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Representations of Reductive Groups over Local Non-Archimedean Fields

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A warning to start with : I have only a few hours to present the representation theory of p-adic groups. For that reason few proofs will be given, and facts are not stated in the order of proofs. Many statements in the text lead to valuable exercices, but for the proofs of the main results I refer you to :

[1] J. Bernstein & A. Zelevinsky, Induced representations of the group GL_n over a p-adic field, Russian Math. Surveys 31 (1976), 1-68.

[2] C. J. Bushnell & G. Henniart, The local Langlands conjecture for $GL(2)$, Grundlehren der math. Wissenschaften vol. 335, Springer (2006).

[3] W. Casselman, Introduction to the theory of admissible representations of p-adic reductive groups, unpublished notes, available on Casselman's web page.

Note that I have not always attributed results, for lack of time to research the history.

1 Smooth representations of locally profinite groups

1.1 In all that follows F is a fixed non-Archimedean locally compact local field. Let us recall briefly what it means.

The field F is endowed with a non-trivial discrete valuation v_F, i.e. a surjective group homomorphism from F^* to Z, which, when extended by $v_F(0) = +\infty$, verifies $v_F(x+y) \geqslant \inf(v_F(x), v_F(y))$, for x and y in F. The integers in F - elements with non-negative valuation- form a ring O_F, which is a local principal ideal domain with fraction field F; its unique maximal ideal P_F consists of the elements with positive valuation.

There is a natural topology on F, for which the subsets of the form $a+P_F^n$, n in $\mathbb{Z}$, form a basis of open neighbourhoods of a in F. This makes F into a topological field. Saying that F is locally compact is tantamount to saying that F is complete and that the residue field $k_F = O_F/P_F$ is finite. Writing q_F for its cardinality, the topology on F is also associated to the normalized absolute value $| \ |_F$ given by $|x|_F = q_F^{-v_F(x)}$ for x in F.

Remark: If K is a number field, or a function field in one variable over a finite field, any non-trivial discrete valuation v on K gives rise to a completion K_v, which is an example of F as above. Taking K to be the field $\mathbb{Q}$ of rational numbers, and v to be the p-adic valuation, we get the field $\mathbb{Q}_p$ of p-adic numbers; if k is a finite field, taking $K = k(X)$ and v given by the order of the zero at 0, we get the field $k((X))$ of Laurent power series in X

with coefficients in k, which is again an example of F as above. It is known that these constructions give all examples.

In the sequel, p will be the residue characteristic of F, that is the characteristic of the finite field k_F.

1.2 If n is a positive integer, we put on F^n the product topology. That procedure gives a topology on the space of square matrices $M(n,F)$, and we put on the group $\mathrm{GL}(n,F)$ the induced topology: all such topologies will be called natural in the sequel. It is easily seen that $\mathrm{GL}(n,F)$ is a topological group, that $\mathrm{GL}(n,O_F)$ is a compact open subgroup, and that the compact open subgroups $1_n + P_F^k M(n,O_F)$, $k \geqslant 1$, give a basis of neighbourhoods of the identity in $\mathrm{GL}(n,F)$.

DEFINITION : *A topological group is locally profinite if there is a basis of neighbourhoods of the identity consisting of compact open subgroups.*

In particular, $\mathrm{GL}(n,F)$ is a locally profinite group. With the induced - natural- topology, any closed subgroup of $\mathrm{GL}(n,F)$ is again locally profinite. A linear F-algebraic group $\underline{G}$ is a Zariski-closed subgroup of some $\mathrm{GL}(n)$ over F: this means that it is defined inside $\mathrm{GL}(n)$ by polynomial equations with coefficients in F. Its group of F-rational points $G = G(F)$ is then closed in $\mathrm{GL}(n,F)$ for the natural topology, hence is locally profinite. In fact that topology does not depend upon the embedding of $\underline{G}$ in some $\mathrm{GL}(n)$: it is the coarsest topology such that all regular algebraic functions from G to F are continuous.

We shall be interested mainly in the case where $\underline{G}$ is reductive, i.e. its maximal unipotent normal subgroup is trivial. We shall say in brief that G is a reductive group over F ; examples are $G = \mathrm{GL}(n,F)$ or $G = \mathrm{SL}(n,F)$.

Let ϖ_F be a uniformizer in F, that is a generator of the maximal ideal P_F of O_F. Then for $G = \mathrm{GL}(n,F)$ and $K = \mathrm{GL}(n,O_L)$ we have the Cartan decomposition: G is the disjoint union of the double cosets KdK where d runs through diagonal matrices with diagonal elements $\varpi_F^{a_i}$, with integers a_i decreasing when i runs from 1 to n.

As K is open in G and compact, each set KdK/K is finite, and consequently G/K is countable. If H is a closed subgroup of G, then H is the disjoint union of the $KdK \cap H$, and the same reasoning shows that $H/K \cap H$ is countable; it follows that H/J is countable for each open compact subgroup J of H.

1.3 Let G be a locally profinite topological group. A representation (π, V) of G on a $\mathbb{C}$-vector space V is simply a group homomorphism π of G into the group of vector-space automorphisms of V.

It is the same as a $\mathbb{C}[G]$-module; such representations form an abelian category, with morphisms the G-equivariant linear maps. We shall use the customary terminology for representations, or $\mathbb{C}[G]$-modules ; for example, a representation (π, V) of G is irreducible if V is non-zero and any non-zero G-invariant subspace of V is V itself; in other words, the corresponding $\mathbb{C}[G]$-module is simple. Sometimes also we abbreviate in saying only π or V instead of (π, V).

A representation (π, V) of G is **smooth** if every vector in V has open stabilizer in G; it is **admissible** if moreover for any open subgroup J of G the vector-space V^J of vectors fixed by J is finite-dimensional.

Smooth representations form a full subcategory $S(G)$ of the category of all representations. Restricting a smooth representation to a closed subgroup gives a smooth representation. Subrepresentations or quotient representations of smooth representations are again smooth. Any representation (π, V) of G has a largest smooth subrepresentation (π^∞, V^∞) where V^∞ consists of the smooth vectors in V , i.e. vectors with open stabilizer in G.

A character of G is a group homomorphism from G to $\mathbb{C}^\times$ with open kernel; it is immediate that characters of G correspond to isomorphism classes of smooth 1-dimensional representations.

If (π, V) is a smooth representation of G and χ a character of G, we define a smooth representation $(\chi \otimes \pi, V)$ of G by $\chi \otimes \pi(g) = \chi(g)\pi(g)$ for g in G.

Remark: let ϕ be a field homomorphism of $\mathbb{C}$ into itself. If V is a $\mathbb{C}$-vector space, we can form the $\mathbb{C}$-vector space $\mathbb{C}\otimes V$, where the tensor product is via ϕ; if (π, V) is a smooth representation of G, then so is $(Id \otimes \pi, \mathbb{C} \otimes V)$. If ϕ is complex conjugation, that new representation is called the complex conjugate of (π, V). It is irreducible if and only if (π, V) is.

Assume for a moment that G is compact, and let (π, V) be a smooth representation of G. Then any vector v in V has open stabilizer in G, and since G is compact that stabilizer has finite index so v generates a finite-dimensional G-invariant subspace of V. On such a finite-dimensional space, some open invariant subgroup J of G acts trivially (take a basis), and G acts through its finite quotient G/J. In particular, G acts semisimply on V; as a consequence there is canonical projection $e_G = e_G^V$ in V with image V^G and kernel $V(G)$ generated by the vectors of the form $\pi(g)v - v$, v in V, g in G.

Return to a general locally profinite group G, and let (π, V) be a smooth representation of G. We have a natural representation of G on the dual vector space V^* given by $g \mapsto {}^t\pi(g^{-1})$; the subspace $V^\vee = (V^*)^\infty$ affords a smooth representation $\pi^\vee$ of G called the **contragredient** of π. In this way we get a contravariant functor from $S(G)$ to itself. As with usual duality the evaluation map from $V^\vee$ to $\mathbb{C}$ associated to a vector v in V gives a morphism of representations from π to $\pi^{\vee\vee}$. If J is a compact open subgroup of G, the vector space $(V^\vee)^J$ is made out of functionals on V trivial on $V(J)$: through the projection e_J above we can identify $(V^\vee)^J$ with the dual of V^J. It follows that the canonical morphism from π to its double contragredient is injective, and is an isomorphism if and only if π is admissible.

Remark: Let (π, V)be a smooth representation of G. It is said to be preunitary if there is a (positive definite) scalar product on V which is G-invariant. If moreover π is admissible, then it is semisimple (exercise): indeed any G-invariant subspace of V has its orthogonal as a G-invariant complement.

1.4 Schur's lemma.

In this subsection G is a locally profinite group such that for some (equivalently: for any) open compact subgroup K, the coset space G/K is countable; as we have seen, that is the case if G is a reductive p-adic group.

Let (π, V) be a smooth representation of G; each vector in V has an open stabilizer in G hence it generates a subrepresentation of countable dimension over $\mathbb{C}$. If the representation π is generated by a countable subset S of V, then V has countable dimension over $\mathbb{C}$; moreover any endomorphism of π is determined by its values on S, so $\mathrm{End}_G(V)$ has countable dimension if S is finite.

Assume that (π, V) is irreducible; then any non-zero vector generates V. The endomorphism algebra $\mathrm{End}_G(V)$ is a division algebra over $\mathbb{C}$; if it is not reduced to scalars, then it contains an extension of $\mathbb{C}$ isomorphic to $\mathbb{C}(X)$ hence cannot have countable dimension over $\mathbb{C}$. We deduce :

Proposition: (Schur's lemma) *Let G be as above, and let (π, V) be a smooth irreducible representation of G. Then all endomorphisms of π are scalar, and the centre $Z(G)$ of G acts on V via a character.*

The latter means that there is a character ω_π of G such that $\pi(z) =$

$\omega_\pi(z) Id_V$ for z in $Z(G)$; that character is called the central character of π.

COROLLARY: *If moreover $G/Z(G)$ is compact, V is finite-dimensional.*

Indeed, a non-zero vector v in V has an open stabilizer J in G; but then the line generated by v is stable by $JZ(G)$, which has finite index in G, and that implies that V has finite dimension.

Exercise : Show that a finite-dimensional irreducible smooth representation of $\mathrm{GL}(n, F)$ is in fact one-dimensional, given by a character of the form $\chi \circ \det$, where χ is a character of F^*.

