{pmatrix}$ is usually denoted $R(p)$; it acts on weight k forms by $R(p)f = \chi(p)f$. The definition of $T(p)$ is extended to all n using multiplicativity $T(m)T(n) = T(mn)$ for $(m,n) = 1$ and

$$T(p^{i+1}) = T(p)T(p^i) + \chi(p)p^{k-1}T(p^{i-1})$$

(and $T(1) = 1$). Using these relations, one can check the following formal identity in the endomorphism ring of $M_k(\Gamma_0(N), \chi)$:

$$\sum_{N\geq 1,(n,N)=1} \frac{T(n)}{n^s} = \prod_{p\nmid N} \frac{1}{1 - T(p)p^{-s} + \chi(p)p^{k-1}p^{-2s}}. \tag{4}$$

A key property of the Hecke operators $T(p)$ for $p \nmid N$ is that they are normal with respect to the Petersson inner product on $S_k(\Gamma_0(N))$:

$$\langle f, g \rangle := \int_{\Gamma_0(N)\backslash\mathbb{H}} f(z)\overline{g(z)}\, y^k \frac{dxdy}{y^2}.$$

(If χ is nontrivial we have $\langle T(p)f, g\rangle := \chi(p)\langle f, T(p)\rangle$.) Thus $\{T(p)\}_p$ is a commuting family of operators on $S_k(\Gamma_0(N), \chi)$ which can be diagonalized, i.e. there is a basis of eigenfunctions. Now a key property of Hecke operators is that for a cusp form $f = \sum_{q\geq 1} a_n q^n$ which is a simultaneous eigenfunction of all $T(p)$ and normalized such that $a_1 = 1$, we have

$$T(p)f = a_p f.$$

Note that there is a unique Hecke eigenform for $SL(2,\mathbb{Z})$ with a given set of eigenvalues $\{a_p\}_p$ as they determine the Fourier expansion (multiplicity one). (This is obviously false in general: $\Delta(z)$ and $\Delta(2z)$ are both in $S_{12}(\Gamma_0(2))$ and have the same Hecke eigenvalues. The Atkin-Lehner theory of newforms defines a subspace of $S_k(\Gamma_0(N), \chi)$ where multiplicity one holds.)

Now consider $\Delta \in S_{12}(SL(2,\mathbb{Z}))$. Since $T(p)$ preserves $S_{12}(SL(2,\mathbb{Z}))$ and this space has dimension 1, $(2\pi)^{-12}\Delta$ is necessarily an eigenvector of each $T(p)$, and the eigenvalue is simply $\tau(p)$ The fact that $T(m) \cdot T(n) = T(mn)$ if $(m, n) = 1$ immediately implies Ramanujan's conjecture on the multiplicativity of τ. This is equivalent to the statement that the Dirichlet series

$$L(\Delta, s) = \sum_{n\geq 1} \frac{\tau(n)}{n^s}$$

has an **Euler product** expansion (by (4)):

$$\sum_{n\geq 1} \frac{\tau(n)}{n^s} = \prod_{p} \frac{1}{1 - \tau(p)p^{-s} + p^{12-1-2s}}.$$

There is a similar Euler product expansion for the Dirichlet series

$$L(f, s) = \sum_{n\geq 1} \frac{a_n}{n^s}$$

associated to any Hecke eigencuspform $f = \sum_{n\geq 1} a_n q^n$. (Thanks to Hecke's basic estimate on Fourier coefficients $a_n = O(n^{k/2})$, the expansion is convergent for $\mathrm{Re}(s) > k$, and the Euler product expansion holds in this region.) This is similar to the L-functions (Hasse-Weil zeta functions) coming from algebraic varieties over number fields, which are defined via an Euler product expansion. (The fact that the local factor at p is a rational function in p^{-s} is quite deep in the geometric context.)

In the next section we will show how to recast the theory in the adelic framework. Some of the advantages are: the action of Hecke operators is more transparent, there is only one cusp to deal with, holomorphic and nonholomorphic forms are treated on an equal footing, ... The disadvantage is that it requires some algebraic group theory and representation theory and is somewhat less concrete.

Recall that the ring of adeles $\mathbb{A}$ of $\mathbb{Q}$ is the restricted direct product of all $\mathbb{Q}_p$ with respect to $\mathbb{Z}_p$, and $\mathbb{Q}$ is embedded diagonally in $\mathbb{A}$. There is a decomposition $\mathbb{A}^\times = \mathbb{Q}^\times \times \mathbb{R}_+ \times \widehat{\mathbb{Z}}^\times$ (strong approximation) and $\widehat{\mathbb{Z}}^\times \cong \varprojlim_N (\mathbb{Z}/N\mathbb{Z})^\times$. A character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ determines a continuous character of $\widehat{\mathbb{Z}}^\times$ and hence a character ω_χ of $\mathbb{A}^\times/\mathbb{Q}^\times$ trivial on $\mathbb{R}_+$.

2. Automorphic forms

2.1. Notation. Fix the following notation for the rest of these notes:

$G = GL(2)$
Z = the centre of G (scalar multiples of the identity)
$B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$
$U = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$
$K_\infty = O(2) = \{ g \in GL(2,\mathbb{R}) : gg^t = Id_2 \}$

For real groups the superscipt $^+$ denotes the identity component:

$Z(\mathbb{R})^+ = \{ aId_2 : a \in \mathbb{R}_+ \}$
$G(\mathbb{R})^+ = \{ g_\infty \in GL(2,\mathbb{R}) : \det(g_\infty) > 0 \}$
$K_\infty^+ = SO(2)$

Lie algebras are denoted by Gothic letters:

$\mathfrak{g}_0 = \text{Lie}\, G(\mathbb{R})^+ = M(2,\mathbb{R})$, $\mathfrak{z}_0 = \text{Lie}\, Z(\mathbb{R})^+$
$\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{C} = \mathfrak{z} \oplus \mathfrak{sl}(2,\mathbb{C})$
$U(\mathfrak{g})$ = the universal enveloping algebra of $\mathfrak{g}$
$Z(\mathfrak{g}) = U(\mathfrak{z}) \otimes \mathbb{C}[\Omega]$ where Ω is the Casimir (defined below)

For finite primes p set:

$K_p = GL(2,\mathbb{Z}_p)$
$G(\mathbb{A}_f) = \prod' G(\mathbb{Q}_p)$ (restricted direct product relative to K_p)
$G(\mathbb{A}) = G(\mathbb{A}_f) \times G(\mathbb{R})$

For $N > 0, N = \prod_p p^{n_p}$ let

$$K_0(N) = \prod_{p\nmid N} K_p \times \prod_{p|N} \left\{ \begin{pmatrix} a_p & b_p \\ c_p & d_p \end{pmatrix} \in K_p : c_p \equiv 0 \bmod p^{n_p}\mathbb{Z}_p \right\}$$

Note that $K_0(N) \cap G(\mathbb{Q}) = \Gamma_0(N)$.

2.2. **From modular forms to functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$.** The first step is to pass from a modular form to a function on $G(\mathbb{R})^+$. Define the factor of automorphy $j(g_\infty, z)$ by

$$j(g_\infty, z) = \det(g_\infty)^{-1/2}(cz + d) \qquad (g_\infty \in G(\mathbb{R})^+, z \in \mathbb{H})$$

This has the cocycle property:

$$j(g_\infty h_\infty, z) = j(g_\infty, h_\infty z) j(h_\infty, z). \tag{5}$$

The group $G(\mathbb{R})^+$ acts transitively on the upper half-plane, with the isotropy at i being $Z(\mathbb{R})K_\infty$. For $f \in M_k(\Gamma_0(N), \chi)$ define a function on $G(\mathbb{R})^+$ by

$$\phi(g_\infty) := f(g_\infty \cdot i) j(g_\infty, i)^{-k}. \tag{6}$$

This is a $Z(\mathbb{R})^+$-invariant function on $G(\mathbb{R})^+$, invariant under $\Gamma_0(N)$ for χ trivial, and satisfying

$$\phi(\gamma g_\infty) = \phi(g_\infty)\chi(d) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \tag{7}$$

in general. This follows from the cocycle property (5), which also gives:

$$\phi(g_\infty k_\theta) = \phi(g_\infty)e^{ik\theta} \quad \text{for } k_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K_\infty^+. \tag{8}$$

The next step is to pass to functions on $G(\mathbb{A})$. Recall that by the strong approximation theorem for $SL(2)$ (cf. [11] for a proof) the product $SL(2, \mathbb{Q})SL(2, \mathbb{R})$ is dense in $SL(2, \mathbb{A})$. So for any open subgroup $U \subset SL(2, \mathbb{A}_f)$ we have

$$SL(2, \mathbb{Q})SL(2, \mathbb{R})U = SL(2, \mathbb{A}).$$

Using the fact that $\mathbb{Q}^\times \mathbb{R}_+ \widehat{\mathbb{Z}}^\times = \mathbb{A}^\times$ we get that for any compact open subgroup $K \subset G(\mathbb{A}_f)$ such that $\det(K) = \widehat{\mathbb{Z}}^\times$, we have

$$G(\mathbb{Q})G(\mathbb{R})^+K = G(\mathbb{A}).$$

Lemma 1. *For $\Gamma = G(\mathbb{Q}) \cap KG(\mathbb{R})^+$ the inclusion $G(\mathbb{R})^+ \subset G(\mathbb{A})$ induces an identification*

$$\Gamma\backslash G(\mathbb{R})^+ = G(\mathbb{Q})\backslash G(\mathbb{A})/K. \tag{9}$$

In the case of trivial character this immediately gives that ϕ is a function on $G(\mathbb{Q})\backslash G(\mathbb{A})/K_0(N)$. To deal with a general character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$ note that it determines a continuous character

$$\chi : K_0(N) \to \mathbb{C}^\times$$

by composing $K_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a$ with χ. Extend ϕ to a function on $G(\mathbb{A})$ by

$$\phi(\gamma g_\infty k) = \phi(g_\infty)\chi(k) \tag{10}$$

for $\gamma \in G(\mathbb{Q}), g_\infty \in G(\mathbb{R})^+, k \in K_0(N)$. This is well-defined because ϕ satisfies (7).

