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Cohomology of Shimura Varieties

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COHOMOLOGY OF SHIMURA VARIETIES

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Abstract. We review very briefly, the Hodge- de Rham theorem on computing cohomology of manifolds via harmonic differential forms and apply it to the case of compact locally symmetric manifolds to obtain the Matsushima formula. We then describe the classification of the representations which can contribute to the Matsushima formula, reviewing the Vogan-Zuckerman theory. We then work out the cohomology of some representations of certain unitary groups.

1. The Hodge-de Rham Theorem

Consider smooth projective varieties $S(\Gamma) = \Gamma\backslash X$ which are (compact) quotients of hermitian symmetric domains X of non-compact type by (congruence) arithmetic groups Γ. These are (connected components of) Shimura varieties. These varieties have a rich structure. In these lectures, we will study the cohomology of these varieties.

Most of the results we describe do not really depend on the fact that the symmetric space is of Hermitian type; we will therefore describe the results first in the general case and later specialise to the Hermitian case.

1.1. **Definition.** A second countable Hausdorff topological space M is a said to be a **manifold** or a $\mathcal{C}^\infty$**-manifold** of dimension n, if M can be covered by a collection $\{U_i : i \in I\}$ of open sets (indexed by a set I) and **homeomorphisms** $\phi_i : U_i \to V_i$ of U_i with an open set V_i in $\mathbb{R}^n$ such that on each intersection $U_i \cap U_j$, the two maps $\phi_i : U_i \to V_i$ and $\phi_j : U_j \to V_j$ "differ" by a smooth map; that is, the map

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

of open sets in $\mathbb{R}^n$ is infinitely differentiable. Each pair (ϕ_i, U_i) is called a **co-ordinate chart**.

1.2. **Examples.** (1) The vector space $\mathbb{R}^n$ is an n- dimensional manifold.

(2) The n-dimensional sphere S^n is an n-dimensional manifold (to see this, one may use the stereographic projection).

(3) The projective space $\mathbf{P}^n(\mathbb{R})$ is an n-dimensional manifold. By definition, $\mathbf{P}^n(\mathbb{R})$ is the space of lines through the origin in $\mathbb{R}^{n+1} = \mathbb{R}^{n+1} \setminus \{0\}/scalars$. For each integer i with $0 \leq i \leq n$, take $U_i = \{(x_0, x_1, \cdots, x_n) : x_i \neq 0\}$, and $\phi_i : U_i \to \mathbb{R}^n$ defined by

$$\phi_i(x_0, \cdots, x_n) = (x_0/x_i, x_1/x_i, \cdots, x_{i-1}/x_i, x_{i+1}/x_i, \cdots, x_n/x_i).$$

One easily checks that (ϕ_i, U_i) give a coordinate chart on $\mathbf{P}^n(\mathbb{R})$.

(4) Open subsets of an n-dimensional manifold M are n-manifolds.

1.3. **Definition.** If M and N are manifolds of dimensions m and n, then a mapping $f : M \to N$ is **smooth** if for each point $p \in M$ and coordinate charts (ϕ_i, U_i) with $p \in U_i \subset M$, and (ψ_j, V_j) with $f(p) \in V_j \subset N$, the composite mapping

$$\phi_i(U_i \cap \psi_j^{-1}(V_j)) \xrightarrow{\phi_i^{-1}} U_i \cap \psi_j^{-1}(V_j) \xrightarrow{f} V_j \xrightarrow{\psi_j} \psi_j(V_j) \subset \mathbb{R}^n,$$

is an infinitely differentiable function as a map from an open set in $\mathbb{R}^m$ to an open set in $\mathbb{R}^n$.

1.4. **Examples.** (1) The maps $x_i : \mathbb{R}^n \mapsto \mathbb{R}$ are smooth.

(2) The map $(x_0, x_1, \cdots, x_n) \mapsto (x_i x_j)_{1 \leq i,j \leq n}$ from $\mathbb{R}^{m+1}$ into $\mathbb{R}^{(m+1)^2}$ yields a smooth map from $\mathbf{P}^n(\mathbb{R})$ into $\mathbf{P}^{m^2+2m}(\mathbb{R})$.