THEOREM: *Let G be a reductive group over F. Then any smooth irreducible representation of G is admissible.*

COROLLARY 1: *The contragredient of any smooth irreducible representation of G is still irreducible.*

CORALLARY 2 : *If (π, V) is a preunitary smooth irreducible representation of G, then the G-invariant scalar product on V is unique up to multiplication by a positive real number.*

We shall indicate the proof of the theorem later, as it requires considerably more material. Note that for an irreducible admissible representation of a locally profinite group G, Schur's lemma is obvious: for any open compact subgroup J of G, the finite-dimensional vector subspace V^J is stable under $\mathrm{End}_G(V)$, which has to be finite-dimensional over $\mathbb{C}$, hence reduced to $\mathbb{C}$. However, Schur's lemma is involved in the proof of the above theorem.

1.5 Unitary representations

For a locally compact topological group G, it is more customary to investigate Hilbert space representations of G: a **unitary** representation of G on a Hilbert space H is a homomorphism π of G into the group of isometries of H, such that the map $G \times H \mapsto H$, $(g, v) \mapsto \pi(g)v$, is continuous. For a locally profinite group G, we can then look at the subspace H^∞; it affords a smooth representation π^∞, and the scalar product on H induces one on H^∞, so that π^∞ is preunitary as a smooth representation. One can show that H^∞ is dense in H, so that H identifies with the completion of H^∞ with respect to its scalar product.

A Hilbert space representation (π, H) as above is said to be topologically irreducible if H is non-zero and any non-zero closed G-invariant subspace of

H is H itself.

THEOREM (Harish-Chandra, Bernstein) : *Let G be a reductive group over F. If (π, H) is a topologically irreducible Hilbert space representation of G, then the smooth representation (π^∞, H^∞) is irreducible.*

In the reverse direction, if (π, V) is a smooth irreducible preunitary representation of G, then the completion $\hat{V}$ of V with respect to its scalar product is topologically irreducible, and $\hat{V}^\infty$ identifies with V as a subspace of $\hat{V}$. As a corollary, we see that $H \mapsto H^\infty$ yields a bijection between isomorphism clases of unitary topologically irreducible representations of G and isomorphism classes of preunitary smooth irreducible representations of G. (Most authors say unitary instead of preunitary for a smooth representation of G).

1.6 Hecke algebras I

Let G be a locally profinite group, and J a compact open subgroup of G. Recall that sending V to V^J gives an exact functor from smooth representations of G to complex vector spaces. The action of G by left translations on $\mathbb{C}[G/J]$ gives a smooth representation of G. If (π, V) is a smooth representation of G, then V^J identifies with $\mathrm{Hom}_J(\mathbb{C}, V)$. By the universal property of tensor products, that identifies in turn with $\mathrm{Hom}_G(\mathbb{C}[G] \otimes_{\mathbb{C}[J]} \mathbb{C}, V)$, and since sending $g \otimes a$ to $a(g \mathrm{mod} J)$ gives a G-isomorphism of $\mathbb{C}[G] \otimes_{\mathbb{C}[J]} \mathbb{C}$ with $\mathbb{C}[G/J]$, we see that V^J is isomorphic to $\mathrm{Hom}_G(\mathbb{C}[G/J], V)$. In particular, V^J is a right module over the algebra of G-endomorphisms of $\mathbb{C}[G/J]$. We write $H(G, J)$ for the opposite algebra, so that V^J is a left module over $H(G, J)$, which we call the Hecke algebra of G with respect to J. Taking V to be $\mathbb{C}[G/J]$ itself, we see that $H(C, J)$, seen as $\mathbb{C}[G/J]^J$, is the subspace $\mathbb{C}[J \backslash G/J]$ of linear combinations of double cosets of J in G.

Exercise: Write the corresponding algebra structure on $\mathbb{C}[J \backslash G/J]$.

In the other direction, if M is any left $H(G, J)$-module, we can form the tensor product $\mathbb{C}[G/J] \otimes M$ over $H(G, J)$, and we get a smooth representation of G; that process is functorial. If we start with a smooth representation π of G on V and we take $M = V^J$, then we have a natural map from the tensor product $\mathbb{C}[G/J] \otimes M$ into V, sending $g \mathrm{mod} J \otimes v$ to $\pi(g)v$; its image is the G-subspace of V generated by V^J. In the other direction, starting with a $H(G, J)$-module M, the space of J-fixed vectors in $\mathbb{C}[G/J] \otimes M$ is , because of the exactness of $V \mapsto V^J$, simply $1_J \otimes M$ again, isomorphic to M as a $H(G, J)$-module.

Exercise: Write the adjointness property of those two functors.

Assume that (π, V) is a smooth irreducible representation of G such that V^J is not zero, and let M be any non-zero submodule of the $H(G,J)$-module V^J. Then the natural image X of $\mathbb{C}[G/J] \otimes M$ in V is non- zero, hence to be V itself since π is irreducible; on the other hand X^J is M so that M has to be all of V^J, and the $H(G,J)$-module V^J is simple.

Conversely let M be a simple $H(G,J)$-module; form $V = \mathbb{C}[G/J] \otimes M$ as above and let U be the union of all G-invariant subspaces Y of V such that $Y^J = 0$; then U is the maximal such subspace. If T is a G-invariant subspace of V strictly containing U, then T^J is non-zero and consequently fills out all of the J-fixed point subspace $1_J \otimes M$ of V; since that subspace obviously generates V as a representation of G, we see that $T = V$. It follows that the G-module V/U is irreducible, and $(V/U)^J$ is isomorphic to M as a $H(G,J)$-module. That way we get a canonical bijection between isomorphism classes of smooth irreducible representations of G with non-zero J-fixed vectors, and isomorphism classes of simple $H(G,J)$-modules.

Remark: Let G be a reductive group over F; then by 1.4 theorem all smooth irreducible representations of G are admissible, so that all simple modules over $H(G,J)$ are finite dimensional over $\mathbb{C}$. One can show actually that, when J is fixed, the dimension of simple $H(G,J)$-modules is uniformly bounded.

Example: Let $G = GL(N,F)$ and $J = GL(N,O_F)$. A smooth irreducible representation (π, V) of G is called unramified if V^J is non-zero; such representations are classified by the simple $H(G,J)$-modules V^J. But here the Hecke algebra $H(G,J)$ is commutative, isomorphic to the algebra of elements in $\mathbb{C}[X_1, X_1^{-1}, X_2, X_2^{-1}, \ldots, X_N, X_N^{-1}]$ invariant under the action of the symmetric group permuting the variables. All its simple modules are one-dimensional, parameterized by N-tuples of non-zero complex numbers up to ordering, or, which amounts to the same, by conjugacy classes of semisimple matrices in $\mathrm{GL}(N, \mathbb{C})$. We shall generalize that example below, and give some more explanation.

Let us return to the situation where G is a general locally profinite group, and J a compact open subgroup of G. We say that J is "typical" (some say "special") if whenever a smooth representation of G is generated by its J-fixed vectors, the same is true for any subrepresentation. If J is typical then sending V to V^J gives an equivalence of the category of smooth representations of G generated by their J-fixed vectors, to the category of

$H(G,J)$-modules, an inverse equivalence being given by sending a $H(G,J)$-module M to $\mathbb{C}[G/J]\otimes M$ as above. That is far stronger than dealing only with the irreducible smooth representations, but it requires J to be typical. For $G = GL(N,F)$, $GL(N,O_F)$ is not typical (if $N > 1$) whereas $1_N + P_F^k M(N,O_F)$ is typical for $k \geqslant 1$. For a general reductive group over F it is known that every neighbourhood of the identity contains a typical subgroup, but their Hecke algebras are usually very difficult to describe and study explicitly.

2 Parabolic induction and cuspidality

2.1 In this chapter G will be a reductive group over F. It is not so easy to construct smooth irreducible representations of G. Let us give two contrasting examples (however, they are in fact closely related, see below 4.4).

The first case is when you take a division algebra D with centre F and finite dimension N^2 over F, and let $G = D^\times$. Then $G/Z(G)$ is compact, and all smooth irreducible representations of G are finite-dimensional; however they are not easy to construct explicitly. The second case is when $G = GL(N,F)$; then by the exercise in 1.5 irreducible smooth representations of G are given by characters or are infinite dimensional, but as we shall see below we have constructions to give some of those infinite-dimensional ones.

For a general group G, there are always, of course, the characters. Of some importance are the so-called unramified characters, which we define now. Let $X(G)$ be the group of algebraic group morphisms of $\underline{G}$ to G_m, defined over F. It is a free abelian group of finite rank. For each χ in $X(G)$, its absolute value $|\chi|_F$ is a character of G, and we let G^1 be the intersection of the kernels of all such characters. It is an open subgroup of G, and it can be shown that it is the subgroup of G generated by all compact subgroups: in any case it clearly contains any compact subgroup of G. The unramified characters of G are those trivial on G^1. For $G = \mathrm{GL}(N,F)$ they are of the form $g \mapsto |\det(g)|_F^s$ for some complex number s. In general G/G^1 can be identified with the dual $Y(G) = \mathrm{Hom}(X(G),\mathbb{Z})$ of $X(G)$, and the group $X_{nr}(G)$ of unramified characters of G with $X(G)\otimes\mathbb{C}^\times$, with $\chi\otimes\lambda$ giving the character $g \mapsto \lambda^{-v_F(\chi(g))}$. In particular $X_{nr}(G)$ can be seen as an affine complex algebraic variety.

One way to construct smooth irreducible representations of G is via parabolic induction, from representations of (Levi subgroups of) parabolic subgroups of G. Parabolic subgroups of G are algebraic subgroups P of G, defined over F (we take the liberty to identify such groups with the group of F-points), such that the quotient G/P is a complete variety; in particular, in the natural topology, G/P is compact. For $G = \mathrm{GL}(N, F)$, they are stabilizers of flags in the vector-space F^N, maximal (proper) parabolic subgroups being the stabilizers of non-zero proper subspaces of F^N. If G is a classical group attached to some form (quadratic, symplectic, hermitian), then maximal (proper) parabolic subgroups are the stabilizers of non-zero totally isotropic subspaces.