Now $G(\mathbb{A})$ acts on itself by both left and right translations. By definition, ϕ is invariant under the left translation action of $G(\mathbb{Q})$. We will consider the right translation actions of

$Z(\mathbb{A}), K_\infty, K_0(N)$, and $G(\mathbb{R})^+$. The action of $G(\mathbb{R})^+$ gives an action of $\mathfrak{g}_0$ and hence of $U(\mathfrak{g})$. Define elements $H, M_+, M_- \in \mathfrak{g}$ by:

$$H = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, M_+ = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, M_- = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

The Casimir operator is

$$\Omega := H^2 + 2M_+M_- + 2M_-M_+.$$

The following lemma summarizes the properties of ϕ:

Lemma 2. *The function ϕ on $G(\mathbb{A})$ associated to $f \in M_k(\Gamma_0(N), \chi)$ is left-invariant under $G(\mathbb{Q})$ and satisfies:*

(i) $\phi(gk) = \chi(k)\phi(g)$ *for* $k \in K_0(N)$
(ii) $\phi(z_\infty g k_\theta) = \phi(g)(e^{i\theta})^k$ *for* $z_\infty \in Z(\mathbb{R})^+, k_\theta \in K_\infty^+$
(iii) $\phi(zg) = \omega_\chi(z)\phi(g)$ *for* $z \in Z(\mathbb{A}), g \in G(\mathbb{A})$
(iv) ϕ *is killed by* M_- *and* $\Omega\phi = k(k-2)\phi$

If $f \in S_k(\Gamma_0(N), \chi)$ *then* ϕ *further satisfies*

(v) $\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \phi(ug)\, du = 0$ *for all* $g \in G(\mathbb{A})$.

Moreover any function on $G(\mathbb{A})$ satisfying these conditions comes from a modular form f of weight k and character χ.

If f is a Maass form of weight k then let $\phi(g_\infty) := \left(\frac{-ci+d}{|ci+d|}\right)^k f\left(\frac{ai+b}{ci+d}\right)$ and set $\phi(\gamma g_\infty k) = \phi(g_\infty)\chi(d)$ for $\gamma g_\infty k \in G(\mathbb{A})$. This function satisfies (i), $\phi(z_\infty g k_\theta) = (e^{i\theta})^{2k}\phi(g)$, (iii), and is $Z(\mathfrak{g})$-finite. If f is a Maass cusp form then ϕ satisfies (v).

Proof. (i) and (ii) are (7) and (8), (iii) follows from (7), so let us consider (iv). The condition $M_-\phi = 0$ is equivalent to the holomorphy of f, as we shall now see. In the coordinates (x, y, a, θ) on $GL(2,\mathbb{R})^+$ given by the decomposition (coming from the Iwasawa decomposition)

$$GL(2,\mathbb{R})^+ = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right\} \times \left\{\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right\} \times \{k_\theta\} \qquad (a, y > 0)$$

the differential operator M_- on functions on $G(\mathbb{R})^+$ is given by

$$M_- = -iye^{-2i\theta}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) + \frac{i}{2}e^{-2i\theta}\frac{\partial}{\partial \theta}$$

(see [7, p. 115f] for this calculation). Note that $\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} = \frac{\partial}{\partial \bar{z}}$ is the Cauchy-Riemann operator. If f is of weight k then it is enough to look at $\phi(g_\infty) = f(g_\infty \cdot i)j(g_\infty, i)^{-k}$ defined by it. Writing $g_\infty = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}k_\theta$ we have

$$\phi(g_\infty) = y^{k/2}e^{ik\theta}f(yi + x)$$

This gives

$$(M_-\phi)(g_\infty) = -2y^{\frac{k}{2}+1}e^{2(\frac{k}{2}-1)i\theta}\frac{\partial f}{\partial \bar{z}} = 0$$

since f is holomorphic.

For the second part of (iv), note that $\exp i\theta H = k_\theta$, so $H\phi = k\phi$ by (ii). Note the relations

$$[H, M_+] = 2M_+, [H, M_-] = 2M_-, [M_+, M_-] = H.$$

Then $\Omega\phi = k^2 + 2M_-M_+\phi = k^2 - 2[M_+, M_-]\phi = (k^2 - 2k)\phi$.

Consider the integral in (v), assuming du has been normalized so that $\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} du = 1$ and $g = g_\infty g_f$. First suppose $g_f = Id$. Writing $g_\infty = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} k_\theta$ using the Iwasawa decomposition we have

$$\begin{aligned} \int_{\mathbb{Q}\backslash\mathbb{A}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx &= \int_{\mathbb{Z}\backslash\mathbb{R}} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} k_\theta\right) dx \\ &= \int_{\mathbb{Z}\backslash\mathbb{R}} f(a^2\, i + x)(a^{-1})^{-k} dx \\ &= a^k \int_{\mathbb{Z}\backslash\mathbb{R}} f(a^2\, i + x) dx. \end{aligned}$$

The last integral is the constant Fourier coefficient at the cusp $i\infty$. At the cusp conjugate to $i\infty$ under $\gamma \in G(\mathbb{Q})$, the same integral with $g_f = \gamma$ gives the constant Fourier coefficient at that cusp. This proves that f is a cusp form if and only if the integral in (v) vanishes (for all $g \in G(\mathbb{A}_f)$.

For the remarks about Maass forms, note that in the coordinates above,

$$\Omega = 4y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - 4y\frac{\partial^2}{\partial x \partial\theta}$$

(cf. [7, p. 198]). So if f is an eigenfunction of

$$\Delta_k = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}$$

and of weight k, then it is an eigenfunction of Ω. In particular, it is $\mathbb{C}[\Omega]$-finite and hence $Z(\mathfrak{g})$-finite by (ii). The proof of (v) given in the holomorphic case applies to Maass cusp forms. □

The properties of ϕ in the lemma motivate the general definition of automorphic form. Note the condition of vanishing at each cusp has been replaced by the condition that a single adelic integral vanishes.

2.3. **Automorphic forms.** Let ω be a character of $Z(\mathbb{A})/Z(\mathbb{Q})$ trivial on $Z(\mathbb{R})^+ = \mathbb{R}_+$. An **automorphic form** with central character ω is a function $\phi : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ which satisfies the following:

(i) for each $g_f \in G(\mathbb{A}_f)$ the function $g_\infty \mapsto \phi(g_f g_\infty)$ is smooth on $G(\mathbb{R})^+$
(ii) ϕ is right K_∞-finite
(iii) ϕ is right-invariant under a compact open subgroup K of $G(\mathbb{A}_f)$
(iv) ϕ transforms according to ω under $Z(\mathbb{A})$
(v) ϕ is $Z(\mathfrak{g})$-finite

(vi) ϕ is of moderate growth

Given (iv), (v) is equivalent to the statement that ϕ is killed by a polynomial in the Casimir Ω. The condition (vi) must be explained. We will define a norm on $G(\mathbb{A})$ as follows: Embed $G(\mathbb{A})$ as a closed subset of $M(2,\mathbb{A}) \times M(2,\mathbb{A}) \cong \mathbb{A}^8$ by $g \mapsto (g, g^{-1})$. Now use the norm on $\mathbb{A}^8$ given by $||(x_1,\ldots,x_8)|| = \prod_v \max_i(|x_i|_v)$ where $|\cdot|_v$ is the absolute value on $\mathbb{Q}_v$ for all places v. In terms of this norm on $G(\mathbb{A})$ the function ϕ is said to be of **moderate growth** if there are c and $n \in \mathbb{Z}$ such that

$$|\phi(g)| \leq c||g||^n \quad \text{for } g \in G(\mathbb{A}).$$

(It might seem that this definition is a little arbitrary but it can be shown that it is independent of the norm, at least for any reasonable norm.) The space of automorphic forms with central character ω is denoted

$$\mathscr{A}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$$

This definition of automorphic form was given in [5] and [6, §10].

