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A short introduction to the trace formula

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1 Introduction

The aim of this note is to give a quick introduction to the Arthur-Selberg trace formula through the case of $GL(2)$. First we recall the "cocompact case" which is easy and serves as a model. Then we introduce Arthur's truncation operator (section 4). The "truncated" regular representation of the group of adelic points $GL(2,\mathbb{A})^1$ on the space $L^2(GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})^1)$ is of trace class. Its trace has two expressions : the first one of spectral nature is given by Langlands' spectral decomposition of the space $L^2(GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})^1)$ (cf. section 5) and the second one of geometric nature is related to conjugacy classes in $GL(2,\mathbb{Q})$ (sections 6, 7 and 8) : this is roughly the trace formula. In section 11, we sketch an application of the trace formula to the Jacquet-Langlands correspondence. In the final section, we give some bibliographical references and give the state of the art of the stable trace formula.

During the text, we often refer the reader to the introductions [21] by Knapp and [19] by Gelbart to Arthur's work. We recommend the nice reports [26] and [27] of Labesse and the synthesis [13] that Arthur made himself of his work.

2 The cocompact case

Let G be a Lie group and Γ a discrete subgroup such that the quotient $\Gamma\backslash G$ is compact. Let us consider the space

$$L^2(\Gamma\backslash G) = \{\phi : \Gamma\backslash G \to \mathbb{C} \;\mid\; \int_{\Gamma\backslash G} |\phi(g)|^2 \, d\dot{g} < \infty\},$$

where dg is a Haar measure on G and $d\dot{g}$ is the quotient measure on $\Gamma\backslash G$. The group G acts on $L^2(\Gamma\backslash G)$ via the right regular representation R and the algebra $C_c^\infty(G)$ of smooth and compactly supported functions on G acts on $L^2(\Gamma\backslash G)$ by right convolution : for $\phi \in L^2(\Gamma\backslash G)$ and $f \in C_c^\infty(G)$, the function $R(f)\phi \in L^2(\Gamma\backslash G)$ is defined by

$$R(f)\phi(x) = \int_G \phi(xy) f(y) \, dy,$$

for $x \in \Gamma\backslash G$.

One can see that $R(f)$ is an integral operator i.e. one can write

$$R(f)\phi(x) = \int_{\Gamma\backslash G} K(x,y)\phi(y)\,d\dot{y}$$

with the kernel

$$K(x,y) = K_f(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y).$$

Since f is compactly supported, it is easy to see that for each (x, y) the sum over γ is of finite support and that the kernel K is a continuous function on $\Gamma\backslash G \times \Gamma\backslash G$. Thus K is square-integrable and $R(f)$ is a Hilbert-Schmidt operator. Moreover, the Hilbert-Schmidt norm of $R(f)$ is nothing else than the L^2-norm of K. Thanks to a theorem of Dixmier-Malliavin [18], we can write f as a finite sum of convolution products of functions f_i and h_i in $C_c^\infty(G)$:

$$f = \sum_i f_i * h_i$$

As a consequence, $R(f)$ is a finite sum of products of Hilbert-Schmidt operators and thus it is a trace class operator. The trace of $R(f)$ is the finite sum of the Hilbert-Schmidt scalar products of $R(h_i)$ and $R(f_i)^*$. We denote by a $*$ the adjoint operator. It is easy to see that $R(f_i)^* = R(f_i^\vee)$ where $f_i^\vee$ is defined by $f_i^\vee(x) = \overline{f_i(x^{-1})}$ for any $x \in G$. One can show that it is also the integral of the kernel K over the diagonal :

$$\mathrm{trace}(R(f)) = \int_{\Gamma\backslash G} K(x,x)\,d\dot{x}.$$

Then one can expand the kernel K according to the set $\mathcal{O}$ of conjugacy classes in Γ. For $\gamma \in \Gamma$, let G_γ and Γ_γ be the centralizers of γ in G and Γ respectively. We get

$$\mathrm{trace}(R(f)) = \sum_{\gamma\in\mathcal{O}} \int_{\Gamma_\gamma\backslash G} f(g^{-1}\gamma g)\,d\dot{g}.$$

Let $\gamma \in \Gamma$. Let us choose a Haar measure on G_γ. We denote $d\bar{g}$ the quotient measure on $G_\gamma\backslash G$. Let us introduce the coefficient

$$a_\Gamma(\gamma) = \mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)$$

and the orbital integral

$$J_\gamma(f) = \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)\,d\bar{g}.$$

The *geometric expansion* of the trace formula is given (in the cocompact case) by

$$\mathrm{trace}(R(f)) = \sum_{\gamma\in\mathcal{O}} a_\Gamma(\gamma)\,J_\gamma(f). \tag{2.1}$$

For the spectral expansion, we need the following proposition.

Proposition 2.1. — *The regular representation R is isomorphic to the completion of a discrete sum of unitary irreducible representations of G*

$$L^2(\Gamma\backslash G) = \widehat{\bigoplus_{\pi\in\Pi(G)}} V_\pi^{m_\Gamma(\pi)} \tag{2.2}$$

where $\Pi(G)$ is the set of equivalence classes of unitary irreducible representations (π, V_π) of G. Moreover, the multiplicity $m_\Gamma(\pi)$ of π in $L^2(\Gamma\backslash G)$ is finite.

Proof. — (Sketch) It relies on some elementary facts about the spectral theory of compact adjoint operators in Hilbert spaces : a non-zero compact adjoint operator in a Hilbert space has always a non-zero eigenvalue. Moreover, the associated eigenvector space is finite dimensional. To prove the existence of the discrete decomposition, it suffices to show that any closed and G-invariant subspace V admits an irreducible closed and G-invariant subspace (the irreducibility is taken in the topological sense). One can approximate the Dirac measure supported in $\{1\}$ by a suitable $f \in C_c^\infty(G)$ such that $f^\vee = f$. In particular, $R(f)$ induces on V a compact (since Hilbert-Schmidt) self-adjoint (since $f^\vee = f$) and non-zero (provided that f is enough close to the Dirac measure). So we have an eigenvector, say $\phi \in V$, associated to a non-zero eigenvalue of $R(f)$. Then we can replace V by V' the closure of the space generated (under G) by ϕ. Now, by using orthogonal projectors, one can see that any orthogonal decomposition of V' has at most d non-zero summands where d is the (finite) dimension of the eigenvector space which contains ϕ. Thus, any component of a maximal orthogonal decomposition of V' is irreducible. In the same manner, one proves that the multiplicities are finite (for more details, see e.g. the proof of theorem 1.5 of [21]). □

From the proposition, we get the *spectral expansion* of the trace formula

$$\operatorname{trace}(R(f)) = \sum_{\pi\in\Pi(G)} m_\Gamma(\pi)\operatorname{trace}(\pi(f)).$$

In conclusion, the trace formula in the cocompact case is the identity for all $f \in C_c^\infty(G)$

$$\sum_{\gamma\in\mathcal{O}} a_\Gamma(\gamma)\, J_\gamma(f) = \sum_{\pi\in\Pi(G)} m_\Gamma(\pi)\operatorname{trace}(\pi(f)).$$

It is a generalization for non-abelian groups of the Poisson summation formula (take $G = \mathbb{R}$ and $\Gamma = \mathbb{Z}$). A fundamental problem is to obtain some informations about the multiplicities $m_\Gamma(\pi)$. The trace formula is one of the most powerful tool to study them : the trace formula “converts” spectral data into (more concrete) geometric data.

