The Fermi-Pasta-Ulam paradox: problems, q-breathers, and going beyond

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Abstract:

In 1955 Fermi, Pasta and Ulam (FPU) reported on the nonequipartition of a nonlinear atomic chain, with initially one normal mode excited.
Modern computational studies show, that on a first, relatively short, time scale the energy is distributed among a few neighbouring modes in modal space, with more distant modes being exponentially weakly excited - i.e., one observes localization in normal mode space. On a much larger second time scale (which was not reachable wih the MANIAC I), the tail modes are slowly growing, and finally the system does equilibrate.
The problem then is to explain i) the exponential localization of energy in modal space on intermediate time scales, and ii) the way the time scales depend on the essential parameters (wave number, energy density).
Despite its strong impact on nonlinear dynamics and statistical physics, the paradox remained essentially unexplained for decades - ideal grounds for the appearance of myths.

Recent studies show that the model allows for exact time-periodic solutions (q-breathers), which are exponentially localized in the space of normal modes. The trajectory initially computed by FPU is a slight perturbation away from an exact q-breather orbit. Consequently most of the key observations related to the FPU problem (localization of energy in normal mode space for long times, recurrence on relatively short times, system size and energy thresholds) are captured by the properties of q-breathers and the phase space flow nearby.
The underlying concept is much more general, and can be easily extended to two-and three-dimensional finite lattices. A scaling approach leads to nontrivial predictions in the limit of infinite system sizes.