It will be necessary to have more notation concerning the parabolic subgroups of G. There is a minimal parabolic P_0 subgroup of G, and it contains a maximal split torus A_0 of G. The group G acts transitively on such pairs (P_0, A_0). (For $G = \mathrm{GL}(N, F)$ you can take the subgroup of upper triangular matrices, and the subgroup of diagonal matrices). The centralizer M_0 of A_0 in G is a reductive group, and if N_0 is the unipotent radical of P_0 (its maximal normal unipotent subgroup), then P_0 is the semi-direct product of M_0 and N_0. In the natural topology, M_0/A_0 is compact, so $M_0/Z(M_0)$ is compact too.

Once a pair (P_0, A_0) has been fixed, the parabolic subgroups of G containing P_0 are called standard. Such a subgroup is of the form $P = MN_0$, where M is the centralizer of a subtorus A of A_0, and P is the semi-direct product of M (its Levi component) and its unipotent radical N, which is contained in N_0. A general parabolic subgroup of G is a conjugate of a standard one, and by conjugation, it also has a Levi decomposition $P = MN$.

2.2 Let P be a parabolic subgroup of G, and $P = MN$ a Levi decomposition. Let (ρ, W) be a smooth representation of M; it can be seen as a representation of P trivial on N, and we can form the space of functions f from G to W such that $f(hg) = \rho(h)f(g)$ for h in P and g in G. The group G acts on that space by left translations, and the space $\mathrm{ind}(P, G, W)$ of smooth vectors in it gives a smooth representation $\mathrm{ind}(P, G, \rho)$ of G which is said to be parabolically induced from ρ. We get a functor from $S(M)$ to $S(G)$ which is exact. It is easy to see that if ρ is admissible then so is $\mathrm{ind}(P, G, \rho)$ (exercise: describe the space of J-fixed points in $\mathrm{ind}(P, G, W)$ for a compact open subgroup J of G).

THEOREM: 1) *If ρ has finite length, then* $\mathrm{ind}(P, G, \rho)$ *has finite length. If ρ is finitely generated, then* $\mathrm{ind}(P, G, \rho)$ *is finitely generated.*

2) *Assume that ρ is irreducible. There is a non-empty Zariski-open subset U of the group of unramified characters of M such that* $\mathrm{ind}(P, G, \rho \otimes \chi)$ *is irreducible for χ in U.*

In the reverse direction, let (π, V) be a smooth representation of G and let $V_N = V/V(N)$ be the maximal quotient space of V on which N acts trivially. Then P still acts on V_N through its Levi quotient M, and the resulting representation (π_N, V_N) is a smooth representation of M. We get that way a functor from $S(G)$ to $S(M)$ called the Jacquet functor (with respect to $P = MN$); it is an exact functor, and takes finitely generated representations to finitely generated representations (use that G/P is compact).

THEOREM : *If (π, V) has finite length, then so has (π_N, V_N); if (π, V) is admissible, so is (π_N, V_N).*

Note that π_N is not in general irreducible when π is. Let us give the example of $G = \mathrm{GL}(2, F)$. The only choice for P distinct from G, up to conjugation, is the Borel subgroup B of upper triangular matrices, and its Levi subgroup T of diagonal matrices. A smooth irreducible representation of T is given by a character, that is a pair $\chi = (\chi_1, \chi_2)$ of characters of $F^\times$. The irreducibility of $\pi = \mathrm{ind}(B, G, \chi)$ depends only on the characters χ_1, χ_2 of $F^\times$, and the Jacquet functor of π always has exactly two irreducible subquotients, the character χ as a quotient, and the character $(\chi_2| \,|_F, \chi_1| \,|_F)$ as subrepresentation. Reducibility occurs only in two cases: the first one is when $\chi_1 = \chi_2$, in which case π has the character $\chi_1 \circ \det$ as a subrepresentation, and the quotient is irreducible (that quotient is written St and called the Steinberg representation of G when χ is trivial; the quotient in general is isomorphic to $\chi_1 \circ \det \otimes St$)). The other case of reducibility occurs when $\chi_1, \chi_2 = | \,|_F^2$; then there is one infinite-dimensional subrepresentation isomorphic to $\chi_1 \otimes | \,|_F^{-1} \circ \det \otimes St$ and, as a quotient, the character $\chi_1| \,|_F^{-1} \circ \det$.

The parabolic induction and Jacquet functors are related by Frobenius reciprocity, which says that if (π, V) is a smooth representation of G and (ρ, W) is a smooth representation of M, then there is a canonical functorial isomorphism of $\mathrm{Hom}_G(\pi, \mathrm{ind}(P, G, \rho)$ to $\mathrm{Hom}_M(\pi_N, \rho)$: if ϕ is an M-morphism of V_N into W, we associate to it the G-morphism $v \mapsto f_v$ of V to $\mathrm{ind}(P, G, W)$, where for g in G $f_v(g)$ is the image under φ of the class of $\pi(g)v$ in V_N (exercise: write the reciprocal map).

We say that a smooth representation (π, V) of G is cuspidal if $V_N = 0$ for all proper parabolic subgroups $P = MN$ of G (because of some obvious transitivity property of parabolic induction and Jacquet functors, the maximal proper parabolic subgroups are enough). If π is irreducible, it means that π is not a subrepresentation of any parabolically induced representation $ind(P, G, \rho)$ where P is proper and ρ is irreducible (use that π_N is finitely generated). It is also equivalent to π not being a subquotient of such a parabolically induced representation, but that is slightly harder to prove. Again by transitivity there is a parabolic subgroup P of G and a cuspidal irreducible smooth representation ρ of the Levi quotient of P such that π (assumed irreducible) is a subrepresentation of $\mathrm{ind}(P, G\, rho)$; of course $P = G$ when ρ is cuspidal. There are some uniqueness properties but they are better expressed after some normalization, see below 2.6.

2.3 Coefficients.

Cuspidality is tied up with properties of the coefficients. If (π, V) is a smooth representation of G, a coefficient of π is a function on G of the form $g \mapsto \lambda(\pi(g)v)$, where v is in V and λ in $V^\vee$; more generally linear combinations of such functions are also called coefficients of π. They should really be thought of as "matrix coefficients" of π, which, we recall, is usually infinite-dimensional. Coefficients are locally constant functions on G, and their asymptotics as g goes "to infinity" reveals much about the representation.

Theorem : *A smooth representation of G is cuspidal if and only if all its coefficients are compactly supported* mod. $Z(G)$.

Before sketching the proof of that theorem let us indicate how it implies that a smooth irreducible representation (π, V) of G is admissible. Because parabolic induction preserves admissibility, it is enough to do it when π is cuspidal. So let J be a compact open subgroup of G and v a non-zero vector in V. As v obviously generates V as a representation of G, the vectors $e_J\pi(g)v$, g in G, generate V^J. We can extract a basis $(e_J\pi(g_i)v)_{i\in I}$ out of that generating set. Define the linear form λ on V to be trivial on $V(J)$ and have value 1 on each v_i, i in I ; then λ is in $V^{\vee J}$ and the coefficient $f(g) = \lambda(\pi(g)v)$ is non-zero on each g_i, i in I. But f is invariant under J and π has a central character, so the support of f contains $Z(G)Jg_i$ for each i in I; as those sets are disjoint and open in G, and the support of f is compact mod. $Z(G)$, it follows that I is finite.

We now sketch the proof of the implication in the theorem which we have just used, the reverse implication is based on similar principles. We first deal with the case where $G = \mathrm{GL}(n, F)$, and then indicate how to modify the proof for the general case.

Let (π, V) be a smooth cuspidal representation of $\mathrm{GL}(n, F)$. Let v be in V, λ in $V^\vee$, and form the coefficient $f : g \mapsto \lambda(\pi(g)v)$. By the Cartan decomposition, it is enough to prove that if $d = \mathrm{diag}(\pi_F^{a_1}, \dots, \pi_F^{a_n})$ (with decreasing integers a_i, and one of the $a_i - a_{i+1}$, $1 \leqslant i \leqslant n-1$, is large enough, then f vanishes on KdK, for $K = \mathrm{GL}(n, O_F)$. As f has only finitely many left and right translates under K, it is enough to show that f vanishes on d (if some $a_i - a_{i+1}$ is large enough). Fix i, $1 \leqslant i \leqslant n-1$, and let P be the block upper triangular subgroup with diagonal blocks of size i and $n-i$; let M be its block diagonal Levi subgroup, and N its unipotent radical, made out of the matrices with upper right corner x in $M(i(n-i), F)$. See N as the union of its compact subgroups N_k, k in $\mathbb{Z}$, where N_k consists of such matrices with all coefficients of x in P_F^k ; write e_k for e_{N_k}. There is certainly an index k such that $e_k v = 0$ (go back to the definition of $V(N)$ which is V itself since π is cuspidal). There is also an index l such that N_l fixes λ. If $a_i - a_{i+1}$ is at least $l - k$ then $dN_k d^{-1}$ is included in N_l and we compute $\lambda(\pi(d)v) = \lambda(e_l \pi(d)v) = \lambda(\pi(d)e_{N'}v)$ where $N' = d^{-1}Nd$; since N' contains N_k we have $e_{N'}v = 0$ so $f(d) = 0$.

The case of a general group G requires more notation. Let us retain the notation of 2.1 above. The maximal split torus A_0 acts on the Lie algebra of N_0 via "roots", i.e. algebraic group morphisms from A_0 to $\mathbb{G}_m$. Then there is a subgroup Γ of G, open and compact mod.$Z(G)$, such that $G = \Gamma A_0 + \Gamma$ where A_0^+ is made out of elements a in A_0 with $|\alpha(a)|_F \leqslant 1$ for each root α of A_0 in N_0. The roots which are not product of two other roots will be called simple. If ϵ is fixed, $0 < \epsilon \leqslant 1$, then the set of a in A_0^+ with $|\alpha(a)|_F \leqslant \epsilon$ for all simple roots α is compact modulo $Z(G)$. A simple root α corresponds to a standard maximal proper parabolic subgroup $P_\alpha = M_\alpha N_0$, where M_α is the centralizer of $\mathrm{Ker}(\alpha)$, and the roots of A_0 in the unipotent radical N_α of P_α are sums of α and other roots. We can exhaust N_α by a sequence of compact open subgroups $N_{\alpha,k}$,k in $\mathbb{Z}$, which $N_{\alpha,k}$ shrinking to 1 as k tends to infinity. Fixing integers k and l, we have $aN_{\alpha,k}a^1$ included in $N_{\alpha,\ell}$ when a in A_0 is such that $|\alpha(a)|_F$ is small enough. Then the proof goes as before.