1.5. **The Cotangent Bundle.** Given a point p in an m-dimensional manifold M, consider the space $\mathcal{O}_p$ of **germs** of smooth functions at p: two complex valued functions defined on an open neighbourhood of p are said to be equivalent if they coincide on a smaller open neighbourhood of p. The space of such equivalence classes forms a commutative $\mathbb{C}$-algebra under the usual addition and multiplication of functions and is denoted $\mathcal{O}_p$; elements of $\mathcal{O}_p$ are called **germs** at p. The ring of germs has a natural maximal ideal $\mathfrak{m}_p$, namely those germs which vanish at p. Then the quotient $T_p^*(M) = \mathfrak{m}_p/\mathfrak{m}_p^2$ is called the **cotangent** space at p to M.

It is easy to see that the dimension of the vector space $\mathfrak{m}_p/\mathfrak{m}_p^2$ is m.

1.6. **Example.** Take $M = \mathbb{R}^m$, and $p = 0 \in M$ the origin. Then it is easily seen that the contangent space is the direct sum:

$$\mathfrak{m}_p/\mathfrak{m}_p^2 = \mathbb{R}dx_1 \oplus \cdots \oplus \mathbb{R}dx_m,$$

where dx_i is the image of the germ x_i in $\mathfrak{m}_p/\mathfrak{m}_p^2$.

We may replace $T^*(M)_p$ above by its dual, and obtain the (complex) **tangent space** $T(M)_p = (\mathfrak{m}/\mathfrak{m}^2)^*$ to M at the point p. We can similarly define the (complexified) **tangent bundle** $T(M)$.

Let $k \leq m$ and $\wedge^k T^*(M)_p$ the k-th exterior power of the vector space $T^*(M)_p$. Denote by $\wedge^k T^*(M)$ the set of pairs (p, ξ) with $p \in M$ and $\xi \in \wedge^k T^*(M)_p$; denote by $\pi : \wedge^k T^*(M) \to M$ the projection to the map $(p, \xi) \mapsto p$. Then $\wedge^k T^*(M)$ has a natural structure of a manifold such that π is a smooth map.

To see this, we first consider the example $M = \mathbb{R}^m$. In that case, it is easy to see that the map $(a_1, a_2, \cdots, a_m) \mapsto a_1 dx_1 + a_2 dx_2 + \cdots a_m dx_m$ is an isomorphism of vector spaces $\mathbb{R}^m \to T^*(\mathbb{R}^m)_p$, which gives an isomorphism

$$T^*(\mathbb{R}^m) = \mathbb{R}^m \times \mathbb{R}^m.$$

Similarly, there is an isomorphism

$$\wedge^k T^*(\mathbb{R}^m) \simeq \mathbb{R}^m \times \mathbb{R}^l$$

with $l = \binom{m}{k}$.

These "local" isomorphisms may be patched together to obtain the manifold structure on $\wedge^k T^*(M)$.

1.7. **Differential Forms.** A smooth function $\omega : M \to \wedge^k T^*(M)$ is called a **differential form of degree** k. For example, if $M = \mathbb{R}^m$, then a differential form may be written as

$$\omega = \sum f_{i_1, i_2, \cdots, i_k} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k},$$

with $f_{i_1, i_2, \cdots i_k}$ a smooth function on $\mathbb{R}^m$. The space of differential forms of degre k on a manifold M is denoted $\Omega^k(M)$. This is a vector space over $\mathbb{R}$ or $\mathbb{C}$ in a natural fashion.

1.8. **Integration.** A manifold is **orientable** if there exists a nowhere vanishing differential form of degree m(=Dimension of the manifold).

