A Unified Approach to Attractor Reconstruction

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Workshop: Perspectives in Nonlinear Dynamics
July 2007

General Problem

We can go from time series to attractor geometry.

"Geometry from time series," N.H. Packard, J.P. Crutchfield, J.D. Farmer et al., Physical Review Letters 45, 712 (1980).

$$x(t_1), x(t_2), x(t_3), x(t_4), \ldots, x(t_N)$$

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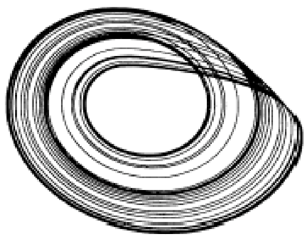
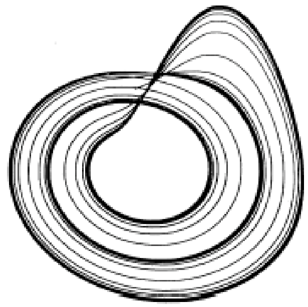


FIG. 1. (x, y) projection of Rossler (Ref. 7).



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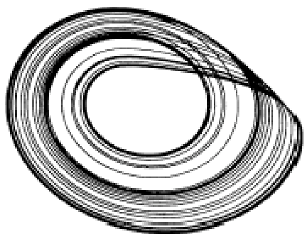

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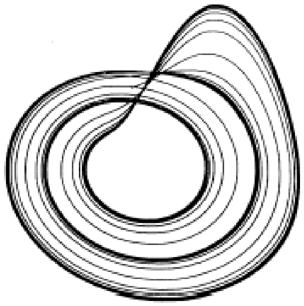

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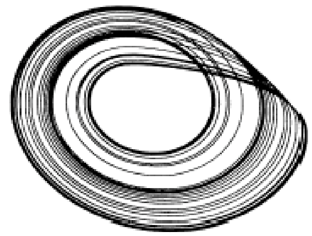

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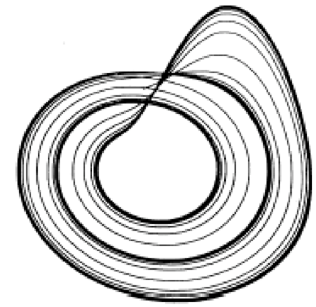

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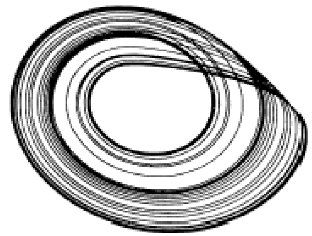

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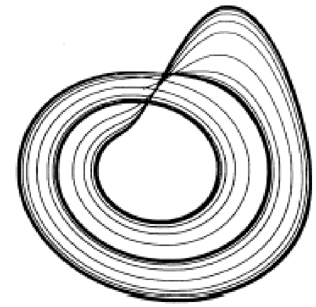

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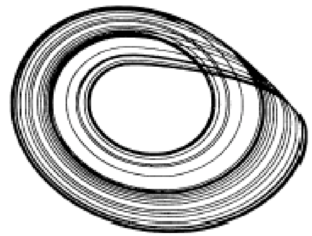

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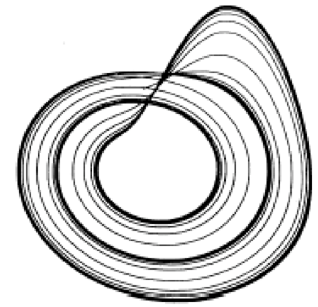

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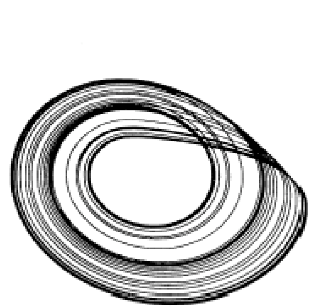

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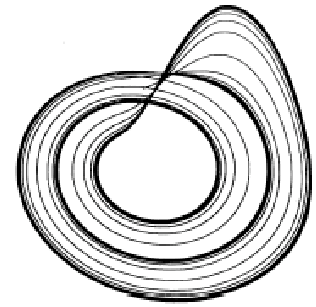

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$$\mathbf{x}(t) \in \mathbf{R}^n \qquad s(t) = h[\mathbf{x}(t)] \qquad \mathbf{x}(t+\tau) = \Phi_\tau[\mathbf{x}(t)]$$

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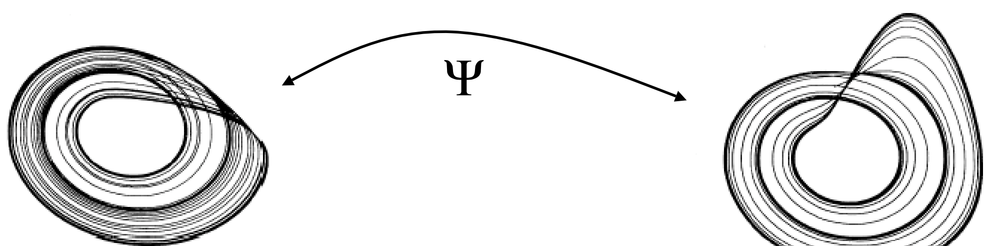

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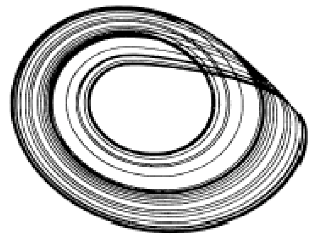

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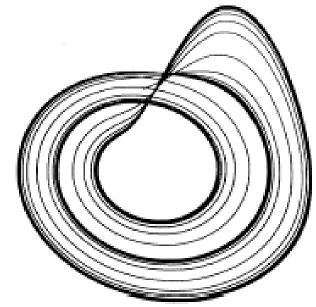

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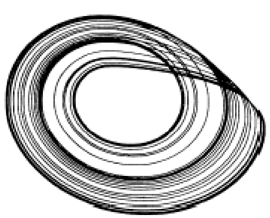
τ ?
Embedding dim ?
Multivariate data?



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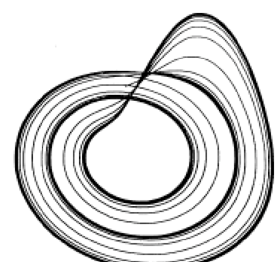

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Current Approaches

Delay time τ ? - autocorrelation or mutual info.

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Multivariate time series? - extend univariate methods?

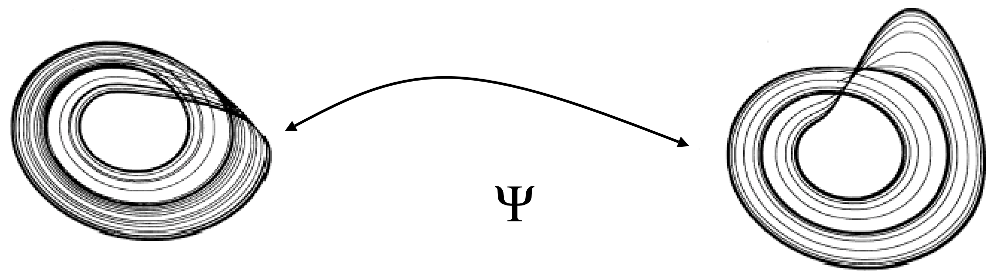
- not rigorous or optimal - heuristic

Theory - The Reconstruction Criterion

What Takens theorem tells us: Make Independent Coordinates

m time series.

$$s_1(t), s_2(t), \ldots, s_m(t)$$



$$\mathbf{v} = \left(s_{j_1}(t+\tau_1), s_{j_2}(t+\tau_2), \ldots, s_{j_k}(t+\tau_k)\right) \longrightarrow \mathbf{v} = \left(s_{j_1}(t+\tau_1), s_{j_2}(t+\tau_2), \ldots, s_{j_d}(t+\tau_d), s_{j_{k+1}}(t+\tau_{k+1})\right)$$

To add a component to **v**, pick time series and τ so new component is **independent** of previous ones.

new component $$s_{j_{k+1}}(t+\tau_{k+1}) \neq f\left(s_{j_1}(t+\tau_1), s_{j_2}(t+\tau_2), \ldots, s_{j_k}(t+\tau_k)\right)$$

Unified approach (τ, d, s_i):

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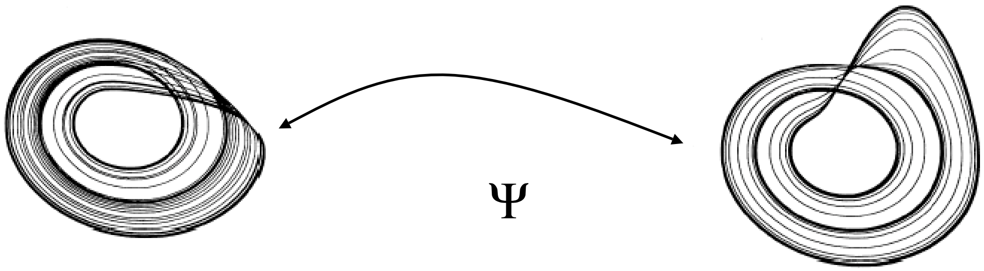

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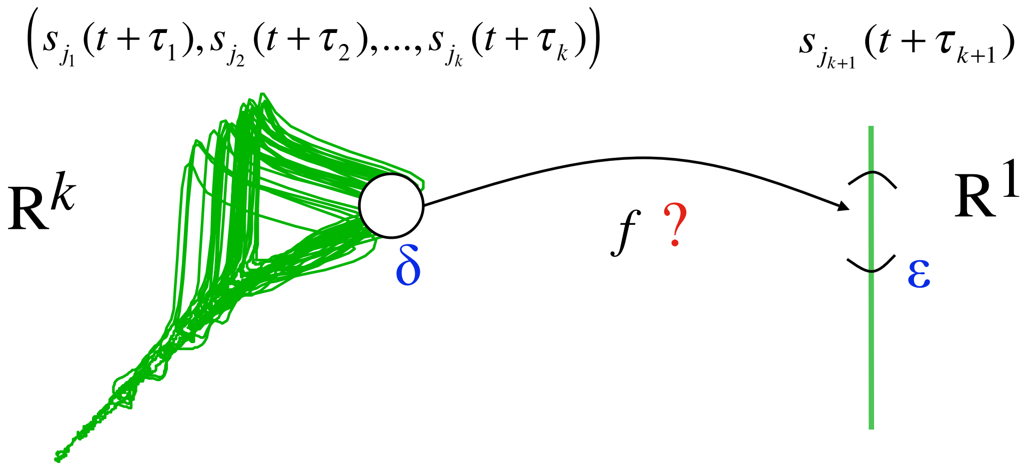
a statistic to check

Statistics for Functional Dependence

The general mathematical criterion: $$s_{j_{k+1}}(t+\tau_{k+1}) \underset{=}{\overset{\neq}{}} f\left(s_{j_1}(t+\tau_1), s_{j_2}(t+\tau_2), \ldots, s_{j_k}(t+\tau_k)\right)$$



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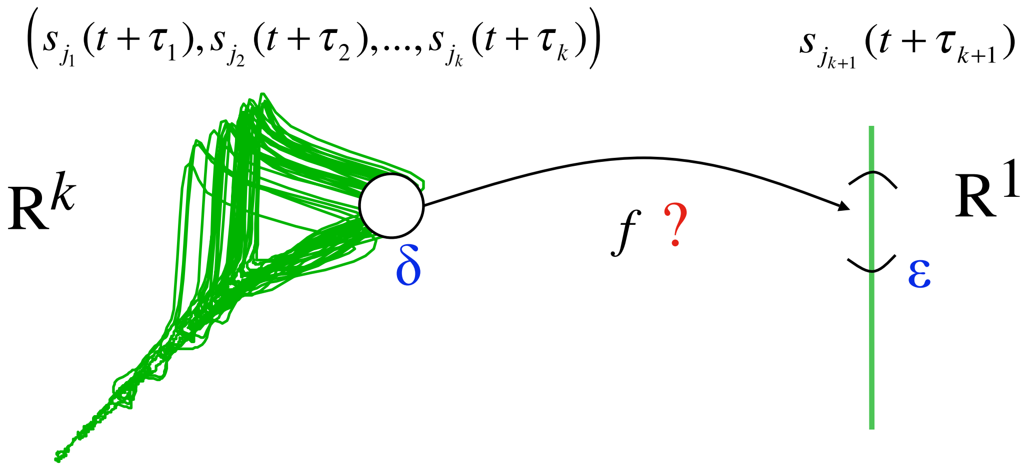

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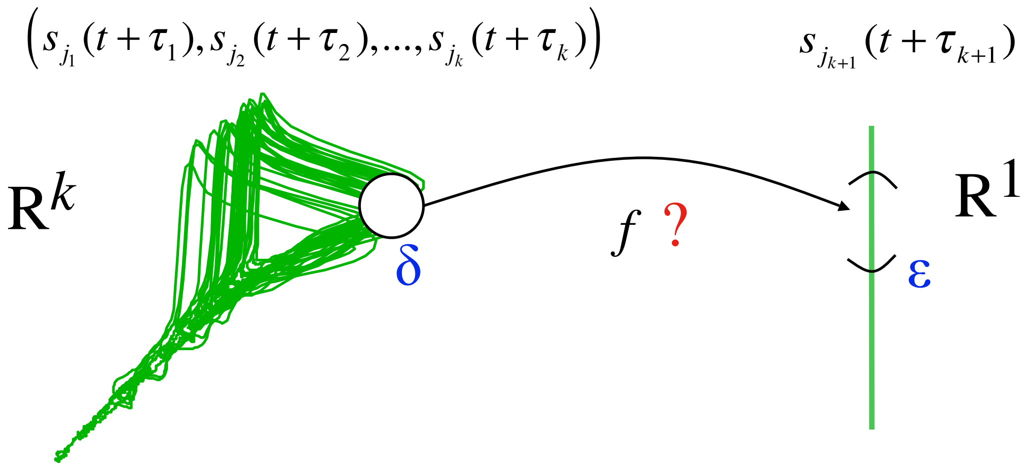

Function Null Hypothesis:
δ points are mapped into ϵ with a probability ***p***. => Binomial distribution of probability of getting *m* points out of *n* into the ε set. Reject the Null when this probability is in the "tail" of the binomial distribution, i.e. very low probability of this happening.

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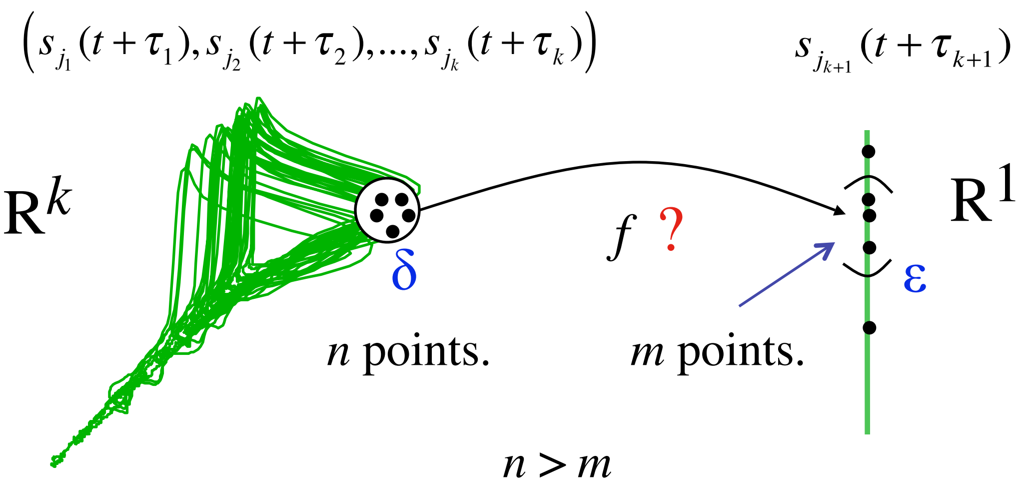

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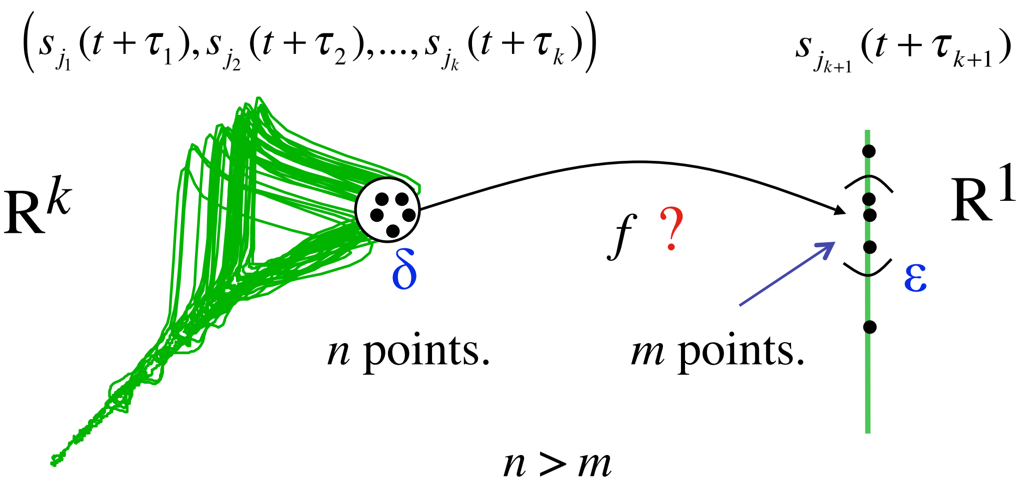

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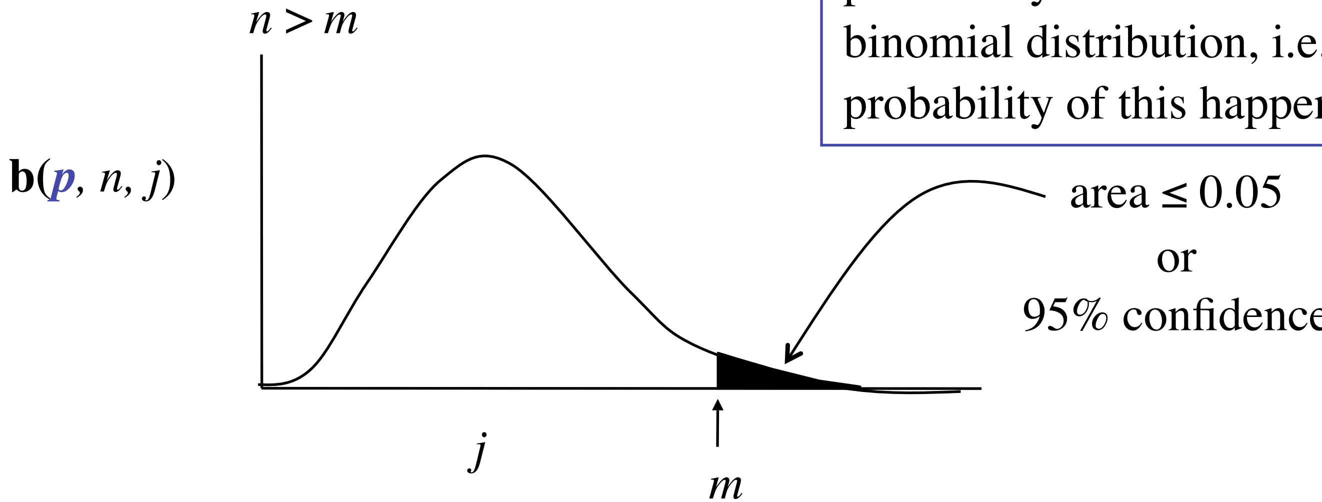

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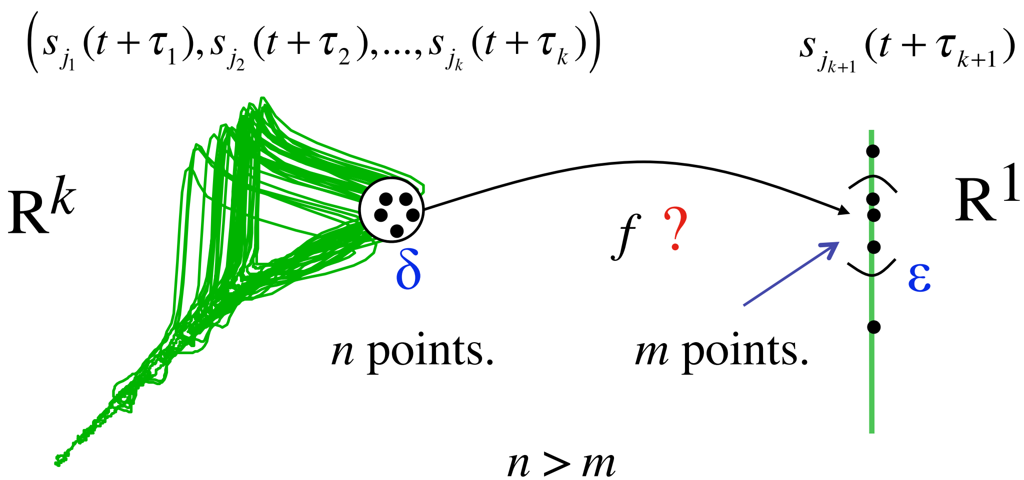