2.4 Construction of cuspidal representations

The only general way to construct smooth irreducible cuspidal representations of G is via compact induction, as we explain now.

Let J be an open subgroup of G, containing $Z(G)$ and compact modulo $Z(G)$. Let (ρ, W) be a smooth irreducible representation of J (then W is finite-dimensional). Form the space $\mathbb{C}[G] \otimes_{\mathbb{C}[J]} W$; it carries a smooth representation of G, called the representation compactly induced from ρ and written $\mathrm{ind}(J, G, \rho)$. For g in G let ρ^g be the representation of $g^{-1}Jg$ given by $\rho^g(x) = \rho(gxg^{-1})$.

Theorem : *Assume that for each g in G but not in J, there is no non-zero $J \cap g^{-1}Jg$ morphism from ρ to ρ^g. Then* $\mathrm{ind}(J, G, \rho)$ *is irreducible and cuspidal.*

Note that the condition on g in the theorem only depends on the double class JgJ.

Example. Let $G = \mathrm{GL}(n, F)$, $J = \mathrm{GL}(n, O_F)$. Assume that ρ is trivial on matrices congruent to 1_n mod. P_F ; then the restriction of ρ to $K = \mathrm{GL}(n, O_F)$ comes via inflation from a representation of the quotient group $\mathrm{GL}(n, k_F)$. Assume that representation is cuspidal i.e., for any integer i between 1 and $n-1$, it has no non-zero fixed vector under the group of upper triangular unipotent block matrices with diagonal blocks of size i and $n-i$. Then the theorem applies.

For $G = \mathrm{GL}(n, F)$ or $G = \mathrm{SL}(n, F)$, Bushnell and Kutzko have given a list of pairs (J, ρ) as in the theorem - including the example just given - such that every smooth irreducible representation of G is of the form $\mathrm{ind}(J, G, \rho)$ for some pair (J, ρ) in the list. That is also true for interior forms of $\mathrm{GL}(n, F)$ (Sécherre and Stevens), for classical groups if p is odd (Stevens), and for any group G provided p is large enough with respect to G (Yu, Kim).

2.5 Normalization of parabolic induction

Parabolic induction has some good properties with respect to unitarity and taking contragredients, but they are better expressed with a normalization, which we now turn to.

Let $P = MN$ be a parabolic subgroup of G. The group M acts by conjugation on N and also on its Lie algebra $\mathrm{Lie}(N)$, and for m in M we let $\delta_P(m)$ be the absolute value of the determinant of the action of m on $\mathrm{Lie}(N)$. Then δ_P is an unramified character of M and on $\mathrm{ind}(P, G, \delta_P)$ there is a non-zero invariant functional, unique up to scalars; we can take it to take non-negative values on non-negative functions, and then it is unique up to multiplication by a positive real number. Choose such a functional λ_P. If (ρ, W) is a smooth representation of M define $i_P^G\rho$ (the normalized parabolic induction of ρ) to be $\mathrm{ind}(P, G, \delta_P^{1/2} \otimes \rho)$. Similarly if (π, V) is a smooth representation of G, define the normalized Jacquet functor $r_P^G\pi$ to be $\delta_P^{-1/2} \otimes \pi_N$. Then we still have Frobenius reciprocity for the functors i_P^G and r_P^G.

If (ρ, W) is a smooth representation of M, we have a pairing between $i_P^G\rho$ and $i_P^G\rho^\vee$ given as follows. Let f be in the space of $i_P^G\rho$ and ϕ in the space of $i_P^G\rho^\vee$; then the function $g \mapsto \phi(g)(f(g))$ is a function in $\mathrm{ind}(P, G, \delta_P)$ and we can apply our chosen functional λ_P to it. One verifies that the pairing thus obtained gives an isomorphism of $i_P^G\rho^\vee$ onto the contragredient of $i_P^G\rho$. Similarly, if ρ is preunitary and $<,>_M$ is an M-invariant scalar product on W then for f, ϕ in the space of $i_P^G\rho$, the function $g \mapsto < \phi(g), f(g) >_M$ is in $\mathrm{ind}(P, G, \delta_P)$ and applying λ_P gives a G-invariant scalar product on $i_P^G\rho$, which is consequently preunitary.

Beware that if (π, V) is a preunitary smooth representation of G, π_N is not in general preunitary, neither is $r_P^G\rho$. That is already seen in the example of $G = \mathrm{GL}(2, F)$, which we now restate in normalized terms.

Let $\chi = (\chi_1, \chi_2)$ be a character of the diagonal subgroup T. Then the Jacquet functor $r_B^G i_B^G\rho$ has only two irreducible subquotients : χ as a quotient and $\chi^w = (\chi_1, \chi_2)$ as a subrepresentation (the extension splits if and only if χ_1 and χ_2 are distinct). Of course $i_B^G\chi$ is preunitary whenever χ is - note also that the contragredient of $i_B^G\chi$ is isomorphic to $i_B^G\chi^{-1}$ -, but there are other cases of unitarity, when $\chi_1 = \eta| \ |_F^s$, $\chi_2 = \eta| \ |_F^s$, with η unitary and $0 < |s| < 1/2$; in those cases $r_B^G i_B^G\chi$ is not preunitary. Note that the cases where $|s| = 1/2$ correspond to reducibility situations : for $s = 1/2$, $i_B^G\chi$ has $\eta \circ \det$ as quotient and $\eta \circ \det \otimes St$ as a subrepresentation, and vice-versa for $s = -1/2$.

Note : The only isomorphisms between the irreducible smooth representations of $\mathrm{GL}(2, F)$ just described are between $i_B^G\chi$ and $i_P^G\chi^w$, the isomorphism actually coming, via Frobenius reciprocity, from the fact that $r_B^G i_B^G\chi$ is then the direct sum of χ and χ^w.

Nevertheless the normalized Jacquet functor possesses some very nice property with respect to taking contragredients, due to Casselman for admissible smooth representations, and to Bernstein in general. With notation as in 2.1 fix a standard parabolic subgroup $P = MN$; then there is a unique parabolic subgroup $\bar{P}$ such that $\bar{P} \cap P = M$; it is called **opposite** to P. Moreover there is a subset I of the set of simple roots such that M is the centralizer in G of the intersections of the kernels of the roots α in I.

Let (π, V) be a smooth representation of G, and use $<,>$ for the natural pairing between $V^\vee$ and V. Then we have a functorial isomorphism ϕ of $r^G_{\bar{P}}\pi^\vee$ onto $(r^G_P\pi)^\vee$; equivalently there a natural pairing $<,>$ P between $r^G_{\bar{P}}\pi^\vee$ and $r^G_P\pi$ which realizes ϕ, and is given as follows: let v be in V, and write $\bar{v}$ for its image in V_N ; let λ be in $V^\vee$, and let $\bar{\lambda}$ be its image in $V^\vee_N$; then we have $< \lambda, \pi(a)v >=< \bar{\lambda}, r^G_P\pi(a)\bar{v} >_P$ for all a in A_0 such that $|\alpha(a)|_F$ is small enough for all roots α in I.

Tied up with those considerations is Bernstein's "second adjunction" theorem :

THEOREM : *For (ρ, W) a smooth representation of M and (π, V) a smooth representation of G, there is a functorial isomorphism of* $\mathrm{Hom}_G(i^G_P\rho, \pi)$ onto $\mathrm{Hom}_M(\rho, r^G_{\bar{P}}\pi)$.

See below 2.11 for the origin of such a functorial isomorphism. (Exercise : deduce from the theorem the preceding result about the contragredient of the Jacquet functor).

2.6 Let us introduce a convenient tool, the Grothendieck group $R(G)$ of finite length smooth representations of G. It can be seen as the free $\mathbb{Z}$-module on the set $A(G)$ of isomorphism classes of smooth irreducible smooth representations of G. A finite length smooth representation π of G has a class $[\pi]$ in $R(G)$, sometimes called the semisimplification of π: the coefficient of an element σ of $A(G)$ in $[\pi]$ is the number of quotients in a Jordan-Hölder series for π which are isomorphic to σ. The support of π or $[\pi]$ is the set of such σ with non-zero coefficient in $[\pi]$.

THEOREM : *Let $P = MN$ and $P' = M'N'$ be parabolic subgroups of G. Let (ρ, W) be a smooth irreducible cuspidal representation of M, and (ρ', W') a smooth irreducible cuspidal representation of M'. Then the following are equivalent :*

(i) *The supports of $i^G_P\rho$ and $i^G_{P'}\rho'$ intersect.*

(ii) $i^G_P\rho$ and $i^G_{P'}\rho'$ *have the same image in $R(G)$.*

(iii) *There is an element g in G such that $M' = gMg^{-1}$ and ρ' is isomorphic to ρ^g.*

We say that the pairs (M, ρ) and (M', ρ') are **equivalent** if condition (iii) of the theorem is satisfied. If π is an irreducible smooth representation of G, then there is a pair (M, ρ) as in the theorem, unique up to equivalence, such that π is in the support of $i_P^G \rho$ for any parabolic subgroup P with Levi subgroup M; in fact it can be proved that given M it is always possible to choose the parabolic subgroup P with Levi subgroup M so that π is a subrepresentation of $i_P^G \rho$. The equivalence class of the pair (M, ρ) is called the **supercuspidal suppor**t of π. If s is such an equivalence class, we write $S_s(G)$ for the full subcategory of $S(G)$ made out of finite length smooth representations of G with all irreducible subquotients having supercuspidal support s. Then the category $S_{fl}(G)$ of finite length smooth representations of G is the direct product, over all equivalence classes s, of the subcategories $S_s(G)$.

More general, but more difficult to prove, is the following decomposition of the entire category $S(G)$, due to Bernstein. Say that two pairs (M, ρ) and (M', ρ') as above are **inertially equivalent** if there is an element g in G and an unramified character χ of M such that the pairs $(M, \chi \otimes \rho)$ and (M', ρ') are equivalent. If s is the inertial equivalence of (M, ρ), write $S(s, G)$ for the full subcategory of $S(G)$ made out of those smooth representations of G with all their irreducible subquotients having supercuspidal support $(M, \chi \otimes \rho)$ for some unramified character χ of M. Then the category $S(G)$ is the direct product of its subcategories $S(s, G)$, where s runs through inertial equivalence classes.