3 The $GL(2)$ case

In the previous section, we studied the cocompact case but this setting is too restrictive : for example, if $G = GL(2,\mathbb{R})$ and Γ is $GL(2,\mathbb{Z})$ or more generally a congruence subgroup the quotient $\Gamma\backslash G$ is not compact. From now on, we will work in an adélic setting : on the one hand, the literature on the Arthur-Selberg trace formula is written for adélic groups and on the other hand it is more convenient to consider $GL(2,\mathbb{Q})$-conjugacy classes rather than $GL(2,\mathbb{Z})$-conjugacy classes.

Let $\mathbb{A}$ be the ring of adèles of $\mathbb{Q}$ (more generally of a number field), $\mathbb{A}^\times$ the multiplicative group of idèles and $\mathbb{A}^1$ the subgroup of idèles of adélic norm 1

$$\mathbb{A}^1 = \{x \in \mathbb{A} \mid |x|_{\mathbb{A}} = 1\}.$$

These groups carry a natural topology for which they are locally compact. Moreover, $\mathbb{Q}$, resp. $\mathbb{Q}^\times$, is a discrete subgroup of $\mathbb{A}$, resp. $\mathbb{A}^1$, and the quotient $\mathbb{Q}\backslash\mathbb{A}$, resp. $\mathbb{Q}^\times\backslash\mathbb{A}^1$, is compact.

Let us denote $G = GL(2,\mathbb{A})$ and $\Gamma = GL(2,\mathbb{Q})$. The topology on $\mathbb{A}$ induces a topology on G for which G is locally compact. Thus G carry a Haar measure dg. Moreover Γ is a discrete subgroup of G but the quotient $\Gamma\backslash G$ is not compact : it is not even of finite volume (for the quotient measure). One can improve this by considering the subgroup G^1 defined by

$$G^1 = \{g \in G \mid |\det(g)|_{\mathbb{A}} = 1\}.$$

It is well-known that $\Gamma\backslash G^1$ is of finite volume (cf. [16]) : nevertheless it is still not compact.

Let K_0 be an open compact subgroup of the group $GL(2,\mathbb{A}_f)$ of finite adèles. The quotient $K_0\backslash G^1/K_0$ is a discrete union of countably many copies of $GL(2,\mathbb{R})^1 = GL(2,\mathbb{R}) \cap G^1$ which is a Lie group. We can therefore define the space $C^\infty(K_0\backslash G^1/K_0)$ of smooth functions on $K_0\backslash G^1/K_0$. Let $C^\infty(G^1)$ be the direct limit

$$C^\infty(G^1) = \varinjlim C^\infty(K_0\backslash G^1/K_0)$$

where the limit is taken over all open compact subgroups K_0 of $GL(2,\mathbb{A}_f)$. Let $C_c^\infty(G^1)$ be the subspace of smooth and compactly supported functions on G^1. As in the previous section, the group G^1 acts by the right regular representation R on the space $L^2(\Gamma\backslash G^1)$ and for $f \in C_c^\infty(G^1)$, the operator $R(f)$ is an integral operator with kernel

$$K(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)$$

with $x, y \in \Gamma\backslash G^1$. Unlike the cocompact case, the kernel K is not square-integrable on $\Gamma\backslash G^1 \times \Gamma\backslash G^1$. It is not even integrable on the diagonal. Let $\gamma \in \Gamma$. Let G^1_γ be the centralizer of γ in G^1. It is easy to see that the coefficient

$$a_\Gamma(\gamma) = \mathrm{vol}(\Gamma_\gamma\backslash G^1_\gamma)$$

is infinite for γ diagonal and non central and that the orbital integral

$$J_\gamma(f) = \int_{G^1_\gamma\backslash G^1} f(g^{-1}\gamma g)\, d\bar{g}$$

is generally infinite for γ a non-trivial unipotent element. So the geometric expansion (2.1) does not have any sense. What is worse is that the discrete spectral decomposition (2.2) is no longer true and $R(f)$ is not a trace class operator. At first glance, there seems to be no hope to have a trace. But, as we shall see, what can be computed is the trace of a truncated avatar of $R(f)$ which still has a rich spectral decomposition.

4 Arthur's truncation operator

4.1. Constant term. — Let B be the subgroup of upper triangular matrices in G. Let N be the subgroup of unipotent matrices in B. The quotient $N \cap \Gamma \backslash N$ is isomorphic to $\mathbb{Q}\backslash\mathbb{A}$ and therefore is compact.

For any $\phi \in L^2(\Gamma\backslash G^1)$. By Fubini's theorem, the function

$$n \in N \cap \Gamma\backslash N \mapsto \phi(nx)$$

belongs to $L^2(N \cap \Gamma\backslash N)$ for almost all $x \in G^1$ and thus also to $L^1(N \cap \Gamma\backslash N)$.

The constant term of ϕ ("along" B) is the function ϕ_B defined for almost all $g \in G^1$ by

$$\phi_B(x) = \int_{N\cap\Gamma\backslash N} \phi(nx)\, dn.$$

We shall say that ϕ is *cuspidal* if $\phi_B = 0$ (almost everywhere). Let

$$L^2_{\text{cusp}}(\Gamma\backslash G^1) = \{\phi \in L^2(\Gamma\backslash G^1) \mid \phi_B = 0 \text{ (a.e.)}\}$$

be the closed subspace of $L^2(\Gamma\backslash G^1)$ generated by cuspidal functions. Note that this subspace is stable by $R(f)$.

4.2. The H-function. — Let T be the subgroup of diagonal matrices in B. Let

$$\Omega = O_2(\mathbb{R}) \prod_{p \text{ prime}} GL(2, \mathbb{Z}_p).$$

This is a maximal compact subgroup. Then we have the Iwasawa decomposition

$$G = BK = NT\Omega.$$

We define a function H from T to $\mathbb{R}$ by

$$H(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}) = \frac{1}{2}\log(|\frac{a}{b}|_{\mathbb{A}}), \tag{4.1}$$

for all $a, b \in \mathbb{A}^\times$. For any $g \in G$, the Iwasawa decomposition enables us to write $g = ntk$ with $n \in N$, $t \in T$ and $k \in \Omega$. We extend the function H to all G by the formula

$$H(g) = H(t). \tag{4.2}$$

We thus obtain a function on G which is invariant on the left by $N(T\cap\Gamma)$ and on the right by Ω.

Let

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

be (a representative) of the non-trivial element of the Weyl group of the diagonal torus. We have the following "positivity lemma".

Lemma 4.1. — *For any $g \in G$, the inequality*

$$-H(g) - H(wg) \geqslant 0$$

is true.

Proof. — It is a simple (local) computation which is left to the reader. (cf. e.g. the computation of the function H in lemma 6.6 of [21]). □

4.3. Reduction theory. — For any $X \in \mathbb{R}$, let τ_X be the characteristic function of the set

$$\{g \in G \mid H(g) > X\}. \tag{4.3}$$

Let T_∞ be the group of diagonal matrices (a, b) with $a, b \in \mathbb{R}_+^\times$. Let T^1 be the subgroup of T of diagonal matrices (a, b) with a and b in $\mathbb{A}^1$. In particular, $T = T^1 T_\infty$.

