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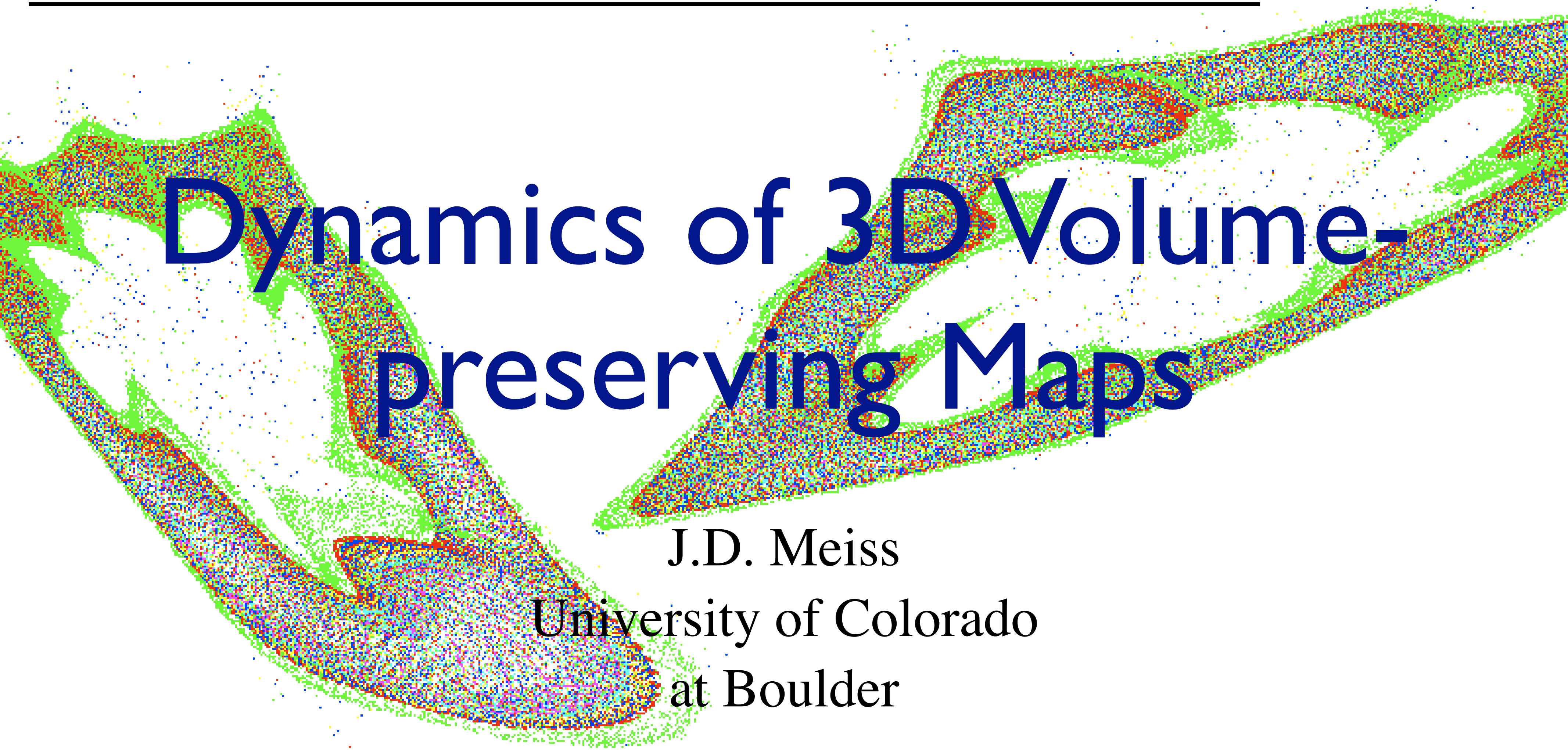


Dynamics of 3D Volume-preserving Maps

J.D. Meiss
University of Colorado
at Boulder

with H. Lomelí, A. Gomez, P. Mullowney, K. Julien, & H. Dullin

Why Maps?

- A *dynamical system* is a rule for evolution on a space of states
  - Dynamical systems can have continuous time (flow) or discrete time (map)
- Maps that preserve volume are common in applications
  - Poincaré sections of Hamiltonian flows
  - Magnetic field line maps $\frac{d\mathbf{x}}{dt} = \mathbf{B}(\mathbf{x},t), \quad \nabla \cdot \mathbf{B} = 0$
    - Structure of field lines in a Plasma confinement device or the magnetosphere
  - Lagrangian particle motion in an incompressible, time dependent flow
    - Geostrophic dynamics
    - Turbulent mixing problems $\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x},t), \ \nabla \cdot \mathbf{u} = 0$

Volume Preserving Maps

Specialize to the 3D case

$$f : \mathbb{R}^3 \to \mathbb{R}^3, \quad \det(Df) = 1$$

$$(x', y', z') = f(x, y, z)$$

where Df is the Jacobian matrix, and $'$ means the new point

A canonical example is the abc map

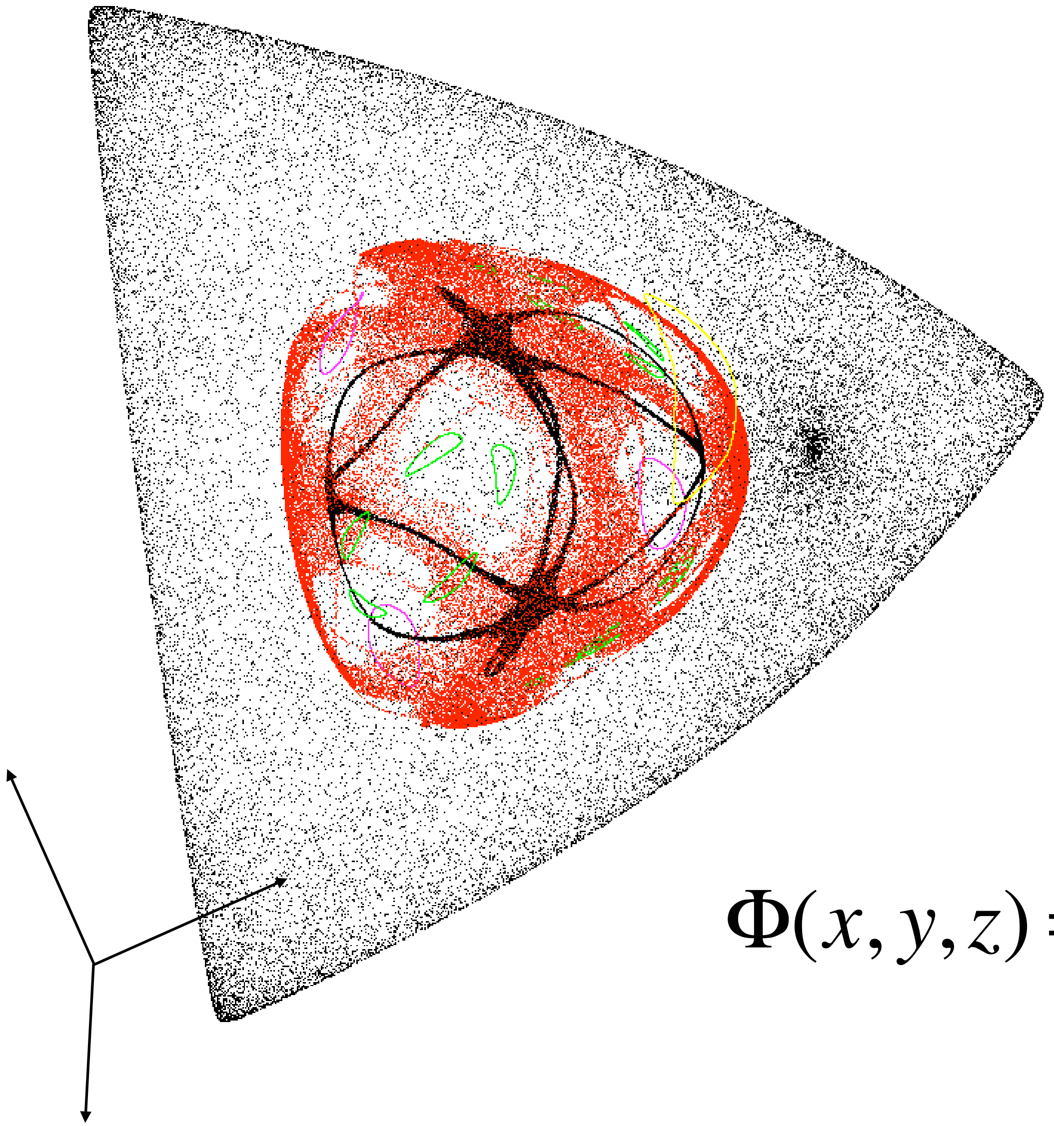
$$x' = x + a\sin(2\pi z) + c\cos(2\pi y)$$