Lemma 3. *An automorphic form is a real analytic function (more precisely, for fixed g_f the function $g_\infty \mapsto \phi(g_f g_\infty)$ is real analytic on $G(\mathbb{R})$).*

Proof. We will show that a function ϕ on $G(\mathbb{R})^+$ which is $Z(\mathbb{R})^+$-invariant, K_∞^+-finite and $Z(\mathfrak{g})$-finite is necessarily real analytic. Since ϕ is $Z(\mathfrak{g})$-finite there is a polynomial P such that $P(\Omega)\phi = 0$. Consider the operator Π_k which projects to the $(e^{i\theta})^k$-eigenspace of K_∞^+:

$$(\Pi_k\phi)(g) := \frac{1}{2\pi}\int_0^{2\pi} \phi(gk_\theta)e^{-ik\theta}d\theta.$$

Since Ω is invariant it commutes with Π_k. On the $(e^{i\theta})^k$-eigenspace, the operator Ω agrees with Δ_k. Hence we have

$$P(\Delta_k)\Pi_k\phi = P(\Omega)\Pi_k\phi = \Pi_k P(\Omega)\phi = 0.$$

The operator $P(\Delta_k)$ has leading term $(-1)^n y^{2n}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^n$, where $n = \deg P$, and is therefore elliptic. By standard regularity results $\Pi_k\phi$ is real analytic. Since ϕ is K_∞^+-finite it is a finite sum $\phi = \sum_{|k|\leq M} \Pi_k\phi$ for some M and hence is real analytic. □

The action of $G(\mathbb{A}_f)$ by right translations evidently preserves $\mathscr{A}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$. The condition of K_∞^+-finiteness (or K_∞-finiteness; they are equivalent) is *not* preserved by right translation under $G(\mathbb{R})$, so there is no representation of $G(\mathbb{R})$. The property of K_∞-finiteness is preserved by the action of $\mathfrak{g}$. (Indeed, consider the generators H, M_-, M_+ of $\mathfrak{g}$ defined earlier. A function ϕ has K_∞^+-weight $(e^{i\theta})^k$ if and only if $H\phi = k\phi$. The commutation relations then imply that M_- lowers the H-weight by 2 and M_+ raises H-weights by 2.) But it is not obvious that the moderate growth condition is preserved by $\mathfrak{g}$. This (somewhat delicate) fact is proved in [5]; for a detailed account tailored to our situation see [1, 2.9]. Here is a sketch of the argument: The key point is that for a K_∞^+-finite $Z(\mathfrak{g})$-finite smooth function ϕ on $G(\mathbb{R})^+$ there is a function $\alpha \in C_c^\infty(G(\mathbb{R}))$ with $\alpha(kgk^{-1}) = \alpha(g)$ for all $k \in K_\infty^+$, such that

$$\phi = \phi * \alpha.$$

Then for $X \in \mathfrak{g}$:

$$X\phi = X(\alpha * \phi) = X\alpha * \phi.$$

Thus $X\phi$ is of moderate growth if ϕ is of moderate growth. To find α, let $J \subset C_c^\infty(G(\mathbb{R})^+)$ be the convolution algebra of functions invariant by conjugation by K_∞^+. Let $\{\alpha_n\}_n$ be an approximate identity in $C_c^\infty(G(\mathbb{R}))$. By averaging over K_∞^+ we can arrange that $\{\alpha_n\} \subset J$. Now the sequence $\phi * \alpha_n$ approximates ϕ uniformly (in the sup norm) on arbitrarily large open subsets of $G(\mathbb{R})$. Suppose $f = \sum_{|k|\leq M} \Pi_k f$. Let W be the sum of the K_∞^+-eigenspaces with character $(e^{i\theta})^k$ for $|k| \leq M$. Then $f * J \subset W$ and so is finite-dimensional and hence closed in the sup norm. Since W is finite-dimensional, there is an open set $U \subset SL(2,\mathbb{R})$ such that restriction of functions $W \to C^\infty(U)$ is injective. It follows that $f \in f * J$ and hence there is an $\alpha \in J$ with $f = f * \alpha$.

The upshot of the previous discussion is that $\mathscr{A}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ is a $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module. Recall that a $(\mathfrak{g}, K_\infty)$**-module** is a vector space V with a $U(\mathfrak{g})$-module structure and a representation of K_∞ such that

(i) every vector is K_∞-finite
(ii) for $X \in \mathrm{Lie}\,(K_\infty)$,

$$X \cdot v = \frac{d}{dt}\bigg|_{t=0} (\exp(tX) \cdot v) \qquad (v \in V)$$

(iii) the actions are compatible:

$$k \cdot X \cdot v = \mathrm{Ad}(k)X \cdot k \cdot v \qquad (k \in K_\infty, X \in \mathfrak{g}, v \in V).$$

The $G(\mathbb{A}_f)$-representation on $\mathscr{A}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ is **smooth**: every vector is fixed under a compact open subgroup. A $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module V is **admissible** if, for any compact open subgroup $K \subset G(\mathbb{A}_f)$ and any irreducible representation σ of $K \times K_\infty$, the space $V[\sigma] := \mathrm{Hom}_{K\times K_\infty}(\sigma, V)$ of vectors which transform according to σ under $K \times K_\infty$ is finite-dimensional.

The fundamental theorem on the space of automorphic forms is the following:

Theorem 1. *(Harish-Chandra) Suppose $I \subset Z(\mathfrak{g})$ is an ideal of finite codimension. Then the space of automorphic forms $\mathscr{A}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)_I$ killed by I is an admissible $(\mathfrak{g}, K_\infty)\times G(\mathbb{A}_f)$-module.*

In particular one may take $I = \ker(\xi)$ the kernel of a character $\xi : Z(\mathfrak{g}) \to \mathbb{C}$. Then $\mathscr{A}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)_I$ consists of ϕ such that $D\phi = \chi(D)\phi$ for $D \in Z(\mathfrak{g})$. Some remarks on the proof of this theorem will be made in the next section.

The **constant term** of $f \in \mathscr{A}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ along U is the function on $B(\mathbb{Q})U(\mathbb{A})\backslash G(\mathbb{A})$ given by

$$g \mapsto \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} f(ug)\, du. \tag{11}$$

A **cuspidal automorphic form** (cusp form for short) with central character ω is an automorphic form f with constant term zero, i.e.:

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} f(ug)\, du = 0 \quad \text{for all } g \in G(\mathbb{A}). \tag{12}$$

(Here du is any Haar measure on $U(\mathbb{A}) \cong \mathbb{A}$.) This condition is stable under the $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-action. ([1]) The submodule of cusp forms is denoted

$$\mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega).$$

A $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module is **irreducible** if it has no proper $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-submodules. The fundamental theorem on the space of cusp forms is the following:

Theorem 2. *(Gelfand–Graev–Piatetskii-Shapiro) The $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module $\mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ is semisimple, i.e. is isomorphic to a(n algebraic) direct sum of irreducible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-modules, each occurring with finite multiplicity.*

Note that $\mathscr{A}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ is definitely *not* semisimple. Some remarks on the proof of this theorem will be made in the next section. An irreducible summand of $\mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ is a **cuspidal automorphic representation**.

Let us check that our definitions do indeed generalize the examples of modular forms we have seen. By Lemma 2, for any weight k, the mapping $f \mapsto \phi$ defines embeddings

$$M_k(\Gamma_0(N), \chi) \hookrightarrow \mathscr{A}(G, \omega_\chi)$$
$$S_k(\Gamma_0(N), \chi) \hookrightarrow \mathscr{A}_0(G, \omega_\chi).$$

(Here ω_χ is the character of $Z(\mathbb{Q})\backslash Z(\mathbb{A})$ associated to χ.) Thus holomorphic modular forms give automorphic forms. The growth condition for Maass forms implies the moderate growth condition, so that the construction for Maass forms in Lemma 2 gives an inclusion of Maass forms in $\mathscr{A}(G, \omega_\chi)$ and of Maass cusp forms in $\mathscr{A}_0(G, \omega_\chi)$. So the definition of automorphic form is general enough to include all our examples. (Moreover, they are all elements of the same space, separated by the actions of K_∞^+, $Z(\mathfrak{g})$ etc.) The various finite-dimensionality statements about spaces of modular forms are implied by Theorem 1 (though not, of course, statements about exact dimensions). The fact that the Hecke operators $T(p)$ can be simultaneously diagonalized on $S_k(SL(2, \mathbb{Z}))$ is contained in Theorem 2 (as we shall see later). The proofs of these theorems use Hilbert-space methods (in the classical setting one sees this in the use of Petersson's inner product to diagonalize the $T(p)$).

2.4. Square-integrable functions and the cuspidal spectrum.

Let $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ denote the Hilbert space of functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ square-integrable with respect to the natural measure, with the right regular representation $(R_g f)(h) = f(hg)$. This representation is a direct integral over characters ω of $Z(\mathbb{Q})\backslash Z(\mathbb{A})$ of the subspaces $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ of functions with central character ω. For a unitary character ω,

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$$

[1] Cuspidal automorphic forms are bounded functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$. In fact, we could have defined cuspidal automorphic forms as smooth K_∞-finite, $Z(\mathfrak{g})$-finite, bounded functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ with central character ω which are right-invariant under some open compact $K \subset G(\mathbb{A}_f)$. Stability under $U(\mathfrak{g})$ follows by the same argument used to prove that the moderate growth condition is $U(\mathfrak{g})$-stable.

is the space of all functions on $G(\mathbb{Q})\backslash G(\mathbb{A})$ with central character ω and which are square-integrable modulo the centre, i.e.

$$\int_{G(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})} |f(g)|^2 dg < \infty.$$

There are natural operators in this space coming from the right regular representation. Define the space of **test functions** to be the tensor product

$$C_c^\infty(G(\mathbb{A})) = C_c^\infty(G(\mathbb{R})) \otimes C_c^\infty(G(\mathbb{A}_f)).$$

Here $C_c^\infty(G(\mathbb{A}_f))$ is the space of locally constant functions of compact support on $G(\mathbb{A}_f)$ and $C_c^\infty(G(\mathbb{R}))$ has its usual meaning. This is an algebra under convolution of functions:

$$(\varphi * \psi)(g) = \int_{G(\mathbb{A})} \varphi(gh^{-1})\psi(h)dh.$$

There is a natural topology on the space of test functions, which we will discuss later. For $\varphi \in C_c^\infty(G(\mathbb{A}))$ and $f \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ define

$$\begin{aligned}(R_\varphi f)(x) &= \int_{G(\mathbb{A})} \varphi(g) f(xg) dg \\ &= \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} \left(\int_{Z(\mathbb{A})} \varphi(zg)\omega(z)dz \right) f(xg)dg\end{aligned} \tag{13}$$

This gives a representation of the convolution algebra $C_c^\infty(G(\mathbb{A}))$ in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$. For functions in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ the constant term integral

$$g \mapsto \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} f(ug)\, du. \tag{14}$$

makes sense as a function (a. e.) on $B(\mathbb{Q})U(\mathbb{A})\backslash G(\mathbb{A})$. A function is called **cuspidal** if its constant term is identically zero (i.e. a. e. in $g \in G(\mathbb{A})$); the subspace of cuspidal L^2 functions is denoted

$$L_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$$

and called the **cuspidal spectrum**. It is preserved by the right translation action of $G(\mathbb{A})$ and hence by the operators R_φ.