Let $\omega \subset NT^1$ be a compact subset. Let $X_1 \in \mathbb{R}$. We define the Siegel set

$$\mathcal{S} = \mathcal{S}(X_1, \omega) = \{x = ptk \in G \mid p \in \omega,\ t \in T_\infty,\ k \in \Omega,\ , \tau_{X_1}(x) = 1\}.$$

The relevance of Siegel sets comes in part from the following lemma. (cf. e.g. proposition 7.10 of [21])

Lemma 4.2. — *If $\mathcal{S}$ is sufficiently large then $G = \Gamma\mathcal{S}$.*

Let ω and $X_1 < 0$ be such that the conclusion of the previous lemma is true. Let $X \in \mathbb{R}$ and

$$\mathcal{S}(X, X_1, \omega) = \{x \in \mathcal{S}(X_1, \omega) \mid \tau_X(x) = 0\}.$$

It is clear that $\mathcal{S}(X, X_1, \omega) \cap G^1$ is compact. Let $F^G(., X)$ be the characteristic function of the projection of $\mathcal{S}(X, X_1, \omega)$ on $\Gamma\backslash G$.

Lemma 4.3. — *For any $g \in G$, the sum*

$$\sum_{\delta \in B\cap\Gamma\backslash\Gamma} \tau_X(\delta g)$$

is finite. If $X > |X_1|$ then for all $g \in G$ we have

$$F^G(g, X) + \sum_{\delta \in B\cap\Gamma\backslash\Gamma} \tau_X(\delta g) = 1.$$

Proof. — We will not prove the first assertion (however cf. e.g. lemma 7.7 in [21]).

The Bruhat decomposition gives the disjoint union

$$\Gamma = (B \cap \Gamma) \cup (B \cap \Gamma) w (B \cap \Gamma).$$

Thus the union of $\{1\}$ and $w(N \cap \Gamma)$ is a set of representatives of the quotient $B \cap \Gamma \backslash \Gamma$. Since the second assertion only depends on the class Γg, we have to consider only two cases : either $\tau_X(g) = 1$ or $\tau_X(\delta g) = 0$ for all $\delta \in \Gamma$. In the first case, by the positivity lemma 4.1, we have for any $n \in N$

$$H(wng) \leqslant -H(ng) = -H(g) \leqslant -X \leqslant X.$$

Thus $\tau_X(\delta g) = 0$ if $\delta \notin B \cap \Gamma$. Moreover it is clear that $F^G(g, X) = 0$.

In the second case, by lemma 4.2, there exists $\delta \in \Gamma$ such that $\delta g \in \mathcal{S}$. Since $\tau_X(\delta g) = 0$ we have $F^G(g, X) = 1$.

□

4.4. Arthur's truncation operator — It is denoted by Λ^X and it depends on a real parameter X. It is defined for any $\phi \in L^2(\Gamma \backslash G^1)$ or $\phi \in C^\infty(\Gamma \backslash G^1)$ by the formula

$$\Lambda^X \phi(x) = \phi(x) - \sum_{\delta \in B \cap \Gamma \backslash \Gamma} \tau_X(\delta x) \phi_B(\delta x)$$

for $x \in G^1$.

Remark. — Thanks to lemma 4.3, we see that the sum over δ can be taken over a finite set and that the constant function 1 satisfies $\Lambda^X(1) = F^G(., X)$ if X is sufficiently large.

Here are the main properties of the truncation operator. (for some details see e.g. [21] §7, for more complete proofs see [2] and perhaps also [32]).

Proposition 4.4. — *For any $\phi \in L^2(\Gamma \backslash G^1)$, we have $\Lambda^X \phi \in L^2(\Gamma \backslash G^1)$. Moreover, the following assertions are true :*

1. *$\Lambda^X \phi = \phi$ for all $\phi \in L^2_{\text{cusp}}(\Gamma \backslash G^1)$;*
2. *$\Lambda^X = \Lambda^X \circ \Lambda^X$;*
3. *Λ^X transforms smooth functions of "uniform moderate growth" into "rapidly decreasing" ones ;*
4. *$\langle \Lambda^X \phi, \psi \rangle = \langle \phi, \Lambda^X \psi \rangle$ for ϕ and $\psi \in C_c^\infty(G^1)$;*
5. *Λ^X extends to a bounded operator on $L^2(\Gamma \backslash G^1)$ which a orthogonal projector.*

Let us denote $\Lambda^X K$, resp. $K \Lambda^X$, the action of Λ^X on the first variable of the kernel K, resp. the second variable. It is easy to check that the operator $\Lambda^X \circ R(f)$ acting on $L^2(\Gamma \backslash G^1)$ is integrable with kernel $\Lambda^X K$. But thanks to the property 3 of the proposition above, one can show that the kernel $\Lambda^X K$ is square-integrable on $\Gamma\backslash G^1 \times \Gamma\backslash G^1$ (see for example the upper bounds for $\Lambda^X K$ in the proof of proposition 3.5 of [19]). We deduce the following theorem.

Theorem 4.5. — *For any $f \in C_c^\infty(G^1)$, the operator $\Lambda^X \circ R(f)$, acting on $L^2(\Gamma\backslash G^1)$ is a Hilbert-Schmidt operator.*

That theorem has the same consequences as in the cocompact case. Before stating them, let us introduce the discrete part of $L^2(\Gamma\backslash G^1)$ that is the closed subspace $L^2_{\text{disc}}(\Gamma\backslash G^1)$ generated by the irreducible subrepresentations of G^1 in $L^2(\Gamma\backslash G^1)$. This is the part of $L^2(\Gamma\backslash G^1)$ whose spectral decomposition looks like the decomposition (2.2).

Corollary 4.6. — *We have the inclusion*

$$L^2_{\text{cusp}}(\Gamma\backslash G^1) \subset L^2_{\text{disc}}(\Gamma\backslash G^1).$$

More precisely, the space $L^2_{\text{cusp}}(\Gamma\backslash G^1)$ is the Hilbert sum of unitary irreducible representations of G^1 with finite *multiplicities.*

Proof. — (Sketch) Let R_{cusp} be the regular representation of G^1 on $L^2_{\text{cusp}}(\Gamma\backslash G^1)$. By property 1 of the proposition 4.4, the operator R_{cusp} is the restriction of the Hilbert-Schmidt operator $\Lambda^X \circ R(f)$ to the closed subspace $L^2_{\text{cusp}}(\Gamma\backslash G^1)$. It is also a Hilbert-Schmidt operator. Then the proof is the same as in the cocompact case. (cf. the proof of proposition 2.1 and for more details, see e.g. the proof of theorem 1.5 of [21]). □

Remark. — The irreducible components (and their multiplicities) of $L^2_{\text{cusp}}(\Gamma\backslash G^1)$ are in the heart of the Langlands' program. In our case and more generally for $GL(n)$ the multiplicities are 0 or 1 (cf. [34]).