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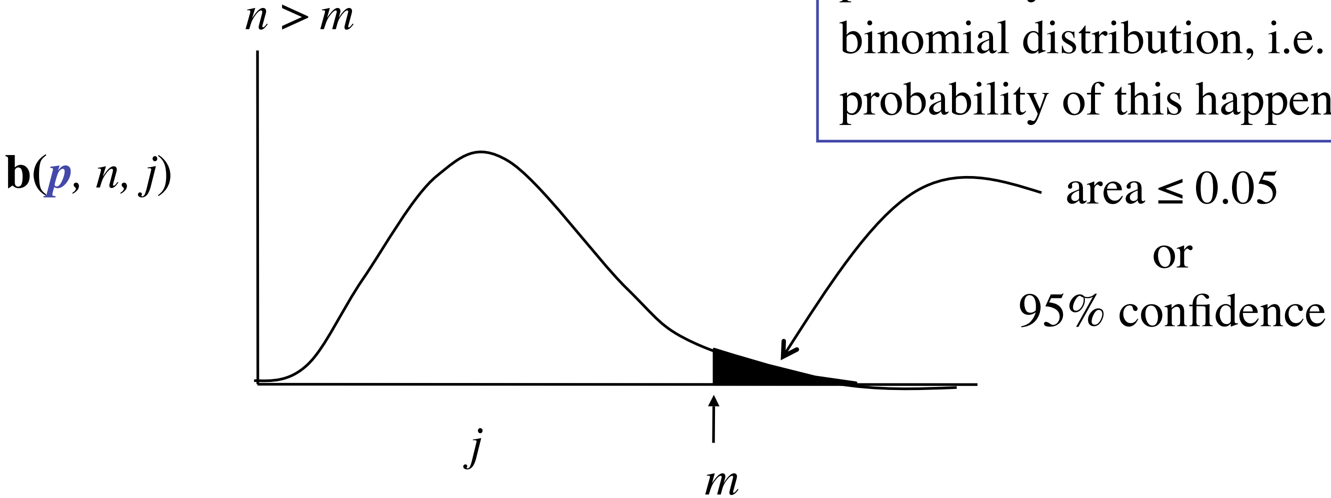

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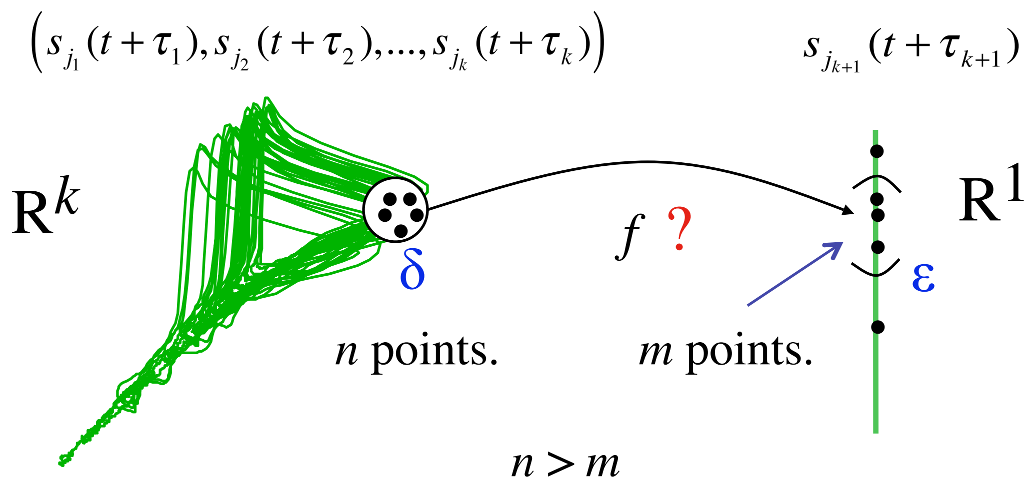

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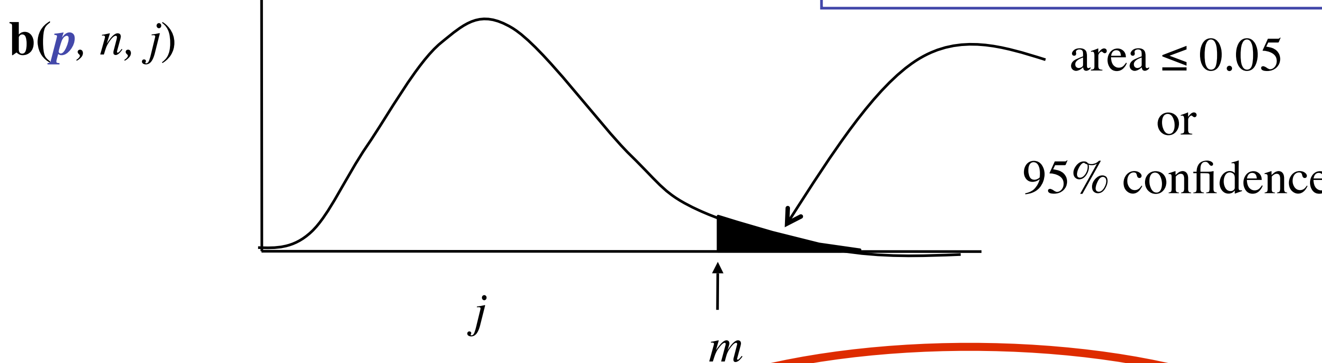

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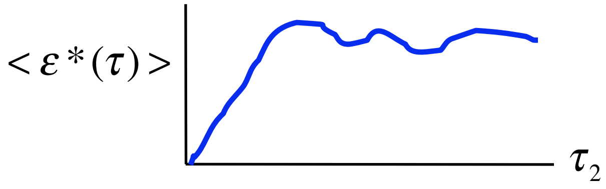
Vary δ and ε to find the smallest ε for which we can reject the Null. Call this ε*. It is the smallest scale at which we can say there is a functional relationship. => **continuity statistic**

Average over time series points **<ε*>**.

$$\left(s_{j_1}(t+\tau_1)\right) \rightarrow s_{j_2}(t+\tau) \Rightarrow \varepsilon^*(\tau_2)$$

$R^1 \rightarrow R^1$



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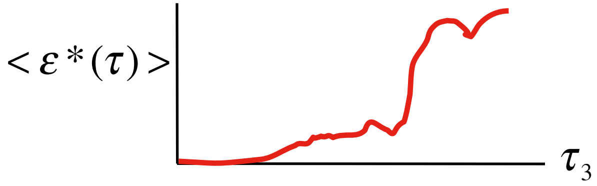

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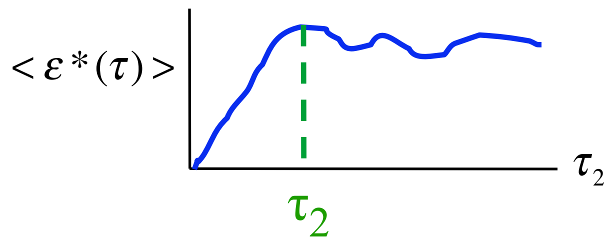

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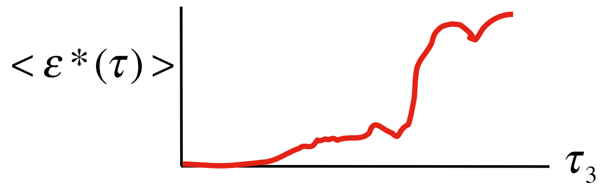

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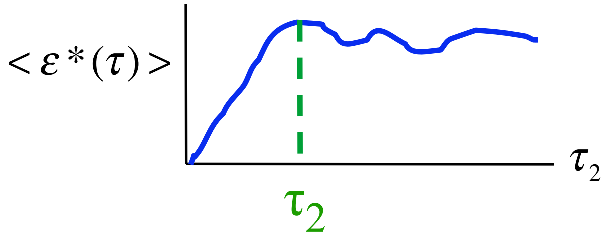
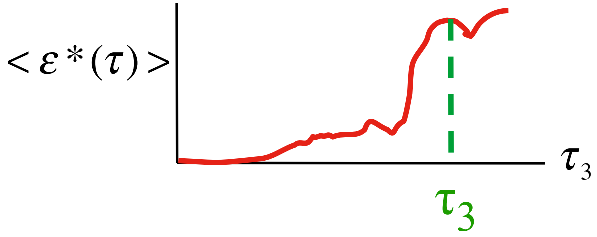


$$\left(s_{j_1}(t+\tau_1), s_{j_2}(t+\tau_2)\right) \rightarrow s_{j_3}(t+\tau) \Rightarrow \varepsilon^*(\tau_3)$$

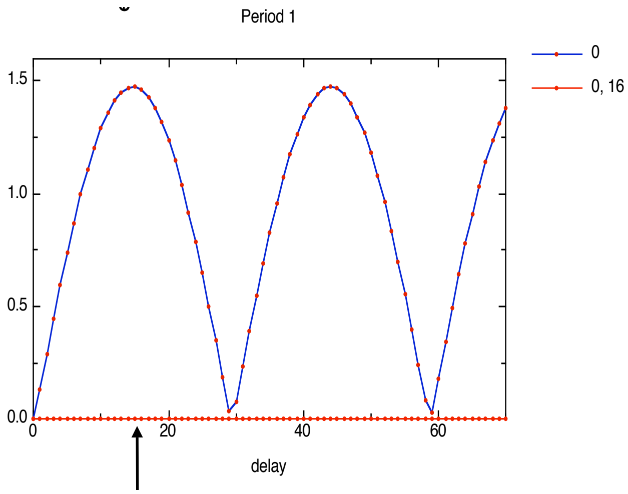
$R^2 \rightarrow R^1$

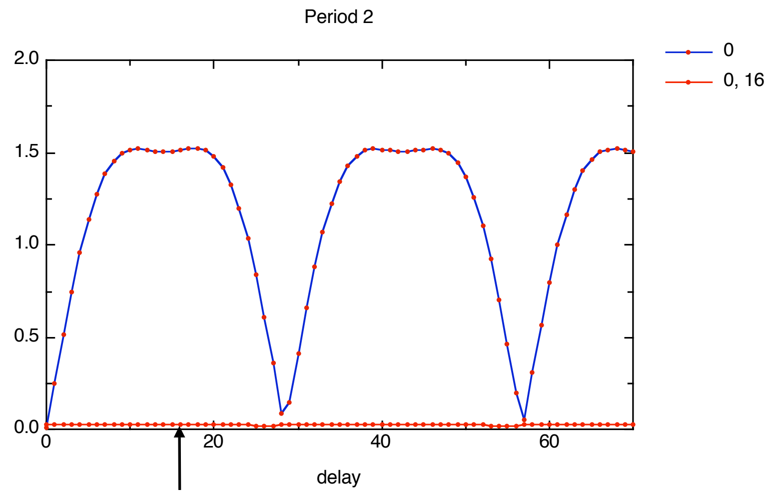


Some examples.

• Periodic Systems

Cont.Stat ε^*





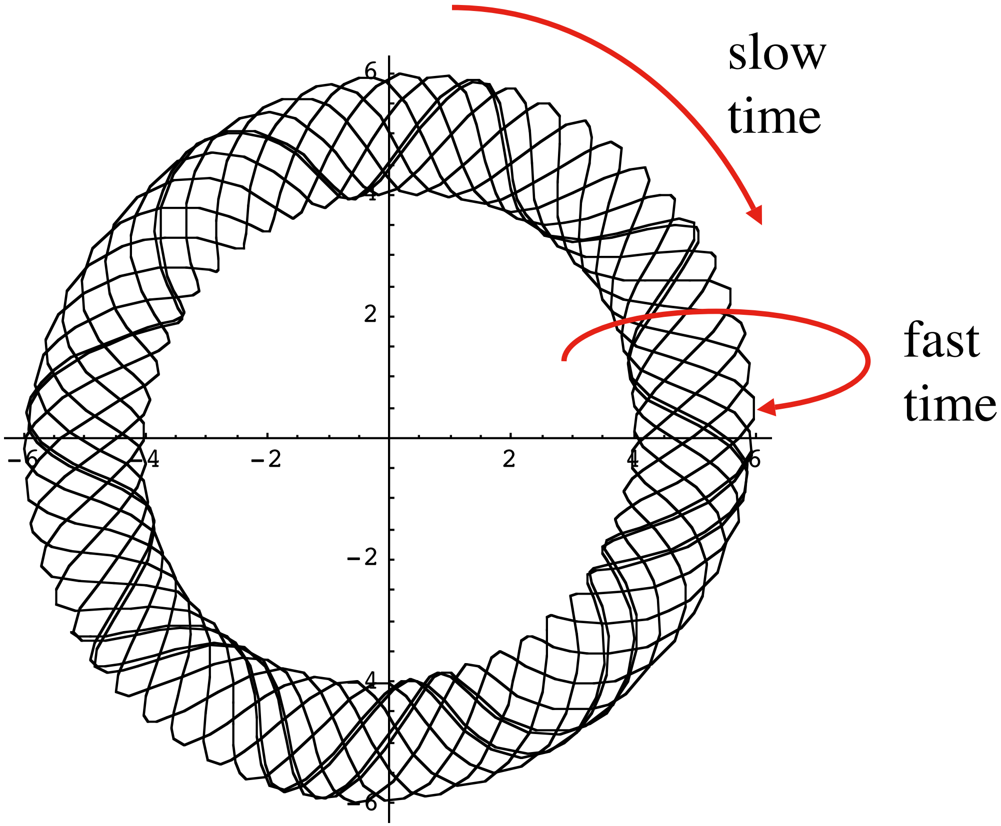
τ=16

τ=16

• Quasiperiodic Systems

Multiple Time Scales

$$s(t) = \left(5 + \cos(2\pi t)\right)\cos(0.8t)$$

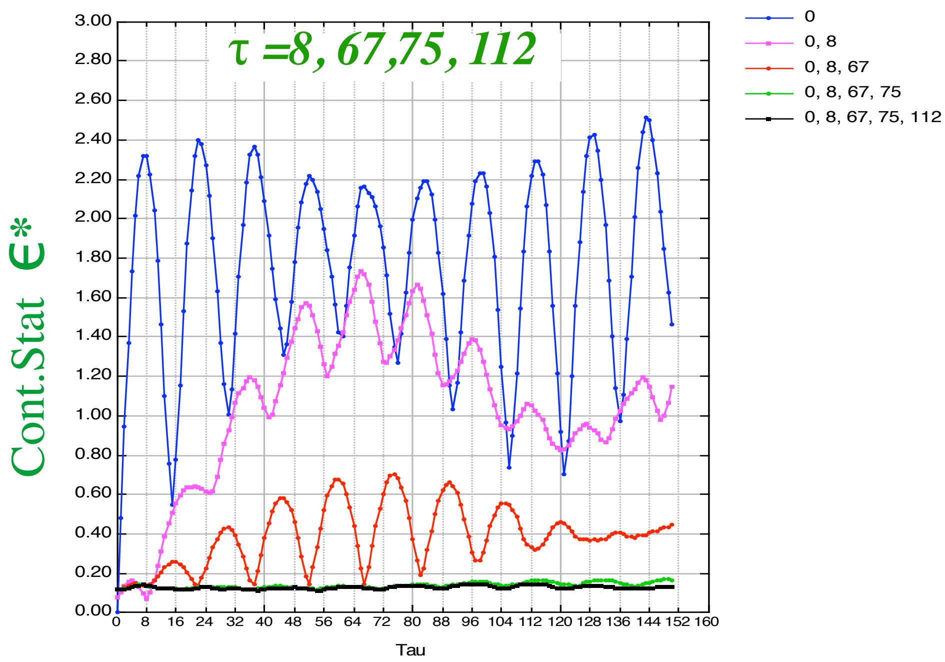


$$\frac{\text{fast}}{\text{slow}} \approx 8$$

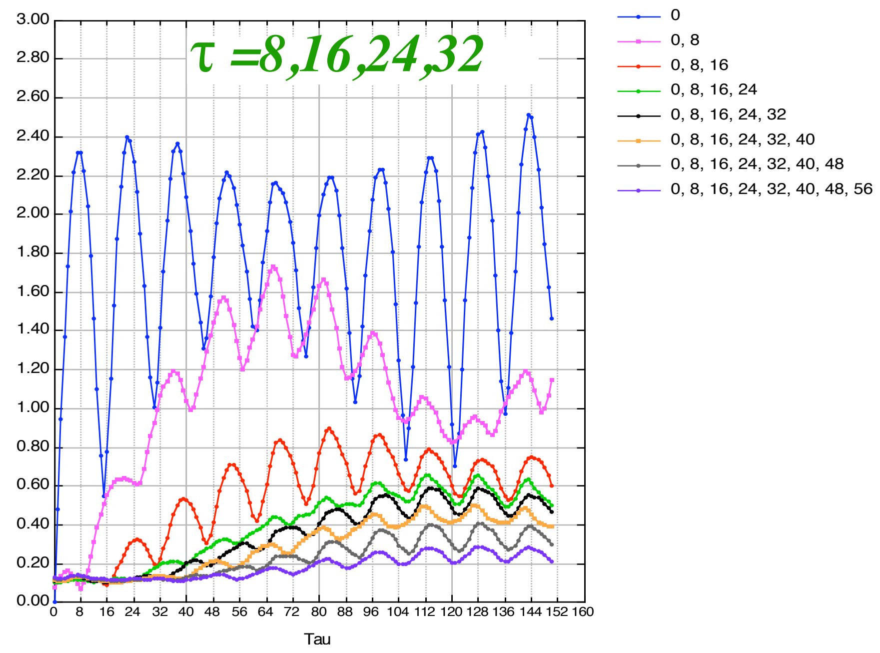
• Quasiperiodic Systems

Multiple Time Scales

Using function statistic



Constant τ (old way)



• Chaotic Systems and large delays

- All statistics (FNN, Redundancy, Continuity) will show minimum functional relationships for large τ in chaotic systems.
- Components of reconstruction vectors become "unrelated" for large τ . Finite data puts an upper bound on the size of τ we can use.