$$y' = y + b\sin(2\pi x') + a\cos(2\pi z)$$

$$z' = z + c\sin(2\pi y') + b\cos(2\pi x')$$

Trace Maps

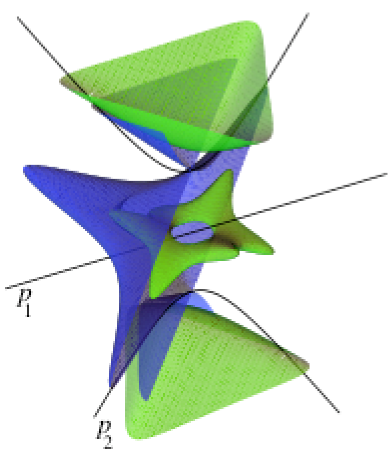
$$(x', y', z') = (y, z, -x + 2yz)$$

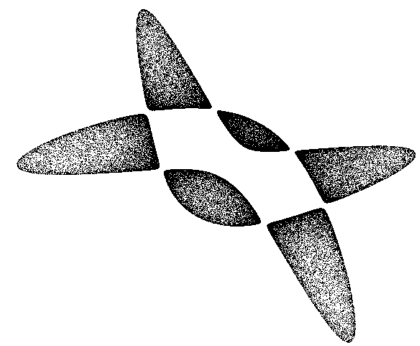


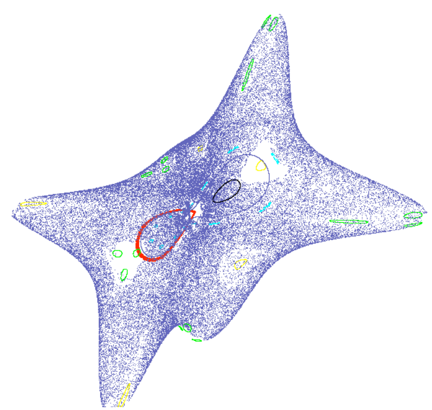
$$\Phi(x, y, z) = x^2 + y^2 + z^2 - 2xyz - 1$$

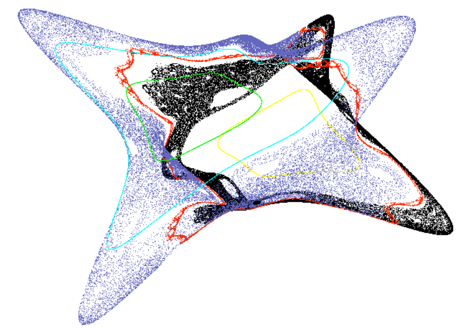
Maps with an Invariant

(Suris, (1989). “Integrable Mappings of the Standard Type.” Func. Anal. & Appl. 23: 74-76.)









A. Gómez and J. D. Meiss (2002). “Volume Preserving Maps with an Invariant.” Chaos 12: 289-299.

Fluid Mixing Models

Mixing vs Diffusion

A passive scalar (blob of die) mixes in a fluid when

- it stretches and folds due to fluid motion
- it diffuses due to Brownian motion

Here we ignore the diffusive time scale and only consider advective mixing

Advective Mixing

A passive scalar follows the fluid:

u(***x***,*t*) is solution of Navier-Stokes PDEs

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t), \quad \mathbf{x}(0) = \mathbf{x}_o$$

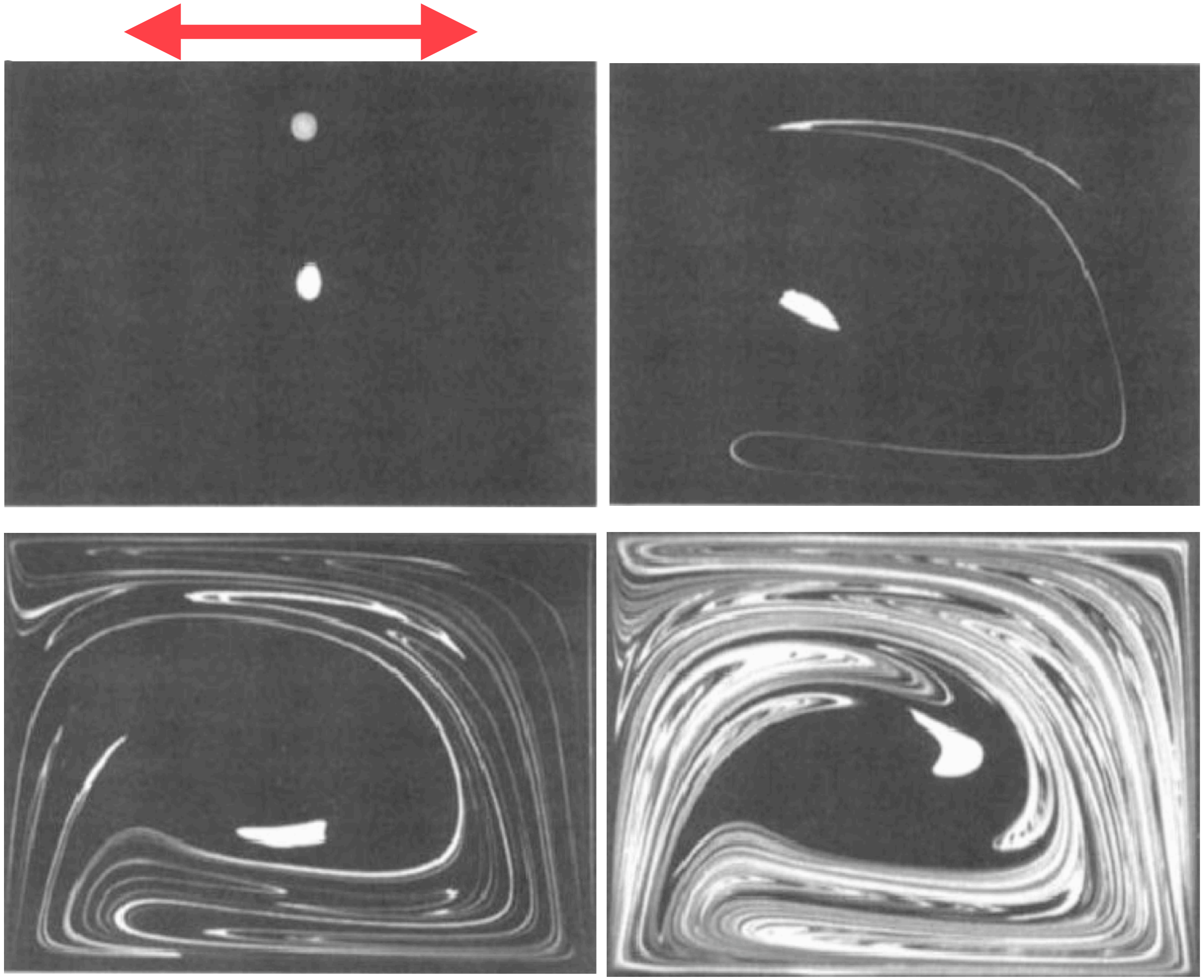
Neglect diffusion if time scale is short enough

We assume that the flow is laminar, so turbulent mixing is not active.

Laminar flows are common for small-scale mixers (MEMs devices, Microbiology devices)