Corollary 4.7. — *The operator $\Lambda^X R(f)\Lambda^X$ is an integral operator with kernel $\Lambda^X K \Lambda^X$. Moreover it is of trace class and its trace is the integral over the diagonal over its kernel*

$$\text{trace}(\Lambda^X R(f)\Lambda^X) = \int_{\Gamma\backslash G^1} \Lambda^X K \Lambda^X(x,x)\, dx. \tag{4.4}$$

Proof. — The first assertion is easy. For the second, the Dixmier-Malliavin's factorization theorem (cf. section 1) enables us to write $\Lambda^X R(f)\Lambda^X$ as a sum of products of the type $\Lambda^X R(g)R(h)\Lambda^X$ with g and h in $C_c^\infty(G^1)$. We have just to remark that the adjoint operator of $R(h)\Lambda^X$ is $\Lambda^X R(h^\vee)$ with $h^\vee(x) = \overline{h(x^{-1})}$ for $x \in G^1$. Thus $\Lambda^X R(f)\Lambda^X$ is a sum of products of Hilbert-Schmidt operators. Hence it is of trace class and its trace is the sum of the Hilbert-Schmidt scalar products of the operators $R(h)\Lambda^X$ and $(\Lambda^X R(g))^*$ which gives the last statement by an easy computation. □

We have dramatically improve the situation : we have now a trace class operator $\Lambda^X R(f)\Lambda^X$. Its trace is the integral of its kernel $\Lambda^X K \Lambda^X$ over the diagonal. Moreover, this trace "includes" the trace of the operator $R_{\text{cusp}}(f)$ which carries interesting arithmetic informations. As in the cocompact case, one can hope to get eventually a geometric decomposition of the integral of $\Lambda^X K \Lambda^X$. But we still need a full spectral decomposition of $L^2(\Gamma\backslash G^1)$ that we shall now review.

5 Langlands' spectral decomposition of $L^2(\Gamma\backslash G^1)$

Let A_∞ be the group of diagonal matrices (a, a^{-1}) with $a \in \mathbb{R}_+^\times$. Let us recall that T^1 is the subgroup of T of diagonal matrices (a, b) with a and b in $\mathbb{A}^1$.

The space $\mathcal{H}$ of measurable functions

$$\phi : \; N(T \cap \Gamma)A_\infty \backslash G^1 \to \mathbb{C}$$

such that for any $x \in G^1$ the function

$$t \in T^1 \mapsto \phi(tx)$$

belongs to $L^2((T \cap \Gamma)\backslash T^1)$ and such that

$$\|\phi\|^2 = \int_K \int_{(T\cap\Gamma)\backslash T^1} |\phi(tk)|^2 \, dtdk < \infty$$

is a Hilbert space.

For any $s \in \mathbb{C}$, we define an induced action R_s of G^1 on $\mathcal{H}$ by the formula

$$R_s(y)\phi(x) = \exp(-(s+1)H(x))\phi_s(xy)$$

where

$$\phi_s(x) = \phi(x)\exp((s+1)(H(x))$$

for any $x, y \in G^1$. The function H has been defined in the previous section. This is a unitary representation of G^1 if $s \in i\mathbb{R}$.

For any $\phi \in \mathcal{H}$, $x \in G^1$ and s in a suitable subset of $\mathbb{C}$, we define the Eisenstein series

$$E(x, \phi, s) = \sum_{\delta \in B\cap\Gamma\backslash\Gamma} \phi_s(\delta x)$$

and the intertwining operator

$$(M(s)\phi)(x) = \exp((s-1)H(x)) \int_N \phi_s(wnx) \, dn.$$

Note that formally

$$E(xy, \phi, s) = E(x, R_s(y)\phi, s)$$

and

$$M(s)R_s(y) = R_{-s}(y)M(s).$$

Theorem 5.1. — *(Langlands' decomposition)*

1. *The series and the integral that define respectively $E(x,\phi,s)$ and $M(s)$ are convergent and analytic for $\mathrm{Re}(s) > 1$. Moreover, they admit a meromorphic continuation to the whole complex plane, which is analytic on $i\mathbb{R}$. They satisfy the functional equations*
$$E(x, M(s)\phi, -s) = E(x,\phi,s)$$
and
$$M(-s)M(s) = 1.$$
2. *Let $\mathcal{F}$ the Hilbert space of functions*
$$F:\ i\mathbb{R} \to \mathcal{H}$$
which satisfy
$$F(-s) = M(s)F(s)$$
and
$$\|F\|^2 = \frac{1}{2}\int_{i\mathbb{R}} \|F(s)\|^2\, ds < \infty$$
The map defined over a dense subspace of $\mathcal{F}$ by
$$F \mapsto \frac{1}{2}\int_{i\mathbb{R}} E(x, F(s), s)\, ds$$
extends to an isometry from $\mathcal{F}$ onto the orthogonal complement denoted $L^2_{\mathrm{cont}}(\Gamma\backslash G^1)$ of $L^2_{\mathrm{disc}}(\Gamma\backslash G^1)$ in $L^2(\Gamma\backslash G^1)$.

Recall that $L^2_{\mathrm{cusp}}(\Gamma\backslash G^1) \subset L^2_{\mathrm{disc}}(\Gamma\backslash G^1)$. Let $L^2_{\mathrm{res}}(\Gamma\backslash G^1)$ be the orthogonal complement of $L^2_{\mathrm{cusp}}(\Gamma\backslash G^1)$ in $L^2_{\mathrm{disc}}(\Gamma\backslash G^1)$. This is the so-called residual spectrum (because it is eventually obtained from residues of Eisenstein series). In fact, in our situation, the residual spectrum admits a very explicit description
$$L^2_{\mathrm{res}}(\Gamma\backslash G^1) = \bigoplus_{\chi} \chi \circ \det$$
where χ belongs to the group of continuous characters of $\mathbb{Q}\backslash\mathbb{A}^1$.

We have an orthogonal decomposition
$$L^2(\Gamma\backslash G^1) = L^2_{\mathrm{cusp}}(\Gamma\backslash G^1) \oplus L^2_{\mathrm{res}}(\Gamma\backslash G^1) \oplus L^2_{\mathrm{cont}}(\Gamma\backslash G^1).$$
Let $f \in C_c^\infty(G^1)$. Each term in the above decomposition is stable by the operator $R(f)$. By restriction, we obtain integrable operators $R_{\mathrm{cusp}}(f)$, $R_{\mathrm{res}}(f)$, and $R_{\mathrm{cont}}(f)$ with kernels respectively denoted by K_{cusp}, K_{res}, and K_{cont}.

Let $\mathcal{B}$ be an orthonormal basis of $\mathcal{H}$. The Langlands' decomposition gives the following expression
$$K_{\mathrm{cont}}(x,y) = \frac{1}{2}\int_{i\mathbb{R}} \sum_{\phi\in\mathcal{B}} E(x, R_s(f)\phi, s)\overline{E(y,\phi,s)}\, ds$$

for any $x, y \in G^1$.