Lemma 4. *Cuspidal automorphic forms are bounded functions, hence:*

$$\mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega) \subset L_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega).$$

Proof. Let us check that ϕ associated to $f \in S_k(\Gamma_0(N), \chi)$ is L^2. The function $|\phi|^2$ is a function on

$$G(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})/K_0(N)K_\infty = \Gamma_0(N)Z(\mathbb{R})^+\backslash G(\mathbb{R})^+/K_\infty^+ = \Gamma_0(N)\backslash\mathbb{H}.$$

Using $dg = y^{-2}dxdyd\theta$ we have

$$\int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} |\phi(g)|^2 dg \sim \int_{\Gamma_0(N)\backslash\mathbb{H}} |f(z)|^k y^k y^{-2} dxdy < \infty.$$

(Note the relation of the L^2 inner product with the Petersson inner product on cusp forms.)

In fact, cusp forms decrease rapidly at infinity. More precisely, if ϕ is a cusp form and $\|\cdot\|$ is a norm on $G(\mathbb{A})$ then for any $n \in \mathbb{Z}$ there is a constant c_n such that

$$|\phi(g)| \leq c_n ||g||^n.$$

For ϕ associated to a holomorphic f this is an easy exercise. □

The theorem of Gelfand–Graev–Piatetskii-Shapiro is:

Theorem 3. *(i) For $\varphi \in C_c^\infty(G(\mathbb{A}))$, the operator R_φ is a compact operator on $L_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$.*

(ii) The space $L_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ decomposes discretely, i.e. it is a Hilbert space direct sum of irreducible closed $G(\mathbb{A})$-submodules, each with finite multiplicity.

We will sketch the proof in several steps. The argument is covered in more detail in [1, 3.2, 3.3] and [7, Chp. XII] and there is a careful treatment in [3] (which works for a general reductive group).

Lemma 5. *Suppose $\mathcal{A}$ is a *-closed algebra of compact operators on a Hilbert space H and assume that $\mathcal{A}$ is nondegenerate (i.e. $\mathcal{A}v \neq \{0\}$ for $v \neq 0$). Then H is a direct sum of closed irreducible subspaces with finite multiplicities.*

Proof of the lemma. Among all collections of mutually orthogonal irreducible closed $\mathcal{A}$-stable subspaces choose a maximal one $\mathcal{C}$ using Zorn's lemma. Replacing H by the orthogonal complement of the closure of the direct sum of all subspaces in $\mathcal{C}$ we may assume that H has no proper closed irreducible subspaces. We will show that this leads to a contradiction, so that the orthogonal complement is zero. First, there is a nonzero normal operator $T \in \mathcal{A}$. (There is some nonzero $T \in \mathcal{A}$ by nondegeneracy, take $T + T^*$ or $(T - T^*)/i$ to get a normal one.) The spectral theorem insures that T has at least one nonzero eigenvalue λ. Now among all closed $\mathcal{A}$-stable subspaces containing a λ-eigenvector choose the one V such the eigenspace $V_\lambda = \{v \in V : Tv = \lambda v\}$ is of minimal dimension. Choose a nonzero $v \in V_\lambda$ and let W be the closure of $\mathcal{A} \cdot v$. We claim that W is $\mathcal{A}$-irreducible. Indeed, suppose $W = W_1 \oplus W_2$ for closed orthogonal $\mathcal{A}$-stable subspaces. Writing $v = v_1 + v_2$ accordingly, we see that $\lambda v = Tv = Tv_1 + Tv_2$ and hence (by orthogonality) $Tv_i = \lambda v_i$ for $i = 1, 2$. Thus $v_i \in V_\lambda$. If both are nonzero either of W_i is $\mathcal{A}$-stable and $\dim(W_i)_\lambda < \dim V$, contradicting the choice of W. Suppose $v = v_1$. Then $W = W_1$ and $W_2 = 0$. This proves W is a closed $\mathcal{A}$-irreducible subspace. Either $W = H$ or W is a proper subspace; either case leads to a contradiction.

For finiteness of multiplicities note that if some $\mathcal{A}$-representation appears infinitely many times then T has infinitely many linearly independent eigenvectors with a common eigenvalue, contradicting the compactness of T. □

Proof that (i) ⇒ (ii) in Thm 3. To apply the lemma to our situation note that $\{R_\varphi\}_\varphi$ is *-closed since $R_\varphi^* = R_{\varphi^*}$ where $\varphi^*(g) = \overline{\varphi(g^{-1})}$. Nondegeneracy follows from the existence of approximate identities in $C_c^\infty(G(\mathbb{A}_f))$.

We have proved (assuming (i)) that the cuspidal spectrum decomposes as a direct sum of closed $C_c^\infty(G(\mathbb{A}))$-submodules with finite multiplicities. This implies the same as $G(\mathbb{A})$-modules. $\square$

Let φ be a test function and let

$$\varphi_\omega(g) = \int_{Z(\mathbb{A})} \varphi(zg)\omega(z)dz.$$

Since $R_\varphi = R_{\varphi_\omega}$ we will simply assume that $\varphi = \varphi_\omega$ and remember that it transforms according to ω^{-1} under $Z(\mathbb{A})$. Now

$$\begin{aligned}(R_\varphi f)(g) &= \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} \varphi(h)f(gh)dh = \int_{Z(\mathbb{A})\backslash G(\mathbb{A})} \varphi(g^{-1}h)f(h)dh \\ &= \int_{U(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})} K(g,h)f(h)dh \qquad (15)\end{aligned}$$

where

$$K(g,h) := \sum_{\gamma\in U(\mathbb{Q})} \varphi(g^{-1}\gamma h).$$

Define a compactly supported smooth function on $\mathbb{A}$ by

$$\varphi_{g,h}(x) := \varphi\left(g^{-1}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}h\right).$$

Let us assume fixed a (unitary) additive character ψ of $\mathbb{A}/\mathbb{Q}$ and let $\widehat{\varphi}_{g,h}$ be the Fourier transform with respect to ψ. Poisson summation gives:

$$\sum_{\alpha\in\mathbb{Q}} \widehat{\varphi}_{g,h}(\alpha) = \sum_{\gamma\in U(\mathbb{Q})} \varphi(g^{-1}\gamma h) = K(g,h) \qquad (16)$$

Now

$$\widehat{\varphi}_{g,h}(0) = \int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(g^{-1}nh)dn.$$

Note that $\widehat{\varphi}_{g,nh}(0) = \widehat{\varphi}_{g,h}(0)$ for $n \in U(\mathbb{A})$. Hence for any $f \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$ we have:

$$\begin{aligned}\int_{U(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})} \widehat{\varphi}_{g,h}(0)f(h)dh &= \int_{U(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})}\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \widehat{\varphi}_{g,nh}(0)f(nh)dndh \\ &= \int_{U(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})}\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \widehat{\varphi}_{g,h}(0)f(nh)dndh \\ &= \int_{U(\mathbb{A})Z(\mathbb{A})\backslash G(\mathbb{A})} \widehat{\varphi}_{g,h}(0)\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} f(nh)dndh\end{aligned}$$

Thus for a cuspidal function f we have

$$\int_{U(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})} \widehat{\varphi}_{g,h}(0)f(h)dh = 0. \qquad (17)$$

By (15),(16), and (17) we have:

$$(R_\varphi f)(g) = \int_{U(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})} \sum_{\alpha \in \mathbb{Q}^\times} \widehat{\varphi}_{g,h}(\alpha)\, f(h) dh \tag{18}$$

for cuspidal f.