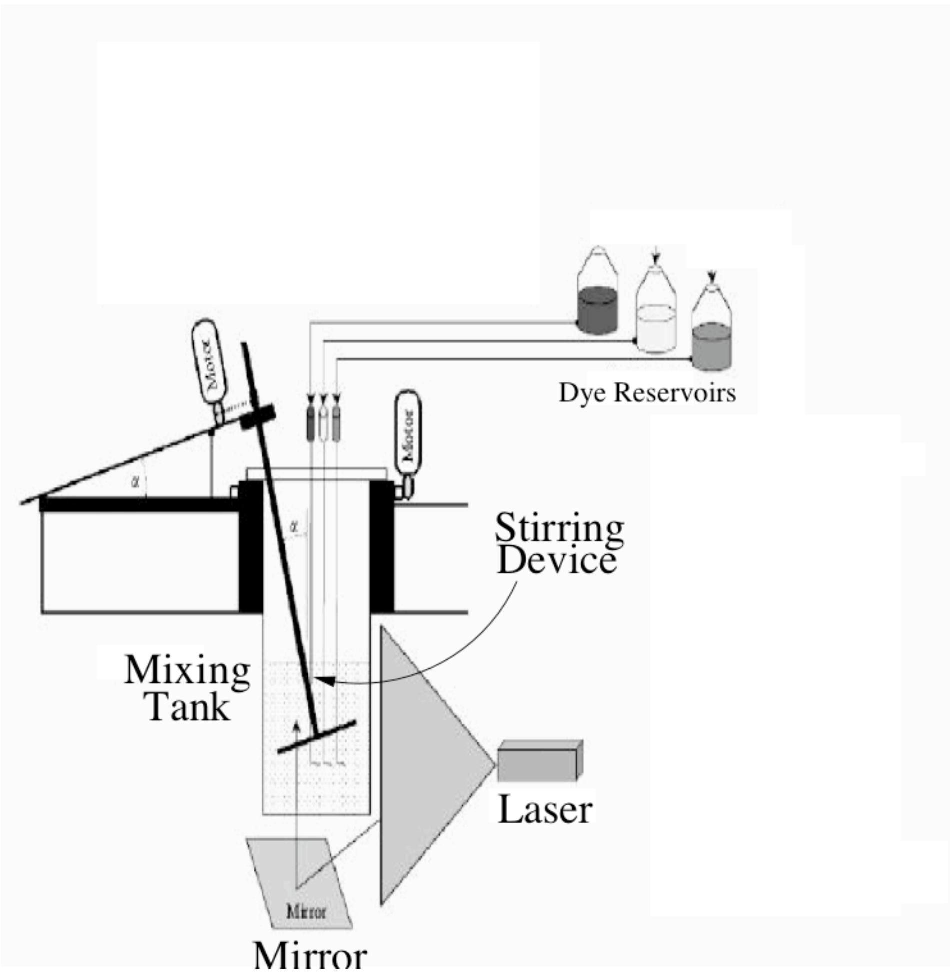
Experimental Advection

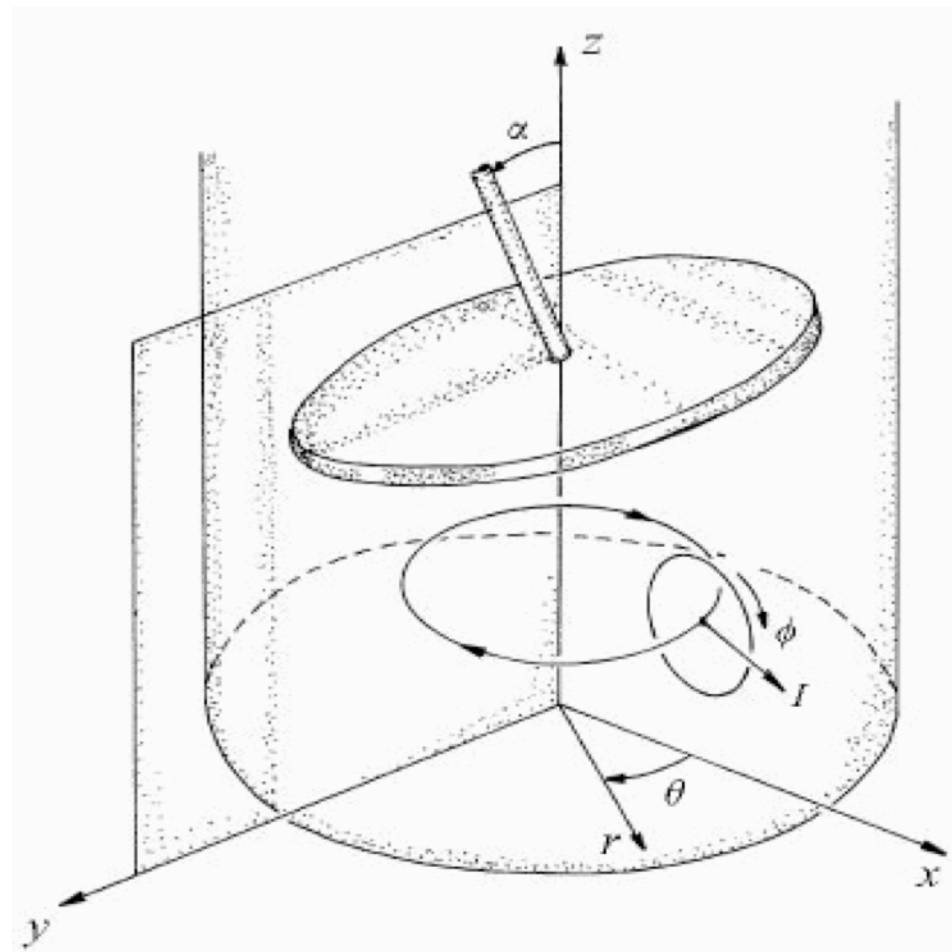
Leong (1989): periodically modulated cavity flow