We have already shown that $\Lambda^X R(f)\Lambda^X$ is of trace class (cf. corollary 4.7). Its kernel $\Lambda^X K \Lambda^X$ satisfies

$$\Lambda^X K\Lambda^X(x,y) = K_{\mathrm{cusp}}(x,y) + \Lambda^X K_{\mathrm{res}}\Lambda^X(x,y)$$
$$+\frac{1}{2}\int_{i\mathbb{R}}\sum_{\phi\in\mathcal{B}}\Lambda^X E(x,R_s(f)\phi,s)\overline{\Lambda^X E(y,\phi,s)}\,ds.$$

The trace of $\Lambda^X R(f)\Lambda^X$ is the integral of its kernel over the diagonal hence

$$\mathrm{trace}(\Lambda^X R(f)\Lambda^X) = \mathrm{trace}(R_{\mathrm{cusp}}(f)) + \mathrm{trace}(\Lambda^X R_{\mathrm{res}}(f)\Lambda^X)$$
$$+\frac{1}{2}\int_{i\mathbb{R}}\sum_{\phi\in\mathcal{B}}\int_{\Gamma\backslash G^1}\Lambda^X E(x,R_s(f)\phi,s)\overline{\Lambda^X E(x,\phi,s)}\,dx\,ds.$$

Note that the inner integral is nothing else than the scalar product of two truncated Eisenstein series, which is not difficult to compute (cf. [21] proposition 7.13). After integration we obtain :

$$\frac{1}{2}\int_{i\mathbb{R}}\sum_{\phi\in\mathcal{B}}\int_{G(\mathbb{Q})\backslash G(\mathbb{A})^1}\Lambda^X E(x,R_s(f)\phi,s)\overline{\Lambda^X E(x,\phi,s)}\,dx\,ds$$
$$=\frac{1}{2}\int_{i\mathbb{R}}\mathrm{trace}(M(s)^{-1}M'(s)R_s(f))\,ds + X\int_{i\mathbb{R}}\mathrm{trace}(R_s(f)))\,ds$$
$$+\frac{1}{4}\mathrm{trace}(M(0)R_0(f)) + r(X)$$

where the last term $r(X)$ vanishes at infinity

$$\lim_{X\to+\infty} r(X) = 0.$$

Finally, we also have

$$\lim_{X\to+\infty}\mathrm{trace}(\Lambda^X R_{\mathrm{res}}(f)\Lambda^X) = \mathrm{trace}(R_{\mathrm{res}}(f))$$

In conclusion, we obtain the theorem.

Theorem 5.2. —*For any $f \in C_c^\infty(G^1)$,*

$$\mathrm{trace}(\Lambda^X R(f)\Lambda^X) = \mathrm{trace}(R_{\mathrm{disc}}(f)) + \frac{1}{4}\mathrm{trace}(M(0)R_0(f))$$
$$+\frac{1}{2}\int_{i\mathbb{R}}\mathrm{trace}(M(s)^{-1}M'(s)R_s(f))\,ds + X\int_{i\mathbb{R}}\mathrm{trace}(R_s(f)))\,ds + \varepsilon(X)$$

with

$$\lim_{X\to+\infty}\varepsilon(X) = 0.$$

Remarks. —

– Besides the trace of $R(f)$ in the discrete spectrum denoted by $\mathrm{trace}(R_{\mathrm{disc}}(f))$, we have an other discrete contribution namely $\mathrm{trace}(M(0)R_0(f))$ which comes from Eisenstein series.
– The continuous contribution $\int_{i\mathbb{R}} \mathrm{trace}(M(s)^{-1}M'(s)R_s(f))\,ds$ is built upon the so-called weighted characters $\mathrm{trace}(M(s)^{-1}M'(s)R_s(f))$ which are not invariant by conjugacy by G.
– the coefficient of X is an invariant distribution : it is the integral of the trace of representations induced from T.

6 The geometric side : the cut-off kernel

As in the cocompact case, we can expect the trace of the operator $\Lambda^X R(f)\Lambda^X$

$$\mathrm{trace}(\Lambda^X R\Lambda^X(f)) = \int_{\Gamma\backslash G^1} \Lambda^X K \Lambda^X(x,x)\,dx$$

to have a geometric decomposition according to conjugacy classes in Γ. In this section, it is our task to provide such a decomposition. Unfortunately, the fact that the truncation operator Λ^X does not act diagonally on the kernel K makes our task harder.

According to Arthur, it is more convenient to introduce before another truncated kernel whose geometric decomposition is more tractable.

Let $f \in C_c^\infty(G^1)$. Let R_B the right regular action of G^1 on

$$L^2(N(T\cap\Gamma)\backslash G^1)$$

As before, one defines a convolution operator $R_B(f)$ which is an integral operator which a kernel denoted K_B. We have the following explicit expression for this kernel

$$K_B(x,y) = \int_N \sum_{\gamma\in T\cap\Gamma} f(x^{-1}\gamma n y)\,dn.$$

for x and $y \in G^1$. Then we can introduce what we call the "cut-off" kernel (to distinguish it from the truncated kernel $\Lambda^X K$)

$$k^X(x,f) = K(x,x) - \sum_{\delta\in B\cap\Gamma\backslash\Gamma} \tau_X(\delta x)K_B(\delta x,\delta x).$$

The function τ was defined in (4.3). Again, by lemma 4.3, the sum over δ can be taken over a finite set.

7 Characteristic polynomials

Instead of conjugacy classes, we are going to expand the cut-off kernel according to characteristic polynomials. Let $\mathfrak{X}$ be the set of characteristic polynomials of elements of Γ. For any $\gamma \in \Gamma$, we denote by χ_γ the characteristic polynomial of γ.

For any $\chi \in \mathfrak{X}$, $x, y \in G^1$, we set

$$K_\chi(x,y) = \sum_{\{\gamma\in\Gamma|\ \chi_\gamma=\chi\}} f(x^{-1}\gamma y)$$

and

$$K_{B,\chi}(x,y) = \int_N \sum_{\{\gamma\in T\cap\Gamma|\ \chi_\gamma=\chi\}} f(x^{-1}\gamma n y)\, dn.$$

Thus we have the expansion

$$k^X(x,f) = \sum_{\chi\in\mathfrak{X}} k^X_\chi(x,f).$$

We can now state the first theorem.

Theorem 7.1. — *For large X, we have*

$$\sum_{\chi\in\mathfrak{X}} \int_{\Gamma\backslash G^1} |k^X_\chi(x,f)|\, dx < \infty.$$

(For a sketch of the proof see [19] lecture II, theorem 4.3)

Thanks to the previous theorem, we can set, for all $\chi \in \mathfrak{X}$, and for large X

$$J^X_\chi(f) = \int_{\Gamma\backslash G^1} k^X_\chi(x,f)\, dx.$$

For applications, it is important to find more explicit expressions for the distributions $J^X_\chi(f)$ and to understand their behaviour in X. Before considering this problem, we need a coarse classification of characteristic polynomials. We introduce the subsets $\mathfrak{X}_{\mathrm{ell}}$ of elliptic characteristic polynomials, $\mathfrak{X}_{\mathrm{par}}$ of parabolic ones and $\mathfrak{X}_{\mathrm{sing}}$ of singular ones. By definition, we have

1. $\mathfrak{X}_{\mathrm{ell}}$ is the subset of irreducible polynomials (over $\mathbb{Q}$) in $\mathfrak{X}$;
2. $\mathfrak{X}_{\mathrm{par}}$ is the subset of polynomials in $\mathfrak{X}$ which have two distincts roots in $\mathbb{Q}$;
3. $\mathfrak{X}_{\mathrm{sing}}$ is the subset of polynomials in $\mathfrak{X}$ which have one double root in $\mathbb{Q}$.