The main point in the proof of (i) of Theorem 3 is that the sup norm of $R_\varphi f$ is bounded in terms of the L^2 norm of f (Lemma 6 below). This will follow from estimates for the sum $\sum_{\alpha\in\mathbb{Q}^\times} \widehat{\varphi}_{g,h}(\alpha)$ appearing in (18). The bounds will be uniform for compact sets of test functions, so we must discuss the topology on the space of test functions. On $C_c^\infty(G(\mathbb{A}_f))$ the topology is simple: It is the direct limit of the (finite-dimensional) spaces of right K-invariant functions with support in S where S varies over compact subsets of $G(\mathbb{A}_f)$ and K varies over compact open subgroups. The topology on $C_c^\infty(G(\mathbb{R}))$ is the direct limit of the locally convex topologies on $C_\Omega^\infty(G(\mathbb{R}))$ (smooth functions with support in Ω) as Ω varies over all compact sets. The locally convex topology on $C_\Omega^\infty(G(\mathbb{R}))$ is given by the seminorms $\varphi \mapsto \sup_{g\in K} |D\varphi(g)|$ for $D \in U(\mathfrak{g})$. (The particular choice of differential operators does not matter in the topology.)

Lemma 6. *Suppose $\varphi \in C_c^\infty(G(\mathbb{A}))$. There is a constant $C = C(\varphi)$ such that*

$$\|R_\varphi f\|_\infty \le C\|f\|_2$$

for $f \in L_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$. The constant C may be chosen uniformly for compact sets of test functions.

Proof. We will need to use Siegel sets, which are approximate fundamental domains for the (left) action of $G(\mathbb{Q})$ on $G(\mathbb{A})$. For $c, d > 0$ let $\mathfrak{S} \subset G(\mathbb{A})$ be the set of $g = g_f g_\infty$ such that

$$g_f \in \mathbb{K} := \prod_p K_p \quad \text{and} \quad g_\infty = \begin{pmatrix} a & \\ & a \end{pmatrix} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} k_\theta \qquad \text{for } \begin{cases} y \ge c \\ 0 \le x \le d. \end{cases}$$

For $c \le \sqrt{3}/2$ and $d \ge 1$,

$$G(\mathbb{A}) = G(\mathbb{Q})\mathfrak{S}.$$

It will suffice to show that for some constant C,

$$\sup_{g\in\mathfrak{S}} |(R_\varphi f)(g)| \le C\|f\|_2.$$

Our task is to estimate the sum $\sum_{\alpha\neq 0} \widehat{\varphi}_{g,h}(\alpha)$ appearing in (18). Write $g \in \mathfrak{S}$ and $h \in G(\mathbb{A})$ as

$$g = \begin{pmatrix} a & \\ & a \end{pmatrix} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} \kappa_g \qquad (y \ge c, x \le d, \kappa_g \in K_\infty^+ \times \mathbb{K})$$

$$h = \begin{pmatrix} z & \\ & z \end{pmatrix} \begin{pmatrix} v & u \\ & 1 \end{pmatrix} \kappa_h \qquad (z, v \in \mathbb{A}^\times, u \in \mathbb{A}, \kappa_h \in K_\infty^+ \times \mathbb{K}).$$

Then

$$\widehat{\varphi}_{g,h}(\alpha) = \int_{\mathbb{A}} \varphi\left(\kappa_g^{-1} \begin{pmatrix} a^{-1}z & \\ & a^{-1}z \end{pmatrix} \begin{pmatrix} y^{-1} & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & t-x+u \\ & 1 \end{pmatrix} \begin{pmatrix} v & \\ & 1 \end{pmatrix} \kappa_h \right) \psi(\alpha t) dt.$$

This is

$$\widehat{\varphi}_{g,h}(\alpha) = \psi(\alpha(x-u))\,\omega(z^{-1}a)\,|y|\,\widehat{F}_{\kappa_g,\kappa_h,y^{-1}v}(\alpha y) \tag{19}$$

where

$$F_{\kappa_g,\kappa_h,y}(t) = \varphi\left(\kappa_g^{-1}\begin{pmatrix}1 & t\\ & 1\end{pmatrix}\begin{pmatrix}y & \\ & 1\end{pmatrix}\kappa_h\right).$$

Since $K\mathrm{supp}(\varphi)K \cap B(\mathbb{R})$ is compact, there is a compact set $C \subset \mathbb{A}^\times$ such that $F_{\kappa_g,\kappa_h,y}(t)$ vanishes identically unless $y \in C$. Thus the functions $F_{\kappa_g,\kappa_h,y}(t)$ lie in a compact set in $C_c^\infty(U(\mathbb{A}))$, and hence in a compact set in the Schwartz space. Their Fourier transforms then lie in a compact set of the Schwartz space. This implies a uniform estimate on the coefficients: For any $N > 0$ there is a c_N such that

$$|\widehat{F}_{\kappa_g,\kappa_h,y^{-1}v}(\alpha)| \leq c_N|\alpha|^{-N}. \tag{20}$$

Replacing $\begin{pmatrix}1 & \alpha\\ & 1\end{pmatrix} \in U(\mathbb{A})$ by its conjugate by $\begin{pmatrix}y & \\ & 1\end{pmatrix}$ gives

$$|\widehat{F}_{\kappa_g,\kappa_h,y^{-1}v}(y\alpha)| \leq c_N|y\alpha|^{-N} = c_N|y|^{-N}|\alpha|^{-N}.$$

Taking $|\cdot|$ in (19) and using this estimate gives

$$\widehat{\varphi}_{g,h}(\alpha)| \leq c_N|y|^{1-N}|\alpha|^{-N}$$

and hence

$$\left|\sum_{\alpha\in\mathbb{Q}^\times}\widehat{\varphi}_{g,h}(\alpha)\right| \leq \sum_{\alpha\neq 0} c_N|y|^{1-N}|\alpha|^{-N} \leq C_N|y|^{-N}.$$

Putting this estimate back in (18) and using the relation of measures $dh = |v|^{-1}d\kappa_h d^\times v du$, we get:

$$\begin{aligned}|R_\varphi f(g)| &\leq C_N|y|^{-N}\int_{\mathbb{Q}\backslash\mathbb{A}}\int_{y^{-1}v\in C}\int_K \left|f\left(\begin{pmatrix}v & u\\ & 1\end{pmatrix}\kappa_h\right)\right| |v|^{-1}\,d\kappa_h d^\times v du\\ &\leq C'\|f\|_1 \qquad (21)\\ &\leq C\|f\|_2 \qquad \text{(since the volume is finite.)}\end{aligned}$$

The constants c_N in (20) can be chosen uniformly for φ in a compact set of test functions, hence so can all subsequent constants. ☐ (Lemma 6)

Proof of (i) of Thm 3. We first check that the family of functions $\mathcal{F} := \{R_\varphi f : \|f\|_2 \leq 1\}$ is equicontinuous. For $X \in \mathfrak{g}$,

$$XR_\varphi f = R_{\varphi_X} f$$

where $\varphi_X \in C_c^\infty(G(\mathbb{R}))$ is defined by

$$\varphi_X(g) = \left.\frac{d}{dt}\right|_{t=0}\varphi(\exp(-tX)g).$$

For X in a compact ball B around $0 \in \mathfrak{g}$, the family φ_X lies in a compact set in $C_c^\infty(G(\mathbb{A}))$ (2), hence there is a uniform pointwise boundedness statement as in the lemma for $\{XR_\varphi f : X \in B, \|f\|_2 \leq 1\}$.

[2] The map $\mathfrak{g} \times C_c^\infty(G(\mathbb{A})) \to C_c^\infty(G(\mathbb{A}))$ by $(X,\varphi) \mapsto \varphi_X$ is continuous.

We would now like to invoke the Arzela-Ascoli theorem (an equicontinuous family of functions on a compact Hausdorff space which is bounded in the sup norm is relatively compact), but the space $G(\mathbb{Q})Z(\mathbb{R})^+\backslash G(\mathbb{A})$ is not compact. One way to get around this is to compactify it by adding cusps. Cuspidal functions decrease rapidly at infinity, so on extending by zero to the cusps they remain continuous. The equicontinuity of the family $\mathcal{F}$ continues to hold. By Arzela-Ascoli the family is compact in the sup norm on the compactification, hence in the L^2 norm on $G(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})$. □

Instead of this, there is a nice argument (cf. Lang [7, Thm 6 on p. 232]) that Lemma 6 implies the stronger statement that R_φ is a Hilbert-Schmidt operator on $L_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$. (Hence R_φ is even of trace class on the cuspidal spectrum, something we have not discussed here.) Alternately (cf. Garrett [3, Prop. 6.1]) one can show directly that an equicontinuous family of bounded continuous functions on $G(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})$ has compact closure in L^2. Both these arguments work for any G (whereas compactifying as above is delicate in general). Aside from this point, the proof of Theorem 3 for a general reductive group requires no new idea (i.e. Poisson summation and Fourier analysis on $U(\mathbb{A})$ (treated as $\cong \mathbb{A}^n$) are enough).

Let us see how to deduce Theorem 2 from Theorem 3 using some results of Harish-Chandra. Let H be an irreducible summand of $L_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$. The subspace

$$H_{K_\infty} \subset H$$

of K_∞-finite vectors in H is an admissible $(\mathfrak{g}, K_\infty)$-module. (This is a general theorem of Harish-Chandra, for a proof in the case of $GL(2,\mathbb{R})^+$, where the K_∞^+-subspaces are even one-dimensional, see [1, Thm 2.4.3 and Corollary].) It has a central character, i.e. there is a character $\chi : Z(\mathfrak{g}) \to \mathbb{C}$ such that $zf = \chi(z)f$ for a K_∞-finite vector f. Now the K_∞-finite functions in H satisfy all the conditions defining automorphic forms except possibly moderate growth. This is purely a question on $G(\mathbb{R})$. By the argument after Lemma 3 we have $\alpha \in C_c^\infty(G(\mathbb{R})$ with $\alpha(kgk^{-1}) = \alpha(g)$ and $\alpha * f = f$ (cf. Lemma 13 of [5] or Lemma 2.3.2 of [1]). This can be shown (ibid.) to imply that f has moderate growth. Thus the K_∞-finite vectors in H are automorphic forms. The space of automorphic forms is the algebraic direct sum of the spaces H_{K_∞} as H runs over closed irreducible summands.