Experimental Advection

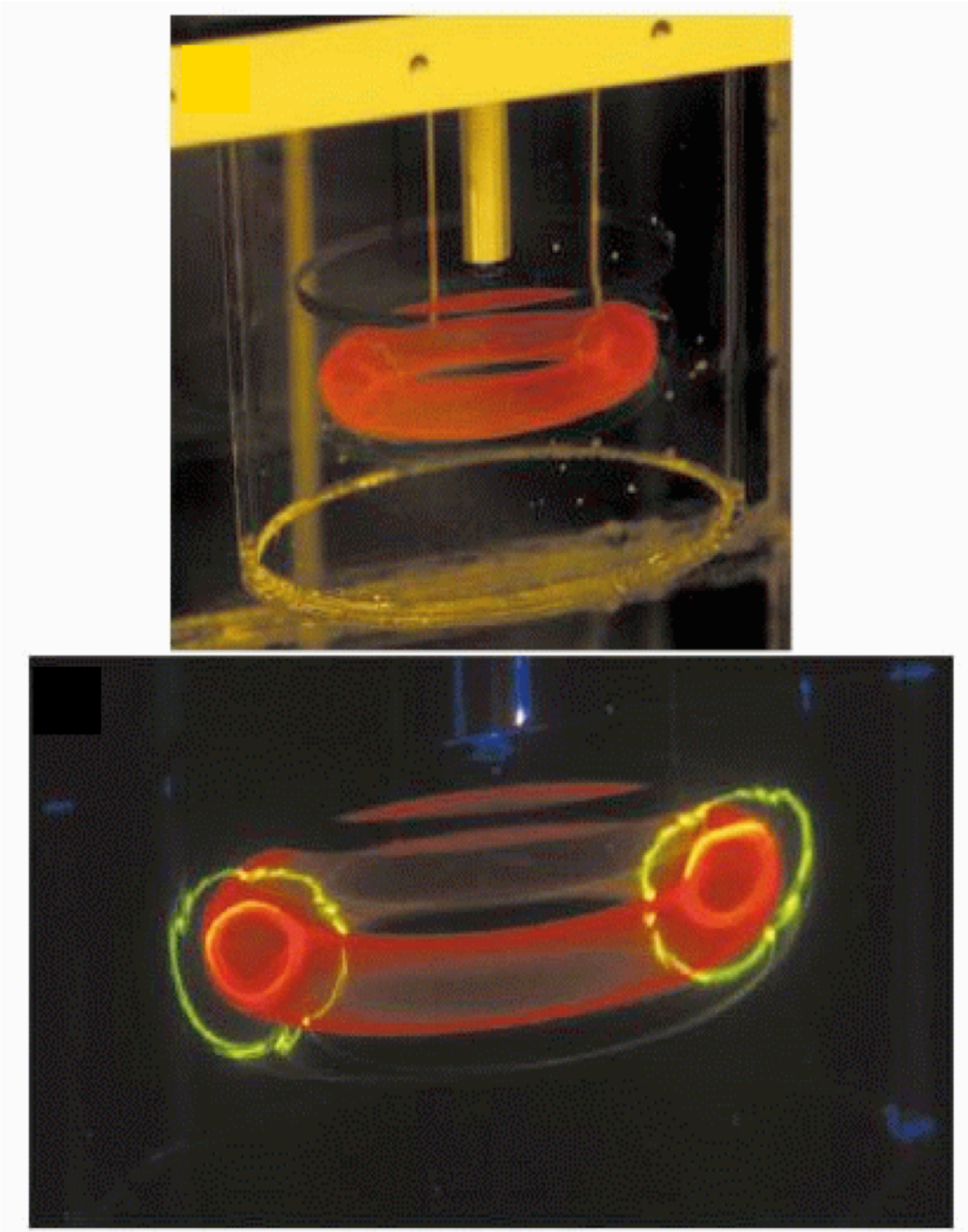
Off-axis stirring

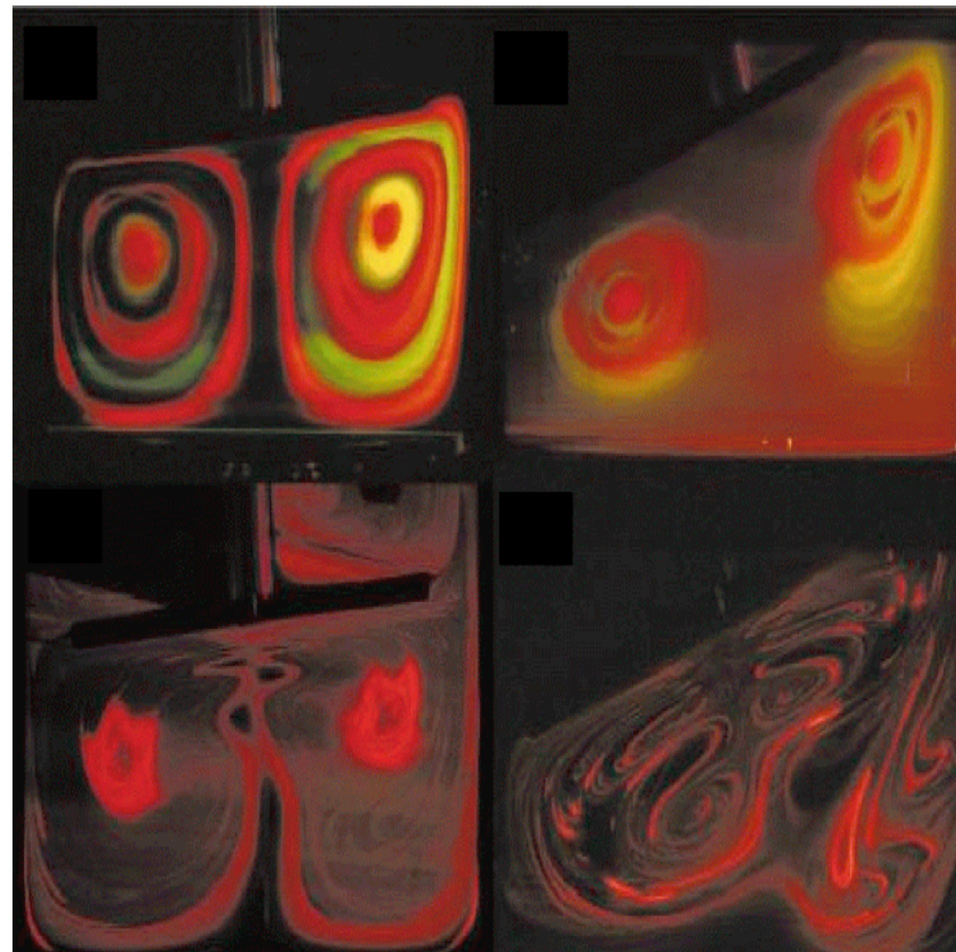




Fountain, et al. (2000). "Chaotic Mixing in a Bounded Three-Dimensional Flow." J. Fluid Mech. 417(265–301).

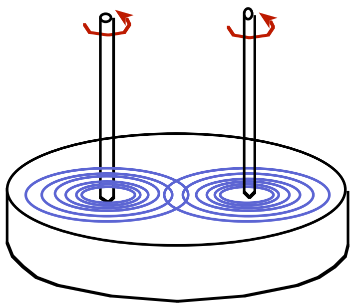
Fountain, Khakhar, Mezic, and Ottino





Aref's Blinking Vortex

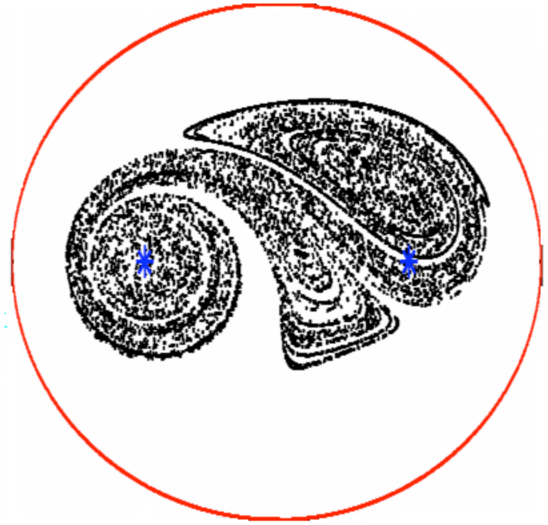
Aref defined the blinking vortex flow by periodically stirring an incompressible, viscous fluid



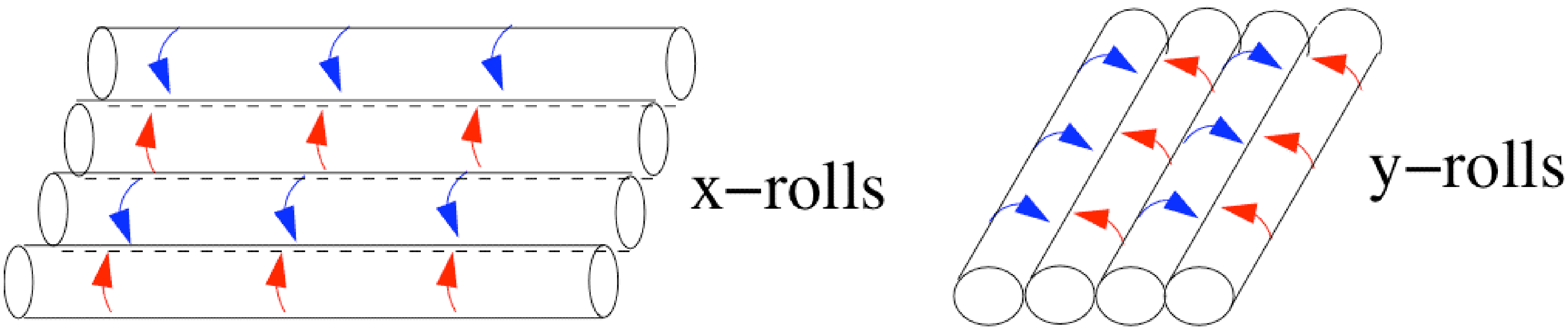
$$\frac{dx}{dt} = u(x,t) = \nabla \psi(x,t) \times \hat{z}$$

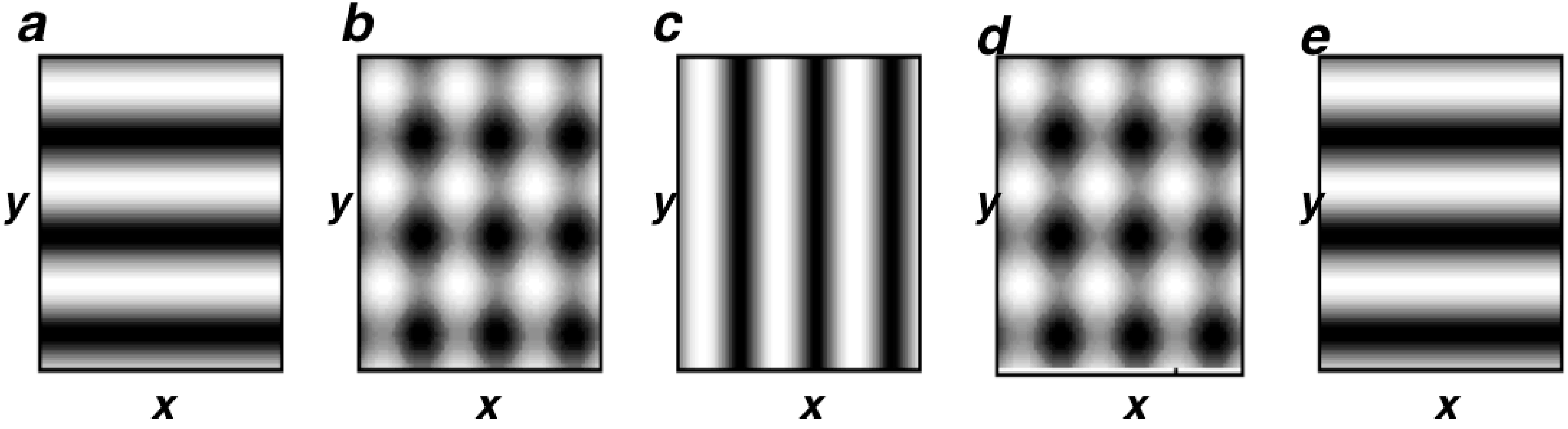
Modeled as point vortices applied for times T_1 and T_2

Explicit map can be obtained.



Roll Switching





Binary Convection

Ethanol-Water mixture in thin layer, heated from below

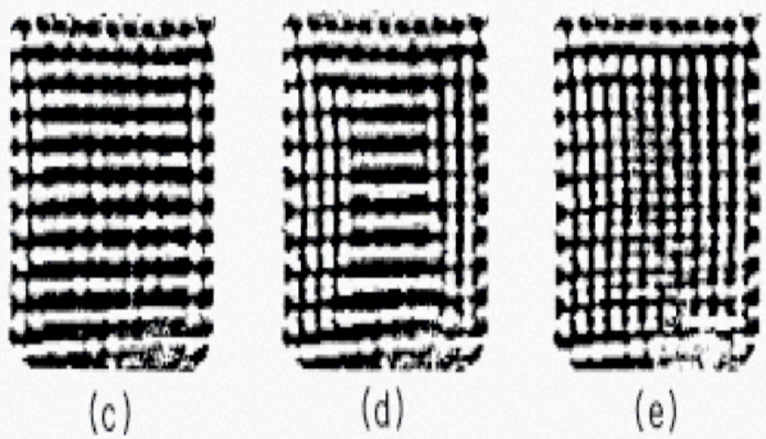


FIG. 10. Flow patterns in $\Gamma=24$ square cell at $T=25\,^{\circ}\mathrm{C}$, 40 wt. % of ethanol: (a) induced "perfect" square grid at $r=1.07$; (b) roll pattern at $r\simeq 2.6$, (c), (d), (e) sequential pictures of the oscillating structure at $r=1.19$. The time difference between pictures is $12.5\tau_{v,T}$.