So we have a disjoint union

$$\mathfrak{X} = \mathfrak{X}_{\mathrm{ell}} \cup \mathfrak{X}_{\mathrm{par}} \cup \mathfrak{X}_{\mathrm{sing}}.$$

Let γ, γ' be elements of Γ. Assume that χ_γ belongs to $\mathfrak{X}_{\mathrm{ell}} \cup \mathfrak{X}_{\mathrm{par}}$. Then γ is semi-simple and regular. Here regular semi-simple just means diagonalizable over an algebraic closure of $\mathbb{Q}$ with two distinct eigenvalues. Moreover $\chi_\gamma = \chi_{\gamma'}$ if and only if γ and γ' are conjugate in Γ. In that sense, we can identify the set $\mathfrak{X}_{\mathrm{ell}} \cup \mathfrak{X}_{\mathrm{par}}$ with the set of conjugacy classes of semi-simple regular elements in Γ.

On the other hand, χ_γ belongs to $\mathfrak{X}_{\mathrm{sing}}$ if and only the semi-simple part of γ in the Jordan decomposition is central (in other terms a scalar matrix). Moreover $\chi_\gamma = \chi_{\gamma'}$ if and only if γ and γ' have the same semi-simple part.

We will now describe the distributions

$$J^X_\chi(f)$$

according to those subsets of $\mathfrak{X}$.

8 Weighted orbital integrals

8.1. Elliptic polynomials — Let $\gamma \in \Gamma$ such that $\chi = \chi_\gamma$ belongs to $\mathfrak{X}_{\rm ell}$. In that case, we have obviously $\gamma \notin T \cap \Gamma$. Thus the χ-part of the cut-off kernel reduces to the χ-part of the kernel :

$$k^X_\chi(x, f) = K_\chi(x, x)$$

does not depend on X. As a consequence, for the elliptic part of the cut-off kernel, we can perform the same calculations as in the cocompact case. Let us state the result.

Proposition 8.1. — *For any $\gamma \in \Gamma$ such that $\chi = \chi_\gamma$ belongs to $\mathfrak{X}_{\rm ell}$, we have*

$$J^X_\chi(f) = \mathrm{vol}(G^1_\gamma \cap \Gamma \backslash G^1_\gamma) \int_{G_\gamma \backslash G} f(g^{-1}\gamma g)\, d\bar{g}$$

where G_γ, resp. G^1_γ, is the centralizer of γ in G, resp. G^1.

Remark. — The centralizer of γ in $GL(2)$ is a torus. The quotient $G^1_\gamma \cap \Gamma \backslash G^1_\gamma$ is in fact compact. So its volume is indeed finite.

8.2. Parabolic polynomials — For elliptic polynomials, we have got orbital integrals $\int_{G_\gamma \backslash G} f(g^{-1}\gamma g)\, d\bar{g}$ as in the cocompact case. For parabolic polynomials, new distributions appear : the so-called weighted orbital integrals. First, we need to define the weight v. For any $g \in G$, we set

$$v(g) = -H(g) - H(wg),$$

where H is the function defined in (4.1) and (4.2). It is an amusing exercise to check that this defines a non-negative function on G which is invariant on the left by T and on the right by K.

We can now state the result.

Proposition 8.2. —*For any* non-scalar *$\gamma \in T \cap \Gamma$ and $\chi = \chi_\gamma$, we have*

$$J^X_\chi(f) = \mathrm{vol}(T \cap \Gamma \backslash T^1) \times \big(2X \int_{T \backslash G} f(g^{-1}\gamma g)\, d\bar{g} + \int_{T \backslash G} f(g^{-1}\gamma g) v(g)\, d\bar{g}\big)$$

where T^1 is the group of diagonal matrices (a, b) with $a, b \in \mathbb{A}^1$.

This proposition is quoted in [19] (cf. lecture IV proposition 1.1) where the reader will find some references.

Remarks. —
- The quotient $T \cap \Gamma \backslash T^1$ is compact hence of finite volume.

- For any $\chi \in \mathfrak{X}_{\text{par}}$, one can find $\gamma \in T \cap \Gamma$ such that $\chi_\gamma = \chi$. Of course, γ is not central. Thus we have a description of $J^X_\chi(f)$ for any $\chi \in \mathfrak{X}_{\text{par}}$.
- The group T is really the centralizer of γ in G. So the integral $\int_{T\backslash G} f(g^{-1}\gamma g)\, d\bar{g}$ is nothing else than the orbital integral associated to γ.

Because of the weight v, the so-called weighted orbital integral $\int_{T(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)v(g)\, d\bar{g}$ is a distribution that is not invariant by conjugacy by G.

8.3. Singular polynomials — There exists a similar but more complicated expression for singular characteristic polynomials. We will not give it (however cf. [19] lecture IV proposition 1.2). The reasons for this complication are easy to track : first a singular characteristic polynomial cannot be identified to a single conjugacy class. Second, the global unipotent integrals do not converge (we already mentionned this fact when we introduced the truncation).

Let us emphasize that $J^X_\chi(f)$ for χ singular is still a polynomial in X of degree at most 1. Moreover, the distributions J^X_χ for χ singular are in some sense in the closure of the distributions J^X_χ for χ elliptic or parabolic.

9 The trace formula

So far, we have introduced the cut-off kernel and we have got a geometric decomposition of its integral in terms of (weighted) orbital integral. The following theorem will enable us to compare the geometric decomposition of the cut-off kernel with the former truncated kernel.

Theorem 9.1. — *Let $f \in C^\infty_c(G^1)$. There exists X_f such that for any $X \geqslant X_f$ we have*

$$\int_{\Gamma\backslash G^1} \Lambda^X K(x,x) = \int_{\Gamma\backslash G^1} k^X(x,f)\, dx.$$

(for a sketch of the proof, see for example [19] p.26)

In theorem 5.2, we saw that the left-hand side is a polynomial in X up to a negligible error term. In section 7 and 8, we saw that the right-hand side is exactly a polynomial in X. Equaling the constant terms (at $X = 0$), we obtain the following corollary.

Corollary 9.2. —*(Arthur's trace formula for $GL(2)$) Let $f \in C^\infty_c(G^1)$. We have the following equality*

$$J_{\text{disc}}(f) + J_{\text{cont}}(f) = \sum_{\chi \in \mathfrak{X}} J_\chi(f)$$

where

– *the discrete part of the spectral side of the trace formula is*

$$J_{\mathrm{disc}}(f) = \mathrm{trace}(R_{\mathrm{disc}}(f)) + \frac{1}{4}\mathrm{trace}(M(0)R_0(f)) \ ;$$

– *the continuous part of the spectral side of the trace formula is*

$$J^X_{\mathrm{cont}}(f) = \frac{1}{2}\int_{i\mathbb{R}} \mathrm{trace}(M(s)^{-1}M'(s)R_s(f))ds \ ;$$

– *for* $\chi \in \mathfrak{X}$, *the geometric distribution* $J_\chi(f)$ *is the value at* $X = 0$ *of* $J^X_\chi(f)$. *Thus,* $J_\chi(f)$ *is an orbital integral for* $\chi \in \mathfrak{X}_{\mathrm{ell}}$ *and a weighted orbital integral for* $\chi \in \mathfrak{X}_{\mathrm{par}}$.

Remark. — As we stated it, the trace formula we obtained does not compute the trace of anything. Nonetheless, it gives a formula for the trace of $R(f)$ in the discrete spectrum in terms of both geometric and spectral data.