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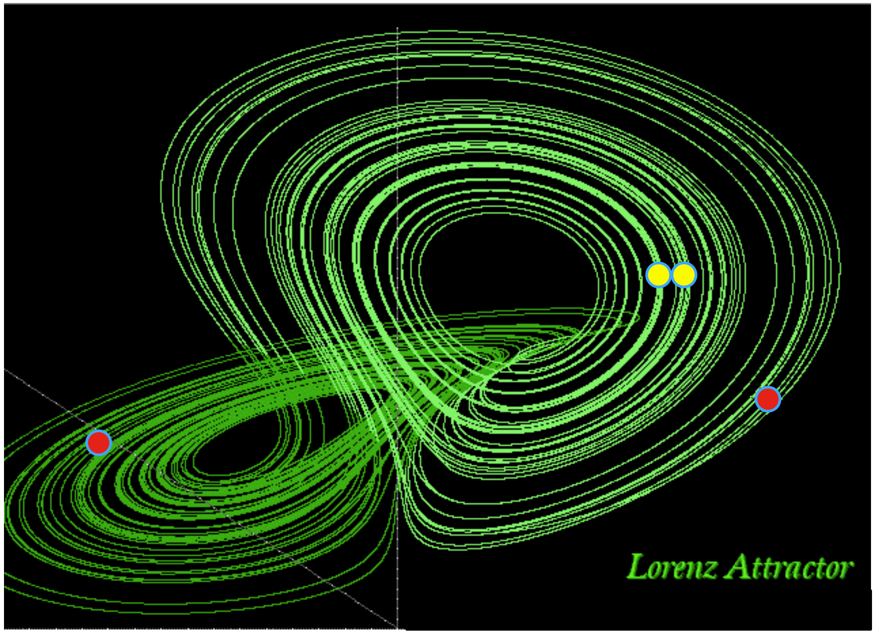

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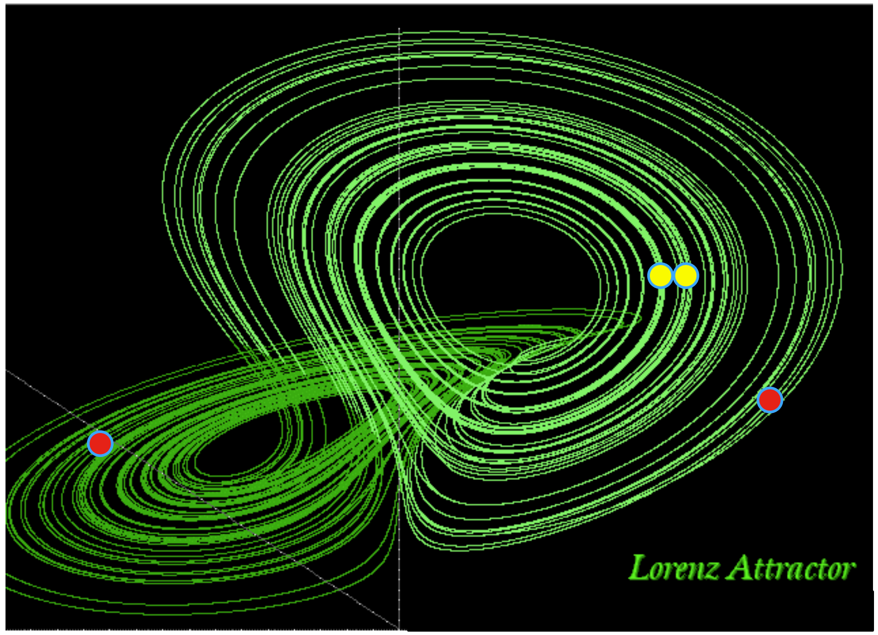
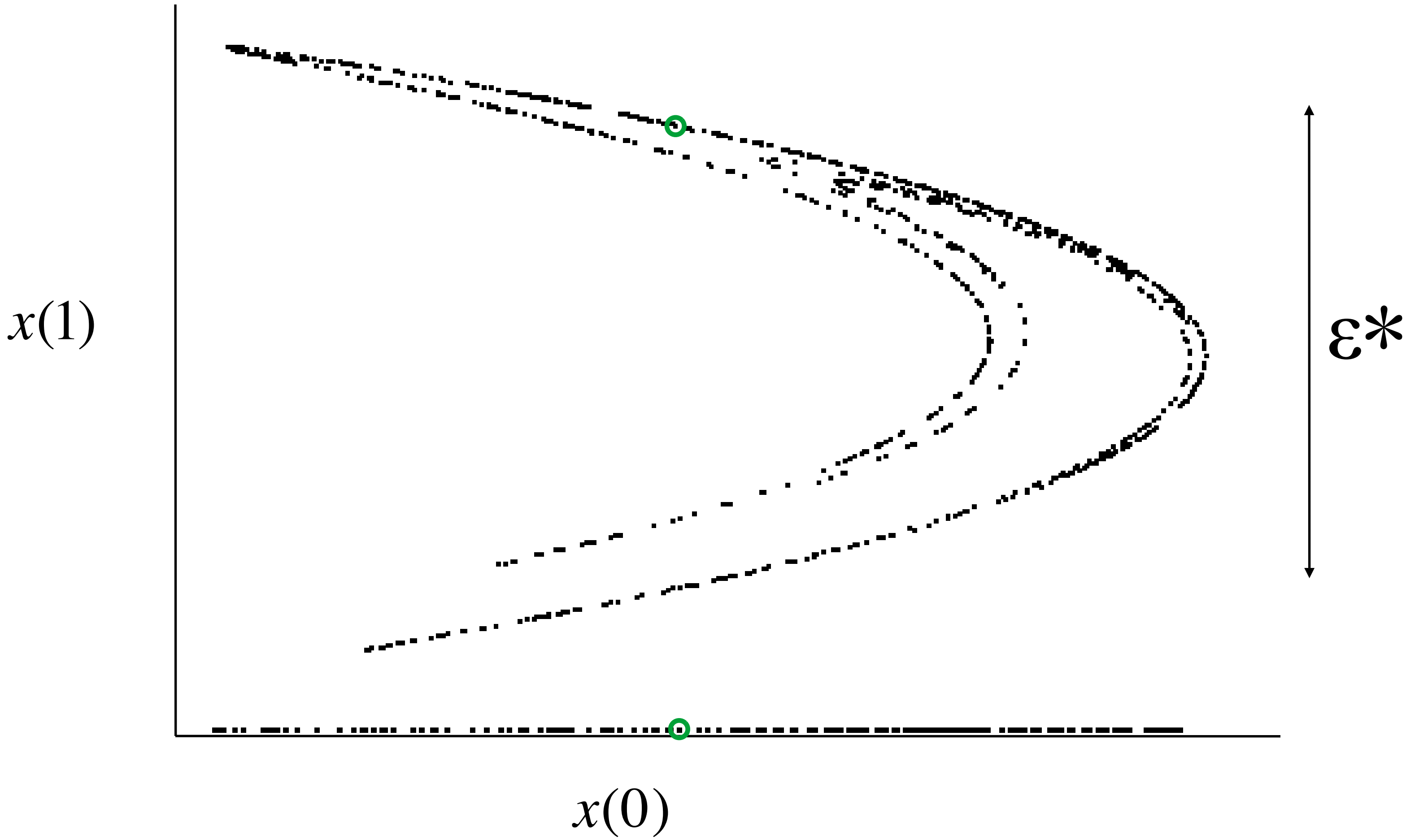


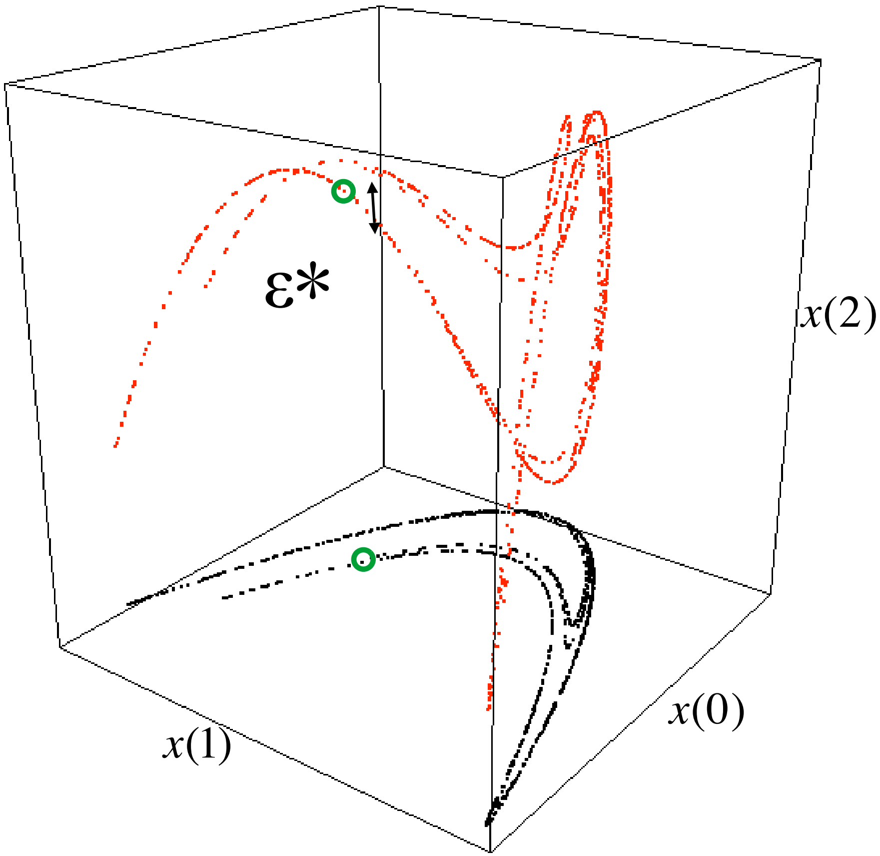
Henon 3D reconstruction

τ_1=0, τ_2=1

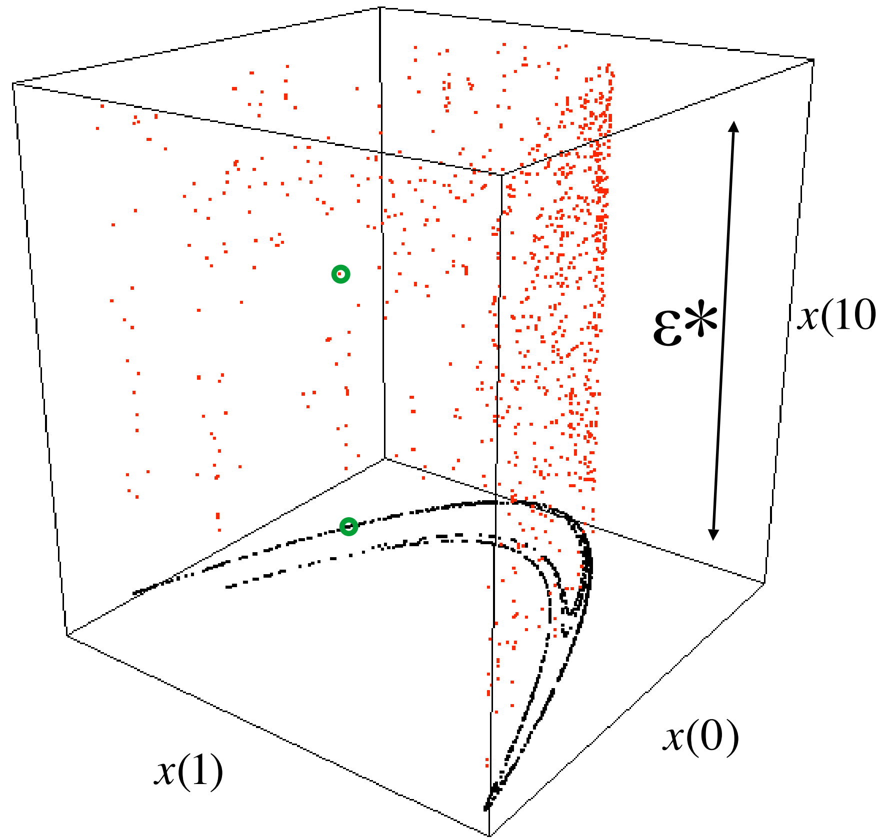


Henon 3D reconstruction

$\tau_1=0, \tau_2=1, \tau_3=2$



$\tau_1=0, \tau_2=1, \tau_3=\underline{10}$

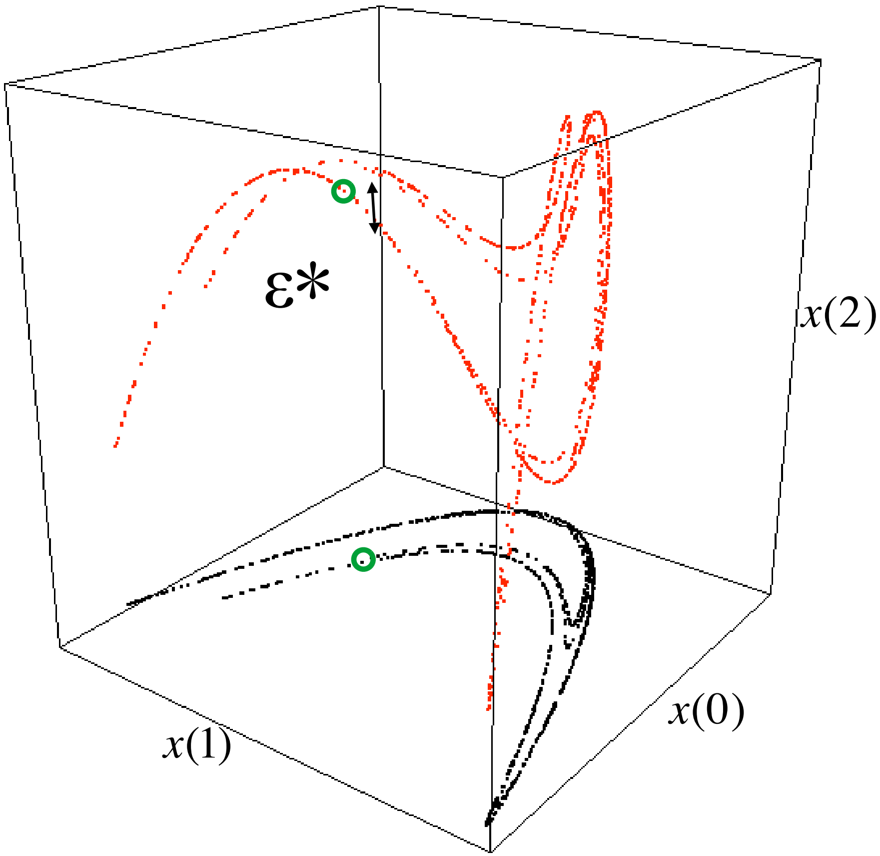


ε^* for $\mathbb{R}^d \rightarrow \mathbb{R}^1$

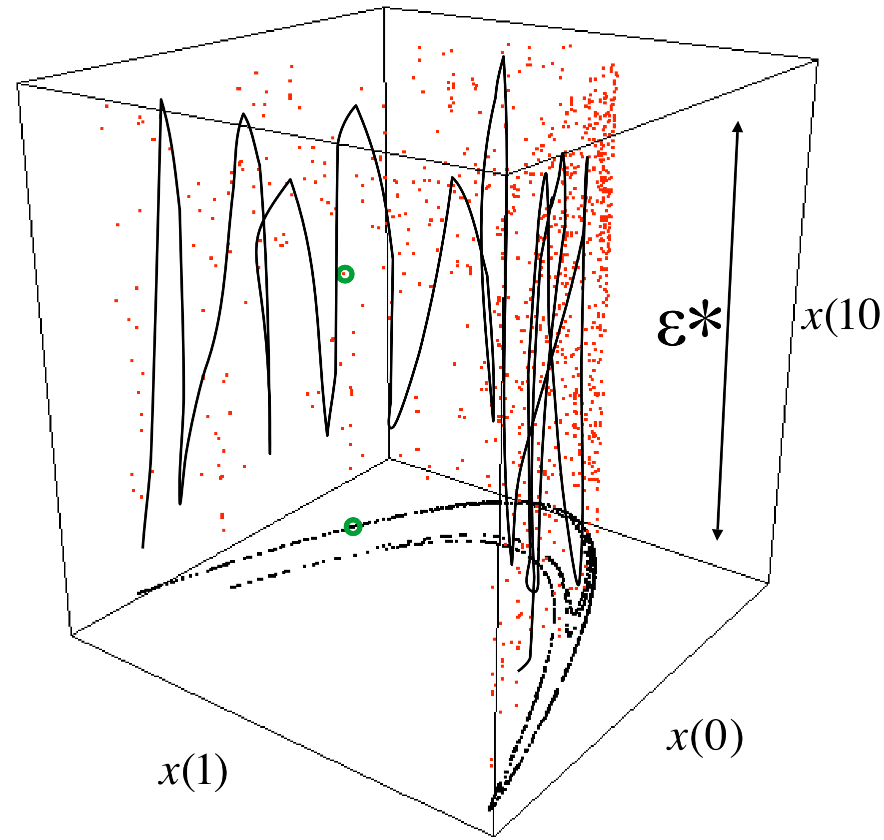
Get NN vector ξ in $\mathbb{R}^{d+1}$

Henon 3D reconstruction

$\tau_1=0, \tau_2=1, \tau_3=2$



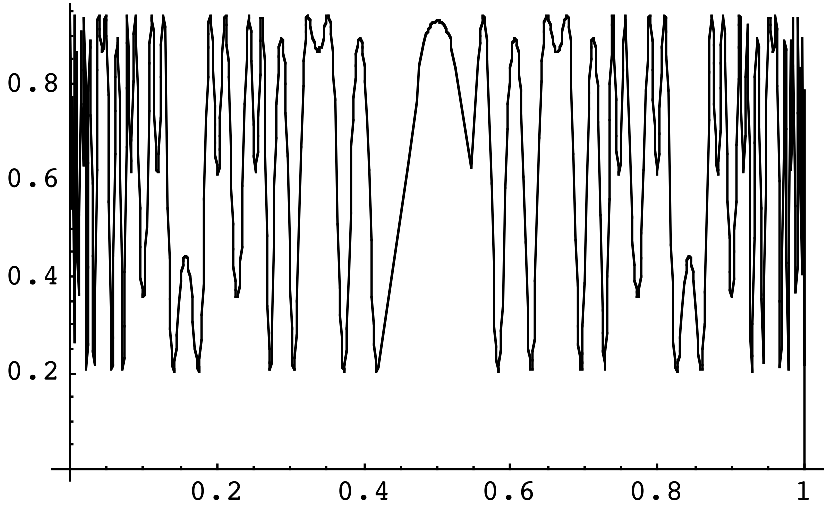
$\tau_1=0, \tau_2=1, \tau_3=\underline{10}$



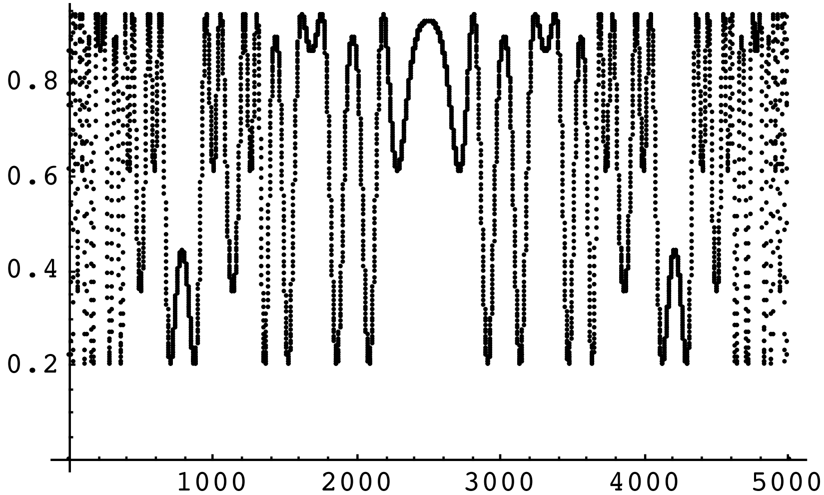
ε^* for $\mathbb{R}^d \to \mathbb{R}^1$

Get NN vector ξ in $\mathbb{R}^{d+1}$

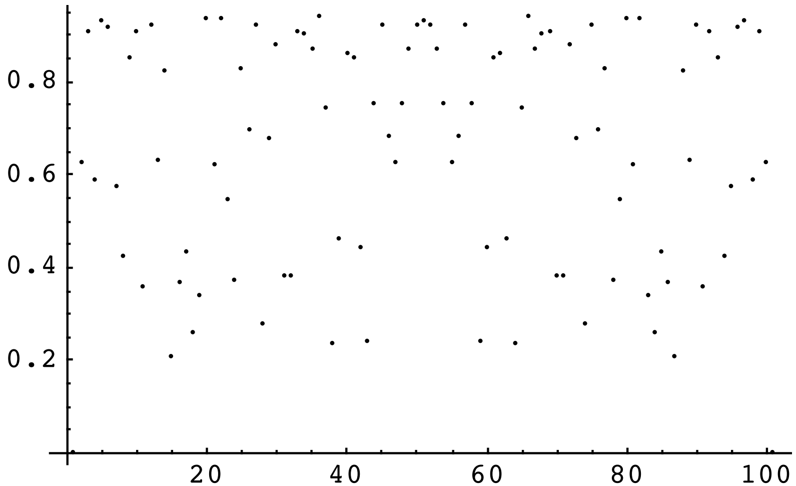
Logistic map, 8 iterations, a=3.77 $ax(1-x)$