2.7 Unramified principal series

Let us examine a particular case of parabolic induction. Let us start with a minimal parabolic P_0 with Levi subgroup M_0. If xhi is an unramified character of M_0 then $i_{P_0}^G \chi$ is not always irreducible but generically so.

It can be shown that there is a particular open compact subgroup I of G such that the following are equivalent, for a smooth irreducible representation (π, V) of g :

(i) V^I is non-zero.

(ii) The supercuspidal support of π is (the class of) (M_0, χ), for some unramified character χ of M_0.

Note that we can replace I by any conjugate in G in condition (i), and that χ in condition (ii) is well-defined up to the action of the normalizer of M_0 in G. For $G = \mathrm{GL}(n, F)$, we can take I to be the Iwahori subgroup, made out of matrices in $K = \mathrm{GL}(n, O_F)$ which are upper triangular modulo P_F.

That relation between unramified principal series $i_{P_0}^G \chi$ and fixed points under compact open subgroups of G goes even further. There is a choice of maximal compact subgroup K of G, called special, such that the following holds for such a special group K :

(i) For each unramified character χ of M_0, the representation $i_{P_0}^G \chi$ has a unique irreducible subquotient $V(\chi)$ with non-zero vectors fixed by K.

(ii) The map $\chi \mapsto V(\chi)$ induces a surjective map from $X_{nr}(M_0)$ onto the set of isomorphism classes of irreducible smooth representations of G with non-zero vectors fixed under K.

(iii) $V(\chi)$ and $V(\chi')$ are equal if and only if χ and χ' are conjugate under the action of the normalizer $N_G(M_0)$ of M_0 in $G*$.

It is not so easy to give a full definition of a special maximal compact subgrou K. For $G = \mathrm{GL}(n, F)$ they are the conjugates of $\mathrm{GL}(n, O_F)$. Relevant properties are that $K \cap M_0$ is the (unique) maximal compact subgroup M_0^1 of M_0, that $G = KP_0$ (Iwasawa decomposition), and that K contains a set of representatives for $N_G(M_0)/M_0$. In general, the group G will have several conjugacy classes of maximal compact subgroups (investigate the cases of $\mathrm{SL}(2, F)$ and $\mathrm{PGL}(2, F)$), and several of those conjugacy classes might consist of special subgroups : all for $\mathrm{SL}(2, F)$ and only the conjugacy class of $\mathrm{PGL}(2, O_F)$ for $\mathrm{PGL}(2, F)$. If the group G is unramified, i.e. its minimal parabolic subgroup M_0 is a torus which splits over an unramified extension of F, then the maximal compact subgroups of maximal volume are special, indeed they are called hyperspecial.

Note: Let $\underline{G}$ be a reductive algebraic group defined over a number field E. View $\underline{G}$ as a closed subgroup of some $\mathrm{GL}(n)$; for a finite place v of E outside a finite set S, the group $\underline{G}$ over the completion E_v is unramified, and $K_v = \underline{G}(E_{v)} \cap \mathrm{GL}(n, O_{E_v})$ is a hyperspecial maximal compact subgroup of $\underline{G}(E_v)$. The same result is valid for a global function field E.

Interlude: We have just mentioned the volume of compact open subgroups of G. If J and J' are two such subgroups of G, then $J \cap J'$ is of finite index in both J and J', and we can say that J has bigger volume than J' if the corresponding index is bigger for J. Another possibility is to introduce an actual measure of the volumes. Let $C_c^\infty(G)$ be the space of locally constant functions from G to $\mathbb{C}$ with compact support. It is easily seen that there is a non-zero linear functional on that space which is invariant under the left translations by elements of G, and that such a functional is unique up to scalar ; it can be chosen to take non-negative values on functions with non-negative values, and then it is unique up to multiplication by a positive real number. Such a functional is called a (left) Haar measure on G ; as G is reductive it can be proved that it is also invariant under the action of elements of G by right translations, so is also a right Haar measure. If X is an open and compact subset of G then its characteristic function is in the space $C_c^\infty(G)$, and its Haar measure is the volume of X. If J is a compact open subgroup of G and J'' an open subgroup of J, then the volume of J is $card(J/J'')$ times the volume of J'' (by G-invariance), so having bigger volume has the same meaning as before.

2.8 The Satake isomorphism

We keep the situation and notation of 2.7, and explain the classification of unramified principal series in a different way.

Recall that unramified characters of M_0 correspond to isomorphism classes of simple modules over the Hecke algebra $H(M_0, M_0^1)$ - which in this case is the group algebra of M_0, M_0^1 with complex coefficients. Note that the normalizer $N_G(M_0)$ of M_0 in G acts on $H(M_0, M_0^1)$, and on its simple modules, via its quotient W by M_0.

On the other hand, smooth irreducible representations of G with non-zero K-fixed vectors correspond to simple modules over the Hecke algebra $H(G, K)$. The map $\chi \mapsto V(\chi)$ of 2.7 is in fact implemented via an algebra homomorphism S from $H(G, K)$ to $H(M_0, M_0^1)$, as we now explain.

Let f be an element of $H(G, K)$; as it is a linear combination of double classes in $K \backslash G / K$, it can be seen as a function on G, bi-invariant under K and with compact support. For a fixed m in M_0, the function $f_m : n \mapsto f(mn)$ on N_0 is locally constant with compact support. Choose on $C_c^\infty(N_0)$ the Haar measure lambda giving volume 1 to $N_0 \cap K$. Then we define a function Sf on M_0 by $Sf(m) = \delta_P(m)^{1/2} \lambda(f_m)$. The main results, due to Satake, are :

(i) The map S gives and algebra isomorphism of $H(G,K)$ onto the algebra of W-invariant elements in $H(M_0, M_0^1)$.

(ii) Let χ be an unramified character of M_0. If we view $\mathbb{C}$ as a simple module over $H(M_0, M_0^1)$ via χ, then we obtain via the map S a simple module over $H(G,K)$, which is associated to $V(\chi)$.

In the case where $G = \mathrm{GL}(n,F)$, $K = \mathrm{GL}(n,O_F)$, with M_0 being the diagonal subgroup, $H(M_0, M_0^1)$ identifies with the algebra of finite type $\mathbb{C}[X_1X_1^{-1}, \ldots, X_n, X_n^{-1}]$, W is the symmetric group S_n permuting the diagonal entries and the variables X_i, and we recover the description of $H(G,K)$ given in 1.6.

2.9 Let us exploit further the parametrization of unramified principal series given by 2.8, which is important for global purposes. Indeed let E be a global field and $\underline{G}$ a reductive group over E. Let π be an automorphic cuspidal representation of $\underline{G}(\mathbb{A}_E)$; then π appears as a (generalized) tensor product, over the places v of E, of representations π_v, where for v finite π_v is a smooth irreducible representation of $G_v = \underline{G}(E_v)$. As mentioned above, outside a finite set of places of E, G_v has a natural hyperspecial maximal compact subgroup K_v ; it comes from the definition of automorphic representations that outside a possibly bigger set of places of E, π_v has a non-zero vector fixed by K_v, so is parametrized by an unramified character of the minimal parabolic subgroup.

Let us revert back to the local case, and our usual notation. Fix a special maximal compact subgroup K of G. As we have seen, irreducible smooth representations of G with non-zero K-fixed vectors are parametrized by $X_{nr}(M_0)$, up to the action of $W = N_G(M_0)/M_0$.

Assume first that $\underline{G}$ is split, i.e. M_0 is a split torus ; we can the introduce a complex reductive group $\hat{G}$ called the dual group of G, with a maximal (split) torus $\hat{T}$, in such a way that the group of characters $X(\hat{T})$ of $\hat{T}$ is identified with the group of cocharacters $Y(M_0)$ of M_0, the group of cocharacters $Y(\hat{T})$ of $\hat{T}$ with the group of characters $X(M_0)$ of M_0, and the Weyl group W with the group $N_{\hat{G}}(\hat{T})/\hat{T}$. Then $X_{nr}(M_0)$, seen as $X(M_0) \otimes \mathbb{C}^\times$, identifies with $\hat{T}(\mathbb{C}) = Y(\hat{T}) \otimes \mathbb{C}^\times$, with the action of W, so that the W-orbits in $X_{nr}(M_0)$ correspond to W orbits in $\hat{T}$, and those are the same as semisimple conjugacy classes in $\hat{G}$. So in this split case, smooth irreducible representations with non-zero K-fixed vectors are parametrized, up to isomorphism, by semisimple conjugacy classes in $\hat{G}$. For $G = \mathrm{GL}(n)$, $\mathrm{SL}(n)$, $\mathrm{PGL}(n)$, $\mathrm{SO}(2n)$, $\mathrm{SO}(2n+1)$, $\mathrm{Sp}(2n)$, we have $\hat{G} = \mathrm{GL}(n, \mathbb{C})$, $\mathrm{PGL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, $\mathrm{SO}(2n, \mathbb{C})$, $\mathrm{Sp}(2N, \mathbb{C})$, $\mathrm{SO}(2n + 1, \mathbb{C})$ respectively.

If G is not split, but is unramified, then M_0 is a torus split over the maximal unramified extension F_{nr} of F, and we can then still concoct a group $\hat{G}$, by using $\underline{G}$ over F_{nr}, and use the action of the Galois group of F_{nr}/F to construct a semidirect product LG of $\hat{G}$ with that Galois group, or rather its variant, the Weil group $W(F_{nr}/F)$ of F_{nr}/F, which is simply the free abelian group generated by the Frobenius automorphism. In that case unramified characters of M_0, up to the action of W, correspond to conjugacy classes of group morphisms from $W(F_{nr}/F)$ to LG, which induce identity when projecting to $W(F_{nr}/F)$ - of course such morphisms are characterized by the image of the Frobenius automorphims, so that in the split case we recover the facts in the preceding discussion. As an example, when G is a unitary group with respect to a Hermitian form on a vector of dimension n over a quadratic unramified extension E of F, the group $\hat{G}$ is $\mathrm{GL}(n, C)$ and $W(F_{nr}/F)$ acts on it through its quotient $\mathrm{Gal}(E/F)$, the non-trivial element acting by $g \mapsto A^t g^{-1} A^{-1}$, where A is the matrix with 1 on the antidiagonal and 0 elsewhere.