10 A simple trace formula

For some more restricted classes of functions, the trace formula is simpler (cf. [19] Lecture V, §2). Here is an useful example. Let V be the set of all places of $\mathbb{Q}$. For $v \in V$, let $\mathbb{Q}_v$ be the completion of $\mathbb{Q}$ at v. Let $G_v = GL(2,\mathbb{Q}_v)$ and T_v the group of diagonal matrices in G_v. By identifying G_v with a subgroup of G in the usual manner, we define the weight function v on G_v. We have the following splitting formula due to Arthur. (see e.g. proposition 1.1 p.46 of [19]). (of course, we tacitly assume some compatibility conditions among the various Haar measures).

Proposition 10.1. — *Let* $\gamma \in T \cap \Gamma$ *and* $f = \otimes_{v\in V} f_v$. *The the global weighted orbital integral*

$$\int_{T\backslash G} f(g^{-1}\gamma g)v(g)\, d\bar{g}$$

is equal to the sum over $v \in V$ *of the product*

$$\int_{T_v\backslash G_v} f_v(g^{-1}xg)v(g)\, d\bar{g} \times \prod_{w\neq v}\int_{T_w\backslash G_w} f_w(g^{-1}xg)\, d\bar{g}.$$

Here almost all the factors of the product are in fact 1. *Moreover for almost all* v, *the local weighted orbital integral*

$$\int_{T_v\backslash G_v} f_v(g^{-1}xg)v(g)\, d\bar{g}$$

is 0 *and the product above is* 0.

Let us fix two distinct places v_1 and v_2 and let us consider functions $f = \otimes_{v\in V} f_v$ such that for $v \in \{v_1, v_2\}$ the *local* orbital integrals of f_v vanish for all $x \in T_v$:

$$\int_{T_v\backslash G_v} f_v(g^{-1}xg)\, d\bar{g} = 0. \tag{10.1}$$

The proposition 10.1 implies the parabolic part of the geometric side of the trace formula vanishes for such functions f. Besides one can show that the singular part reduces to a sum over the set Z of central elements in Γ (cf. [19] lecture V, §2). More precisely, we obtain for the geometric side the expression

$$\sum_{\chi\in\mathfrak{X}_{\mathrm{ell}}} J_\chi(f) + \mathrm{vol}(\Gamma\backslash G^1)\sum_{z\in Z} f(z).$$

The vanishing asumption (10.1) implies that the trace of f_v in the induced representations is 0 for $v\in\{v_1,v_2\}$. But we have also splitting formulae for the spectral terms (see e.g. proposition 1.3 p.48 of [19]). As consequence we can also simplify the spectral side of the trace formula. Finally, the trace formula reduces for such functions f to the equality

$$\mathrm{trace}(R_{\mathrm{disc}}(f)) = \sum_{\chi\in\mathfrak{X}_{\mathrm{ell}}} J_\chi(f) + \mathrm{vol}(\Gamma\backslash G^1)\sum_{z\in Z} f(z). \tag{10.2}$$

11 Application : the Jacquet-Langlands correspondence

One of the most striking applications of the trace formula is to obtain some special cases of what is known as Langlands' functoriality. Such applications are based on a comparison of different trace formulae for different groups. Let us discuss a typical case : the Jacquet-Langlands correspondence. Let $G' = D^\times$ be the group of invertible elements of a quaternion algebra D over $\mathbb{Q}$. We will use a $'$ to denote objects relative to G'. Let $G'(\mathbb{A})^1$ be the subgroup of $G'(\mathbb{A})$ of elements $g\in G'(\mathbb{A})$ such that $|\mathrm{Nrd}(g)|_{\mathbb{A}} = 1$ where Nrd is the reduced norm. Then the quotient $G'(\mathbb{Q})\backslash G'(\mathbb{A})^1$ is compact. Therefore using the same method as in the section 2, we can prove that for any test function f' on $G'(\mathbb{A})^1$

$$\mathrm{trace}(R'(f')) = \sum_{\chi} J'_\chi(f') \tag{11.1}$$

where as usual R' is the regular representation of $G'(\mathbb{A})^1$ on $L^2(G'(\mathbb{Q})\backslash G'(\mathbb{A})^1)$. On the right-hand side, there is a sum indexed by conjugacy classes χ in $G'(\mathbb{Q})$ of $J'_\chi(f')$ which is a global orbital integral times a volume. We can still classify the elements of $G'(\mathbb{Q})$ according to their characteristic polynomials. Moreover since there are no non-trivial unipotent element in $G'(\mathbb{Q})$, two elements in $G'(\mathbb{Q})$ which have the same characteristic polynomial are in fact conjugate. Moreover there are no element in $G'(\mathbb{Q})$ which has a parabolic characteristic polynomial. The characteristic polynomial map induces a bijection between the set of conjugacy classes in $G'(\mathbb{Q})$ and the set of elliptic or singular characteristic polynomials. In this way, the formula (11.1) can be written

$$\mathrm{trace}(R'(f')) = \sum_{\chi\in\mathfrak{X}_{\mathrm{ell}}} J'_\chi(f) + \mathrm{vol}(G'(\mathbb{Q})\backslash G'(\mathbb{A})^1)\sum_{z\in Z'(\mathbb{Q})} f'(z), \tag{11.2}$$

where Z' is the center of G' wich we identified to the center Z of $GL(2,\mathbb{Q})$. Of course, $Z'\simeq Z$. This formula looks like the simple trace formula (10.2) we obtained for $GL(2)$.

We would like to compare the geometric parts of these formulae of G and G' for enough pairs of test functions (f', f). We normalize the Haar measures on G and $G'(\mathbb{A})$ by taking the Tamagawa measures (cf. [19] p.53). These measures decompose locally in a product of Haar measures. Let f' be a smooth compactly supported function on $G'(\mathbb{A})^1$ such that $f' = \otimes_{v\in V} f'_v$. For $v \in V$, we denote G'_v the group $G'(\mathbb{Q}_v)$. Let S be the finite set of places v where G_v and G'_v are not isomorphic (note that $|S| \geqslant 2$). Thus for the places $v \notin S$, we may and we will take $f_v = f'_v$. For the places $v \in S$, we will use a function f_v which satisfies the following proposition ("existence of the transfer").

Proposition 11.1. — *Let $v \in V$. For any test function f'_v on G'_v there exists a test function f_v such that*

1. *for any $\gamma \in T_v$ the orbital integral of f_v vanishes*

$$\int_{T_v\backslash G_v} f_v(g^{-1}\gamma g)\, d\bar{g} = 0 \ ;$$

2. *for any $\gamma \in G_v$ and $\gamma' \in G'_v$ with matching characteristic polynomials we have*

$$\int_{G_{\gamma,v}\backslash G_v} f_v(g^{-1}\gamma g) d\bar{g} = \int_{G'_{\gamma',v}\backslash G'_v} f'_v(g^{-1}\gamma' g) d\bar{g}$$

(when γ and γ' match, their centralizers $G_{\gamma,v}$ and $G'_{\gamma',v}$ are isomorphic which enables us to take compatible Haar measures on $G_{\gamma,v}$ and $G'_{\gamma',v}$.)