Now we make some remarks about the proof of Harish-Chandra's finiteness theorem. Fix a compact open $K \subset G(\mathbb{A}_f)$ and an irreducible representation σ of $K \times K_\infty$. We must show that $\dim \mathscr{A}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)_\chi[\sigma] < \infty$ where $\chi : Z(\mathfrak{g}) \to \mathbb{C}$. The first step, which we will not discuss in detail, is a reduction using a theorem of Langlands to showing that

$$\dim \mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)_\chi[\sigma] < \infty. \tag{22}$$

Consider the irreducible summands $\{H_i\}_i$ of $L_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$. By general representation-theoretic results it is known that for a fixed χ and $\sigma_\infty \in \widehat{K}_\infty$, there are only finitely many H_i which contain the K_∞-type σ_∞ and have infinitesimal character χ. Since $\dim H_i[\sigma_\infty] < \infty$ we get (22). (For another proof see [5].)

These proofs give no information on the exact dimensions (which can be computed for classical holomorphic cusp forms of weight ≥ 2 using a little geometry).

2.5. **Other groups, groups over number fields.** For a reductive algebraic group G over $\mathbb{Q}$ the definition of automorphic form is exactly the same: a function on $G(\mathbb{Q})\backslash G(\mathbb{A})$ satisfying (i)–(vi). The moderate growth condition is imposed with respect to any reasonable norm on $G(\mathbb{R})$ (use a faithful finite-dimensional representation to embed $G(\mathbb{A})$ as a closed subset of $\mathbb{A}^N$, then take the standard norm on $\mathbb{A}^N$). A cuspidal automorphic form is an automorphic form for which the integral (12) vanishes when U is the unipotent radical of any standard rational parabolic subgroup. With these definitions, the results so far are valid as stated. For a reductive group G over a number field F, one works with the restriction of scalars $R_{F/\mathbb{Q}}G$, which is reductive over $\mathbb{Q}$.

3. Automorphic representations

The discussion thus far, though couched in adelic terms, has made use of analysis at the real places. The analysis at the p-adic places is the analysis of Hecke operators.

3.1. An **automorphic representation** is an irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module which is isomorphic to a subquotient of $\mathscr{A}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$. (The admissibility requirement is redundant by Harish-Chandra's finiteness theorem.) Note that an automorphic representation is not a representation of $G(\mathbb{A})$ at all! A **cuspidal automorphic representation** is an irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module which is equivalent to a submodule of $\mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$. Notice that cuspidal automorphic representations are unitary (rather, unitarizable). Theorem 2 implies that

$$\mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega) = \oplus_\pi \mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)[\pi]$$

where π runs over cuspidal automorphic representations and the π-isotypic component $\mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)[\pi]$ is a direct sum of finitely many copies of π.

Just as a continuous idele class character $\chi : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ has a factorization as $\chi = \otimes_v \chi_v$ over all places, an irreducible admissible representation of $G(\mathbb{A})$ has a factorization into representations of the local groups $G(\mathbb{Q}_p)$. Before discussing this factorization we review some local representation theory.

3.2. **Spherical representations of** $GL(2, \mathbb{Q}_p)$**.** Fix a prime p and consider $K_p = GL(2, \mathbb{Z}_p)$ and $G_p := G(\mathbb{Q}_p)$. Recall that for quasi-characters χ_1, χ_2 of $\mathbb{Q}_p^\times$ the (normalized) induced representation $\mathrm{Ind}(\chi_1, \chi_2)$ is the space of locally constant functions $f : G_p \to \mathbb{C}$ satisfying

$$f(bg) = \delta(b)^{1/2}\chi_1(b_1)\chi_2(b_2)f(g) \qquad \text{for } b = \begin{pmatrix} b_1 & * \\ & b_2 \end{pmatrix} \in B_p$$

where $\delta(b) = |b_1/b_2|$ is the modular character. If χ_1 and χ_2 are unitary characters then $\mathrm{Ind}(\chi_1, \chi_2)$ is unitary. Note that the central character is $\omega = \chi_1\chi_2$.

An admissible representation of G_p is called **spherical** (or **unramified**) if it contains a K_p-fixed vector. Consider the induced representation $\mathrm{Ind}(\chi_1, \chi_2)$ where χ_1 and χ_2 are unramified characters. They can be written as:

$$\chi_i(x) = t_i^{\mathrm{ord}_p(x)} \qquad (x \in \mathbb{Q}_p^\times)$$

for $(t_1, t_2) \in (\mathbb{C}^\times)^2$. In this case (χ_1, χ_2) is trivial on $B_p \cap K_p$ and restriction of functions to K_p gives a K_p-isomorphism

$$\mathrm{Ind}(\chi_1, \chi_2) \cong C^\infty(B_p \cap K_p \backslash K_p).$$

It follows that $\mathrm{Ind}(\chi_1, \chi_2)$ is spherical and contains a unique K_p-fixed vector up to scalars. The spherical representations are parametrized by points of the complex torus $\widehat{T} := (\mathbb{C}^\times)^2$ modulo $(t_1, t_2) \mapsto (t_2, t_1)$:

Theorem 4. *(cf.* [1, Thm 4.6.4]*) An irreducible admissible spherical representation of G_p is isomorphic to one of the following:*

(i) *For $(t_1, t_2) \in (\mathbb{C}^\times)^2$ with $t_1 t_2^{-1} \neq p^{\pm 1}$ the representation*

$$\pi(t_1, t_2) := \mathrm{Ind}(\chi_1, \chi_2)$$

where $\chi_i(x) = t_i^{\mathrm{ord}_p(x)}$.

(ii) *For $(t_1, t_2) \in (\mathbb{C}^\times)^2$ with $t_1 t_2^{-1} = p^{\pm 1}$,*

$$\pi(t_1, t_2) = \chi \circ \det$$

(the one-dimensional subquotient of $\mathrm{Ind}(\chi_1, \chi_2)$).

The only isomorphisms between these representations are $\pi(t_1, t_2) \cong \pi(t_2, t_1)$.

The **spherical Hecke algebra** at p is the convolution algebra

$$\mathcal{H}_p := \mathcal{H}(G(\mathbb{Q}_p), K_p)$$

of compactly supported K_p-biinvariant functions on $G(\mathbb{Q}_p)$. There is a unit: if we fix the Haar measure so that K_p has volume one the unit is the characteristic function of K_p. In general $\mathcal{H}_p$ is spanned by the characteristic functions of double cosets $K_p g_p K_p$. The spherical Hecke algebra $\mathcal{H}_p$ is commutative. ([3]) If (π, V) is an irreducible spherical representation of G_p then V^{K_p} is an irreducible representation of $\mathcal{H}_p$. So (π, V) gives a character of $\mathcal{H}_p$, the character $\lambda_\pi : \mathcal{H}_p \to \mathbb{C}$ by which $\mathcal{H}_p$ acts on V^{K_p}. The character λ_π determines π and every character of $\mathcal{H}_p$ is a λ_π. Together with the previous theorem this identifies the characters of $\mathcal{H}_p$ with $\widehat{T}/W$ where $W = \mathbb{Z}/2\mathbb{Z}$ acts by $(t_1, t_2) \mapsto (t_2, t_1)$.

Theorem 5. *(Satake) The spherical Hecke algebra $\mathcal{H}_p$ is identified with the ring*

$$\mathbb{C}[\widehat{G}]^{\widehat{G}} = \mathbb{C}[\widehat{T}]^W$$

of conjugation-invariant polynomial functions on $\widehat{G} = GL(2, \mathbb{C})$. The set of unramified representations is identified with the set $\widehat{T}/W$ of semisimple conjugacy classes in $\widehat{G}$.

---

[3]Here is a quick proof using Gelfand's trick: The Cartan decomposition

$$G_p = K_p \left\{ \begin{pmatrix} p^{n_1} & \\ & p^{n_2} \end{pmatrix} : n_1 \geq n_2 \right\} K_p$$

shows that the characteristic functions of double cosets of the elements $\begin{pmatrix} p^{n_1} & \\ & p^{n_2} \end{pmatrix}$ for $n_1 \geq n_2$ form a basis of $\mathcal{H}_p$. Matrix transposition gives a vector space automorphism $\iota : \mathcal{H}_p \to \mathcal{H}_p$ which can be checked to be an antiinvolution of the algebra, i.e. $\iota(f_1 * f_2) = \iota(f_2) * \iota(f_1)$ for $f_1, f_2 \in \mathcal{H}_p$. But ι fixes the basis elements pointwise, so it is the identity. This shows that $\mathcal{H}_p$ is isomorphic to its opposite, hence commutative. The same proof works for $GL(n, \mathbb{Q}_p)$.