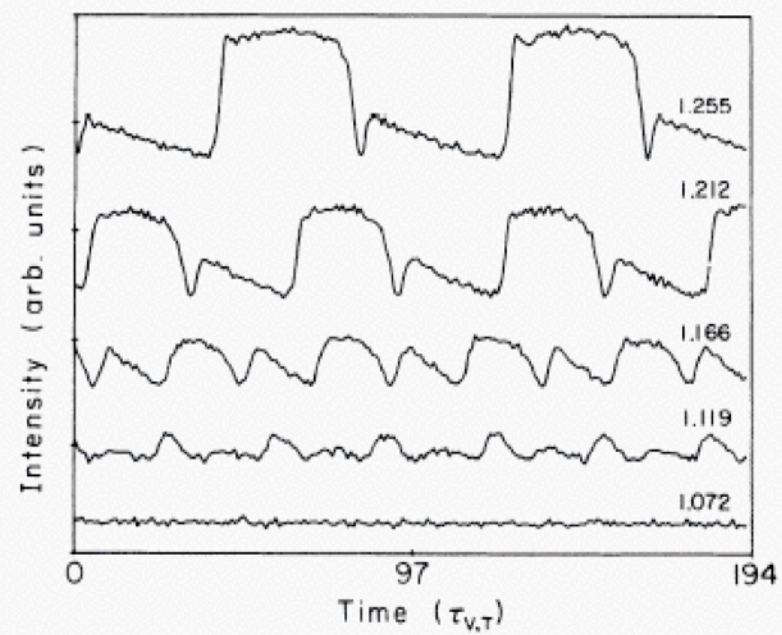


FIG. 11. The light intensity of the shadowgraph at a chosen location in the large square cell with 40 wt. % of ethanol for five different values of r. The numbers given on the figure are averaged values of r.

3D Stirring Model

Incompressible fluid written as a sum of stream functions:

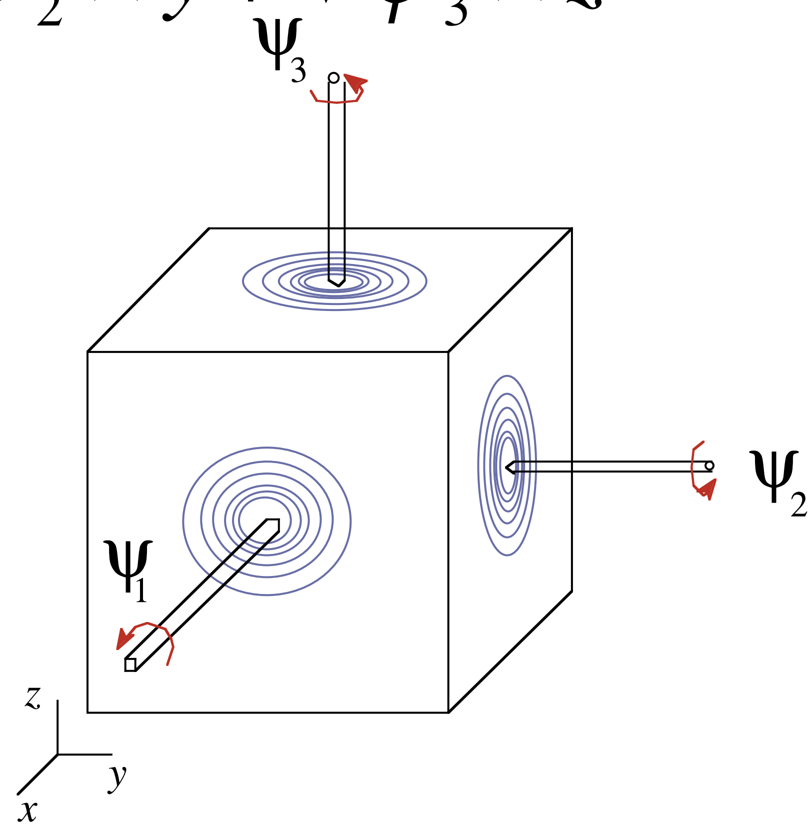
$$\frac{d\mathbf{x}}{dt} = \mathbf{u}(\mathbf{x},t) = \nabla\psi_1 \times \hat{x} + \nabla\psi_2 \times \hat{y} + \nabla\psi_3 \times \hat{z}$$

Roll form:

$$\psi_1 = A(t)g(y)h(z)$$

$$\psi_2 = B(t)f(x)h(z)$$

$$\psi_3 = C(t)f(x)g(y)$$



Mullowney, Julien, & Meiss (2005). "Blinking rolls: chaotic advection in a 3D flow with an Invariant." SIAM J. Appl. Dyn. Sys. 4: 159-186.

Surprise!

There is an Invariant!

$$J = f(x)g(y)h(z)$$

for any functions A(t), B(t), C(t) and f,g,h!

$$0 = \frac{dJ}{dt} = v \cdot \nabla J \quad \Rightarrow \quad v = \nabla J \times E$$

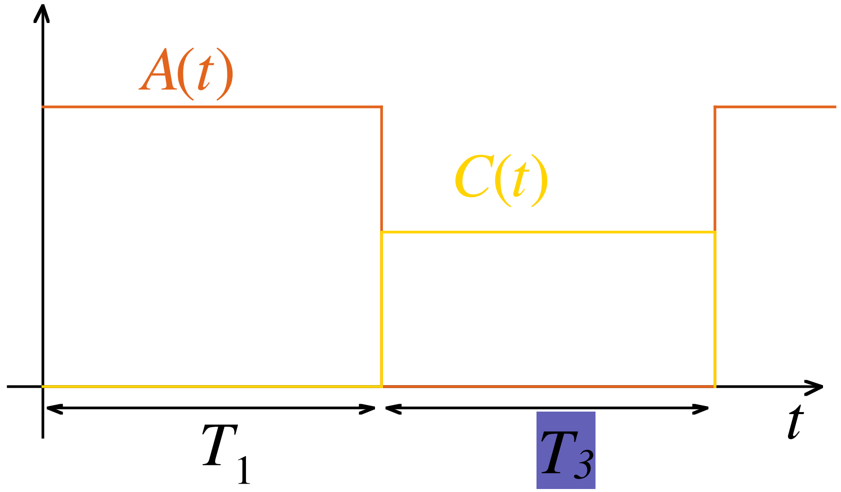
$$0 = \nabla \cdot v = -\nabla J \cdot (\nabla \times E)$$

$$E = \left(\frac{A}{f}, \frac{B}{g}, \frac{C}{h} \right)$$

Blinking Rolls

Periodic rolls
$$\psi_1 = A(t)\cos(y)\cos(z)$$
$$\psi_3 = C(t)\cos(x)\cos(y)$$

with on/off stirring protocol



Blinking Roll Map

For a single roll, the flow can be obtained analytically. For $\psi_1 = \cos(y)\cos(z)$ the time *T* flow is:

$$\Phi^{(1)}{}_T(x,y,z) = \begin{pmatrix} x \\ \sin^{-1}\left(\dfrac{\sin(y)\,\mathrm{cn}(T)\,\mathrm{dn}(T) - \sin(z)\cos^2(y)\,\mathrm{sn}(T)}{1-\sin^2(y)\,\mathrm{sn}(T)}\right) \\ \sin^{-1}\left(\dfrac{\sin(z)\,\mathrm{cn}(T)\,\mathrm{dn}(T) + \sin(y)\cos^2(z)\,\mathrm{sn}(T)}{1-\sin^2(z)\,\mathrm{sn}(T)}\right) \end{pmatrix}$$

Here sn,cn and dn are Jacobi Elliptic functions with modulus

$$k = \sqrt{1-\cos^2 y \cos^2 z}$$

Blinking Roll Map: Two Orthogonal Rolls

Composition of flows for ψ_1 and ψ_3 give the two roll map:

$$f = \Phi_{T_1}^{(1)} \circ \Phi_{T_3}^{(3)} = \Phi_{T_1}^{(1)} \circ R^{-1} \circ \Phi_{T_3}^{(1)} \circ R$$