If M is a compact manifold witha fixed orientation, and $\omega \in \Omega^m(M)$, then define the integral of ω over M by constructing first a **partition of unity** of M. This is a collection of functions ϕ_i of smooth functions on M with compact support such that the support lies in a coordinate neighbourhood U_i of M. We assume (one can always secure such a system of co-ordinate charts, provided M is orientable) that the co-ordinate charts are such that they take the standard orientation on $\mathbb{R}^n$ into the given orientation on M. Our assumptions ensure that there is always a partition of unity of M (even one with only finitely many ϕ_i). The integral of a compactly supported smooth function on an open set U in $\mathbb{R}^m$ is just the definition of the Lebesgue integral of the function on the open set. The integral of an m-form ω' on U is by definition the Lebesgue integral of f where $\omega' = f dx_1 \wedge \cdots \wedge dx_m$. Now the integral of ω on M is the sum over i of the integrals of $\omega\psi_i$ over the coordinate neighbourhood U_i. One checks, using the Jacobian formula for change of variables on $\mathbb{R}^m$, that these definitions are independent of the choice of the partition of unity and of the coordinate neighbourhoods of M.

1.9. **Vector Fields.** Similarly, a **vector field** is a section $X : M \to T(M)$ of the tangent bundle $T(M)$. We can view differential operators as smoothly varying alternating forms on the space of vector fields. Vector fields may be thought of as operators on the space of smooth functions on M: a smooth function $f \in \mathcal{C}^\infty(M)$ yields a map $df : T(M) \to T(\mathbb{R}) = \mathbb{R} \times \mathbb{R}$ since it pulls back smooth functions on $\mathbb{R}$ to smooth functions on M and pulls back elements of the maximal ideal $\mathfrak{m}_{f(p)}$ into $\mathfrak{m}_p$. If $X : M \to T(M)$ is a vector field, then $Xf : M \to M \times \mathbb{R}$ is the composite of the second projection $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ followed by $df \circ X$.

If X and Y are vector fields, then the commutator operators $XY - YX$ on smooth functions turns out to be a vector field denoted $[X, Y]$, and is called the **Lie Bracket** of X and Y.

1.10. **The Operator d.** Given a differential form $\omega \in \Omega^k(M)$ of degree k, define the differential form $d\omega \in \Omega^{k+1}(M)$ by the formula

$$d\omega(X_0, X_1, \cdots, X_k) = \sum_{1 \le i \le m} (-1)^i X_i \omega(X_0, \cdots, X_{i-1}, X_{i+1}, \cdots, X_k) +$$

$$\sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \cdots X_{i-1}, X_{i+1}, \cdots, X_{j-1}, X_{j+1}, \cdots, X_k)$$

where $[X_i, X_j]$ is the Lie bracket of X_i and X_j. Thus, d is a linear map from $\Omega^k(M)$ to Ω^{k+1}. It can be proved that $d^2 = 0$ (this is a purely local statement, and can be verified on $\mathbb{R}^m$). The elements of degree k in the space

$$\frac{ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{Im(d : \Omega^{k-1}(M) \to \Omega^k(M))}$$

form the k-th **de-Rham cohomology group** of the manifold M; it is denoted $H^k(M)$.

(1) (Poincaré Lemma) The k-th de-Rham cohomology groups oi $\mathbb{R}^m$ are all zero, except the zero-th one, which is $\mathbb{R}$.

(2) (de Rham) If M is a compact manifold, the the de-Rham cohomology groups are all finite dimensional.

(3) The de Rham cohomology is naturally isomorphic to the singular cohomology of the manifold M, and is thus a topological invariant M.

Differential forms which are in the kernel of the boundary operator are called **closed** forms and those that are in the image of d are the **exact** forms. One says that two closed forms ω and η are **cohomologous** if their difference is a coboundary.

1.11. **Riemannian manifolds.** We will now assume that M is a manifold such that for each point $p \in M$, there exists an inner product $<,>_p$ in the tangent space $T(M)_p$ which varies smoothly in p. That is, every pair of vector fields X, Y on M, the inner product $p \mapsto < X_p, Y_p >_p = < X, Y > (p)$ is a smooth function. The pair $(M, <,>)$ is called a **Riemannian manifold**.

The metric yields a metric on the contangent space $T^*(M)_p$ and one on its exterior powers as well. If M is a compact manifold which is **orientable** i.e. has a nowhere vanishing section of $\wedge^m T^*(M)$ (m=dimension of M), then we get an associated metric on the space $\Omega^k(M)$ of differential forms as follows.