The conjugacy class $\left\{\begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix}\right\}$ in $GL(2,\mathbb{C})$ associated to $\pi(t_1,t_2)$ is called the **Satake parameter** of $\pi(t_1,t_2) \cong \pi(t_2,t_1)$. The appearance of $GL(2,\mathbb{C})$ here is no accident; it is the Langlands dual group of $GL(2,\mathbb{Q}_p)$. (In general, the Satake parameter of a spherical representation of a split reductive p-adic group $G(\mathbb{Q}_p)$ is a semisimple conjugacy class in the complex dual group $\widehat{G}$.) Note that

$$\mathbb{C}[\widehat{T}]^W = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]^W = \mathbb{C}[t_1 + t_2, t_1 t_2, t_1^{-1} t_2^{-1}].$$

Let

$$\mathbb{T}_p = \text{characteristic function of } K_p \begin{pmatrix} p & \\ & 1 \end{pmatrix} K_p = K_p \begin{pmatrix} 1 & \\ & p \end{pmatrix} K_p$$

$$\mathbb{R}_p = \text{characteristic function of } K_p \begin{pmatrix} p & \\ & p \end{pmatrix} K_p.$$

Lemma 7. *$\mathbb{T}_p$ and $\mathbb{R}_p$ act on the spherical vector in $\pi(t_1,t_2)$ by $p^{1/2}(t_1+t_2)$ and $p(t_1t_2)$ respectively. It follows that $\mathcal{H}_p$ is generated by $\mathbb{T}_p$ and $\mathbb{R}_p^{\pm 1}$.*

Proof. It is enough to calculate the action of $\mathbb{T}_p$ on the spherical vector in $\mathrm{Ind}(\chi_1,\chi_2)$. Let ϕ_0 be the spherical vector, normalized so $\phi_0(e) = 1$. The action of $\mathbb{T}_p$ is given by:

$$(\mathbb{T}_p\phi)(g) = \int_{K_p \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) K_p} \phi(gh)dh.$$

Since we know that ϕ_0 is an eigenfunction of $\mathbb{T}_p$ it is enough to compute $(\mathbb{T}_p\phi_0)(e)$. The double coset $K_p \begin{pmatrix} 1 & \\ & p \end{pmatrix} K_p$ decomposes as:

$$K_p \begin{pmatrix} p & \\ & 1 \end{pmatrix} K_p = \coprod_{0\le b\le p-1} \begin{pmatrix} p & -b \\ & 1 \end{pmatrix} K_p \coprod \begin{pmatrix} 1 & \\ & p \end{pmatrix} K_p. \tag{23}$$

Using the right K_p-invariance of ϕ_0 we have:

$$\begin{aligned}(\mathbb{T}_p\phi)(e) &= \int_{K_p \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) K_p} \phi_0(h)dh \\ &= \sum_{0\le b\le p-1} \phi_0\left(\begin{pmatrix} p & -b \\ & 1 \end{pmatrix}\right) + \phi_0\left(\begin{pmatrix} 1 & \\ & p \end{pmatrix}\right). \\ &= p|p|^{1/2} t_{1,p} + |p^{-1}|^{1/2} t_{2,p} = p^{1/2}(t_{1,p} + t_{2,p}).\end{aligned}$$

The calculation for $\mathbb{R}_p$ is easier. □

When are the representations $\pi(t_1,t_2)$ unitary (rather, preunitary)? The induction we have used preserves unitarity, so this is certainly true if $|t_1| = |t_2| = 1$. However, there are other unitary representations. A necessary condition for unitarity is that

$$p^{-1/2} \le |t_i| \le p^{1/2} \qquad (i = 1,2). \tag{24}$$

(If the central character t_1t_2 is trivial then this is equivalent to $|t_i| \le p^{1/2}(i = 1,2)$.) The representation is **tempered** if $|t_1| = |t_2| = 1$.

3.3. **Discrete series for $G(\mathbb{R})$.** For each integer $k \geq 2$ the discrete series representation π_k is a certain infinite-dimensional unitary representation of $G(\mathbb{R})$ on a Hilbert space. We will define the associated $(\mathfrak{g}, K_\infty)$-module DS_k. This is spanned as a $\mathbb{C}$-vector space by vectors

$$\dots, v_{-k-4}, v_{-k-2}, v_{-k}, v_k, v_{k+2}, v_{k+4}, \dots$$

K_∞ acts by:

$$k_\theta \cdot v_m = (e^{i\theta})^m v_m \qquad \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \cdot v_m = v_{-m}.$$

$\mathfrak{g}$ acts by:

$$H \cdot v_m = v_m, \quad M_+ \cdot v_m = \frac{k+m}{2} v_{m+2}, \quad M_- \cdot v_m = \frac{k-m}{2} v_{m-2}.$$

These formulae define an irreducible $(\mathfrak{g}, K_\infty)$-module. (As a $(\mathfrak{g}, K_\infty^+)$-module it splits into two non-isomorphic $(\mathfrak{g}, K_\infty^+)$-modules.) For a unitary representation of $G(\mathbb{R})$ on a Hilbert space of functions which gives rise to DS_k see [7, Chp. IX] or [1, 2.5]. (Again, this representation breaks up as a sum of two non-isomorphic representations of $GL(2,\mathbb{R})^+$.) For a list of the other irreducible $(\mathfrak{g}, K_\infty)$-modules for $GL(2,\mathbb{R})$ see [7, Chp. VI] or [1, Thms 2.5.4,2.5.5]. Notice that $M_- v_k = 0 = M_+ v_{-k}$ and that Ω acts by the scalar $k(k-2)$. In fact, these properties characterize the $(\mathfrak{g}, K_\infty)$-module DS_k uniquely.

3.4. **Factorization.** Suppose that we are given, for each prime p, an irreducible admissible representation (π_p, V_p) of $G(\mathbb{Q}_p)$. Suppose that for almost all p this representation is spherical and suppose we are given a K_p-fixed vector $v_p^0 \in V_p^{K_p}$. Then we can form the restricted tensor product

$$V_f := \otimes_p' V_p := \varinjlim_S \otimes_{p \in S} V_p$$

where, for $S \subset T$, the inclusion $\otimes_{p \in S} V_p \hookrightarrow \otimes_{p \in T} V_p$ is given by $v \mapsto x \otimes (\otimes_{p \in T-S} v_p^0)$. This is an admissible representation of $G(\mathbb{A}_f)$. A different set of choices $\{w_p^0 \in V_p^{K_p}\}$ which differs from $\{v_p^0\}$ in only finitely many places gives an naturally isomorphic representation. If V_∞ is an admissible $(\mathfrak{g}, K_\infty)$-module then $V := V_\infty \otimes V_f$ is an admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module.

The following theorem was proved in [6] for $GL(2)$ and by Flath in general. A complete proof for $GL(2)$ can be found in [1, §3.3].

Theorem 6. *(i) If $\{(\pi_p, V_p)\}_p$ and (π_∞, V_∞) are (unitary) irreducible admissible as above then the representation $\pi = \pi_f \otimes \pi_\infty$ on $V = V_f \otimes V_\infty$ is an irreducible admissible (unitary) $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module.*

(ii) If (π, V) is an irreducible admissible (unitary) $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module then there exist irreducible admissible (unitary) representations (π_p, V_p), almost all of them unramified, and an irreducible (unitary) $(\mathfrak{g}, K_\infty)$-module (π_∞, V_∞) such that (π, V) is isomorphic to the restricted tensor product $\pi = \otimes_p \pi_p \otimes \pi_\infty$ on $V = \otimes_p' V_p \otimes V_\infty$.

3.5. **Multiplicity one.** Recall the classical result that a Hecke eigenform is determined by its eigenvalues with respect to the Hecke operators $T(p)$ $(p \nmid N)$. The representation-theoretic version of this is due to Jacquet-Langlands [6] and Casselman and Miyake:

Theorem 7. *(Strong multiplicity one) Suppose that $\pi_1 = \otimes_v \pi_{1,v}$ and $\pi_2 = \otimes_v \pi_{2,v}$ are cuspidal automorphic representations of $GL(2)$ such that $\pi_{1,v} \cong \pi_{2,v}$ for almost all v. Then $\pi_1 \cong \pi_2$ and $\mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)[\pi_1] = \mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)[\pi_2]$.*

The proof uses the theory of Whittaker models, which is the representation-theoretic translation of the use of Fourier expansions (cf. [1, §3.5]). Strong multiplicity one holds for $GL(n)$ but is false in general.

3.6. **Modular forms and automorphic representations.** Let us revisit the classical theory of cusp forms using representation theory. Consider the map

$$S_k(\Gamma_0(N),\chi) \hookrightarrow \mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$$

defined in Lemma 2. Functions in the image are invariant under

$$K^N := \prod_{p \nmid N} K_p,$$

i.e. they are contained in $\mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)^{K^N}$. This space has an action of the (commutative) spherical Hecke algebra away from N: ([4])

$$\mathcal{H}^N := \otimes'_{p\nmid N} \mathcal{H}_p$$

acts by convolution of functions on G_p. Thus for each $p \nmid N$ the elements of the Hecke algebra act. If

$$S_k(\Gamma_0(N),\chi) \ni f \longleftrightarrow \phi \in \mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)^{K^N}$$

then the classical and adelic actions are related by:

$$\begin{aligned} p^{-1/2}\mathbb{T}_p\phi &\longleftrightarrow p^{-\frac{k-1}{2}}T(p)f \\ p^{-1}\mathbb{R}_p\phi &\longleftrightarrow R(p)f. \end{aligned} \tag{25}$$

(This requires translating between adelic double cosets and classical double cosets and is left to the reader.)