Remark: Let $G = \mathrm{SL}(2, F)$; its diagonal torus T has a unique order 2 unramified character ω, and the parabolically induced representation $i_B^G \omega$ (where B is the upper trianguar subgroup), is the direct sum of two irreducible preunitary representations. Those two representations are not isomorphic to one another, and they can be distinguished by their fixed points under (hyper)special maximal compact subgroups: there are two conjugacy classes of such, $K = SL(2, O_F)$ has non-zero fixed points in one of the components π_1, not in the other π_2, and conversely the other group K', conjugate (in $\mathrm{GL}(2, F)$) to K by the antidiagonal matrix with antidiagonal coefficients 1 and a uniformizer of F, has non-zero fixed points in π_2 but not in π_1. Both π_1 and π_2 are parametrized by the homomorphism of $W(F_{nr}/F)$ to $\mathrm{LG} = \mathrm{PGL}(2, \mathbb{C}) \times W(F_{nr}/F)$ which send the Frobenius automorphism to the element of LG with first component the image in $\mathrm{PGL}(2, \mathbb{C})$ of the diagonal matrix $\mathrm{diag}(1, -1)$: those two representations are said to be in the same L-packet.

2.10 To classify all irreducible smooth representations of a general reductive group G over F is more complicated. Fixing a separable algebraic closure

$\bar{F}$ of F, we can form the Weil group $W_F = W(\bar{F}/F)$ consisting of those elements in the Galois group acting on the residue field of $\bar{F}$ as an integral power of Frobenius. There is an L-group LG which is a semi-direct product of $\hat{G}$ with W_F (in the unramified case of 2.8, this is obtained by pulling back the L-group of 2.9 via the projection of $W(\bar{F}/F)$ onto $W(F_{nr}/F)$), and L-packets of (isomorphism classes of) irreducible smooth representations of $\hat{G}$ are parametrized by (certain) continuous group homomorphisms from $W_F \times \mathrm{SU}(2, \mathbb{C})$ to LG such that the composite homomorphism to W_F restricts to the identity on W_F, conjugate group homomorphisms parametrizing the same class of representation.

For $G = \mathrm{GL}(n, F)$ the L-group is simply the direct product of $\mathrm{GL}(n, \mathbb{C})$ and W_F, the L-packets are singletons, and the parametrization is a one-to-one correspondence between semisimple continuous homomorphisms from $W_F \times \mathrm{SU}(2)$ to $\mathrm{GL}(n, \mathbb{C})$, up to conjugation by $\mathrm{GL}(n, \mathbb{C})$ (or, which amounts to the same thing, continuous semisimple complex representations of $W_F \times \mathrm{SU}(2)$ of dimension n, up to isomorphism), and isomorphism classes of irreducible smooth representations of $\mathrm{GL}(n, F)$. The irreducible smooth cuspidal representations of $\mathrm{GL}(n, F)$ correspond to the irrreducible continuous representations of $W_F \times \mathrm{SU}(2)$ of dimension n which are trivial on $\mathrm{SU}(2)$.

For $G = \mathrm{GL}(1, F)$ the correspondence is between characters of W_F and characters of $G = F^\times$: that correspondence is given by class-field theory. For $G = \mathrm{GL}(n, F)$, $n > 1$, the correspondence is a theorem (due to M. Harris and R. Taylor), and there is a complete characterization via invariants called the L - and ϵ- factors - more about that below in chapter 4. For a general group G, the L-packets are not singletons in general, and the parametrization is still conjectural (a conjecture formulated by Langlands) in most cases, except, of course, that for unramified groups G and unramified principal series representations we have the correspondence in 2.8.

Remark: the appearance of $\mathrm{SU}(2, \mathbb{C})$ in the conjecture might be surprising. In fact that group occurs only via its continuous complex representations, the point being that for each positive integer there is an irreducible continuous representation of that dimension, and only one up to isomorphism. So the literature offers other versions, where for example $\mathrm{SU}(2)$ is replaced with $\mathrm{SL}(2, \mathbb{C})$, and we consider only representations which are analytic on $\mathrm{SL}(2, \mathbb{C})$.

Beware that switching from one model to another involves some normalization. A convenient version for $G = \mathrm{GL}(n, F)$ is that we are looking at pairs (σ, N), where σ is a continuous semisimple representation of W_F over a complex vector space, say V, of dimension n, and N is a nilpotent endomorphism of V such that for g in W_F projecting to a Frobenius endomorphism in $W(F_{nr}/F)$ we have $\sigma(g)N\sigma(g^{-1}) = q_F N$. Such pairs are taken only up to isomorphisms, with the obvious notion of isomorphisms.

For $G = \mathrm{GL}(2, F)$ for example, if $\chi = (\chi_1, \chi_2)$ is a character of the diagonal subgroup T, the parabolically induced representation $i_B^G \chi$, if irreducible, corresponds to the pair $(\sigma, 0)$, where σ is the direct sum of the two characters of W_F corresponding to χ_1 and χ_2 via class field theory. In the reducible cases, the nilpotent part of the pair is trivial for characters of $\mathrm{GL}(2, F)$, non-trivial for the infinite-dimensional component : for example the trivial character of $\mathrm{GL}(2, F)$ corresponds to $(\sigma_1, 0)$, where σ_1 is the direct sum of the two characters of W_F corresponding to $|\ |_F^{-1/2}$ and $|\ |_F^{1/2}$, whereas the Steinberg representation St corresponds to the pair (σ_1, N), with N a non trivial nilpotent.

The version with such pairs (σ, N) is in fact more natural. It stems from the fact that the natural representations of W_F are not complex representations, but representations on vector spaces over l-adic fields, coming from etale cohomology. When l is different from p at least, it is possible to translate such l-adic representations into complex ones, at the cost of introducing the nilpotent operator N (see for example the corresponding chapter in reference (2)).

2.11 The geometric lemma

To conclude this long chapter, let us explain an important argument in the proof of the Theorem in 2.6, which will crop up again in chapter 3. Let us revert to the situation of 2.6. Investigating the parabolically induced representations $i_P^G \rho$ means in particular knowing when two such representations are isomorphic, so that we want to understand $\mathrm{Hom}_G(i_{P'}^G \rho', i_P^G \rho)$, which by Frobenius reciprocity is $\mathrm{Hom}_M(r_P^G i_{P'}^G \rho', \rho)$. In that case the "geometric lemma" gives a natural filtration of $r_{P'}^G i_P^G \rho$, for any smooth representation ρ of M, and identifies the successive quotients of the filtration.

The first step is to restrict to P the representation $i_{P'}^G \rho$; clearly we are lead to considering double cosets $P'gP$. Such cosets are finite in number, and one can choose an ordering $X(1), \ldots, X(r)$ such that the union of $X(1), \ldots, X(i)$ is open in the union of $X(1), \ldots, X(i+1)$, for $i = 1, \ldots, r-1$. For example for $G = \mathrm{GL}(2, F)$, and $P = B$ the upper triangular subgroup there are only two double cosets $X(1) = BwB$ and $X(2) = B$ with $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The space V of $i_{P'}^{G}\rho$ can be filtered by P-invariant subspaces $V(i)$ where $V(i)$ is made out of the functions in V with support in the union of $X(1), \ldots, X(i)$. The quotient space $V(i)/V(i+1)$ (for $i = 1, \ldots, r$), putting $V(0) = 0$) can be identified with a space $W(i)$ of functions on $X(i)$ with values in the space Y of ρ', with the obvious action of P. Write $X(i) = P'gP$ for some g in G, and put $Q = g^{-1}P'g \cap P$; defining $\sigma(q) = \delta_P^{-1/2}\rho(gqg^{-1})$ for q in Q, we can identify $W(i)$ with the space of functions f from P to Y, tranforming along σ under left translations by elements of Q, and with compact support modulo Q, that space carrying a representation $\mathrm{ind}(Q, P, \sigma)$ under the action of P by right translations.

The second step is to identify the Jacquet functor of that representation of P . We can choose the coset representative g in the coset $P'gP$ so that :

(i) the subgroup QN of P is a parabolic subgroup of G, with Levi subgroup $M_Q = g^{-1}P'g \cap M$;

(ii) $g^{-1}P'g \cap M$ is a parabolic subgroup of M with unipotent radical $g^{-1}N'g \cap M$ and Levi subgroup M_Q ;

(iii) $M' \cap gPg^{-1}$ is a parabolic subgroup of M'.

One can then apply the following lemma :

Lemma: *Let P be a locally profinite group, Q a closed subgroup of P, N an invariant closed subgroup of G which is the union of its compact subgroups, and assume QN is closed in G. Then there is a specific unramified character δ of $Q/Q \cap N$ such that for each smooth representation (σ, U) of Q we have a natural isomorphism of the N-coinvariants of* $\mathrm{ind}(Q, P, \sigma)$ *with* $\mathrm{ind}(QN/N, P/N, \delta\sigma_{Q\cap N})$.

Here the representation $\sigma_{Q\cap N}$ is obtained from σ by taking coinvariants under $Q \cap N$, it is a representation of Q trivial on $Q \cap N$, which we see as a representation of QN/N trivial on N ; the induced representation has the usual meaning. The character δ is obtained by looking at the action of Q by conjugation on Haar measures on N and $Q \cap N$.

Keeping careful track of the different characters δ , one checks that $r_P^G W(i)$ identifies with $i^M_{g^{-1}P'g\cap M} \circ Adg \circ r^{M'}_{gPg^{-1}\cap M}\sigma$.

As a first application take $P = P'$ and take for $\rho = \rho'$ a smooth cuspidal representation of M. Then of course $r^M_{M\cap gPg^{-1}}\rho$ is zero unless gPg^{-1} contains M. A consequence of such remarks is that $r_P^G i_P^G \rho$ has a filtration with quotients $Ad(g) \circ \rho$, where g normalizes M.

As a second application replace P,P' by $\bar{P}$ and P, where $\bar{P}$ is the parabolic subgroup opposite to P with respect to M. Then $\bar{P}P$ is open in G, we can take it to be $X(1)$ in the notation above, and we get that $r_{\bar{P}}^G i_P^G \rho$ contains ρ naturally as a subrepresentation, for any smooth representation ρ of M. If π is a smooth representation of G, applying the functor $r_{\bar{P}}^G$ gives a natural homomorphism of $\mathrm{Hom}_G(i_P^G\rho, \pi)$ into $\mathrm{Hom}_M(r_{\bar{P}}^G i_P^G\rho,\ r_{\bar{P}}^G\pi)$, and composing with the inclusion of ρ just mentioned, we get a natural homomorphism from $\mathrm{Hom}_G(i_P^G\rho, \pi)$ into $Hom_M(\rho, r_{\bar{P}}^G\pi)$. It is an isomorphism (not easy to prove!) which realizes the adjointness of i_P^G and $r_{\bar{P}}^G$.