The dual of $\wedge^k T^*(M)_p$ is identifiable with $\wedge^{m-k} T(M)_p$; thus a metric on $T^*(M)_p$ defines an isomorphism (of $T(M)$ with $T^*(M)$ and hence) $*$ of the exterior power $\wedge^k (T^*(M)_p$ with $\wedge^{m-k} T^*(M)_p$. Consequently we have an isomorphism $* : \Omega^k(M) \to \Omega^{m-k}(M)$. If ω, η are two differential forms in $\Omega^k(M)$, define their inner product by $< \omega, \eta >= \int_M \omega \wedge * \eta$.

With this inner product, define the **adjoint** δ of the boundary operator d; this is a mapping $\delta : \Omega^k(M) \to \Omega^{k-1}(M)$. Define the Laplacian of M by $\Delta = d\delta + \delta d$. More precisely, this is $d_{k-1}\delta_k + \delta_{k+1} d_k : \Omega^k \to \Omega^k$. the kernel of the Laplacian is called the space of **harmonic** forms. Harmonic forms are automatically closed and are not exact. We have the Hodge Theorem:

Theorem 1. *On a compact orientable Riemannian manifold M, every closed form is cohomologous to a unique harmonic differential form.*

2. Locally Symmetric Spaces

We will now specialise to the case when the manifold in question is the quotient of a symmetric space by a discrete group of automorphisms.

2.1. **Notation.** Let G denote the a linear semi-simple Lie group and K a maximal compact subgroup, It can be shown that G/K is a manifold. Suppose that $\Gamma \subset G$ is a discrete subgroup, which is torsion-free and such that the quotient $\Gamma\backslash G$ is compact. It can then be proved that Γ operates properly discontinuously and freely on G/K. The quotient $\Gamma\backslash G/K$ is a **locally symmetric manifold** which -under our assumptions- is compact.

Let $\mathfrak{g}$ denote the complexified Lie algebra of G. The group G is parallelizable; that is, there are n vector fields ($n = dim(G)$) on G which are linearly independent at every point. Such are provided by elements of $\mathfrak{g}$ which can be thought of as left invariant vector fields.

2.2. **Metric defined by the Killing Form.** On the real Lie algebra $Lie(G)$, we have the operators $ad(X)$ with $X \in Lie(G)$. Consider the form $\kappa(X,Y) = trace(ad(X)ad(Y)$ with $X, Y \in Lie(G)$. This is the **Killing form** on $Lie(G)$. Fix a maximal compact subgroup K of G. Then on $Lie(K)$ the Killing form κ is **negative definite**. Let $\mathfrak{p}_0$ denote the orthogonal complement of $Lie(K)$ in $Lie(G)$. The restriction of κ to $\mathfrak{p}_0$ is **positive definite**.

Thus, The Killing form is non-degenerate on $\mathfrak{p}$ and defines a metric on $\mathfrak{p}_0$ the real tangent space. This metric is invariant under conjugation under K and hence we have a metric on G/K which is G invariant.

Let $\mathfrak{k}$ and $\mathfrak{g}$ be the complexified Lie Algebras of K and G respectively. Let $\mathfrak{p}$ denote the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. It is easily seen that the Lie Brackets $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

Consider the quotient $\Gamma\backslash G$; this is a manifold and we have the quotient map $\Gamma\backslash G \to \Gamma\backslash G/K$ (this is automatically a smooth map); elements of $\mathfrak{g}$ still give linearly independent vectors in $\Gamma\backslash G$ and define vector fields. Hence the tangent bundle is trivial. Hence so is the cotangent bundle : the cotangent bundle is simply the product $T^*(\Gamma\backslash G) = \Gamma\backslash G \times \mathfrak{g}^*$. Consequently, the space $\wedge^k T^*(\Gamma\backslash G)$ is isomorphic to $\Gamma\backslash G \times \wedge^k \mathfrak{g}^*$, and the space of differential forms is therefore

$$\Omega^k(\Gamma\backslash G) = \wedge^k \mathfrak{g}^* \otimes \mathcal{C}^\infty(\Gamma\backslash G) = Hom(\wedge^k \mathfrak{g}, \mathcal{C}^\infty(\Gamma\backslash G).$$