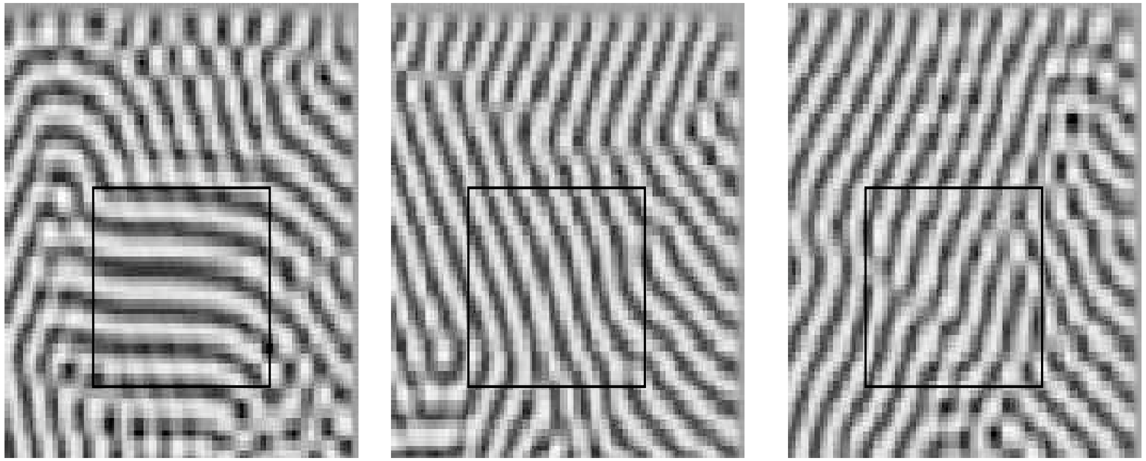
where R is rotation by $\pi/2$ about y

Küppers-Lortz Instability

Convection in a rotating fluid layer

Assume that rotation is small (centripetal acceleration << gravity).

Onset of instability: rolls grow, then loose stability to new rolls at an angle $\theta \approx 120^{o}$



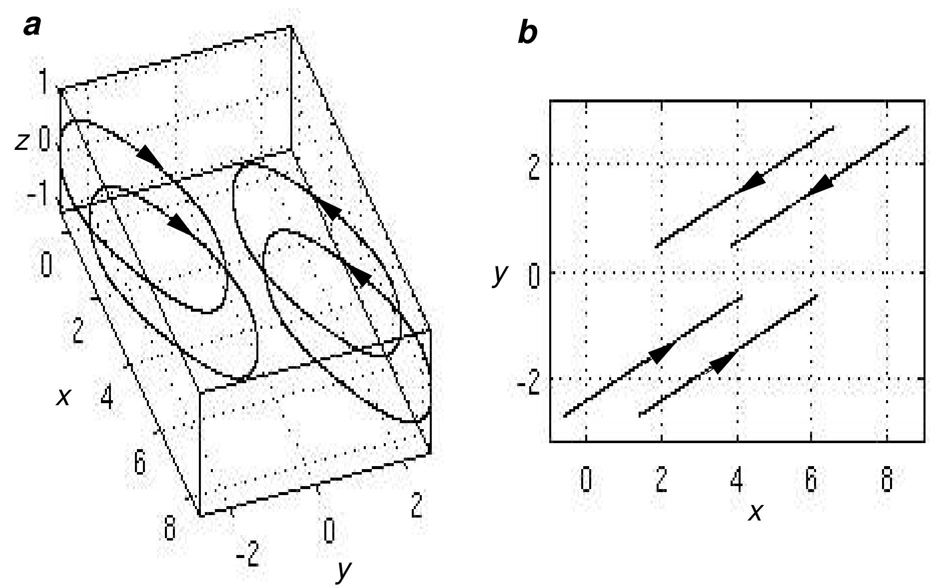
Küppers-Lortz Instability

Model using Boussinesq equations

Instability gives tilted rolls

$$\mathbf{u}_\eta = -(\tan\eta \sin y \sin z, -\sin y \sin z, -\cos y \cos z)$$

where $\tan\eta$ is proportional to the rotation rate



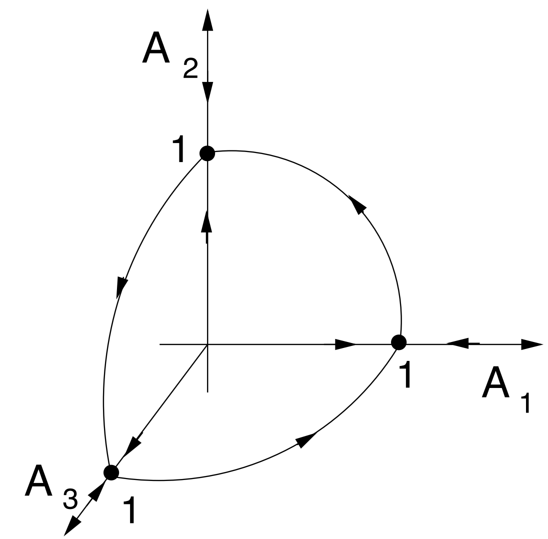
Busse-Heikes Model

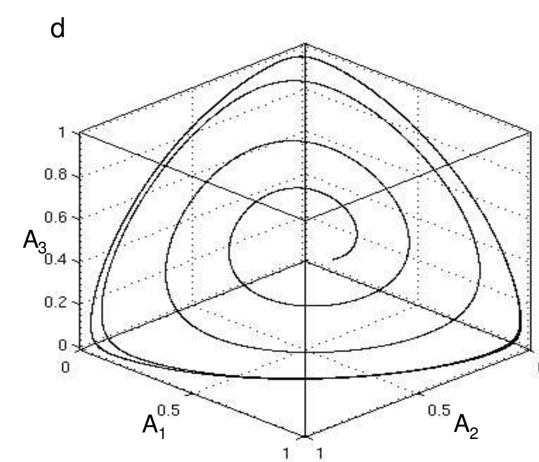
$$\mathbf{u}(\mathbf{x},t) = A_1(t)\mathbf{u}_\eta(\mathbf{x}) + A_2(t)R\mathbf{u}_\eta(R^T\mathbf{x}) + A_3(t)R^T\mathbf{u}_\eta(R\mathbf{x})$$

$$\dot{A}_1 = A_1\left(1-|A_1|^2 - \alpha|A_2|^2 - \beta|A_3|^2\right)$$

$$\dot{A}_2 = A_2\left(1-|A_2|^2 - \alpha|A_3|^2 - \beta|A_1|^2\right)$$

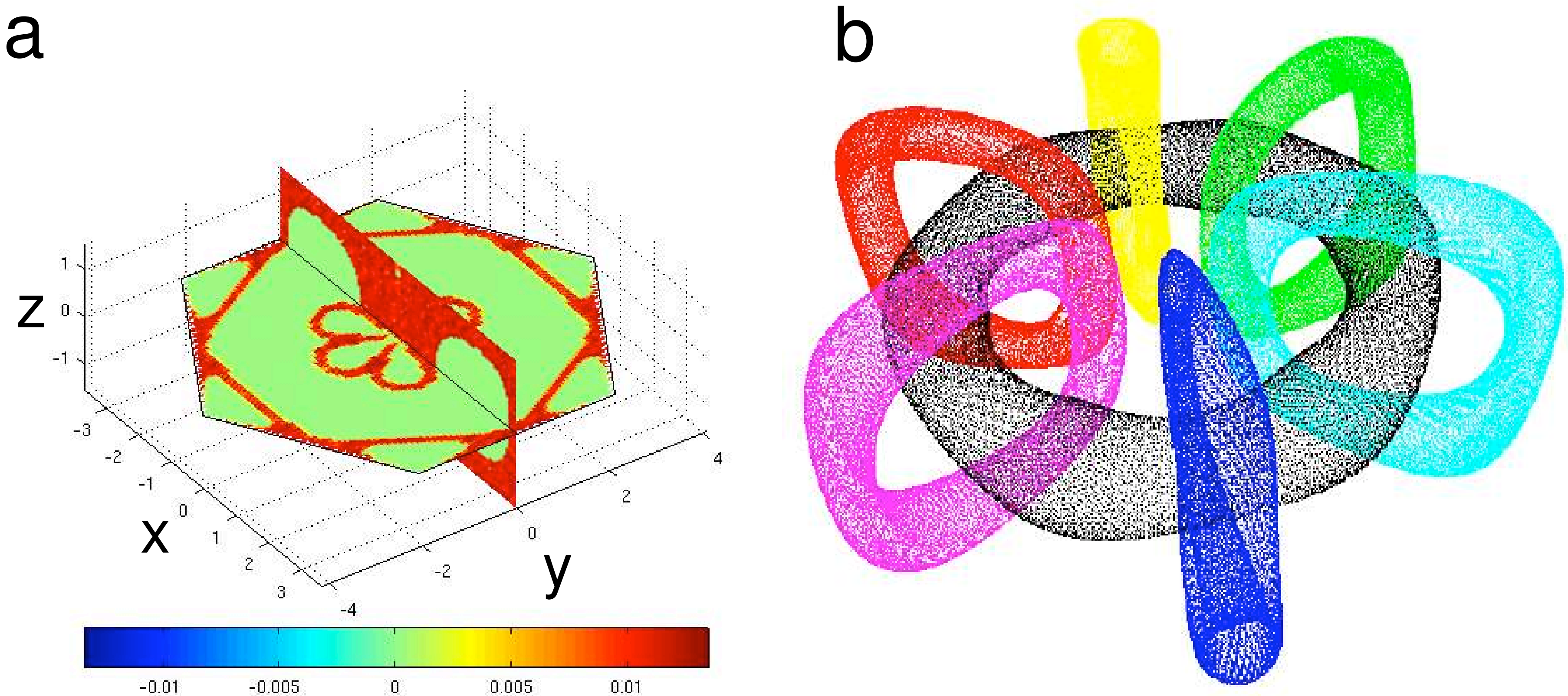
$$\dot{A}_3 = A_3\left(1-|A_3|^2 - \alpha|A_1|^2 - \beta|A_2|^2\right)$$