The function $f = \otimes_{v\in V} f_v$ is smooth and compactly supported. Using the simple form (10.2) of the trace formula for G and f and the formula (11.2) above, we write

$$\mathrm{trace}(R_{\mathrm{disc}}(f)) - \mathrm{trace}(R'(f')) = \mathrm{vol}(\Gamma\backslash G^1) \sum_{z\in Z} f(z) - \mathrm{vol}(G'(\mathbb{Q})\backslash G'(\mathbb{A})^1)) \sum_{z\in Z'(\mathbb{Q})} f'(z).$$

With the identification $Z \simeq Z'(\mathbb{Q})$, it is possible to show that we have for the pair (f, f') the equality : $f(z) = f'(z)$ for all $z \in Z$ (this equality is true locally only up to a Kottwitz sign which disappears globally). Thanks to Weyl's integration formula, it is easy to show that the contributions of the 1-dimensional subrepresentations to $\mathrm{trace}(R_{\mathrm{disc}}(f))$ and $\mathrm{trace}(R'(f'))$ match. Let us denote R'_0 the regular representation of $G'(\mathbb{A})^1$ on $L^2_0(G'(\mathbb{Q})\backslash G'(\mathbb{A})^1)$ the subspace of $L^2(G'(\mathbb{Q})\backslash G'(\mathbb{A})^1)$ orthogonal to all the invariant 1-dimensional subspaces. We obtain

$$\mathrm{trace}(R_{\mathrm{cusp}}(f)) - \mathrm{trace}(R'_0(f')) = (\mathrm{vol}(\Gamma\backslash G^1) - \mathrm{vol}(G'(\mathbb{Q})\backslash G'(\mathbb{A})^1)) \sum_{z\in Z} f(z).$$

But this equality is possible only if both sides are zero. Let us explain the rough idea which comes from functional analysis : take $v \notin S$ an "unramified" place. Then the formula above is an equality of distributions for f_v in the Hecke algebra. But the left-hand side is a discrete sum of unramified unitary representations whereas the right-hand side is a continuous integral of tempered representations against the Plancherel measure (thanks to the Plancherel formula). By Riesz uniqueness theorem, both distributions are zero. As a consequence, we have

$$\mathrm{vol}(\Gamma\backslash G^1) = \mathrm{vol}(G'(\mathbb{Q})\backslash G'(\mathbb{A})^1) \tag{11.3}$$

and

$$\mathrm{trace}(R_{\mathrm{cusp}}(f)) = \mathrm{trace}(R'_0(f')). \tag{11.4}$$

From the last equality, we can eventually prove that there is a natural injective map from the irreducible constituents of $L^2_0(G'(\mathbb{Q})\backslash G'(\mathbb{A})^1)$ to the irreducible constituents of $L^2_{\mathrm{cusp}}(G'(\mathbb{Q})\backslash G'(\mathbb{A})^1)$ such that for $v \notin S$ the local components at v match. Moreover it is possible to characterize the image of the map (cf. [19] lecture VI). This is the global Jacquet-Langlands correspondence.

12 Final comments

Arthur has developped the trace formula for any reductive connected group (over a number fields) for more than twenty-five years. Without being exhaustive, we can mention several important steps. The reader can find the “coarse form” of the trace formula in the papers [1] for the geometric side and [2] for the spectral side (which relies on the Langlands’ spectral decomposition cf. [29] or [32] for an adelic setting). The papers [9, 6, 5] are devoted to express the geometric distribution in terms of (local) weighted orbital integral. The papers [3, 4] do the same job for the spectral distributions and the weighted characters.

In the papers [7, 8], Arthur deduces from the non-invariant trace formula an invariant one. As a byproduct, he establishes a very useful “simple” trace formula (in the spirit of the section 10). In [14], Arthur and Clozel prove the general comparison between the trace formula for $GL(n)$ and one of its inner form [14]. Using their result, Badulescu recently succeeded in proving a general global Jacquet-Langlands correspondence (cf. [15]).

When one tries to generalize the Jacquet-Langlands correspondence to other reductive groups, one has to face a difficulty. For any reductive group one can still define a notion of characteristic polynomial. Moreover, two groups which are inner forms of each other share the same set of characteristic polynomials. But the fibers of the characteristic polynomial map are usually not ordinary conjugacy classes but rather stable conjugacy classes (roughly speaking the stable conjugacy is the conjugacy over an algebraic closure of the base field). The two notions of conjugacy are the same for $GL(n)$ but in general they do not coincide. So, if we want to compare the geometric sides of the trace formulae for two groups which are inner forms of each other, we have before to express these geometric sides in terms of stable conjugacy classes. As we saw it for $GL(2)$ in section 11, the comparison is based on a transfer of orbital integrals. In the most general situation, the comparison should be based on a transfer of stable orbital integrals. Therefore we have to express the trace formula on a group in terms of stable distributions (those which are in the weak closure of stable orbital integrals). At first glance, it is not possible. In fact, what should be possible to do is to write the trace formula for a group G in terms of a stable part and an instable part which is a sum of stable trace formulae for a family of groups called endoscopic groups : these groups have the same rank as G but a smaller dimension. This is basically Langlands' strategy (cf. [30]). In return, the stabilization of the trace formula should give the fonctorialities of endoscopic type.

The works of Langlands [30] and Kottwitz [22, 23] achieved this program for the elliptic part of the geometric side of the trace formula but under two assumptions : the first one is the existence of the transfer for stable orbital integrals (in the spirit of proposition 11.1 above). The second one is the so-called fundamental lemma : it is a precise version of the tranfer at unramified places which says that the transfer of the unit of a Hecke algebra is again the unit of a Hecke algebra. Kottwitz used the stabilization of the elliptic part to prove that the Tamagawa numbers are the same for inner forms (cf. [24] : this is a generalization of the equality (11.3) above).

In a subsequent work (cf. [10, 11, 12] and many other ancillary papers), Arthur achieved the stabilization of the whole trace formula : his work is also based on two assumptions namely the existence of the transfer and a generalized fundamental lemma extended to weighted orbital integrals. The existence of the transfer at archimedean places is due to Shelstad ([35, 36]). Waldspurger showed how one can deduce the transfer from the fundamental lemma ([38]). So the cornerstone of the theory is the fundamental lemma and its weighted version. Some cases were known by works of Hales, Waldspurger (for $SL(n)$) and Weissauer. A breakthrough was the introduction of a geometric approach due to Goresky-Kottwitz-MacPherson and Laumon. In this way, Laumon-Ngô [31] proved the fundamental lemma for unitary groups. Recently Ngô has found a proof for the general case. In general, the weighted fundamental lemma remains a conjecture (however cf. [39] for $GSp(4)$). We also emphasize that when one tries to compare the trace formula to a Lefschetz trace formula one often has to use the stable form of the trace formula : this is the case when one tries to express Hasse-Weil zeta functions of some Shimura varieties in terms of automorphic L-functions (cf. for an example Clozel's report [17]) on Kottwitz work).

To finish, we mention that there is a twisted version of the trace formula (unpublished notes of Clozel-Labesse-Langlands). In full generality, a twisted stable trace formula was not yet established (however see the works of Kottwitz-Shelstad [25], Labesse [28], Renard [33] and Waldspurger [37]). Conjecturally, such a formula should give functorialities for the classical groups to $GL(n)$ (work in progress by Arthur, cf. §30 of [13]). Arthur and Clozel have already used a twisted trace formula to prove the cyclic base change for $GL(n)$ (cf. [14]). To conclude, let us mention that Jacquet has introduced interesting variants of the Arthur-Selberg trace formula (cf. e.g. [20]).

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