Note that G operates on both sides of this isomorphism (on $\Gamma\backslash G$ by right translations, and on $\mathfrak{g}$ by inner conjugation) and that the isomorphism respects the G-action. The d-operator takes the form

$$d\omega(X_0, \cdots, X_k) = \sum_{0\le i\le k} (-1)^i X_i \omega(X_0, , \cdots, \overset{i}{\wedge}, \cdots, X_k)$$

$$+\sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_0, \cdots, \overset{i}{\wedge}, \cdots, \overset{j}{\wedge}, \cdots, X_k)$$

We can now identify $\Omega^k(\Gamma\backslash G/K) = Hom_K(\wedge^k(\mathfrak{g}/\mathfrak{k}), \mathcal{C}^\infty(\Gamma\backslash G))$. This is a sub-complex of the complex $\Omega^*(\Gamma\backslash G)$ of differential forms on $\Gamma\backslash G$. Hence, if $X_i \in \mathfrak{p}$, we get

$$d\omega(X_0, \cdots, X_k) = \sum_{0\le i\le k} (-1)^i X_i \omega(X_0, \cdots, X_k).$$

for all differential forms on $\Gamma\backslash G/K$ identified as above.

2.3. $(\mathfrak{g}, K)$**-cohomology.** Suppose now that π is a $(\mathfrak{g}, K)$-module and consider the complex $C^k = \oplus_k(\wedge^k\mathfrak{g}^* \otimes \pi)$ with values in π. the differentials $d : C^k \to C^{k+1}$ is defined by

$$d\omega(X_0, \cdots, X_k) = \sum_i (-1)^i \pi(X_i)(\omega(X_0, \cdots, X_{i-1}, X_{i+1}, \cdots, X_k)).+$$

$$\sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_0, \cdots, X_{i-1}, X_{i+1}, \cdots, X_{j-1}, X_{j+1}, \cdots, X_k).$$

The complex C_k has a subcomplex $\oplus_k C_0^k$, where $C_0^k = Hom_K(\wedge^k\mathfrak{g}/\mathfrak{k}, \pi)$.

The cohomology of this sub-complex C_0^k is by definition the $(\mathfrak{g}, K)$-cohomology of π.

2.4. **Example.** The above calculation shows that the de-Rham cohomology of $\Gamma\backslash G/K$ is the $(\mathfrak{g}, K)$-cohomology of the module $\mathcal{C}^\infty(\Gamma\backslash G)$.

The metric on G/K is G invariant and hence gives a metric on $\Gamma\backslash G/K$. We may construct the Lapliacian on the differential forms with respect to this metric.

Theorem 2. *(Matsushima-Kuga) Under the identification of the complex* $\Omega^k(\Gamma\backslash G/K)$ *with* $Hom_K(\wedge^k\mathfrak{p}, \mathcal{C}^\infty(\Gamma\backslash G))$*, the Laplacian on the left becomes the action of the Casimir of* $\mathfrak{g}$ *on the right hand side.*

For a proof, see [Bo-Wa].

Corollary 1. *(The Matsushima-Kuga Formula) The de-Rham cohomology group of* $\Gamma\backslash G/K$ *is isomorphic to the vector space* $H^k(\Gamma\backslash G/K) = Hom_K(\wedge^k\mathfrak{p}, \mathcal{C}^\infty(\Gamma\backslash G)(0))$ *where* $\mathcal{C}^\infty(\Gamma\backslash G)(0)$ *is the space of functions killed by the Casimir.*

This is a corollary of the Hodge Theorem together with the calculation of the Laplacian.

Remark. If π is a unitary $(\mathfrak{g}, K)$-module, then one can define the Laplacian on the complex defining the cohomology in a similar way, and the Matsushima-Kuga calculation on the complex associated to π then implies that the Laplacian is related to the Casimir in the same way.