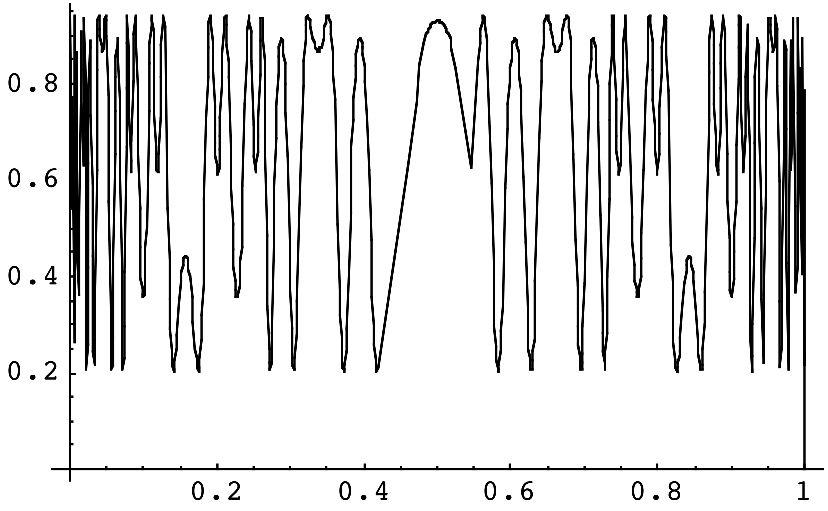
5000 point sampling



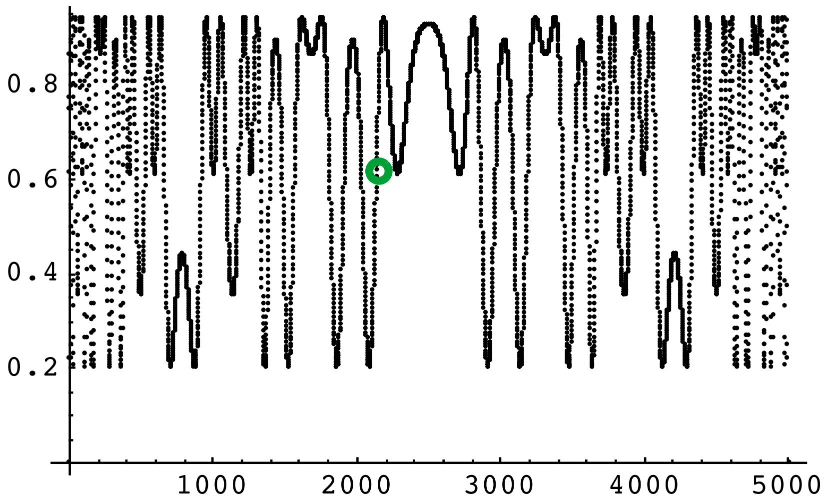
100 point sampling



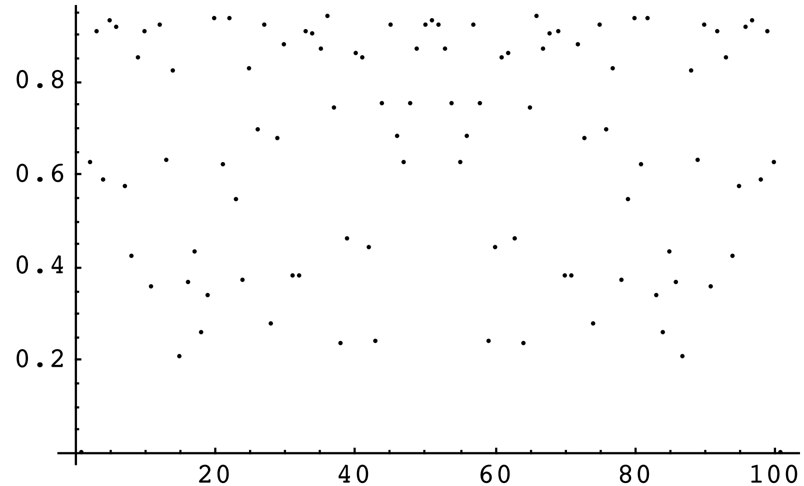
Logistic map, 8 iterations, *a*=3.77 $ax(1-x)$



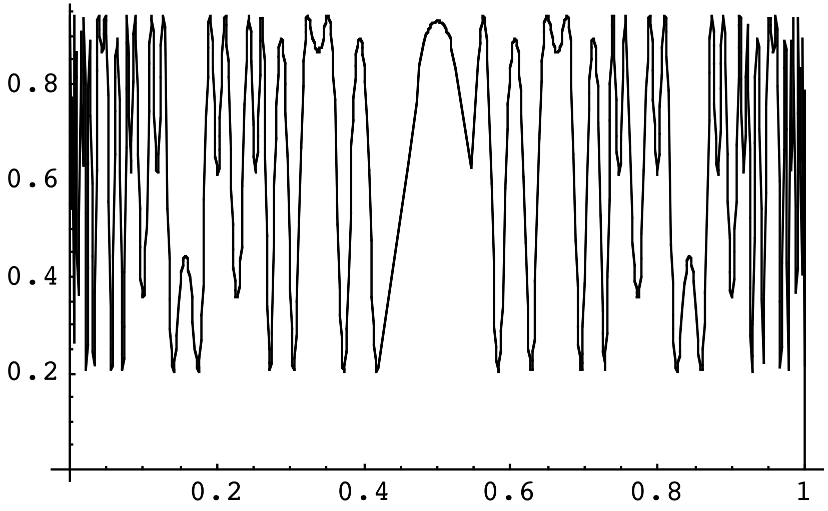
5000 point sampling



100 point sampling



Logistic map, 8 iterations, a=3.77 $ax(1-x)$



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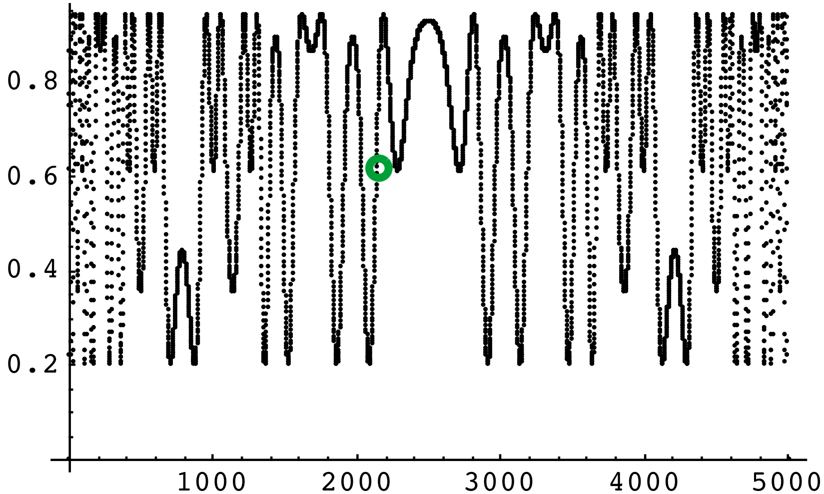
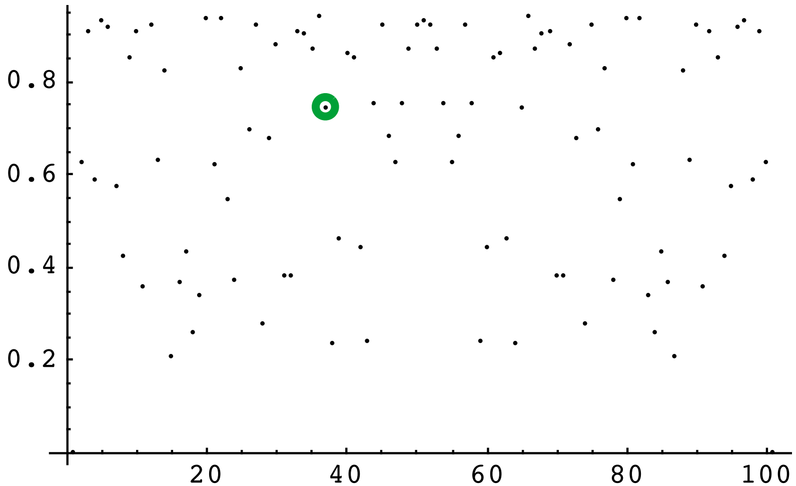
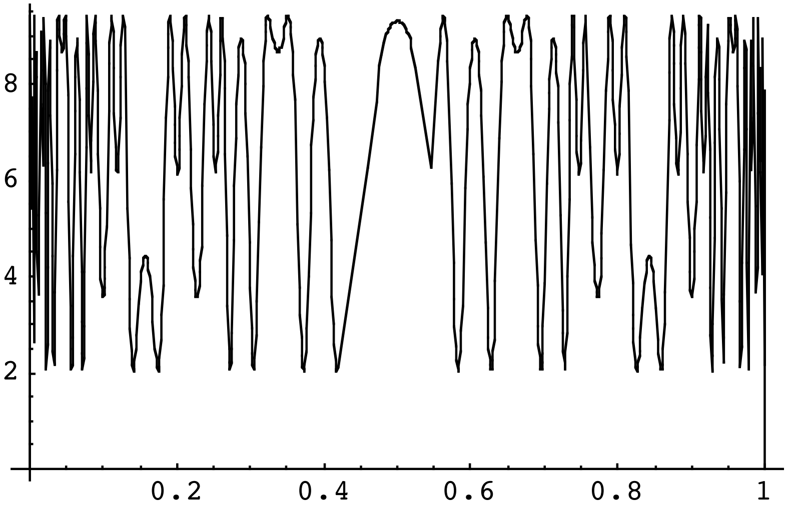


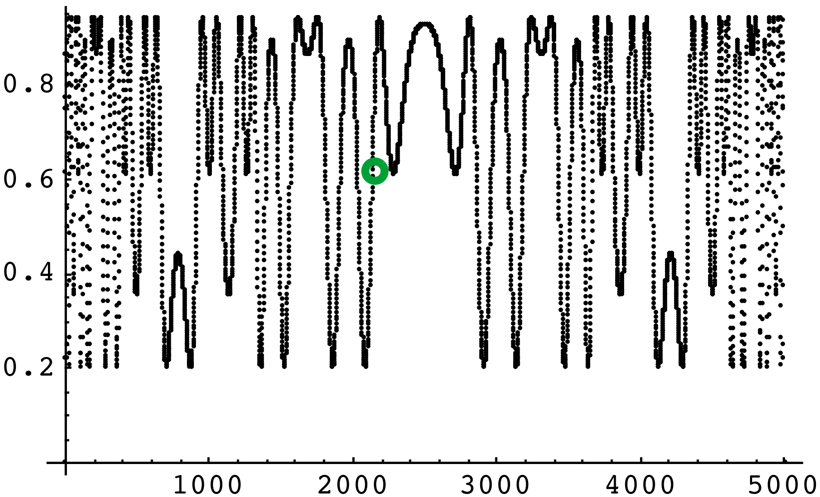
100 point sampling



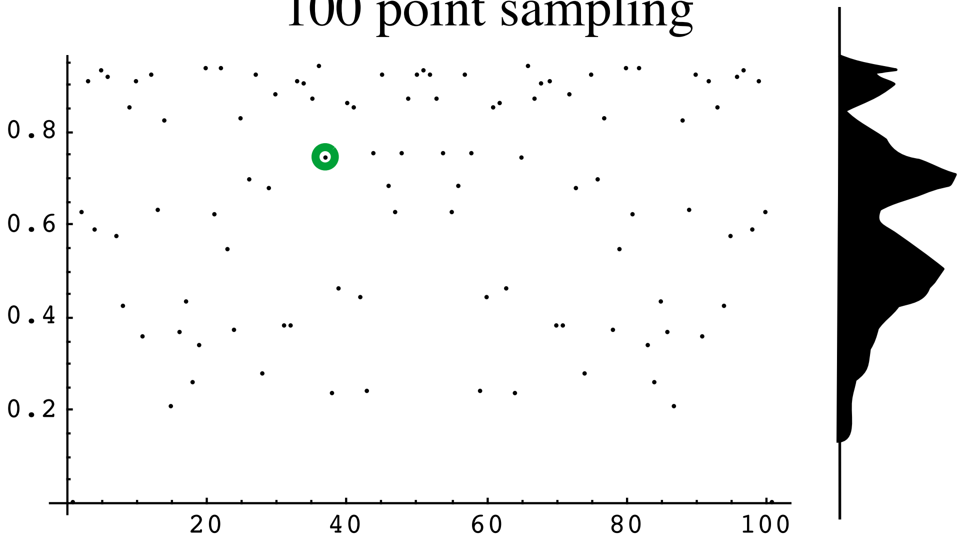
Logistic map, 8 iterations, a=3.77 $ax(1-x)$



5000 point sampling



100 point sampling



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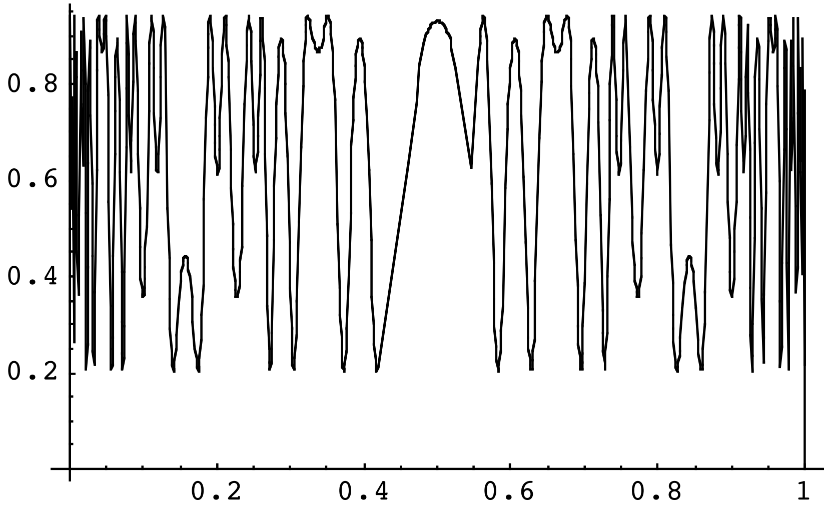
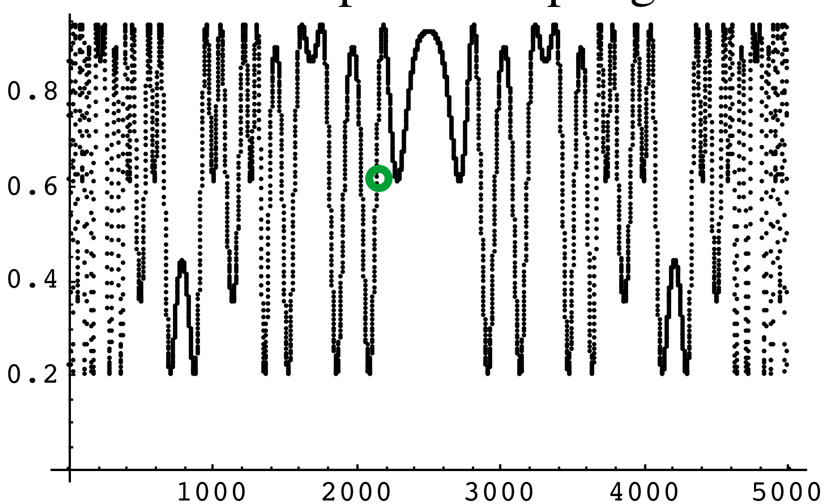
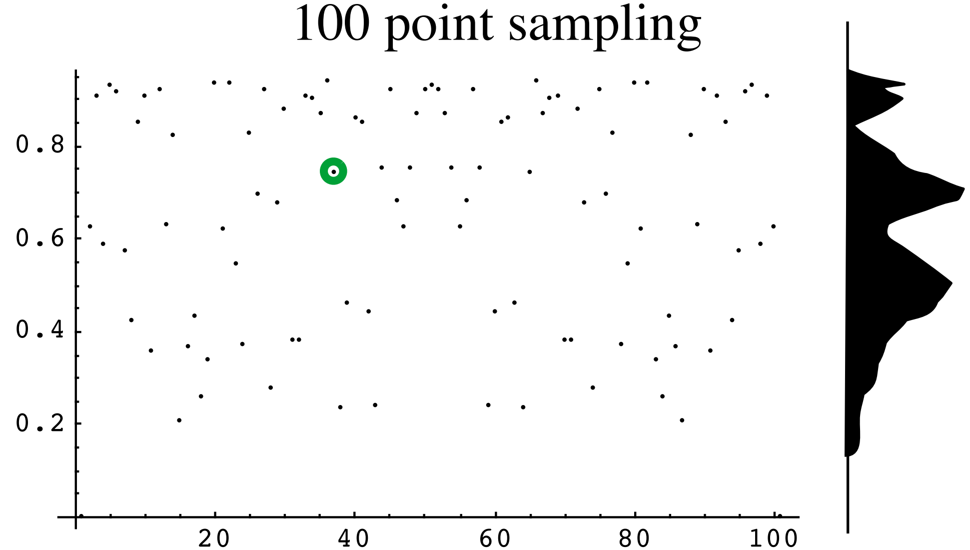


- "Scalar observations from a class of high-dimensional chaotic systems: Limitations of the time delay embedding," H. Kantz and E. Olbrich, Chaos 7 (3), 423 (1997).

5000 point sampling



100 point sampling



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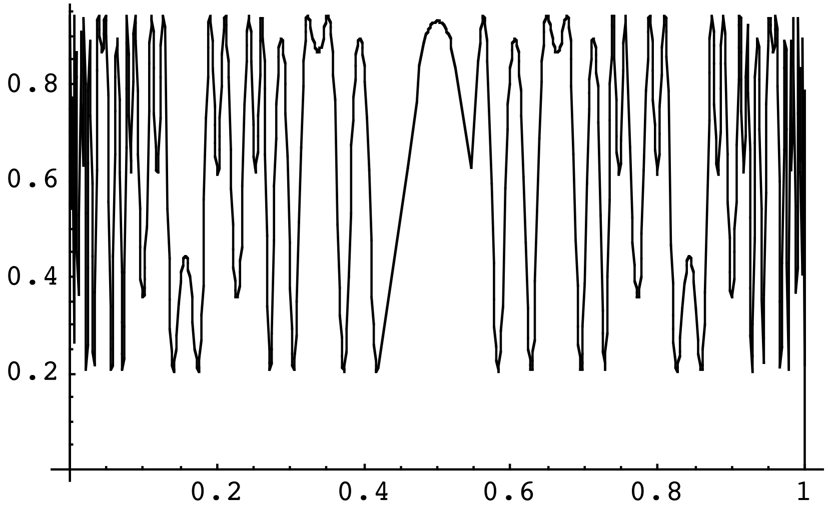

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- "Inferring chaotic dynamics from time-series: On which length scale determinism becomes visible.," E. Olbrich and H. Kantz, Physics Letters A 232, 63 (1997).