Chapter 3 The Langlands classification

3.1 Asymptotics of coefficients

As we have seen, among the irreducible smooth representations of G, the cuspidal representations are characterized by the fact that their coefficients have compact support mod.$Z(G)$.

More generally the asymptotics of the coefficients of an irreducible smooth representation π of G are governed by the normalized Jacquet functors $r_P^G\pi$ when $P = MN$ runs through parabolic subgroups of G - by conjugation we can restrict to standard parabolic subgroups.

More precisely, if A is a maximal split torus in the centre of M (we can take A contained in A_0 if M contains M_0) the finite length representation $r_P^G\pi$ of M is the direct sum over characters χ of A of the generalized eigenspaces $r_P^G\pi[\chi]$, where that space is the intersection of the kernels of $(r_P^G\pi(a) - \chi(a))^k k$ for k large enough, a running through A. Only a finite number of those generalized eigenspaces are non-zero, those for which χ is the central character of some irreducible subrepresentation of $r_P^G\pi$; we call them the central characters of $r_P^G\pi$. They control the asymptotics of the coefficients of π.

3.2 Discrete series

Apart from cuspidal representations, the most important notion is that of square integrable (smooth irreducible) representations, also called discrete series representations of G. A smooth irreducible representation π of G is said to be **square integrable** if :

(i) its central character is unitary ;

(ii) the absolute value $|f|$ of any coefficient f of π is square integrable on $G/Z(G)$, with respect to a Haar measure on that quotient - it does not matter which.

A discrete series representation (π, V) of G is preunitary. Indeed take a non-trivial functional λ in $V^\vee$; then the map which to a vector v in V associates the function $\phi_v : g \mapsto \lambda(\pi(g)v)$ is a G-embedding of V into the space of functions on G transforming via ω_π under tranlations by $Z(G)$ and the absolute value of which is square integrable on $G/Z(G)$. On that space there is a natural scalar product which is G-invariant :

$$< \phi, \psi >= \int_{G/Z(G)} \bar{\phi}(g)\psi(g)dg$$

If π is cuspidal, it is a discrete series as soon as ω_π is unitary.

We say that π is essentialy square integrable if $\chi \otimes \pi$ is square integrable for some character χ of G (which we can take to have real positive values). If π is cuspidal, it is essentially square integrable. For a general group G, it is not so easy to construct the remaining discrete series; see below for the case of $\mathrm{GL}(n, F)$.

A criterion for square integrability :

A smooth irreducible representation π of G is square integrable if and only if :

(i) its central character is unitary ;

(ii) for each parabolic subgroup $P = MN$ of G, all central characters χ of $r_P^G\pi$ verify $|\chi(a)|_F < 1$ whenever a in A contracts N.

Here we say that a **contracts** N if for each root α of A in Lie(N) we have $|\alpha(a)|_F < 1$. We can restrict to standard parabolic subgroups of G if we want.

3.3 Tempered representations

If we weaken condition (ii) above to a large inequality, we get the notion of tempered representation. In fact it can be shown that a smooth irreducible representation of G is **tempered** if and only if all its coefficients have "moderate growth", meaning they are bounded by a multiple of a natural "size function" on G.

For us the relevant properties are the following :

(i) a smooth irreducible tempered representation is preunitary.

(ii) If $P = MN$ is a parabolic subgroup of G and ρ a smooth irreducible tempered representation of M, then the preunitary representation $i_P^G\rho$ is the orthogonal direct sum of finitely many smooth irreducible tempered representations of G.

(iii) Any smooth irreducible tempered representation π of G is a subrepresentation of $i_P^G\rho$ where ρ is an irreducible smooth tempered representation of M ; the Levi subgroup M is determined by π up to conjugation in G, and once M is fixed, ρ is determined up to isomorphism.

The way to prove that is to define a "weak" Jacquet functor $(r_P^G)^0$, cutting only the part of r_P^G where the central characters have absolute value 1, and use a variant of the geometric lemma.

3.4 The Langlands classification

Every smooth irreducible representation of G can be obtained form tempered representations of Levi subgroups, in the following manner.

Here it is convenient to fix a minimal parabolic subgroup $P_0 = M_0N_0$ of G with A_0 being the maximal split torus in the centre of M_0. To each simple root α of A_0 in Lie(N_0) is attached a coroot $\alpha^\vee$ which is an algebraic group morphism of $F^\times$ into A_0.

Let $P = MN$ be a standard parabolic subgroup containing P_0, with Levi subgroup M containig M_0. A character χ of M with positive real values is said to be **positive** (with respect to N) if for every simple root α of A_0 in $\mathrm{Lie}(N)$ we have $|\chi \circ \alpha^\vee(\pi_F)|_F < 1$. Let ρ be a smooth irreducible tempered representation of M, and χ a character of M with positive real values which is positive with respect to N. Then $i_P^G \chi \otimes \rho$ has a unique irreducible quotient $J(P, \chi, \rho)$.

Any smooth irreducible representation of G is isomorphic to such a quotient, where P, χ and the isomorphism class of ρ are determined.

3.5 For $G = \mathrm{GL}(n, F)$, all discrete series can be constructed from cuspidal representations in the following manner. Let r be a divisor of n,$n = rs$, and let ρ be a smooth irreducible cuspidal representation of $\mathrm{GL}(s, F)$. Form the representation $\rho(r) = \rho \otimes \rho| \ |_F \otimes \cdots \otimes \rho| \ |_F^{r-1}$ of $\mathrm{GL}(s, F)^r$, seen as a block diagonal subgroup M of $\mathrm{GL}(n, F)$. Form the parabolically induced representation $i_P^G \rho(r)$, where P is the upper triangular parabolic subgroup of G with Levi subgroup M. It has a unique irreducible quotient $L(\rho, r)$ which is essentially square integrable, and square integrable if and only if $\omega_\rho| |_F^{(r-1)/2}$ is unitary. A discrete series representation of G is of the form $L(\rho, r)$ where the integer r is unique and ρ is unique up to isomorphism. For $G = \mathrm{GL}(2, F)$, $L(| \ |_F^{-1/2}, 2)$ is the Steinberg representation.

Always for $G = \mathrm{GL}(n, F)$, a smooth irreducible square integrable representation of a Levi subgroup of G induces irreducibly to G, so the classification of tempered representations of G is obvious.

To explicit the Langlands clssification, fix the upper triangular subgroup P_0 as minimal parabolic subgroup, with the diagonal subgroup $M_0 = A_0$ as Levi subgroup. Let $P = MN$ be a standard parabolic subgroup of G ; then M is a product of groups $\mathrm{GL}(n_1, F), \ldots, \mathrm{GL}(n_t, F)$ as blocks along the diagonal. A positive real valued character χ of M is of the form $(m_1, \ldots, m_t) \mapsto \chi_1 \circ \det(m_1) \cdots \chi_t \circ \det(m_t)$, and χ is positive with respect to N if and only if $\chi_i/\chi_{i+1}(\pi_F) < 1$ for $i = 1, \ldots, t-1$.

For $G = \mathrm{GL}(2, F)$, the trivial representation of G is the unique quotient of $i_B^G \chi$ where χ is the character $(a, b) \mapsto |a/b|^{1/2}$.

A different way of expressing the classification for $\mathrm{GL}(n, F)$ has been obtained by Zelevinsky, an approach which actually gives more precise results. A segment is a pair (ρ, r) where ρ is an isomorphism class of smooth irreducible cuspidal representation of some $\mathrm{GL}(s, F)$, and rs is its length. If (ρ, r)

is a segment and (ρ', r') another segment, say that the first one precedes the second if $\rho' = \rho||_F^t$, where t is a positive integer verifying $r - r' < t \leqslant r + 1$.

Let $(\rho_1, r_1), \cdots, (\rho_u, r_u)$ be segments of length $l_1, \ldots, l_u$ of sum n, and such that for $1 \leqslant i < j \leqslant u$, (ρ_i, r_i) does not precede (ρ_j, r_j). Let ρ be the upper triangular subgroup of G with diagonal blocks of size $l_1 \cdots, l_u$, forming the Levi subgroup M. Form $i_P^G \rho$ where ρ is the tensor product $L(\rho_1, r_1) \otimes \cdots \otimes L(\rho_u, r_u)$.

Then it has a unique irreducible quotient written $L((\rho_1, r_1), \cdots (\rho_u, r_u))$. Every smooth irreducible representation π of G is of that form and the segments are determined by π up to ordering.

Chapter 4 Odds and ends

4.1 The Hecke algebra

Let us choose a Haar measure dg on G, which we see as integration on G and write $f \mapsto \int f(g)dg$. On the space of locally constant complex functions on G with compact support we then have the convolution product $(f, \phi) \mapsto f \times \phi$ given by $f \times \phi(h) = \int f(hg^{-1})\phi(g)dg$ for h in G. That gives an associative algebra written $H(G)$ and called the Hecke algebra of G (with respect to dg).

To a compact open subgroup J of G is attached an idempotent e_J in $H(G)$ which is the characteristic function of J divided by its volume. For f in $H(G)$ we have $f \times e_J = f$ if and only if f is invariant under right translations by elements of J, and similarly on the left. Thus $e_J H(G) e_J$ is a subalgebra of $H(G)$ with unit e_J and consists of functions which are bi-invariant under J. One verifies that the map which to a double coset JgJ associates its characteristic function divided by the volume of J induces an algebra isomorphism of $H(G, J)$ onto $e_J H(G) e_J$.

Let V be a complex vector space. The Haar measure dg induces a linear map from $C_c^\infty(G) \otimes V$ to V ; identifying that tensor product with the space $C_c^\infty(G; V)$ of locally constant maps from G to V with compact support, we get an integration map $f \mapsto \int f(g)dg$ on $C_c^\infty(G; V)$.