Moreover, in the space of cocycles, one may choose representatives from the space of the orthogonal complement of the space of coboundaries. If π is irreducible, these spaces are all finite dimensional and the whole space is a direct sum of these spaces. Further, the orthogonal complement consists of harmonic forms as is easily seen.

In conclusion, we have, for a unitary representation with $\pi(C) = 0$, the cohomology is computed as

$$H^*(\mathfrak{g}, K, \pi) = Hom_K(\wedge^*\mathfrak{p}, \pi).$$

For example, if $\pi = \mathbb{C}$ is the *trivial* $(\mathfrak{g}, K)$-module, then the Kuga calculation shows that

$$H^*(\mathfrak{g}, K, \mathbb{C}) = Hom_K(\wedge^*\mathfrak{p}, \mathbb{C}) = (\wedge^*\mathfrak{p}^*)^K,$$

where the K-superscript denotes the space of K-invariants. Note that if $\mathfrak{g}_u = Lie(K) \oplus \mathfrak{p}_0$, then $\mathfrak{g}_u$ is a real Lie subalgebra of the complex Lie algebra $\mathfrak{g}$ and is the Lie algebra of the compact dual G_u (a maximal compact of $G(\mathbb{C})$) of the group G. Under a G_u invariant metric metric on G_u/K, we have - using the de-Rham theorem for the compact manifold G_u/K- that

$$H^*(\mathfrak{g}, K, \mathbb{C}) = H^*(G_u/K).$$

Remark. The cohomology groups $H^*(\mathfrak{g}, K, \pi)$ can be shown to be the Ext groups $Ext^*_{(\mathfrak{g},K)}(\mathbb{C}, \pi)$ where $\mathbb{C}$ is the trivial $(\mathfrak{g}, K)$-module. This easily implies that the centre of the universal enveloping algebra acts the same way as it does on the trivial module, provided the representation has non-vanishing cohomology.

It follows that the action of the centre of the enveloping algebra on $H^*(\Gamma\backslash G/K)$ is trivial, which means that

$$Hom_K(\wedge^*\mathfrak{p}, \mathcal{C}^\infty(\Gamma\backslash G)(0)) = Hom_K(\wedge^*\mathfrak{p}, \mathcal{C}^\infty(\Gamma\backslash G/K)(00))$$

where (00) refers to the space of functions which are annihilated by all elements of the centre. This shows that these functions on the right hand side of the Matsushima formula are **automorphic forms**.

Recall that the space $L^2(\Gamma\backslash G)$ is a Hilbert space direct sum of irreducible representations π each occurring with a finite multiplicity denoted $m(\pi)$.
We can then write the Matsushima formula in the form

$$H^k(\Gamma\backslash G/K) = \oplus m(\pi) H^k(\mathfrak{g}, K, \pi),$$

where the sum is over those π such that the Casimir $\pi(C) = 0$. Note that this is an **algebraic** direct sum.

We have "reduced" the problem of cohomology of $\Gamma\backslash G/K$ to two questions: determine the representations π with cohomology, and determine when they occur in $L^2(\Gamma\backslash G)$. The second question will not be dealt with in these notes.

3. Cohomological Representations

We seek to determine the representations π of $G(\mathbb{R})$ which contribute to the Matsushima formula. In particular, we wish to find representations π of $G(\mathbb{R})$ which are unitary and such that the Casimir acts by zero. Furthermore, we want that $H^*(\mathfrak{g}, K, \pi) \neq 0$. By our computations, this is the same as

$$H^*(\mathfrak{g}, K, \pi) = Hom_K(\wedge^* \mathfrak{p}, \pi).$$

(It is then a consequence that any element of the centre of the enveloping algebra acts on π by a scalar, namely the scalar by which it acts on the trivial representation.

There is a complete characterisation (the Vogan-Zuckerman Theory) of these representations. To describe the final result, we first need to introduce some notation.

To motivate the definitions below, we first state the result for the group $U(p,q)$. Fix positive integers $a_1, a_2, \cdots, a_l$ and $b_1, \cdots b_l$ with $\sum a_i \leq p$ and $\sum b_i \leq q$. Define the groups

$$L = \prod_{1\leq i\leq l} U(a_i, b_i)) \text{ and} \widehat{X}_L = \prod U(a_i + b_i)/U(a_i) \times U(b_i)).$$

Let $R = pq - \sum a_i b_i$. The result for $U(p,q)$ states the following.