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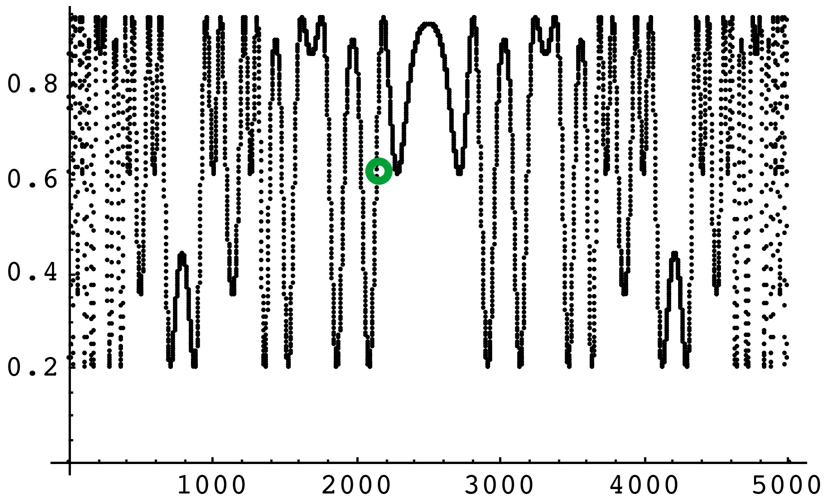
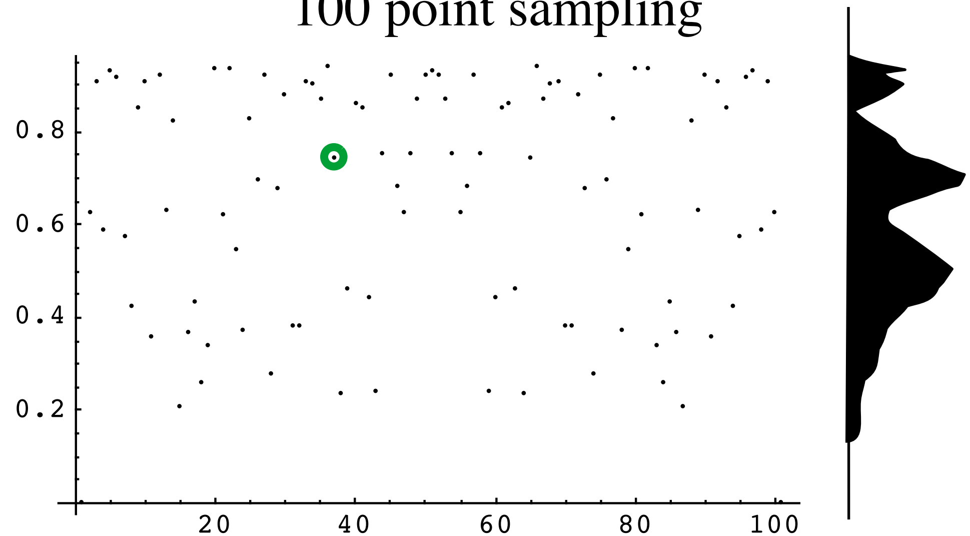


100 point sampling

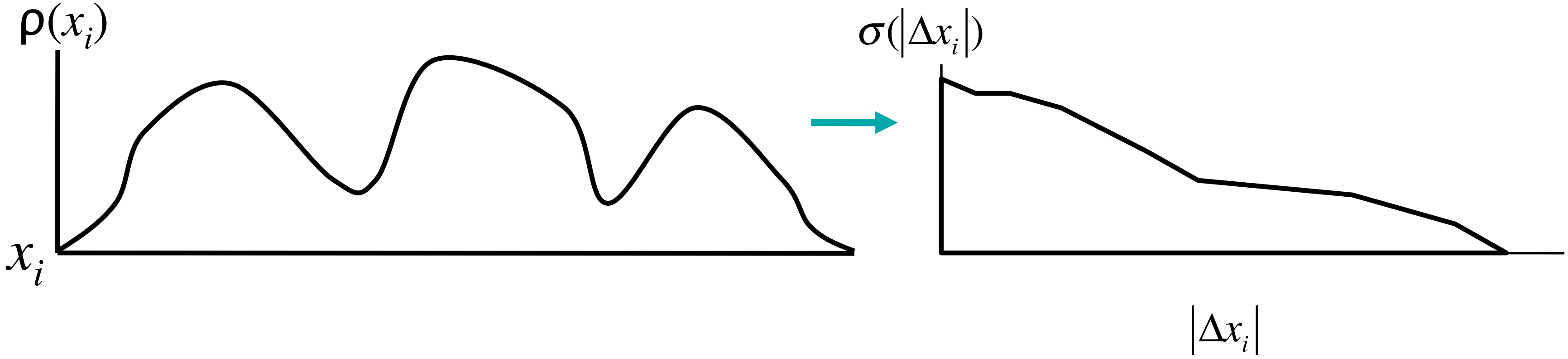


• Chaotic Systems and large delays

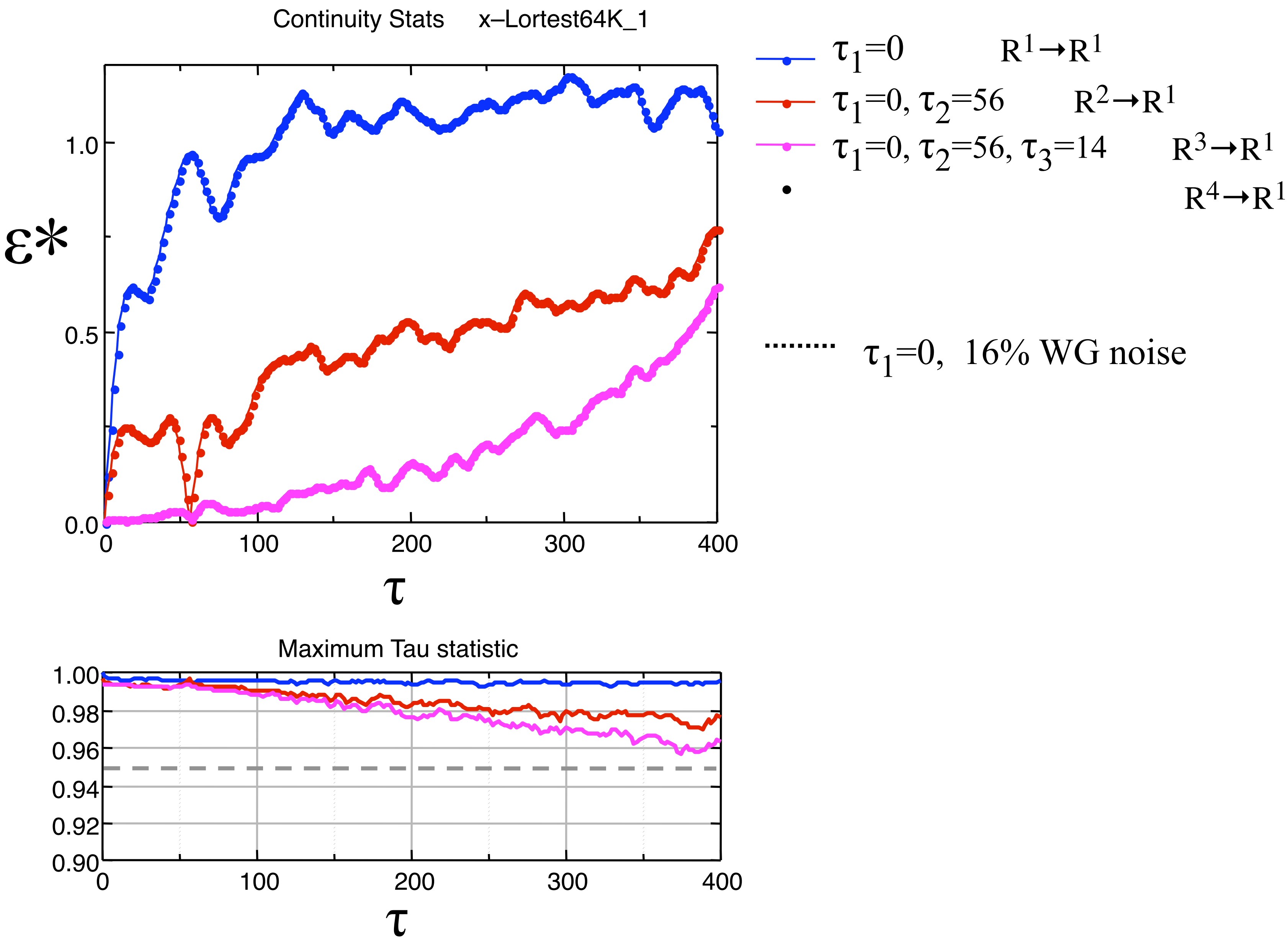
Null Hypothesis: One component of the reconstruction vector is statistically independent of all others.

Maximum τ Statistic: Test components of differences between **nearest neighbors** ξ against the distribution of differences between randomly chosen points in the time series distribution

Δx= NN vector



Lorenz *x* time series reconstruction



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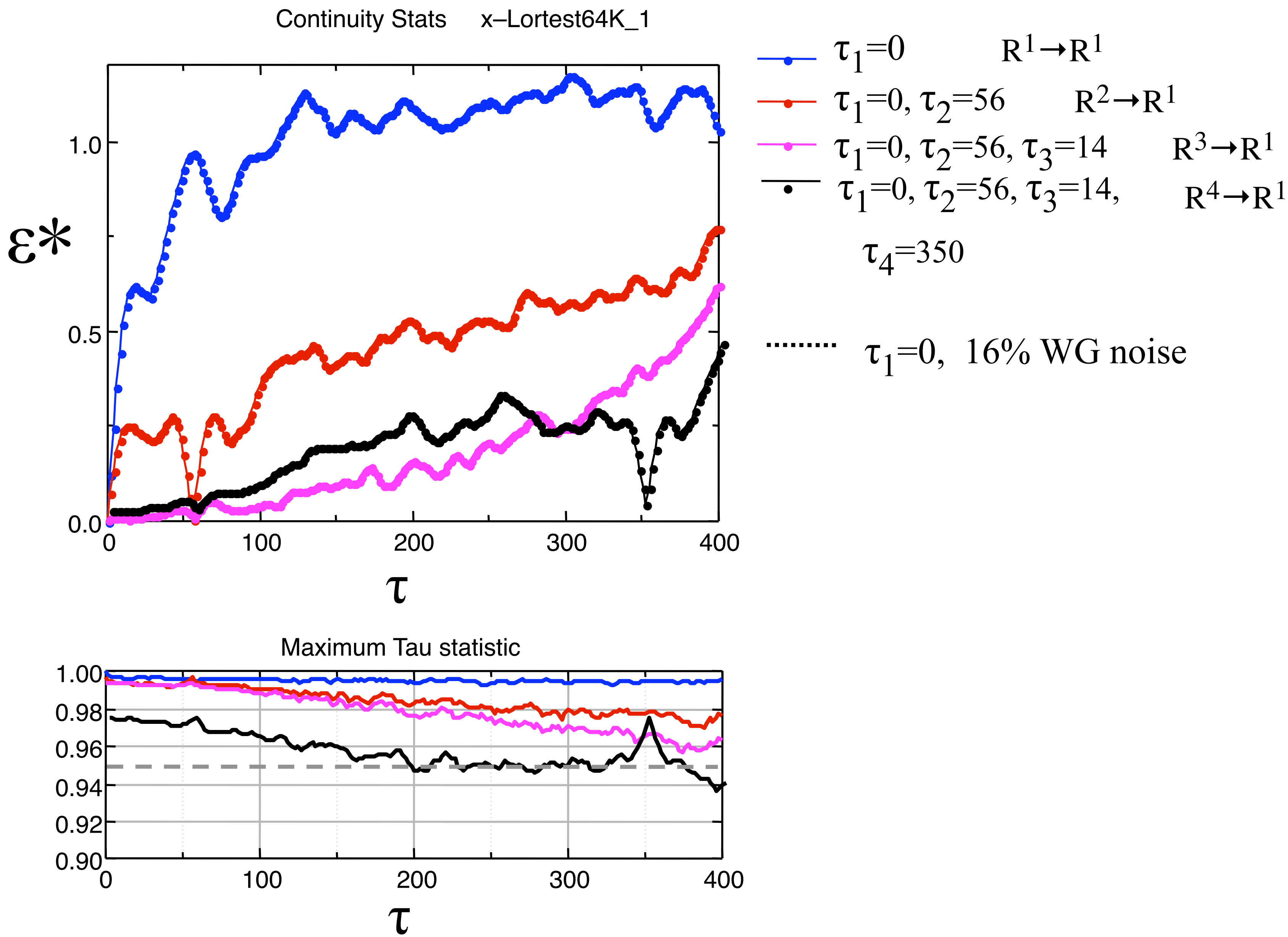

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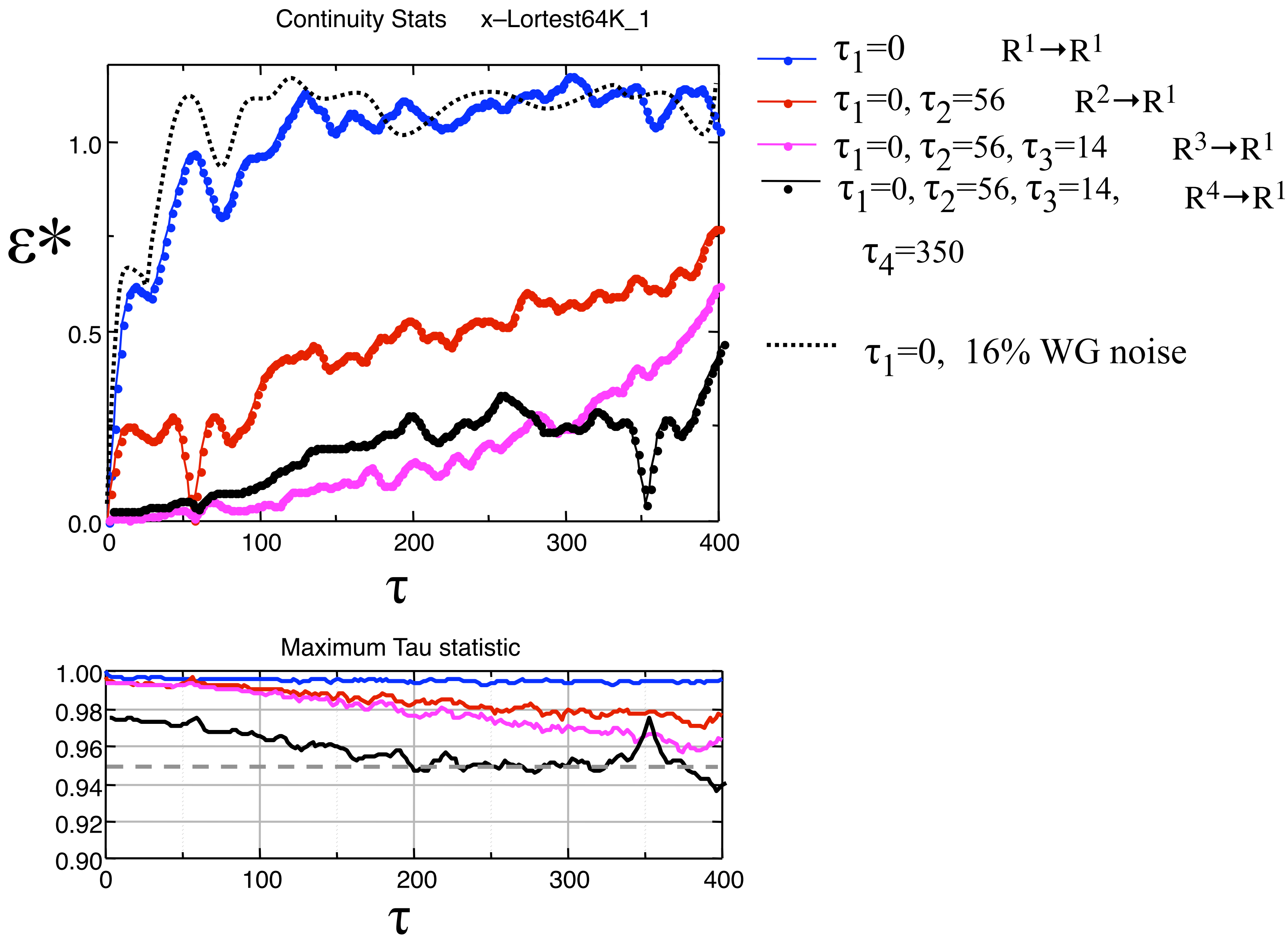
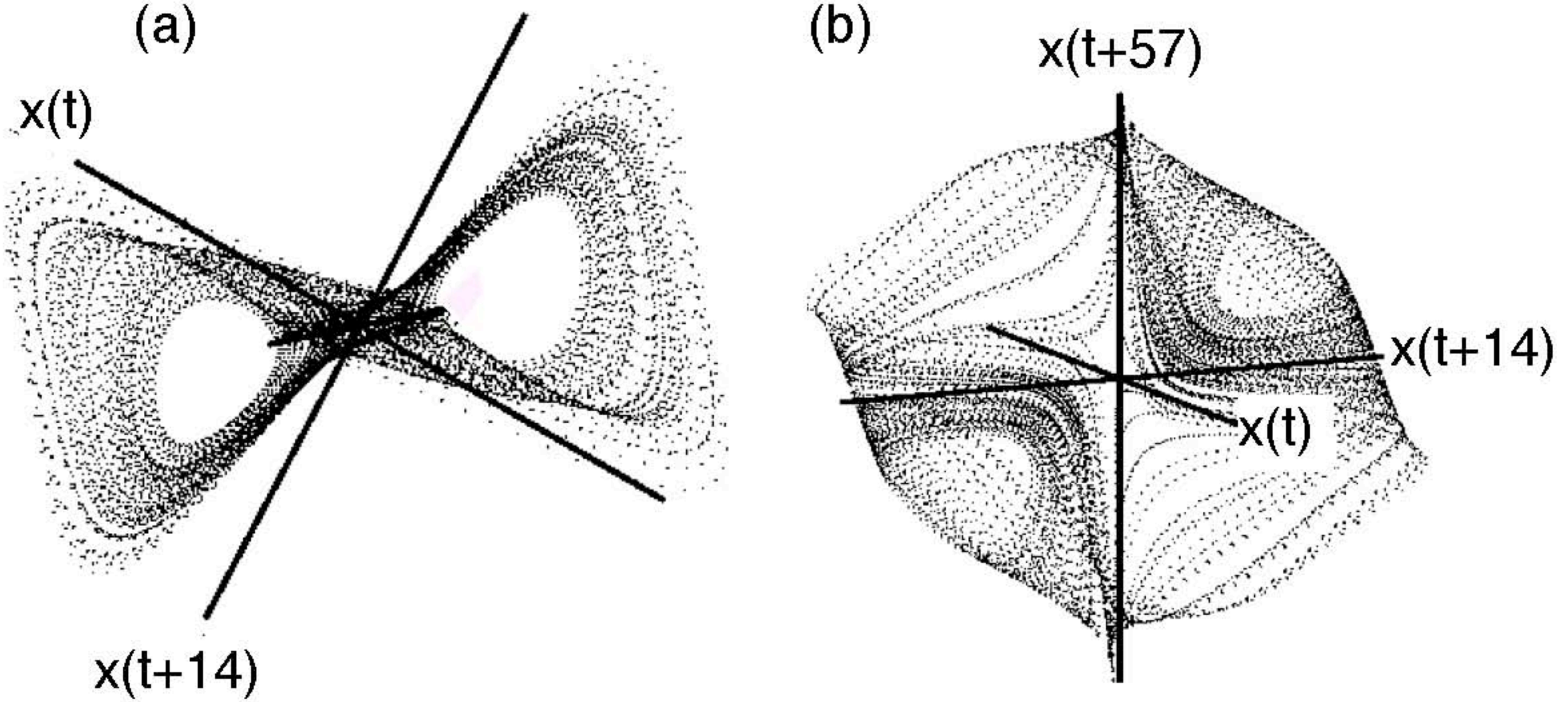
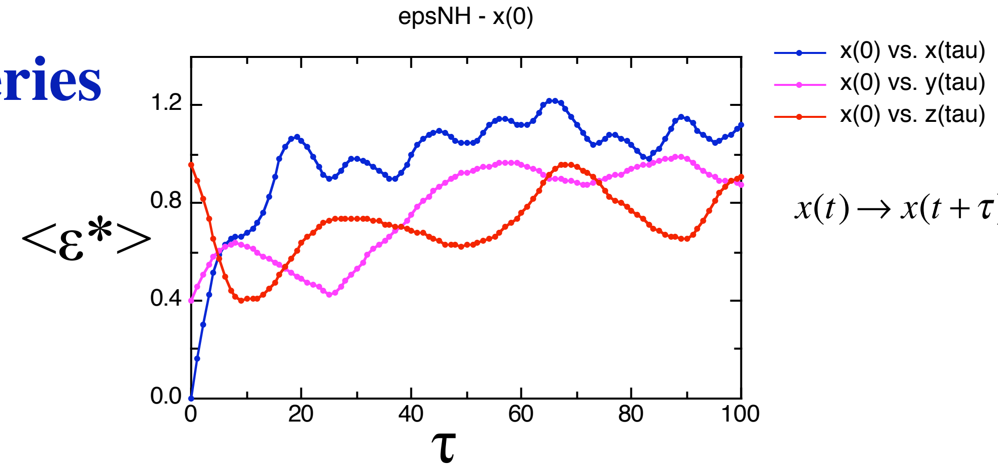


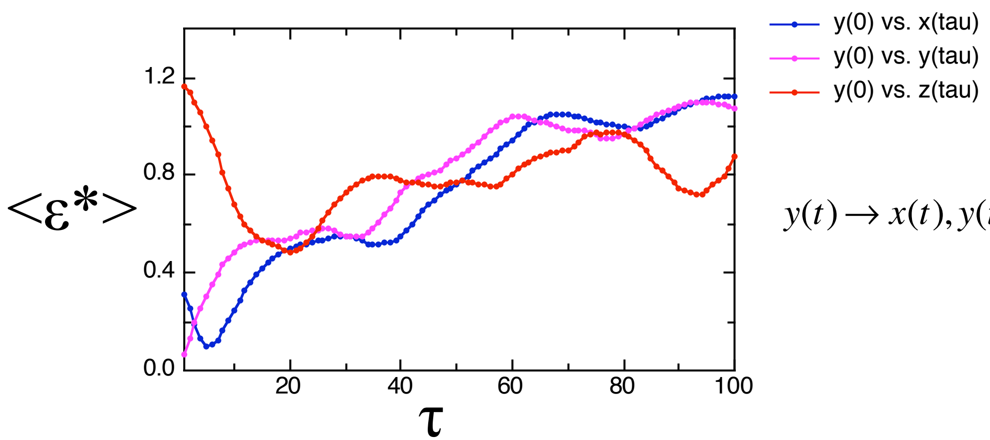

(a)
x(t)
x(t+14)
(b)
x(t+57)
x(t+14)
x(t)