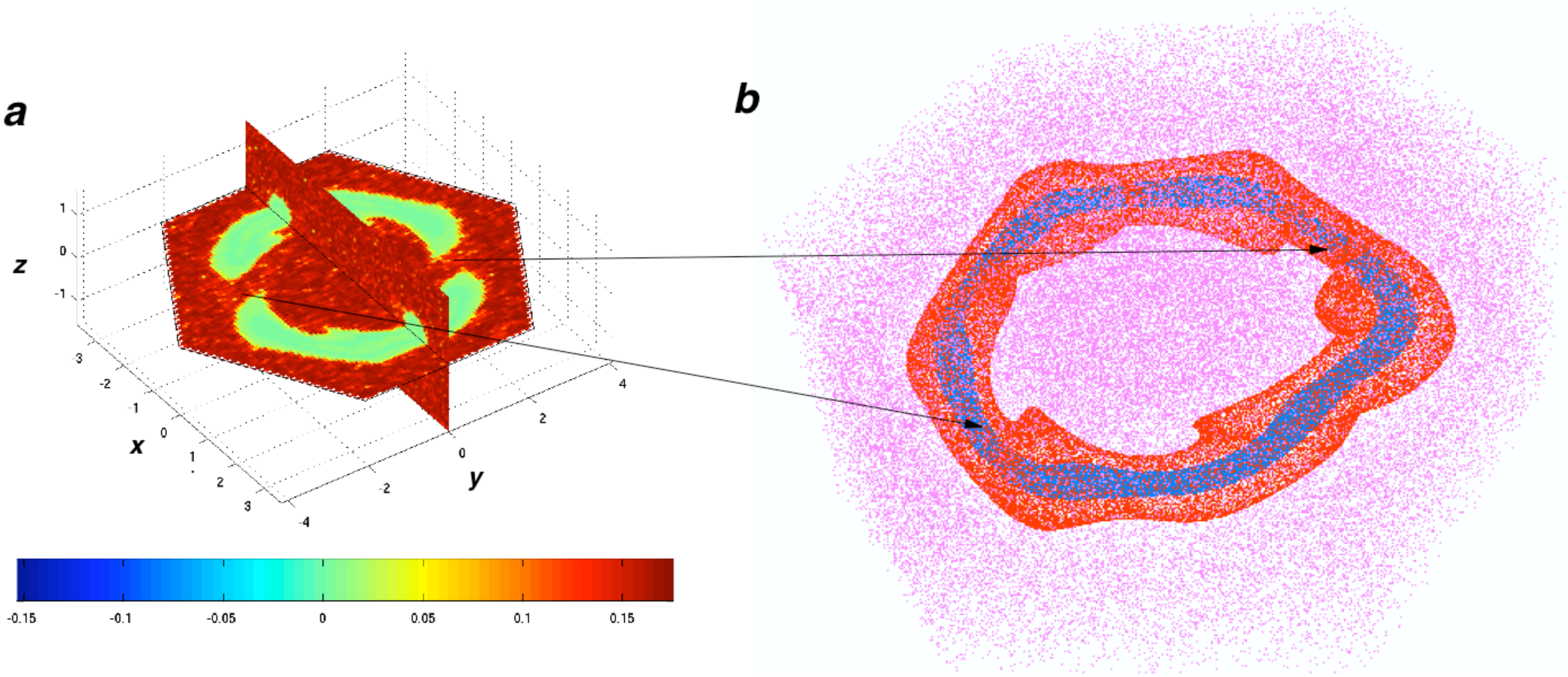
T = 0.2

$$T = \frac{\tau_{switch}}{\tau_{roll}}$$



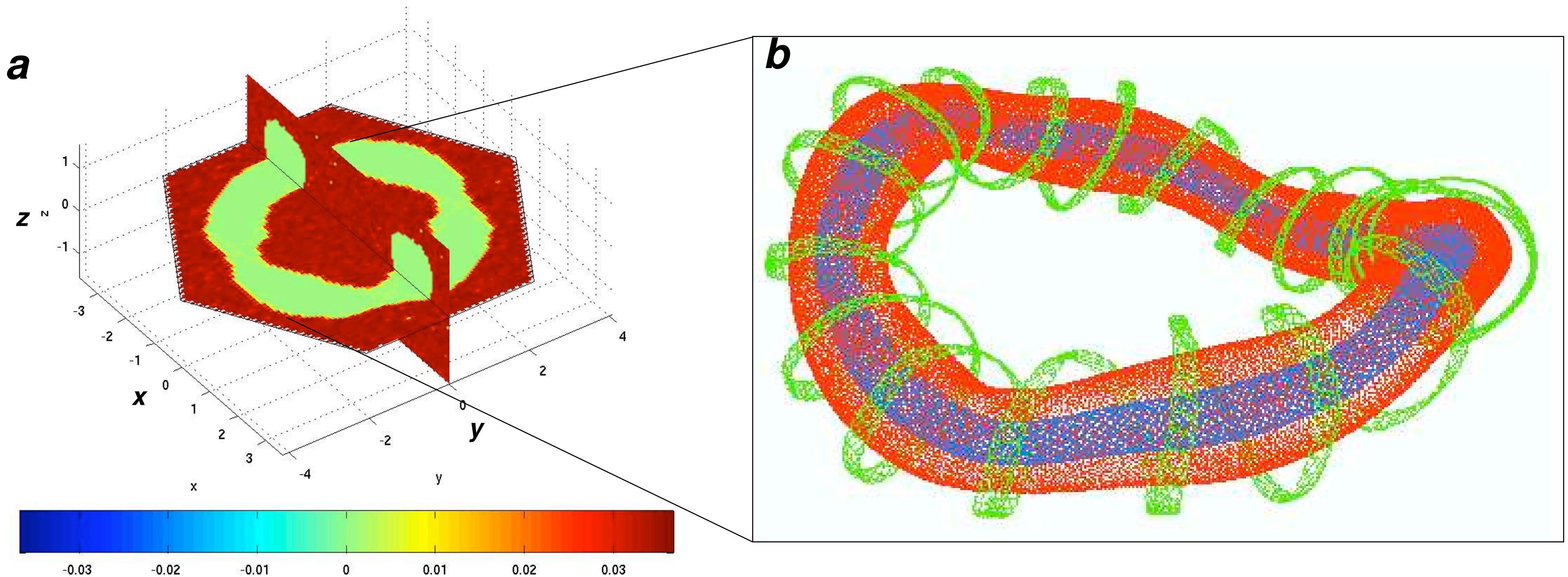
T = 0.5

$$T = \frac{\tau_{switch}}{\tau_{roll}}$$

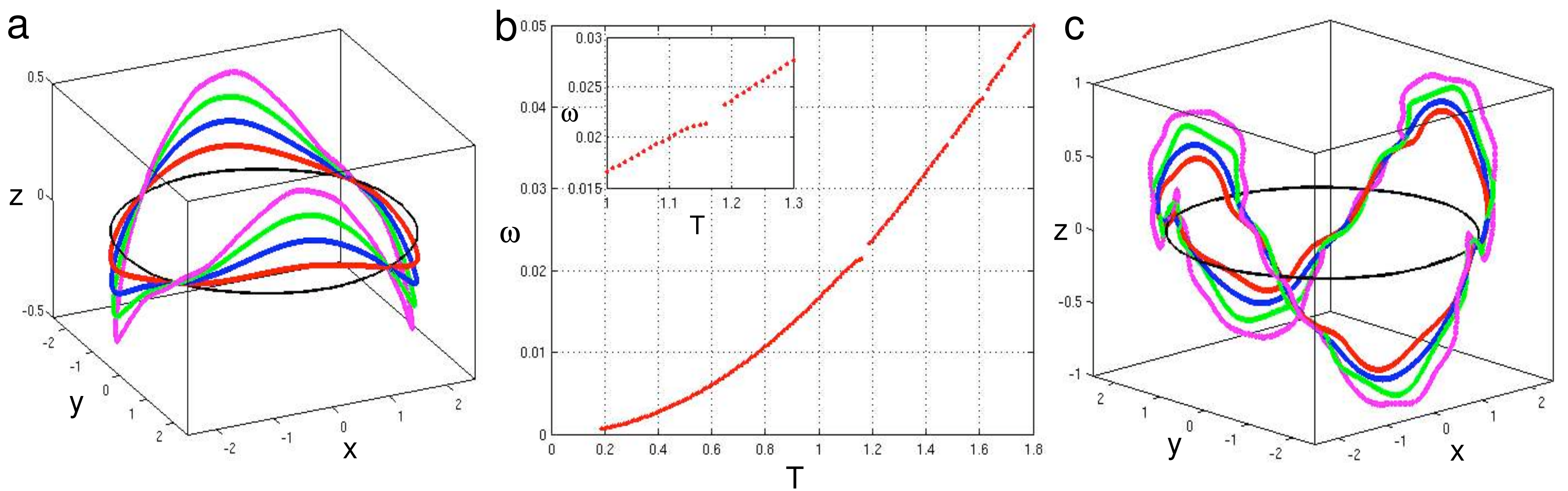


T = 1.0