Theorem 3. *If $G = U(p,q)$ and π is a cohomological representation, there exist integers a_i and b_i as above such that the $(\mathfrak{g}, K)$ cohomology of π is given by*

$$H^i(\mathfrak{g}, K, \pi) = H^{i-R}(\widehat{X}_L),$$

where the latter is the de-Rham cohomology of the space in question.

The description is easier when G/K is a Hermitian symmetric domain, and we will assume this. In particular, G/K is a complex manifold such that the connected component G^0 of identity of G (mod centre) is the group of holomorphic automorphisms of G/K. If we assume that G^0 is a simple Lie group (has no connected normal subgroups) this is equivalent to the connected component $Z(K)^0$ of the centre $Z(K)$ being isomorphic to S^1. For example, when $G = Sp_g$, the maximal compact subgroup is $K = U(g)$ has centre the scalar unitary matrices, i.e. S^1. If $G = SU(p,q)$ then the maximal compact $K = S(U(p) \times U(q)$ has centre S^1.

Recall that if π is irreducible, unitary and cohomological, then

$$H^*(\mathfrak{g}, K, \pi) = Hom_K(\wedge^* \mathfrak{p}, \pi).$$

Thus, a related question is : what are the common K-types between a cohomological representation and $\wedge^* \mathfrak{p}$? It tuns out the common K-types are of a very special type (termed the canonical K-types). These are described in terms of certain θ-stable parabolic subalgebras, where θ is a Cartan Involution.

If $z \in Z(K)^0 = S^1$, then the Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{p}^+ \oplus \mathfrak{p}^- \oplus \mathfrak{k}$ the z, the z^{-1} and 1 eigenspaces. Fix a maximal torus T of K. Then by the foregoing, T is also a maximal torus in G. Since $Lie(T)$ acts by imaginary eigenvalues, $iLie(T)$ acts by real eigenvalues. Fix a Borel subalgebra $\mathfrak{b}_K$ of $\mathfrak{k}$ containing $\mathfrak{t}$. We have the root space decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}_\alpha$ where α runs through the roots of T. Fix a system of positive roots of $\mathfrak{g}$ such the roots occurring Fix $X \in iLie(T)$ such that $\alpha(X) \geq 0$ for all positive roots in $\mathfrak{b}_K$. Define $\mathfrak{q} = \mathfrak{q}(X) = \mathfrak{g}^X \oplus_{\alpha(X)>0} \mathfrak{g}_\alpha$ $\mathfrak{l} = \mathfrak{l}(X) = \mathfrak{g}^X$ and $\mathfrak{u} = \mathfrak{u}(X) = \oplus_{\alpha(X)>0} \mathfrak{g}_\alpha$. Then $\mathfrak{q}$ is a parabolic subalgebra of $\mathfrak{g}$ and $\mathfrak{l}$ and $\mathfrak{u}$ are the Levi part and the nil-radical of $\mathfrak{q}$. Note also that the Cartan involution leaves $\mathfrak{q}, \mathfrak{l}$ and $\mathfrak{u}$ stable. Let $R = dim(\mathfrak{u} \cap \mathfrak{p}$. Define $e(\mathfrak{q}) = \wedge^R \mathfrak{u} \cap \mathfrak{p}$, a line in $\wedge^r \mathfrak{p}$. Define the K-span of $e(\mathfrak{q})$ to be $V(\mathfrak{q})$. It can be easily proved that $V(\mathfrak{q})$ is irreducible.