Fix $f \in S_k(\Gamma_0(N),\chi)$ and consider the $(\mathfrak{g},K_\infty)\times G(\mathbb{A}_f)$-submodule generated by ϕ. By the semisimplicity theorem and multiplicity one it is a direct sum of inequivalent representations. Now $V(\phi)$ is irreducible precisely when f is a eigenform for all $T(p), p \nmid N$. (However, f is not unique giving rise to $V(\phi)$.) The function ϕ has $M_-\phi = 0, K_\infty$-weight k and $\Omega\phi = k(k-2)\phi$. By the properties characterizing DS_k we must have $\pi_\infty = \mathrm{DS}_k$.

To go in the other direction, i.e. from a cuspidal automorphic representation π to a cusp form f which generates it, we need a result of local representation theory. For $r > 0$ let $K_{p,0}(r) = \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in K_p : \mathrm{ord}_p(c) \geq r\right\}$. A character ω_p of $\mathbb{Q}_p^\times$ which is trivial on

[4]This is a restricted tensor product: linear combinations of elements $\otimes_{p\nmid N} f_p$ where f_p is the unit of $\mathcal{H}_p$ at almost all places.

$1 + p^r\mathbb{Z}_p$ (i.e. a character of conductor r) defines a character of $K_{p,0}(r)$ by $\omega_p(k) = \omega_p(a)$. The following theorem from [2] shows that there is a good notion of conductor $c(\pi_p)$:

Theorem 8. *For an irreducible admissible representation (π_p, V_p) of G_p with central character ω_p, there is a minimal $c(\pi_p) \geq c(\omega_p)$ such that $V_p^{K_{p,0}(r),\omega_p} := \{v \in V : \pi_p(k)v = \omega_p(k)v \; \forall k \in K_{p,0}(r)\}$ is nonzero. The dimension of $V_p^{K_{p,0}(c(\pi)),\omega_p}$ is one.*

The conductor is zero for spherical representations, one for special representations, and ≥ 2 for supercuspidals. Using this we can find a cusp form in a cuspidal automorphic representation which generates it. Let $\pi = \otimes_p \pi_p \otimes \pi_\infty$ be a cuspidal automorphic representation. Let

$$N = N(\pi) := \prod_p p^{c(\pi_p)}.$$

Let ω be the central character of π and χ the associated character of $K_0(N)$. By multiplicity one, Theorem 8 and Lemma 2, the space of functions ϕ satisfying

(i) $\phi \in \mathscr{A}_0(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)[\pi]$
(i) $M_-\phi = 0$
(ii) $\phi(gk) = \chi(k)\phi(g)$ for $k \in K_0(N)$

has dimension one and lies in the image of $S_k(\Gamma_0(N), \chi)$. Let f be the normalized form ($a_1 = 1$) in $S_k(\Gamma_0(N), \chi)$ spanning this space. (An $f \in S_k(\Gamma_0(N), \chi)$ that arises from cuspidal π in this way, or any multiple of it, is called a (normalized) **newform**.) If π_p is the (spherical) local component of π for $p \nmid N$ and $\begin{pmatrix} t_{1,p} & \\ & t_{2,p} \end{pmatrix}$ is its Satake parameter, the classical Hecke operators are given by:

$$\begin{aligned} T(p)f &= a_p f = p^{\frac{k-1}{2}}(t_{1,p} + t_{2,p})f \\ R(p)f &= \chi(p)f = t_{1,p}t_{2,p}f. \end{aligned}$$

Since $t_{1,p}t_{2,p} = \omega_\pi(p)$, we have the following explicit relation between Satake parameters and classical Hecke eigenvalues a_p

$$\begin{aligned} 1 - a_p p^{-\frac{k-1}{2}} p^{-s} + \chi(p)p^{-2s} &= (1 - t_{1,p}p^{-s})(1 - t_{2,p}p^{-s}) \\ &= \det\left(Id - \begin{pmatrix} t_{1,p} & \\ & t_{2,p} \end{pmatrix} p^{-s}\right). \end{aligned} \tag{26}$$

The Ramanujan-Petersson conjecture (proved by Deligne for holomorphic cusp forms ca. 1972) states that $|a_p| \leq 2p^{\frac{k-1}{2}}$. By (26) this is equivalent to $|t_{1,p}| = |t_{2,p}| = 1$, i.e. the local component π_p is tempered. This statement is expected to hold for all automorphic representations on $GL(2)$ and even on $GL(n)$ (but does not hold for other groups).

3.7. **Euler products and automorphic L-functions.** Let S be a finite set of primes containing ∞. Suppose that we are given a family of Satake parameters for $p \notin S$, i.e semisimple conjugacy classes

$$\Phi_p = \left\{ g \begin{pmatrix} t_{1,p} & \\ & t_{2,p} \end{pmatrix} g^{-1} \right\} \subset GL(2, \mathbb{C}).$$

Define a function of $s \in \mathbb{C}$ by the **Euler product** over places not in S:

$$\prod_{p \notin S} \frac{1}{\det(1 - p^{-s}\Phi_p)}. \tag{27}$$

It evidently depends only on the conjugacy class Φ_p. The same definition works if $\Phi_p \subset GL(n, \mathbb{C})$ are given. Of course, this product need not converge anywhere as the Φ_p are arbitrary. If we are given a homomorphism $\rho : GL(2, \mathbb{C}) \to GL(N, \mathbb{C})$ for some N we can form the Euler product as in (27) using the semisimple conjugacy classes $\{\rho(\Phi_p)\}_p$ in $GL(N, \mathbb{C})$.

Now let π be an irreducible admissible unitary representation of $G(\mathbb{A})$ (e.g. π cuspidal automorphic). There is a factorization $\pi = \otimes_v \pi_v$. Let S be the set of places at which π_p is not spherical. Langlands [8, 9] associated to π the Euler product (27)

$$L^S(\pi, s) = \prod_{p \notin S} \frac{1}{\det(1 - p^{-s}\Phi_p)}. \tag{28}$$

where Φ_p is the Satake parameter of π_p (cf. [8, 9]). Since π_p is unitary for all p we have the bound (24) on the absolute values of eigenvalues of Φ_p and hence the Euler product converges in a right half-plane. ([5]) Given a representation $\rho : GL(2, \mathbb{C}) \to GL(N, \mathbb{C})$ one has the Euler product:

$$L^S(\pi, \rho, s) = \prod_{p \notin S} \frac{1}{\det(1 - p^{-s}\rho(\Phi_p))},$$

also convergent in a right half-plane.

When π is an automorphic unitary representation (e.g. π cuspidal) these **automorphic L-functions** (are expected to) have many good properties. The basic problems about these are: meromorphic continuation of $L^S(\pi, \rho, s)$, completion of the L-function by adding appropriate factors for $p \in S$ (including gamma factors at ∞) and establishing functional equations. (For standard L-functions for $GL(2)$ these were solved in [6] using the Whittaker model; much is now known in general.) A deeper problem (an instance of Langlands' functoriality principle [9]) is to show that for suitable ρ, the L-function $L(\pi, \rho, s)$ is itself $L(\Pi, s)$ for some automorphic representation Π of $GL(N, \mathbb{C})$. Another set of deep problems is whether the L-functions coming from algebraic geometry are (products of) automorphic L-functions.

REFERENCES

[1] D. Bump, *Automorphic Forms and Representations*, Cambridge Univ. Press, 1997.
[2] W. Casselman, On some results of Atkin and Lehner, *Math. Annalen* 201 (1973), 301–314.

[5]For f for $f \in S_k(\Gamma_0(N), \chi)$ the L-function agrees with the (shifted, partial) classical L-function $L^N(f, s) = \sum_{(n,N)=1} a_n n^{-s}$. Indeed, if $f \in S_k(\Gamma_0(N), \chi)$ is the normalized Hecke eigenform associated to π, then (26) implies

$$L^S(s, \pi) = \prod_{p \nmid N} (1 - a_p p^{-\frac{k-1}{2}} p^{-s} + p^{-2s})^{-1} = \sum_{(n,N)=1} a_n n^{-s-\frac{k-1}{2}} = L^N\left(f, s + \frac{k-1}{2}\right).$$

[3] P. Garrett, Decomposition of cusp forms, notes (April 2007), available at `http://www.math.umn.edu/~garrett/`
[4] S. Gelbart, *Automorphic forms on adele groups*, Princeton Univ. Press, 1975.
[5] Harish-Chandra, *Automorphic Forms on Semisimple Lie Groups*, LNM 62, Springer, NY 1968.
[6] H. Jacquet and R. P. Langlands, *Automorphic forms on GL(2)*, LNM 114, Springer, NY 1970, available at `http://sunsite.ubc.ca/DigitalMathArchive/Langlands/`
[7] S. Lang, $SL_2(\mathbb{R})$, Springer, 1985.
[8] R. P. Langlands, *Euler products*, Yale University, 1967, available at `http://sunsite.ubc.ca/DigitalMathArchive/Langlands/`
[9] R. P. Langlands, Problems in the theory of automorphic forms, in *Lectures in Modern Analysis and Applications*, LNM 170, 18–86. available at `http://sunsite.ubc.ca/DigitalMathArchive/Langlands/`
[10] J.-P. Serre, *A Course in Arithmetic*, Springer, NY, 1973.
[11] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton Univ. Press., Princeton, NJ, 1971.