• Multivariate Time Series

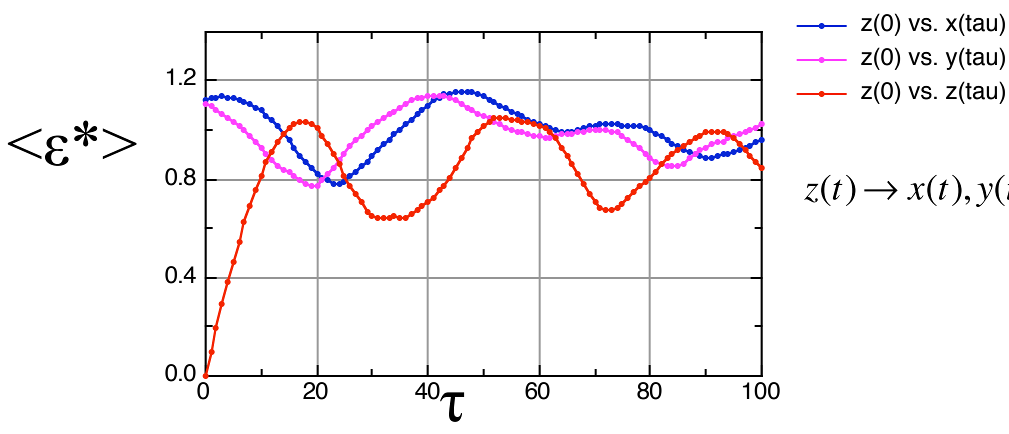
Lorenz: $x(t), y(t), z(t)$



$x(t) \rightarrow x(t+\tau), y(t), z(t)$



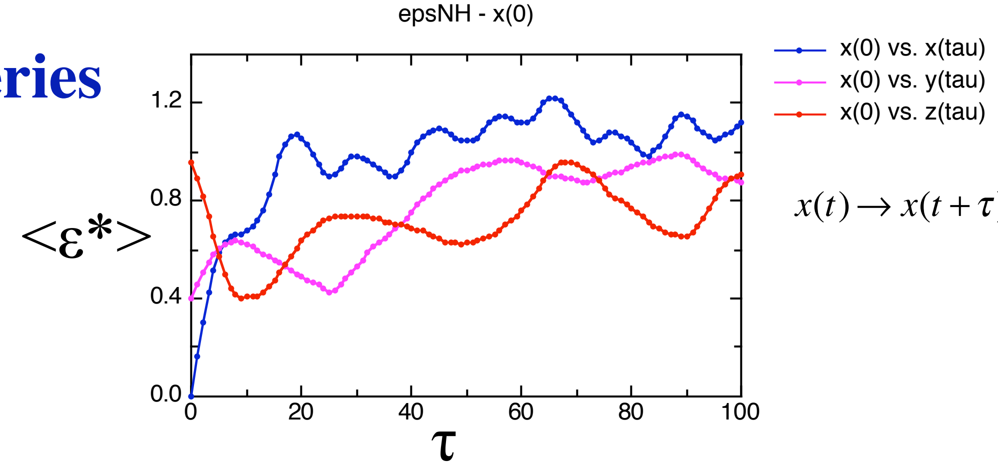
$y(t) \rightarrow x(t), y(t+\tau), z(t)$



$z(t) \rightarrow x(t), y(t), z(t+\tau)$

• Multivariate Time Series

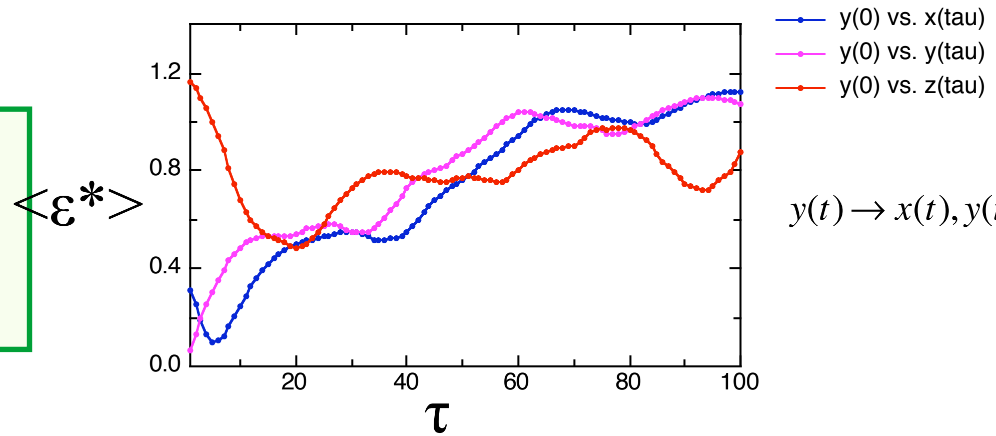
Lorenz: $x(t), y(t), z(t)$



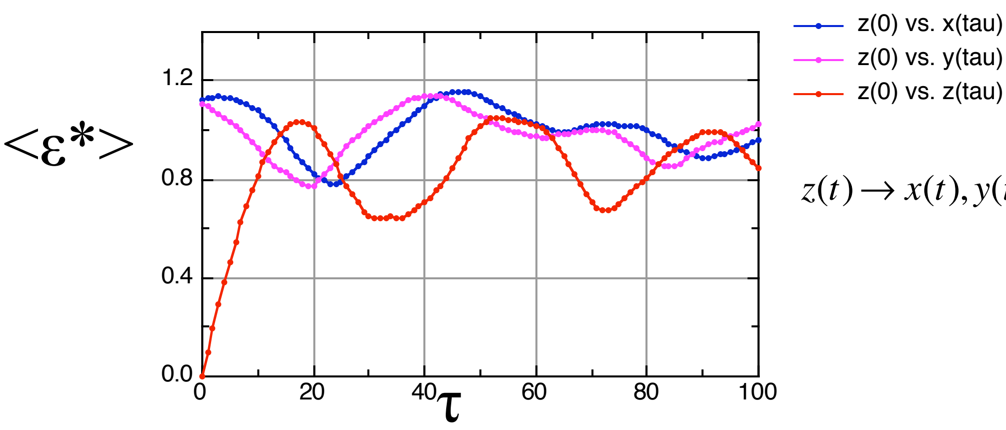
$x(t) \rightarrow x(t+\tau), y(t), z(t)$

preferred reconstruction:

$x(t), x(t+\tau), z(t)$



$y(t) \rightarrow x(t), y(t+\tau), z(t)$

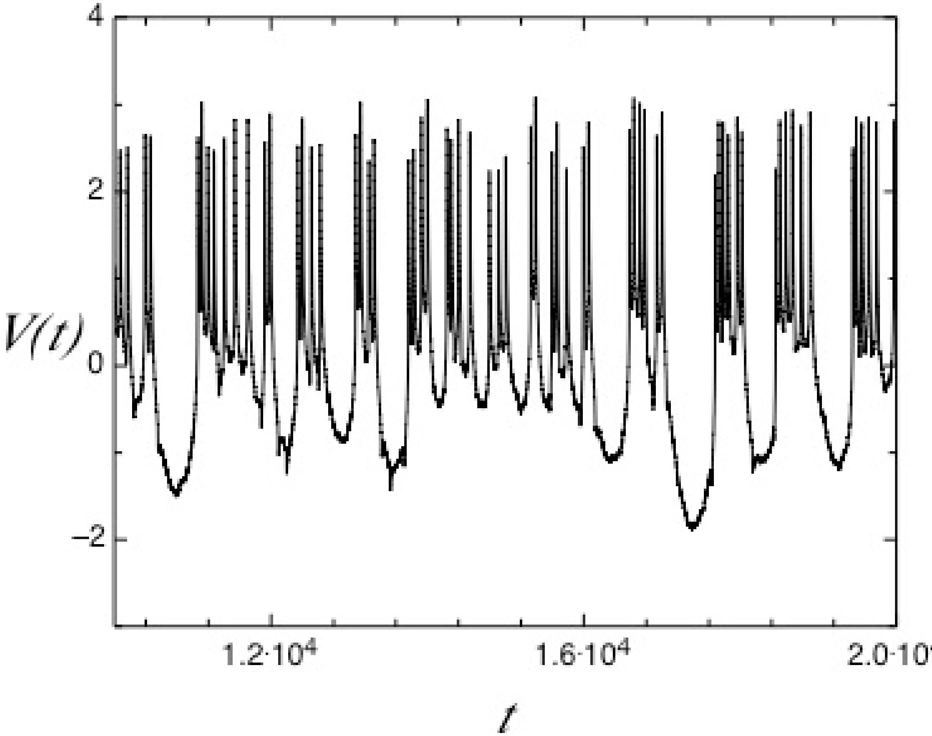


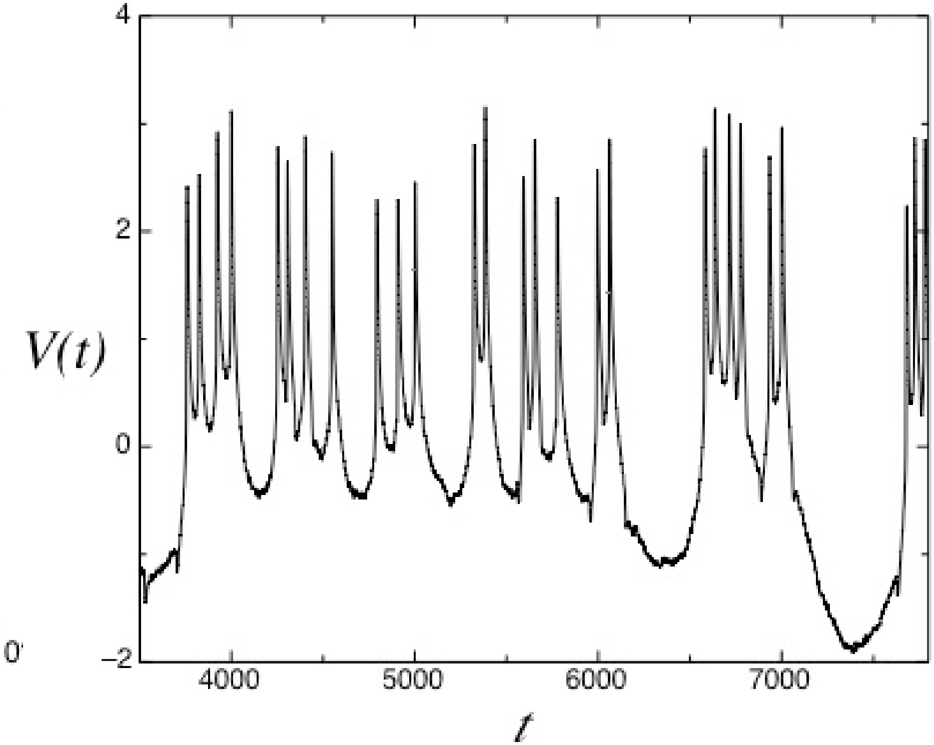
$z(t) \rightarrow x(t), y(t), z(t+\tau)$

Neuronal time series. Lobster stomatogastric ganglia. CPG for swallowing.

INLS - UCSD

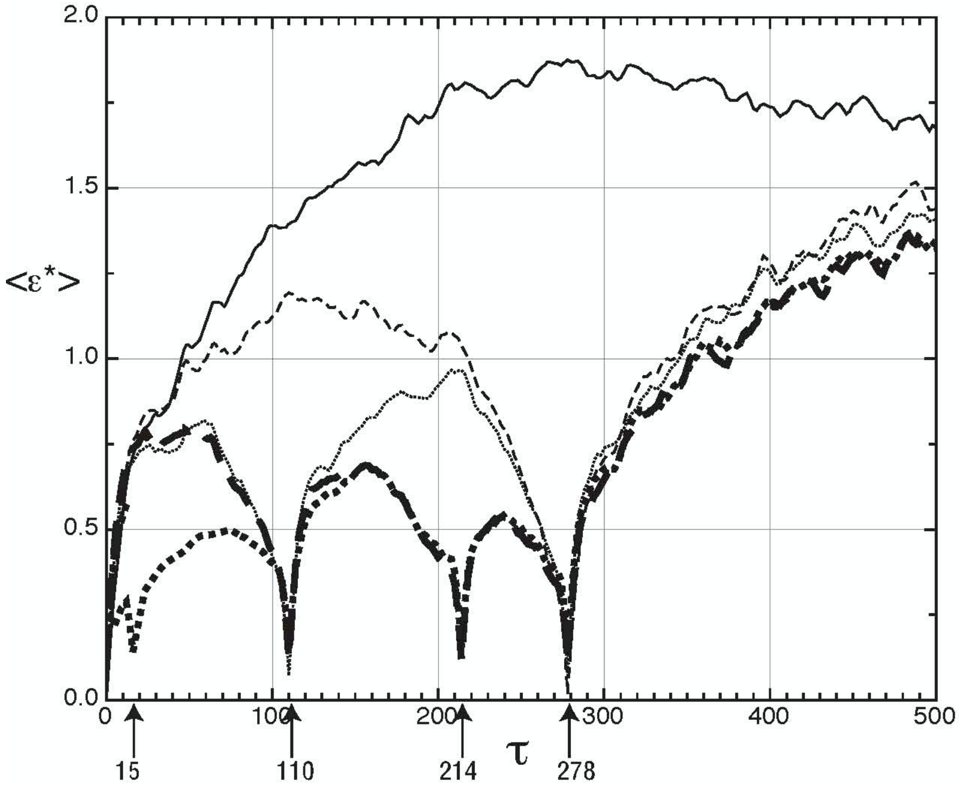
R.C. Elson, A.I. Selverston, R. Huerta et al., Physical Review Letters 81 (25), 5692 (1998).
J.L. Hindmarsh and R.M. Rose, Proceedings of the Royal Society of London B221, 87 (1984).
Martin Falcke, Ramón Huerta, Mikhail I. Rabinovich et al., Biological Cybernetics 82, 517 (2000).



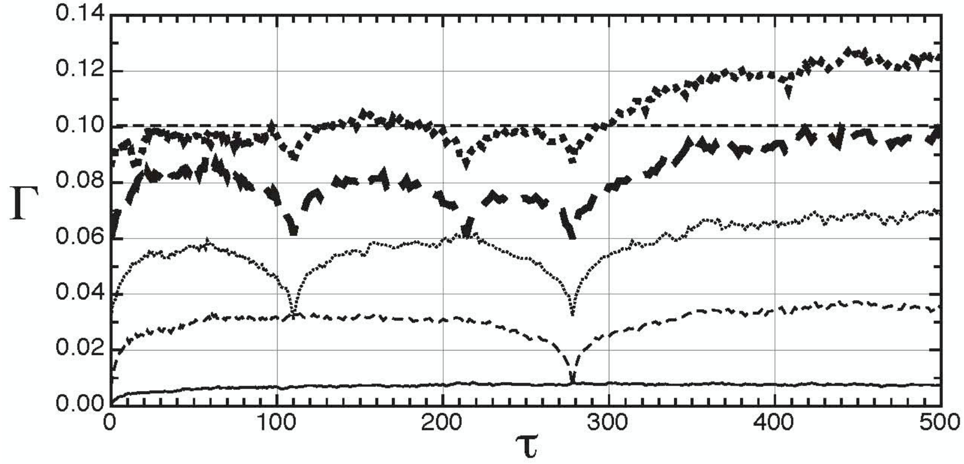


Neuronal time series.

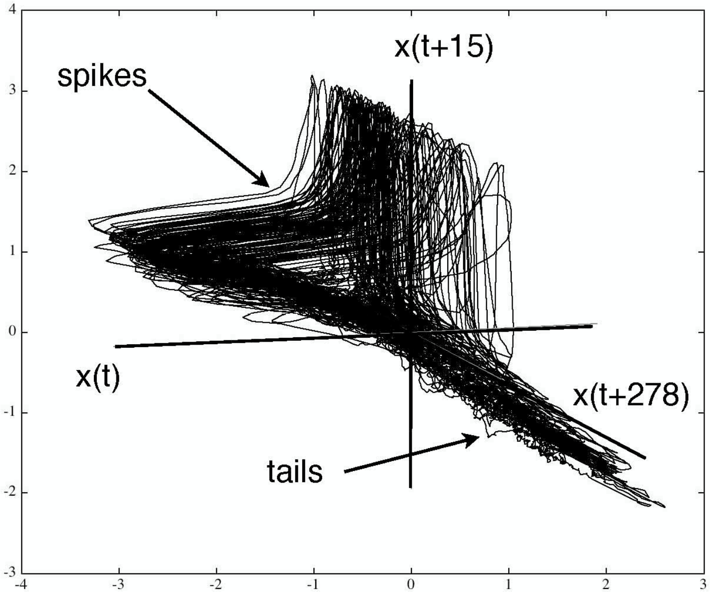
Continuity statistic.

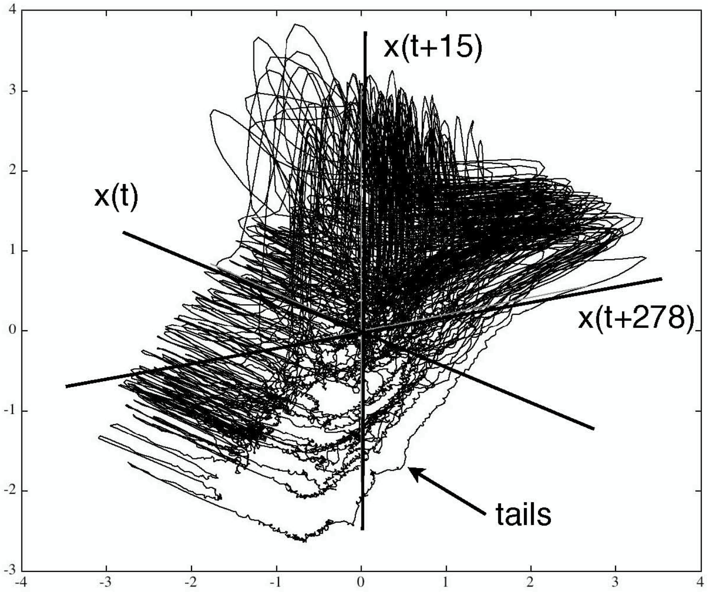


Under-embedding statistic.



Neuronal attractor
(projected into 3D)



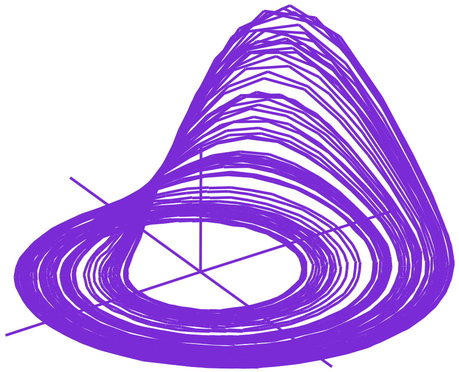


Conclusions & Remarks

- A unified approach <ε*> that offers solutions of
  - delay or advances - multiple time scales
  - embedding dimension - able to give 'minimal' dimension
  - multivariate time series - choice of which to use
- A geometric view of the effect of large delays: an undersampling of a highly folded manifold.
- The Maximum τ statistic offers a reasonable stopping point for delay times.
- Here we used a "greedy algorithm".
  Optimal reconstruction: a combinatorial problem. Test all combinations of possible time series and delays.

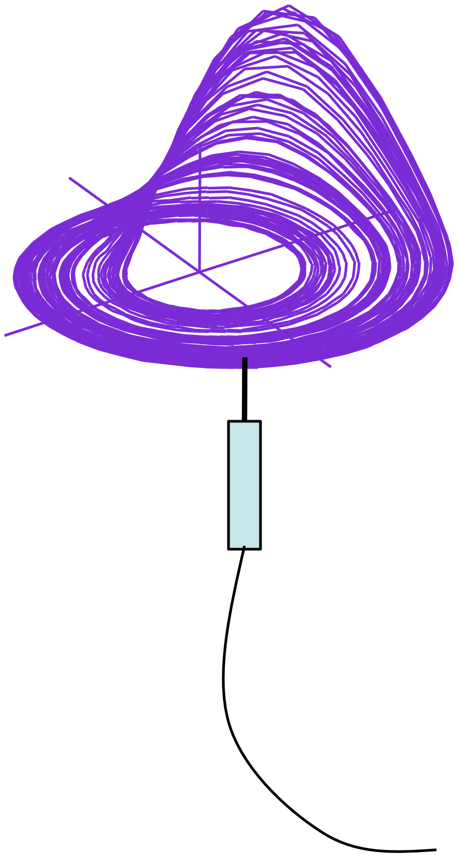
A Unified Approach to Attractor Reconstruction,
CHAOS **17** to appear March 2007

The End



Multidimensional state space and attractors capture the geometry of a dynamical system (*x,y,z*).