Let $W(\mathfrak{q})$ denote the isotypic of the representation $V(\mathfrak{q})$ in $\wedge^*\mathfrak{p}$. It can easily be proved that as a K-module, $W(\mathfrak{q})$ is generated by the vectors $(\wedge^*\mathfrak{l}\cap\mathfrak{p})^{L\cap K}\wedge e(\mathfrak{q})$ (the latter are highest weight vectors for the Borel subalgebra $\mathfrak{b}_K$). We have then the following theorem

Theorem 4. *(Kumaresan-Parthasarathy) Given π, there exists a unique $V(\mathfrak{q})$ such that $Hom_K(\wedge^*\mathfrak{p},\pi) = Hom_K(W(\mathfrak{q}),\pi)$. That is, the only K-type of π common with $\wedge^*\mathfrak{p}$ is $V(\mathfrak{q})$.*

Moreover, the cohomology is

$$H^*(\mathfrak{g},K,\pi) = H^{*-R}(\mathfrak{l}, L\cap K,\mathbb{C}) = H^{*-R}(\widehat{X_L}).$$

Theorem 5. *(Vogan-Zuckerman) To each cohomological representation π there is a unique canonical K-type $V(\mathfrak{q})$ occurring in π.*

Given a parabolic $\mathfrak{q}$ as before, there is a unique cohomological representation $\pi = A_{\mathfrak{q}}$ which contains $V(\mathfrak{q})$ as a canonical K-type.

Corollary 2. *(Vanishing Theorems) Let $R_G = inf dim(u(\mathfrak{q})\cap\mathfrak{p}$ where the infimum runs over all the θ-stable parabolic subalgebras of $\mathfrak{g}$. If π is a non-trivial irreducible representation of $\mathfrak{g}, K$ then $H^i(\mathfrak{g},K,\pi)=0$ for all integers $i<R_G$.*

In particular, if $\mathbb{R}$-rank of G is at least two, then $H^1(\Gamma\backslash G/K)=0$ (Kazhdan's Theorem). The same holds even if G is not locally isomorphic to $SU(n,1)$ or $SO(n,1)$.

Remark. The construction of these representations $A_{\mathfrak{q}}$ was done by different methods: one by Parthasarathy imitated the Enright Varadarajan construction of discrete series. Later Zuckerman constructed these using derived functors. It can also be proved ([Wong]) that the Dolbeaux complex of a suitable line bundle on G/L (L being the Levi subgroup of the parabolic $Q\subset G(\mathbb{C})$ in question - it is easy to show then that G/L is a complex manifold on which G operates by holomorphic automorphisms) at the degree $S = dim(K/L\cap K)$ has the property that the coboundaries are a closed subset of the space of cocycles, and that (K-finite vectors in) the Dolbeaux cohomology space is the module $A_{\mathfrak{q}}$.

4. Examples

Set $G = U(p,q)$ and $K = U(p)\times U(q)$. Let T=diagonals in K. Then, $iLie(T)$ consists of dagonal matrics with real entries. Fix the Borel sugroup B_K of the complex group $K_{\mathbb{C}}$ (Th complexified group is $GL(p,\mathbb{C})\times GL(q,\mathbb{C})$) to be the group of upper trangular atrices in $K_{\mathbb{C}}$. We assume that the eignvalues of $ad(X)$ are non-negative on the Lie algera of B_K.

Write $X = (a_1, a_2, \cdots a_p;_1, \cdots, b_q)$. Our assumptions imply that $a1 \geq a_2 \geq \cdots \geq a_p$ and similarly the b_j are decreasing. We divide the $i's$ and the $j's$ into subsets $I-1, \cdots, I_r$ and $J_1, \cdots, J_r$ (possibly empty) such that for all $1 \leq \mu \leq r$, we have $a_i = b_j \ \forall i \in I_\mu$ and $\forall j \in J_\mu$. It is then immediate that the centraliser of X in G is the product group

$$L = \prod_{1 \leq \mu \leq r} U(p_\mu, q_\mu).$$

Moreover, the compact dual of the symmetric space of L is

$$\prod U(p_\mu + q_\mu)/U(p_\mu) \times U(q_\mu) = \prod Gr_{p_\mu}(\mathbb{C}^{p_\mu + q_\mu})$$

is a product (over the μ's) of grassmannians of p_μ planes in the $p_\mu + q_\mu$ dimensional complex vector space.

This implies the result for $U(p, q)$ stated earlier.

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