Let (π, V) be a smooth representation of G. Let f be in $H(G)$ and v in V. Then the map $v \mapsto f(g)\pi(g)v$ is locally constant with compact support; integrating, we get a map $(f, v) \mapsto \pi(f)v = \int f(g)\pi(g)vdg$ which makes V into a $H(G)$-module - we write $\pi(f)$ for the operator $v \mapsto \pi(f)v$. That module is non-degenerate in the sense that for every vector v in V there is a compact open subgrop J of G such that $e_J v = v$.

It is easily seen that the previous procedure yields an isomorphism from the category $S(G)$ of smooth representations of G onto the category of non-degenerate $H(G)$-modules.

Remark : if J is a compact open subgroup of G, the idempotent e_J acts on the space V of a smooth representation of G as the canonical J-invariant projector with image V^J. The subalgebra $e_J H(G) e_J$ of $H(G)$ stabilizes V^J and via the isomorphism of $H(G,J)$ onto $e_J H(G) e_J$ we recover the action of $H(G,J)$.

4.2 Traces

Let (π, V) be an admissible representation of G. If f in $H(G)$ is bi-invariant under a compact open subgroup J, the operator $\pi(f)$ on V takes values in the finite-dimensional space V^J, hence its trace is well-defined.The linear form $\mathrm{tr}\pi : f \mapsto \mathrm{tr}(\pi(f))$ is an invariant distribution on G that is a linear form on $H(G)$ which is invariant under the action of G on $H(G)$ by conjugation.

If we have a family (π_i, V_i) of smooth irreducible representations of G; having distinct isomorphism classes, then, for a given compact open subgroup J of G, the spaces V_i^J which are non-zero give simple modules over $H(G,J)$ with distinct isomorphism classes. We deduce that the linear forms $\mathrm{tr}\pi_i$ are linearly independent.

4.3 Characters

There is a way to define the character of an irreducible smooth representation of G as a class function on G as is done for finite groups. But because smooth irreducible representations of G are usually infinite-dimensional, such character-functions are not defined on the whole of G. We let G_{reg} be the set of regular semisimple elements in G, i.e. semisimple elements with centralizer of minimal dimension: for $\mathrm{GL}(n,F)$ they are the elements with n distinct eigenvalues in an algebraic closure. The set G_{reg} is open and dense in G, even Zariski-open.

Theorem (Harish-Chandra): *let π be an irreducible smooth representation of G. Then there exists a unique locally constant function χ_π on G_{reg} such that for f in $H(G)$ we have* $\mathrm{tr}\pi(f) = \int_{G_{\mathrm{reg}}} \chi_\pi(g) f(g) dg$ *whenever f has* *support in* G_{reg}.

Note that for such an f the integral makes sense since the support of f is compact in G_{reg} : we simply extend the integrand by 0 outside G_{reg}. Also by uniqueness χ_π is clearly a class function on G_{reg}, and it does not depend on the choice of the Haar measure dg.

When F has characteristic zero, the character function has nice properties :

THEOREM (Harish-Chandra): *Assume that the field* F *has characteristic zero. Let* π *be a smooth irreducible representation of* G. *then* χ_π *is a locally* L^1 *function on* G *and for any* f *in* $H(G)$*we have* $\mathrm{tr}\pi(f) = \int_G \chi_\pi(g)f(g)dg$.

COROLLARY : *Let* π_i *be inequivalent smooth irreducible representations of* G. *Then the character-functions* χ_{π_i} *are linearly independent.*

The proof involves control of $\chi_\pi(g)$ as g approaches $G - G_{\text{reg}}$. That analysis is more difficult in positive characteristic because of possiby inseparable elements. But the theorem is still true for $G = \mathrm{GL}(n, F)$ in positive characteristic (Lemaire), which also gives the corollary (due to Rodier in that case).

4.4 The Jacquet-Langlands correspondence

Characters, or variants, are very important in the formulation of natural relations between representations of different reductive groups over F, an area known as Langlands functoriality. The first example is the "Jacquet-Langlands" correspondence between $G = \mathrm{GL}(rs, F)$ and $G' = \mathrm{GL}(r, D)$ where D is a central division algebra over F with dimension s^2.

THEOREM : *There is a unique bijection* $\pi \mapsto \pi'$ *between isomorphism classes of discrete series representations of* G *and* G', *such that* $\chi_\pi(g) = (-1)^{n-r}\chi_{\pi'}(g')$ *whenever* g *in* G_{reg} *and* g' *in* G'_{reg} *have the same characteristic polynomial.*

The correspondence for general r and s is due to Deligne, Kazhdan and Vignéras in characteristic zero, and to Badulescu in positive characteristic. It is obtained by global means. One chooses a global field K with a place v such that $K_v = F$, and a global division algebra B isomorphic to $M(r, D)$ at the place v. Then for almost all places w of K, B_w is isomorphic to

$M(rs, K_w)$. Using the trace formula, one sets up a correspondence $\Pi' \mapsto \Pi$ between automorphic discrete representations of B^* and $GL(rs)$ over the adele ring of K, such that, at almost all places w of K, where the two groups are isomorphic, the representations Π'_w and Π_w are isomorphic too. The local correspondence at v (that is, over F) is obtained by relating Π_v and Π'_v.

4.5 Whittaker models

Let us assume that G is a quasi-split reductive group over F, i.e. that the Levi subgroup M_0 of a minimal parabolic subgroup P_0 is a torus. A smooth character of the unipotent radical N_0 of P_0 is **non-degenerate** if its stabilizer in P_0 is $Z(G)N_0$. For $G = \mathrm{GL}(n, F)$ and P_0 the upper triangular subgroup, the non-degenerate characters of N_0 are those of the form $(n_{ij}) \mapsto \Sigma\theta_i(n_{ii+1})$ where the sum runs form 1 to $n-1$ and θ_i is a smooth non-trivial character of F. In that case they are all conjugate to each other under P_0.

Fix a non-degenerate smooth character Θ of N_0. let (π, V) be a smooth representation of G. A functional λ on V is a Whittaker functional for π (with respect to θ) if we have $\lambda(\pi(n)v) = \theta(n)\lambda(v)$ for v in V, n in N_0. In other words λ is an N_0-morphism of π into the representation θ of N_0 on $\mathbb{C}$.

Theorem: *Let π be a smooth irreducible representation of G. Then the space of Whittaker functionals for π has dimension* 0 *or* 1.

We say that π is θ-generic if the dimension is 1. Of course it is then θ'-generic for any conjugate of θ in P_0, so for $G = \mathrm{GL}(n, F)$ we simply say generic. Moreover for $G = \mathrm{GL}(n, F)$ any smooth irreducible cuspidal representation is generic. Even more important, local components of automorphic cuspidal global representations of $\mathrm{GL}(n)$ are generic. In terms of the Zelevinsky classification in Chapter 3, the classes of smooth irreducible generic representations of $\mathrm{GL}(n, F)$ are the $L((\rho_1, n_1), \ldots, (\rho_u, n_u))$, where none of the (ρ_i, n_i) precedes any other - in which case the quotient L is actually the full induced representation.

Returning to a general G, if λ is a Whittaker functional on a smooth representation π of G, we get a G-equivariant map from V to the space $C^\infty(G, \theta)$ of locally constant functions f on G such that $f(ng) = \theta(n)f(g)$ for n in N_0 and g in G. If lambda is non-zero and π is irreducible it is an embedding of π into a concrete space of functions, which by the theorem is uniquely determined by π. That space is called the Whittaker model of π

(with repect to θ).

4.6 L- and ϵ-factors

It is often possible to use the Whittaker model from a smooth irreducible generic representation of G to construct invariants of the representation π. Let us simply give the important example of general linear groups.

Fix a non-trivial character ψ of F, which gives a non-degenerate character θ for the upper triangular unipotent subgroup of any $\mathrm{GL}(n, F)$ (take all θ_i to be ψ in the above formula). Let π_1 be a smooth irreducible generic representation of $\mathrm{GL}(n_1, F)$, and π_2 be a smooth irreducible generic representation of $\mathrm{GL}(n_2, F)$. Then using the Whittaker models for π_1 and π_2 it is possible to concoct invariants $L(\pi_1 \times \pi_2, s)$ and $\epsilon(\pi_1 \times \pi_2, s, \psi)$ which enter the formulation of the Langlands correspondence. Actually there are two constructions of such invariants, the Rankin-Selberg convolution appproach due to Jacquet, Piatetski-Shapiro and Shalika, and the local coefficient approach of Shahidi, but Shahidi proved they give the same formulas. The L-function is of the form $P(q_F^{-s})^{-1}$ where P is a polynomial with complex coefficients and value 1 at 0. When π_1 and π_2 are cuspidal it is very easy to compute, as it is the product of $(1 - \chi(\pi_F) q_F^{-s})^{-1}$ where χ runs through unramified characters of $F^\times$ such that $\chi \circ \det \otimes \pi_1^\vee$ is isomorphic to π_2. The ϵ-factor is more subtle : it is a monomial in q_F^{-s} whose value at 0 is related to Gauss sums. One version of the Langlands correspondence for $GL(n, F)$ proved by Harris and Taylor via the geometry of Shimura varieties is the following :

Theorem: *There is a unique family of maps π_n, for positive integers n, from the set $G_F^0(n)$ of isomorphism classes of irreducible smooth representations of W_F of dimension n, to the set of isomorphism classes of smooth irreducible cuspidal repesentations of* $\mathrm{GL}(n, F)$*, such that* :

(i) *for $n = 1$ π_1 is given by class field theory* ;

(ii) *for σ_1 in $G_G^0(n_1)$ and σ_2 in $G_F^0(n_2)$ we have*
(L) $L(\pi_{n_1}(\sigma_1) \times \pi_{n_2}(\sigma_2), s) = L(\sigma_1 \otimes \sigma_2, s)$ *and*
(ϵ) $\epsilon(\pi_{n_1}(\sigma_1) \times \pi_{n_2}(\sigma_2), s, \psi) = \epsilon(\sigma_1 \otimes \sigma_2, s, \psi)$ *for all non-trivial characters ψ of F.*

The maps π_n are bijections.

The L-factor on the right is the Artin L-factor, whereas the ϵ-factor on the right is the one defined by Deligne and Langlands : for $n = 1$ they both originate in Tate's thesis, but ironically the two sides of the inequality are very different generalizatons of Tate's thesis, which turn out to be the same after all ! (see Rajan's lectures for the summer school).