Measurements of physical systems come from sensors that rarely measure the dynamical variables directly and usually are smaller in number than the number of dynamical variables.



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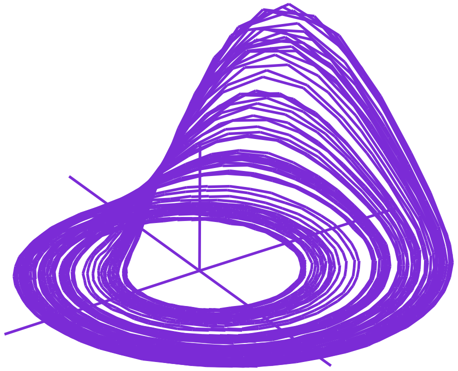
$$s(t_1), s(t_2), s(t_3), \ldots, s(t_N)$$



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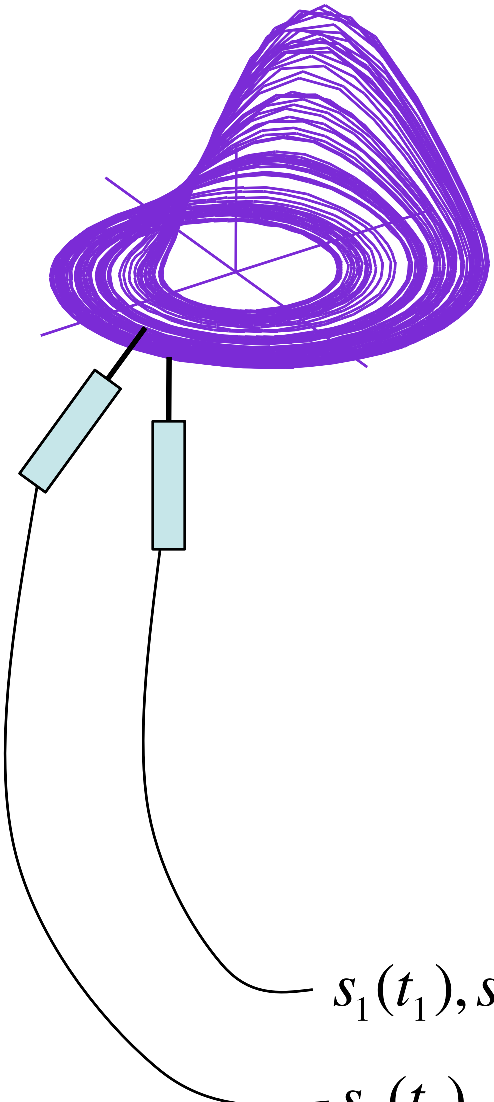

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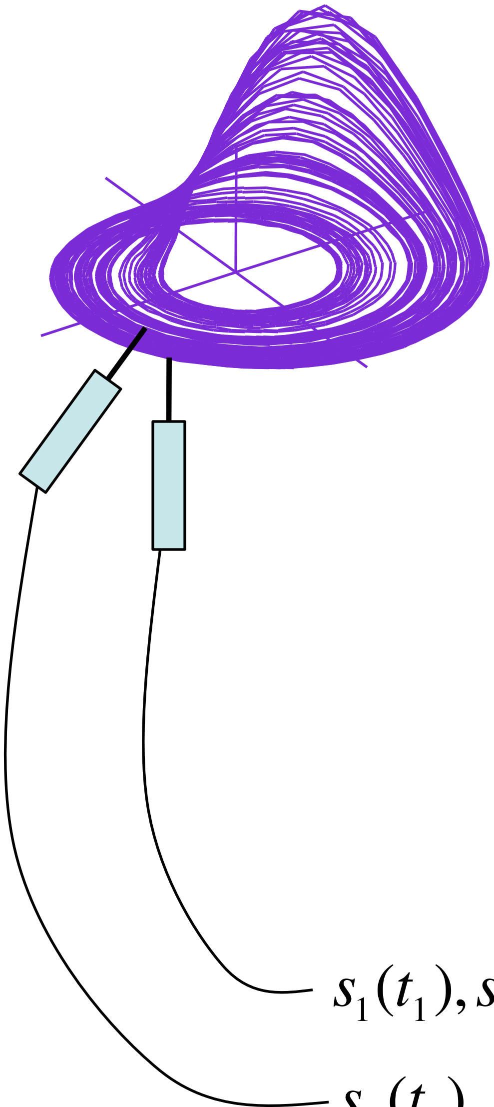
$$s_1(t_1), s_1(t_2), s_1(t_3), \ldots, s_1(t_N)$$

$$s_2(t_1), s_2(t_2), s_2(t_3), \ldots, s_2(t_N)$$



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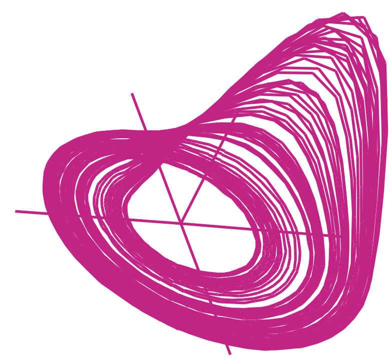
$$s_2(t_1), s_2(t_2), s_2(t_3), \ldots, s_2(t_N)$$

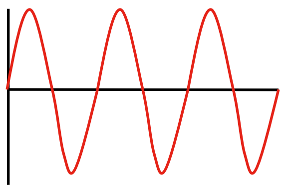
? →



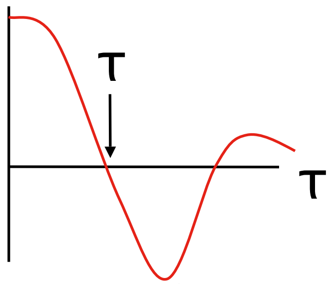
Issues in Reconstruction and current approaches

- Finding the time delay - heuristic approaches to independence.

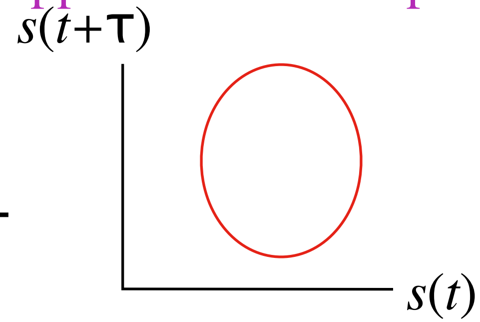
Auto correlation.



time series

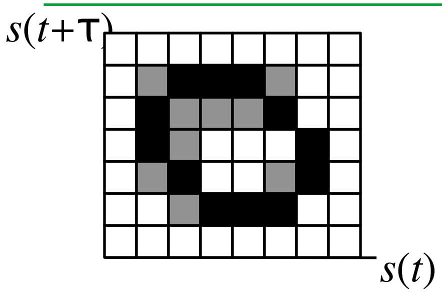


autocorrelation



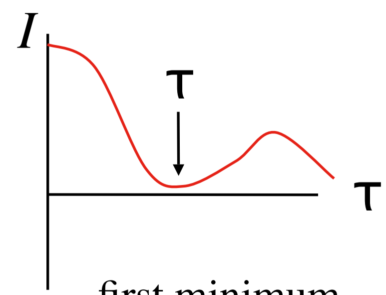
phase plot

Mutual information.



$$I(\tau) = \sum_{ij} P_{ij} \ln\left[P_{ij}\right] - 2\sum_{i} P_i \ln\left[P_i\right]$$

A.M. Fraser and H.L. Swinney,
Physical Review A 33, 1134 (1986).



first minimum

Weaknesses.

2D only, good for $s(t)$ vs. $s(t+\tau)$, but what about $(s(t), s(t+\tau))$ vs. $s(t+2\tau)$?

Autocorrelation may not go to zero.

Mutual information is symmetric and requires choice of bin size.

How to handle multiple time scales? Unsolved problem.

- Finding the embedding dimension.

False Nearest Neighbors (FNN).

M.B. Kennel, R. Brown, and H.D.I. Abarbanel, Physical Review A 45, 3403 (1992).
M.B. Kennel and H.D.I. Abarbanel, Physical Review E 47, 3057 (1993).
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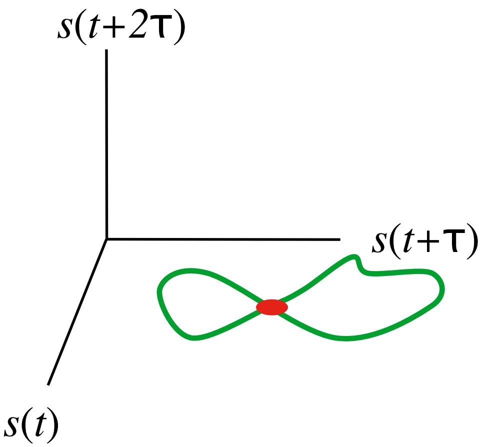

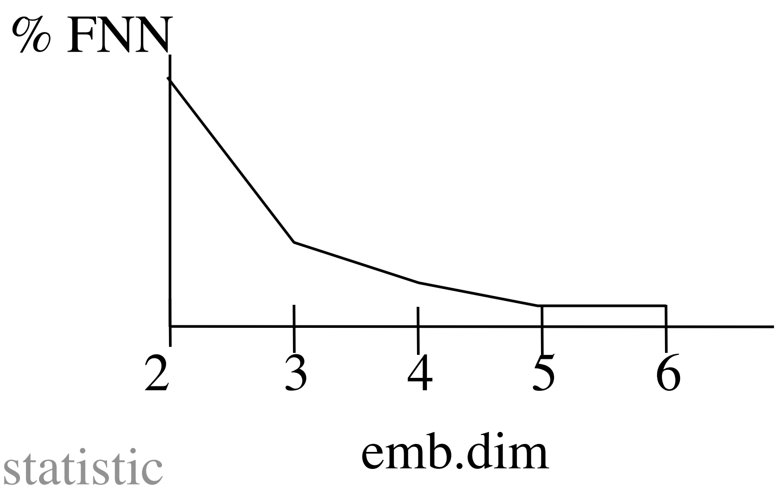

Note: like a (dis)continuity statistic

Weakness.

Weaknesses.

Necessary to pick a scale (threshold): ~ 1 std is recommended - why?

When to stop adding components? Chaotic signals always generate FNN for large enough τ

Suggested implementation has same τ for each emb. dim

- Finding the embedding dimension.

False Nearest Neighbors (FNN).

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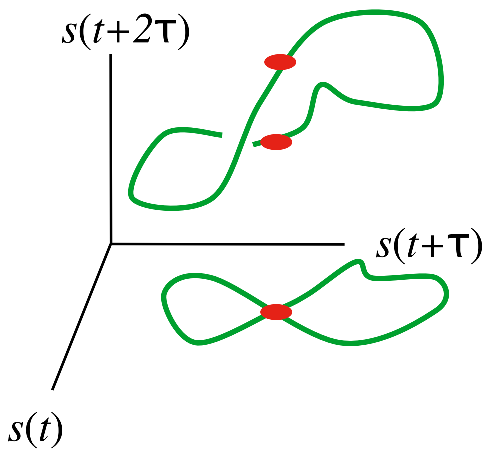
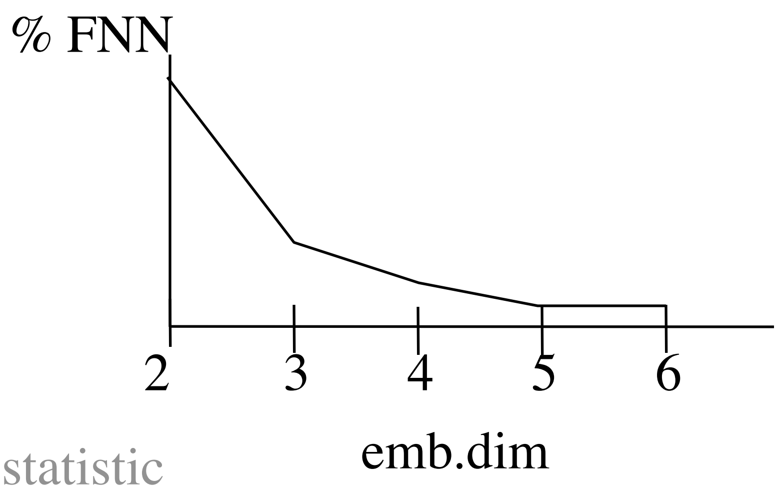
M.B. Kennel and H.D.I. Abarbanel, Physical Review E66, 026209 (2002).





Note: like a (dis)continuity statistic

Weaknesses.

Necessary to pick a scale (threshold): ~ 1 std is recommended - why?

When to stop adding components? Chaotic signals always generate FNN for large enough τ

Suggested implementation has same τ for each emb. dim

Issues in Reconstruction and current approaches

- Finding the embedding dimension.

False Nearest Neighbors (FNN).

M.B. Kennel, R. Brown, and H.D.I. Abarbanel, Physical Review A 45, 3403 (1992).
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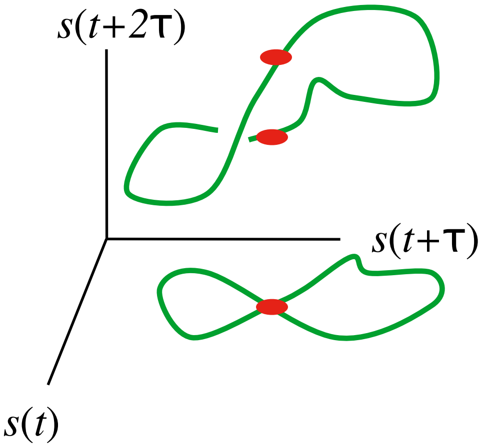
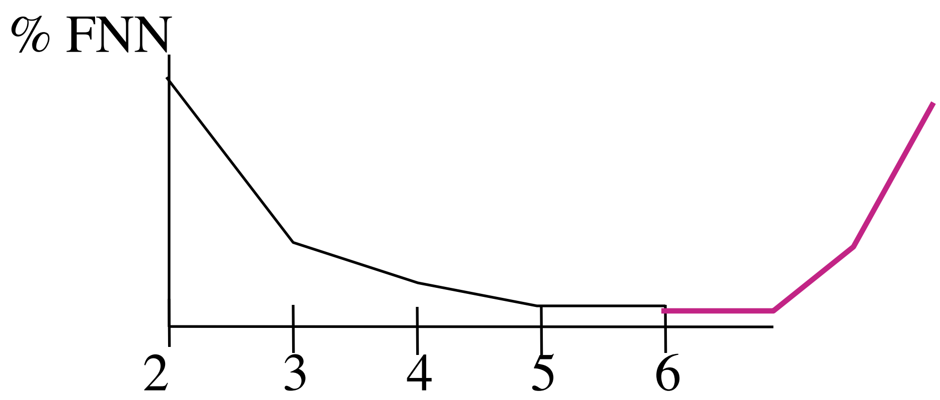




Note: like a (dis)continuity statistic

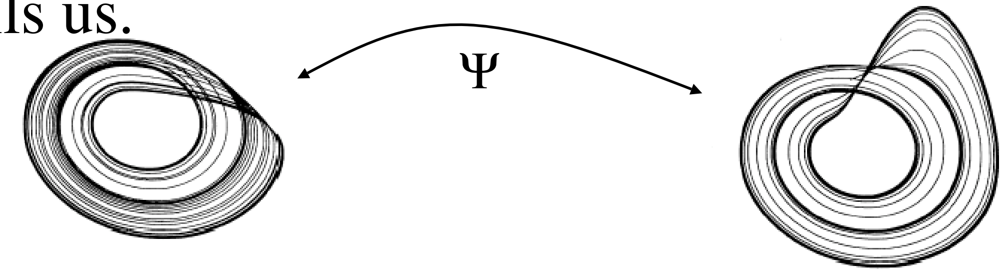
Weaknesses.

Necessary to pick a scale (threshold): ~ 1 std is recommended - why?

When to stop adding components? Chaotic signals always generate FNN for large enough τ (unexplored problem)

Suggested implementation has same τ for each emb. dim

What Takens theorem tells us.



$$s_1(t), s_2(t), \ldots, s_m(t)$$

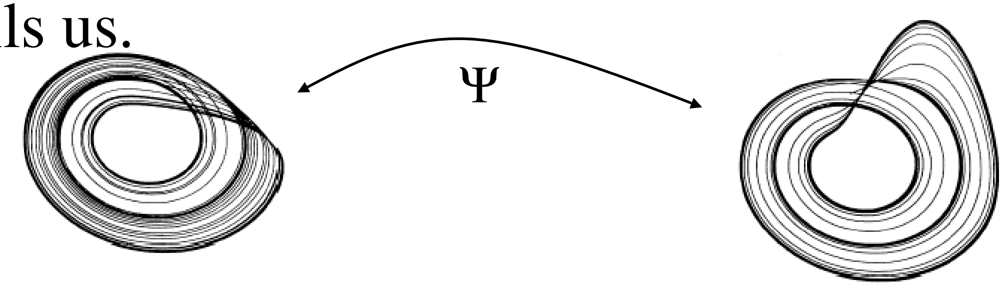
$$\downarrow$$

$$\mathbf{v} = \left(s_{j_1}(t+\tau_1), s_{j_2}(t+\tau_2), \ldots, s_{j_k}(t+\tau_k)\right) \longrightarrow \mathbf{v} = \left(s_{j_1}(t+\tau_1), s_{j_2}(t+\tau_2), \ldots, s_{j_d}(t+\tau_d), s_{j_{k+1}}(t+\tau_{k+1})\right)$$

To add a component to **v**, pick time series and τ independent of previous ones.

new component $$s_{j_{k+1}}(t+\tau_{k+1}) \neq f\left(s_{j_1}(t+\tau_1), s_{j_2}(t+\tau_2), \ldots, s_{j_k}(t+\tau_k)\right)$$

What Takens theorem tells us.



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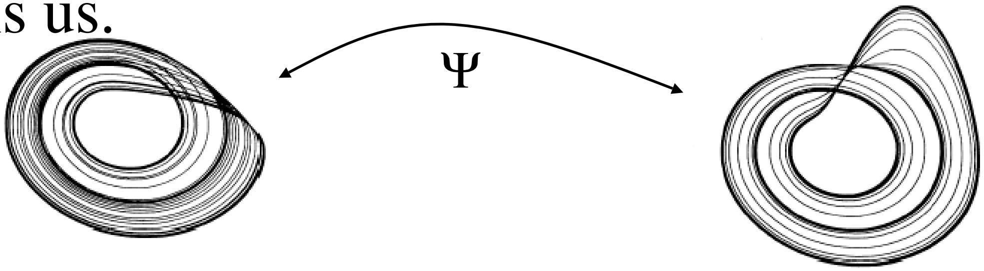
$$s_{k+1} = f\left(s_1, s_2, \ldots, s_k\right)$$

$$\det\frac{\partial\Psi}{\partial\mathbf{x}} = \begin{vmatrix} \frac{\partial s_1}{\partial x_1} & \frac{\partial s_1}{\partial x_2} & \cdots & \frac{\partial s_1}{\partial x_n} \\ \frac{\partial s_2}{\partial x_1} & \frac{\partial s_2}{\partial x_2} & \cdots & \frac{\partial s_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial s_{k+1}}{\partial x_1} & \frac{\partial s_{k+1}}{\partial x_2} & \cdots & \frac{\partial s_{k+1}}{\partial x_n} \end{vmatrix} = \begin{vmatrix} \frac{\partial s_1}{\partial x_1} & \frac{\partial s_1}{\partial x_2} & \cdots & \frac{\partial s_1}{\partial x_n} \\ \frac{\partial s_2}{\partial x_1} & \frac{\partial s_2}{\partial x_2} & \cdots & \frac{\partial s_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i=1}^{k}\frac{\partial s_{k+1}}{\partial s_i}\frac{\partial s_i}{\partial x_1} & \sum_{i=1}^{k}\frac{\partial s_{k+1}}{\partial s_i}\frac{\partial s_i}{\partial x_2} & \cdots & \sum_{i=1}^{k}\frac{\partial s_{k+1}}{\partial s_i}\frac{\partial s_i}{\partial x_n} \end{vmatrix}$$

What Takens theorem tells us.



$$s_1(t), s_2(t), \ldots, s_m(t)$$

$$\downarrow$$

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NO increase in rank by adding s_k

If we have enough reconstruction components (= d)

$$\mathbf{x}(t_n) = \Psi'\Big(\big(s_{j_1}(t+\tau_1), s_{j_2}(t+\tau_2), \ldots, s_{j_d}(t+\tau_d), s_{j_{d+1}}(t+\tau_{d+1})\big)\Big)$$

$$= \Psi\Big(\big(s_{j_1}(t+\tau_1), s_{j_2}(t+\tau_2), \ldots, s_{j_d}(t+\tau_d)\big)\Big)$$

$$\big(s_{j_1}(t+\tau_1), s_{j_2}(t+\tau_2), \ldots, s_{j_{d+1}}(t+\tau_{d+1})\big) = \Psi'^{-1} \circ \Psi\big(s_{j_1}(t+\tau_1), s_{j_2}(t+\tau_2), \ldots, s_{j_d}(t+\tau_d)\big)$$

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$$\Rightarrow \quad s_{j_{d+1}}(t+\tau_{d+1}) = f\left(s_{j_1}(t+\tau_1), s_{j_2}(t+\tau_2), \ldots, s_{j_d}(t+\tau_d)\right)$$

If we have enough reconstruction components (= d)

$$\mathbf{x}(t_n) = \Psi'\left(\left(s_{j_1}(t+\tau_1), s_{j_2}(t+\tau_2), \ldots, s_{j_d}(t+\tau_d), s_{j_{d+1}}(t+\tau_{d+1})\right)\right)$$

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The mathematical idea behind FNN

- Multivariate time series

$$s_1(t_1), s_1(t_2), s_1(t_3), \ldots, s_1(t_N)$$

$$s_2(t_1), s_2(t_2), s_2(t_3), \ldots, s_2(t_N)$$

Typical Approaches.

Find a τ_i for each series and have some recipe for mixing time-delayed components together.

$$\mathbf{v}(t) = \left(s_1(t), s_1(t+\tau_1), s_2(t), s_1(t+2\tau_1), s_2(t+\tau_2), \ldots,\right)$$

Weaknesses.

Not rigorous.

Not optimal - which time series and τ for next component are best?

What do we mean by 'best' choices of time series?

- Multivariate time series

$$s_1(t_1), s_1(t_2), s_1(t_3), \ldots, s_1(t_N)$$

$$s_2(t_1), s_2(t_2), s_2(t_3), \ldots, s_2(t_N)$$

Typical Approaches.

Find a τ_i for each series and have some recipe for mixing time-delayed components together.

$$\mathbf{v}(t) = \left(s_1(t), s_1(t+\tau_1), s_2(t), s_1(t+2\tau_1), s_2(t+\tau_2), \ldots,\right)$$

"Unifying framework for synchronization of coupled dynamical systems,"
S. Boccaletti, L.M. Pecora, and A. Pelaez, Physical Review E 63 (6), 066219/1 (2001).
"Dynamics from a multivariate time series,"
L. Cao, A. Mees, and JK. Judd, Physicsa D 121, 75 (1998).

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Not rigorous.

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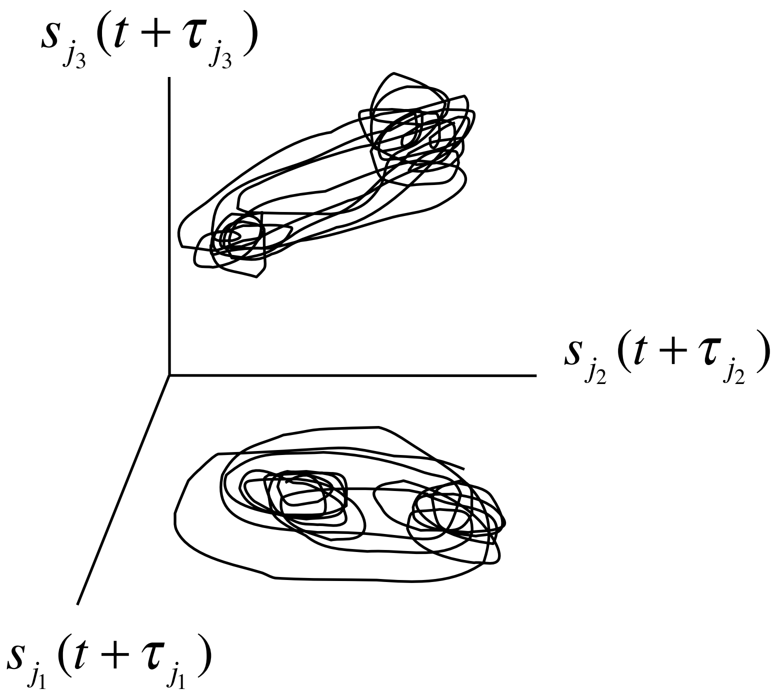
What do we mean by 'best' choices of time series?

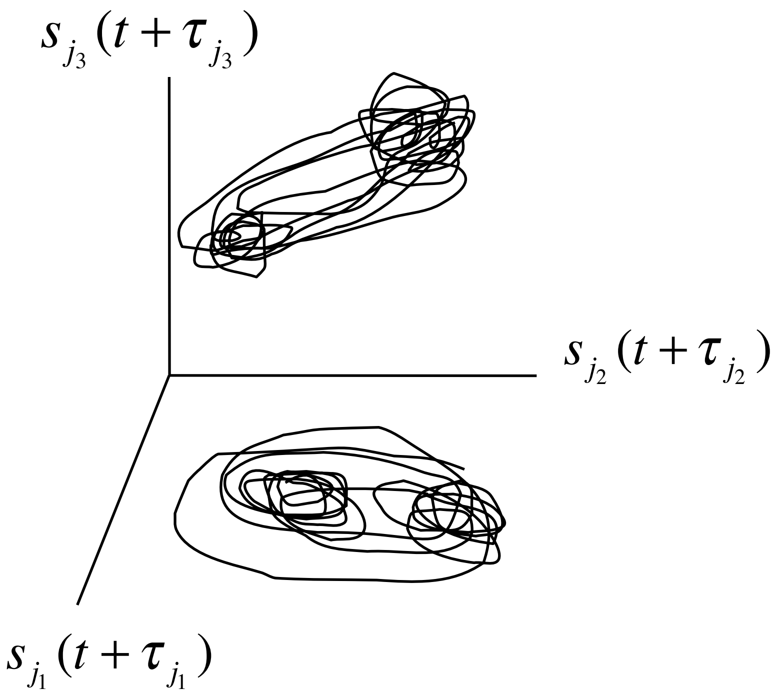
$$s_{j_{n+1}}(t+\tau_{j_{n+1}}) \stackrel{?}{=} f\left(s_{j_1}(t+\tau_{j_1}), s_{j_2}(t+\tau_{j_2}), \ldots, s_{j_n}(t+\tau_{j_n}), \ldots,\right)$$



$$f : \mathbf{R}^n \to \mathbf{R}^1$$

- **FNN**

$$s_{j_{n+1}}(t+\tau_{j_{n+1}}) \overset{?}{=} f\left(s_{j_1}(t+\tau_{j_1}), s_{j_2}(t+\tau_{j_2}), \ldots, s_{j_n}(t+\tau_{j_n}), \ldots,\right)$$



$$f : \mathbf{R}^n \rightarrow \mathbf{R}^1$$

- **FNN**

Necessary to pick a scale (threshold)

• Redundancy (marginal)

$$f : \mathbf{R}^n \to \mathbf{R}^1$$

$$R(x_1,\ldots,x_n;x_{n+1}) = \begin{array}{l} -\sum_{j_1 j_2 \cdots j_{n+1}} p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1})\log_2 p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) \\ +\sum_{j_1 j_2 \cdots j_{n+1}} p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1})\log_2 p_{j_1 j_2 \ldots j_n}(x_1,\ldots,x_n) p_{j_{n+1}}(x_{n+1}) \end{array}$$

• Redundancy (marginal)

$$f : \mathbf{R}^n \to \mathbf{R}^1$$

Multidimensional version of Mutual Information

$$R(x_1,\ldots,x_n;x_{n+1}) = \begin{array}{l} -\displaystyle\sum_{j_1 j_2 \cdots j_{n+1}} p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) \log_2 p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) \\ +\displaystyle\sum_{j_1 j_2 \cdots j_{n+1}} p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) \log_2 p_{j_1 j_2 \ldots j_n}(x_1,\ldots,x_n) p_{j_{n+1}}(x_{n+1}) \end{array}$$

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$$+ \sum_{j_1 j_2 \cdots j_{n+1}} p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) \log_2 \underline{p_{j_1 j_2 \ldots j_n}(x_1,\ldots,x_n)}\,\underline{p_{j_{n+1}}(x_{n+1})}$$

$$x_{n+1} = f(x_1,\ldots,x_n) \quad \Rightarrow \quad p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) = p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n)\delta(x_{n+1} - f(x_1,\ldots,x_n))$$

• Redundancy (marginal)

$$f : \mathbf{R}^n \to \mathbf{R}^1$$

Multidimensional version of Mutual Information

$$R(x_1,\ldots,x_n;x_{n+1}) = \begin{array}{l} -\sum_{j_1 j_2 \cdots j_{n+1}} p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) \log_2 p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) \\ +\sum_{j_1 j_2 \cdots j_{n+1}} p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) \log_2 p_{j_1 j_2 \ldots j_n}(x_1,\ldots,x_n) p_{j_{n+1}}(x_{n+1}) \end{array}$$

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$R(x_1;x_2) =$ Mutual Information

• Redundancy (marginal)

$$f : \mathbf{R}^n \rightarrow \mathbf{R}^1$$

Multidimensional version of Mutual Information

$$R(x_1,\dots,x_n;x_{n+1}) = -\sum_{j_1 j_2 \cdots j_{n+1}} p_{j_1 j_2 \dots j_{n+1}}(x_1,\dots,x_n,x_{n+1}) \log_2 p_{j_1 j_2 \dots j_{n+1}}(x_1,\dots,x_n,x_{n+1}) + \sum_{j_1 j_2 \cdots j_{n+1}} p_{j_1 j_2 \dots j_{n+1}}(x_1,\dots,x_n,x_{n+1}) \log_2 p_{j_1 j_2 \dots j_n}(x_1,\dots,x_n) p_{j_{n+1}}(x_{n+1})$$

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$R(x_1;x_2) =$ Mutual Information

- Necessary to pick a bin size

  Remedy (?) fixed bin "mass"

• Redundancy (marginal)

$f : \mathbf{R}^n \to \mathbf{R}^1$

Multidimensional version of Mutual Information

$$R(x_1,\ldots,x_n;x_{n+1}) = \begin{array}{l} -\sum_{j_1 j_2 \cdots j_{n+1}} p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) \log_2 p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) \\ + \sum_{j_1 j_2 \cdots j_{n+1}} p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) \log_2 p_{j_1 j_2 \ldots j_n}(x_1,\ldots,x_n) p_{j_{n+1}}(x_{n+1}) \end{array}$$

$$x_{n+1} = f(x_1,\ldots,x_n) \quad \Rightarrow \quad p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) = p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n)\delta(x_{n+1} - f(x_1,\ldots,x_n))$$

$R(x_1;x_2) =$ Mutual Information

- Necessary to pick a bin size

  Remedy (?) fixed bin "mass"

- Inherits some problems from MI?

• Redundancy (marginal)

$$f : \mathbf{R}^n \to \mathbf{R}^1$$

Multidimensional version of Mutual Information

$$R(x_1,\ldots,x_n;x_{n+1}) = -\sum_{j_1 j_2 \cdots j_{n+1}} p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) \log_2 p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) + \sum_{j_1 j_2 \cdots j_{n+1}} p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) \log_2 p_{j_1 j_2 \ldots j_n}(x_1,\ldots,x_n) p_{j_{n+1}}(x_{n+1})$$

$$x_{n+1} = f(x_1,\ldots,x_n) \quad \Rightarrow \quad p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n,x_{n+1}) = p_{j_1 j_2 \ldots j_{n+1}}(x_1,\ldots,x_n)\delta(x_{n+1} - f(x_1,\ldots,x_n))$$

$R(x_1;x_2) =$ Mutual Information

- Necessary to pick a bin size

  Remedy (?) fixed bin "mass"

- Inherits some problems from MI?

- Large dimension=> large bin number?

  Apparently not, just bins on attractor.

Consequence of the Null:

Calculate the probability distribution of the time series.

Joint prob. distribution of two independently chosen points

$$\rho(s_1, s_2) = \rho(s_1)\rho(s_2)$$

Joint prob. distribution of one point and the difference $\xi = s_1 - s_2$

$$\sigma(s_1, \xi) = \rho(s_1, \xi - s_1)$$

Prob. distribution of the difference $\xi = s_1 - s_2$

$$\sigma(\xi) = \int \sigma(s_1, \xi)\, ds_1$$

Prob. distribution of $|\xi| = |s_1 - s_2|$

$$\sigma(|\xi|) = \sigma(\xi) + \sigma(-\xi)$$

Prob. that $|\xi| < a$

$$P(|\xi| < a) = \int_0^a \sigma(|\xi|) d\xi$$

Possible candidates for "a" in the above.

a= max {δ vector components and ϵ component used to get ϵ* }

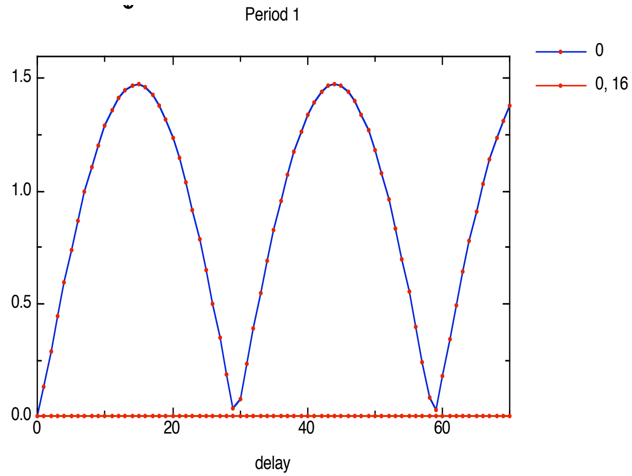
a=||**v**|| = magnitude of vector **v** made from δ vector components and ϵ component

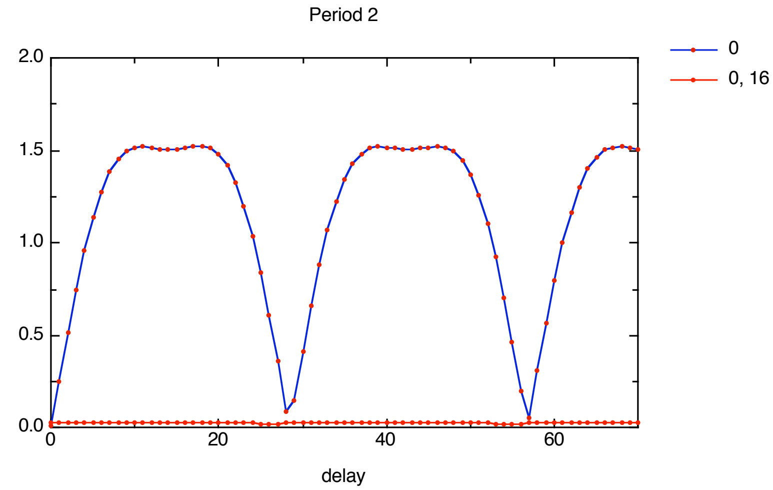
since ||**v**|| > max$\{v_i\}$

Then while plotting ϵ* vs. delay also plot 1-$P(|\xi|<a)$ vs. delay to see when the delay is too long.   E.g. (just suggestive, not done yet for real time series),

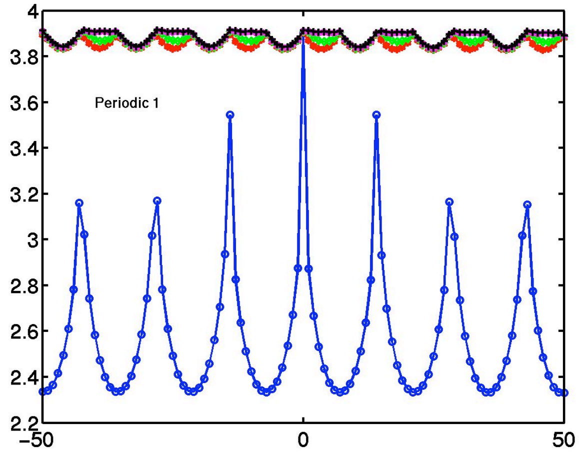
• Periodic Systems

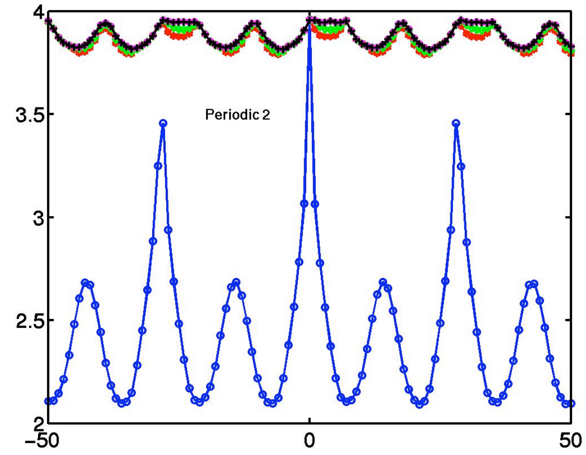
Cont.Stat ϵ*





Redundancies





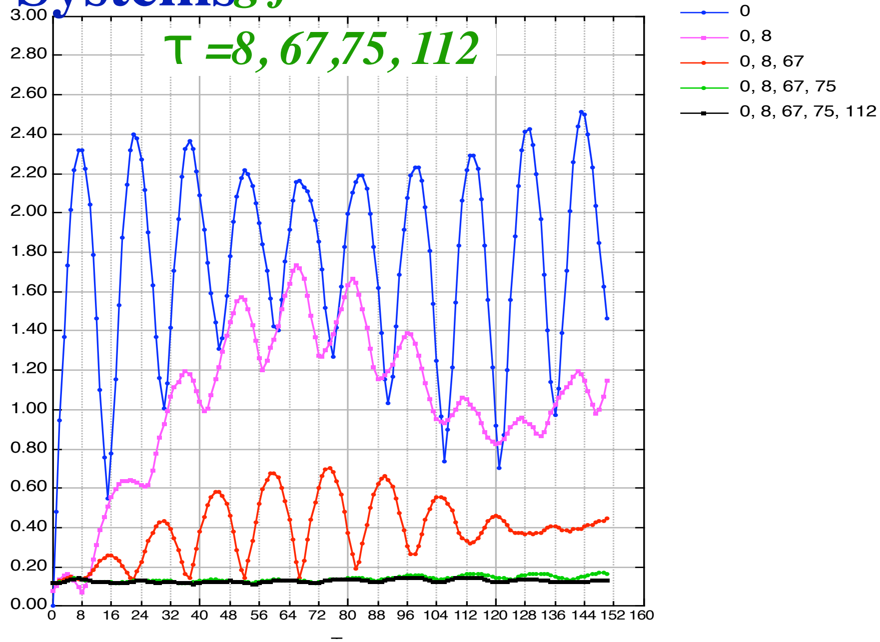
• Quasiperiodic Systems

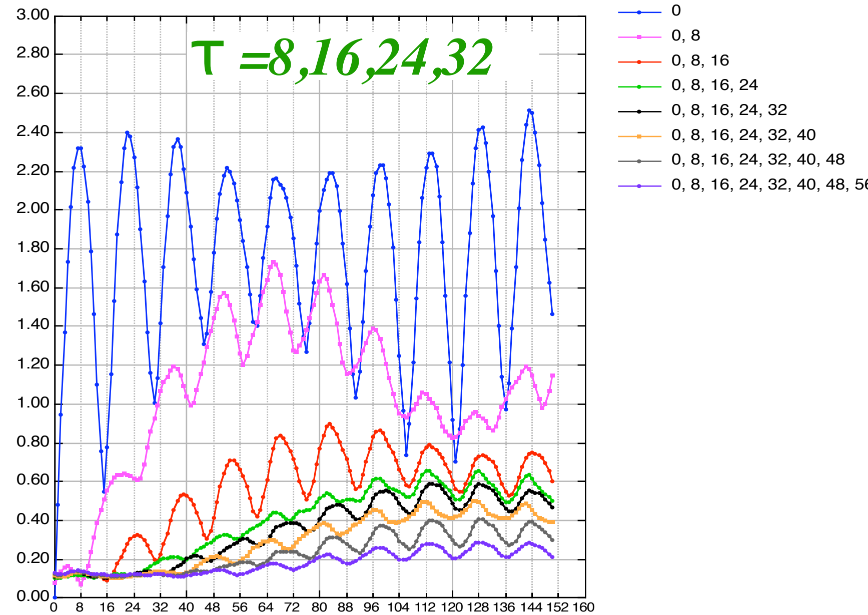
Multiple Time Scales

Using function statistic

Constant τ (old way)

Cont.Stat ϵ*





Redundancies