$$T = \frac{\tau_{switch}}{\tau_{roll}}$$



Elliptic Invariant Circles

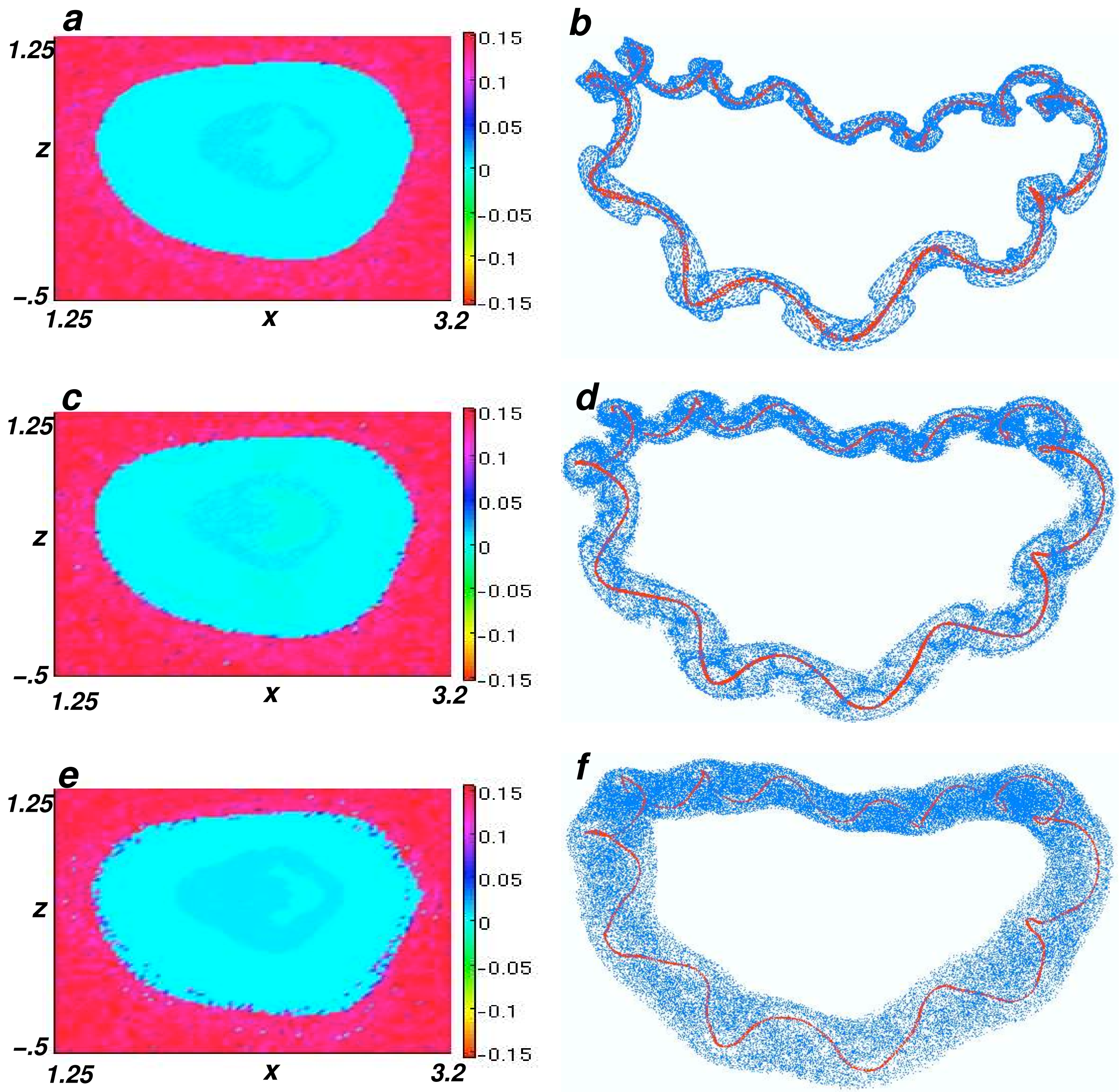


Circle Bifurcations

T = 1.16

T = 1.164

T = 1.168

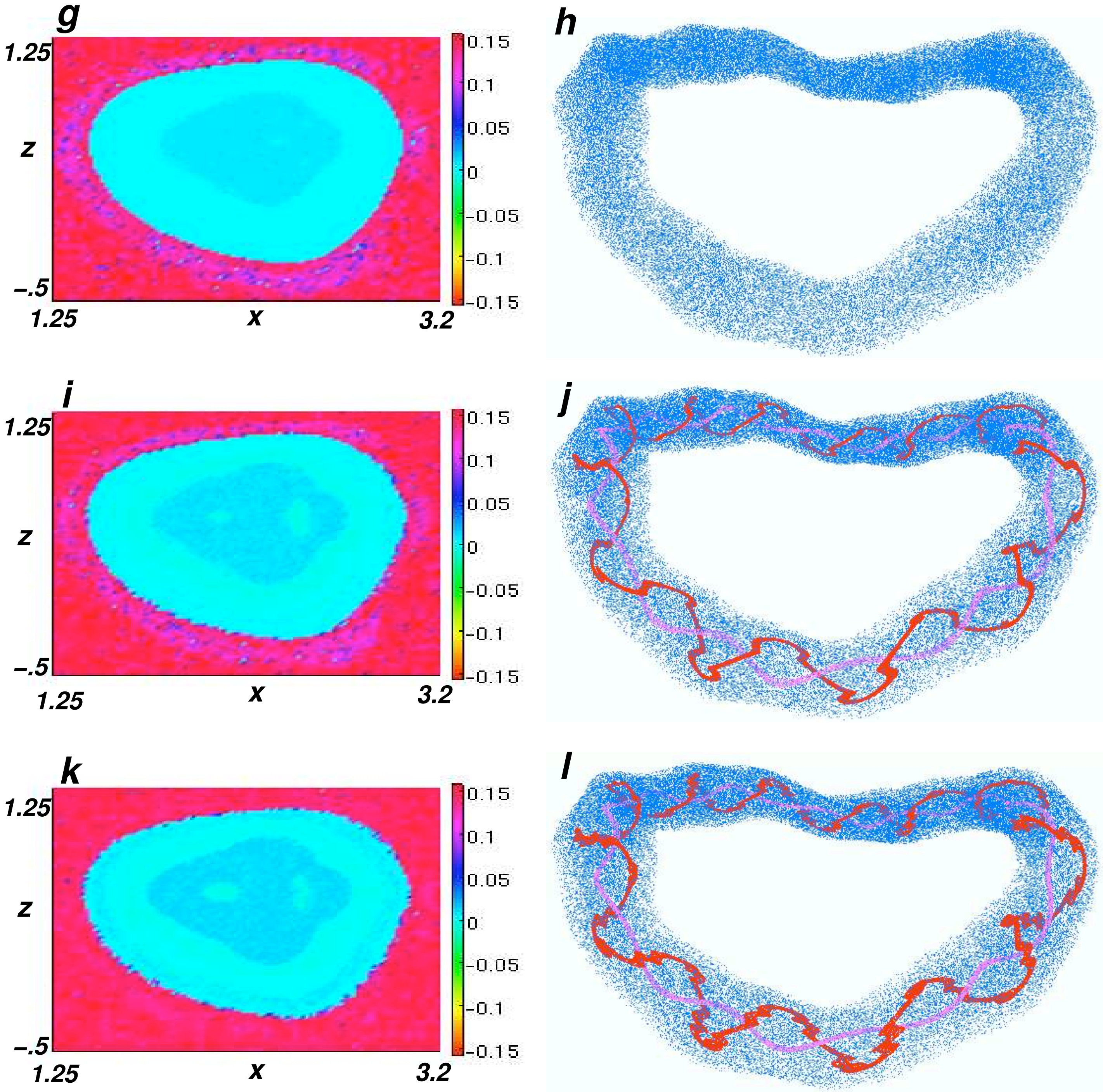


Circle Bifurcations

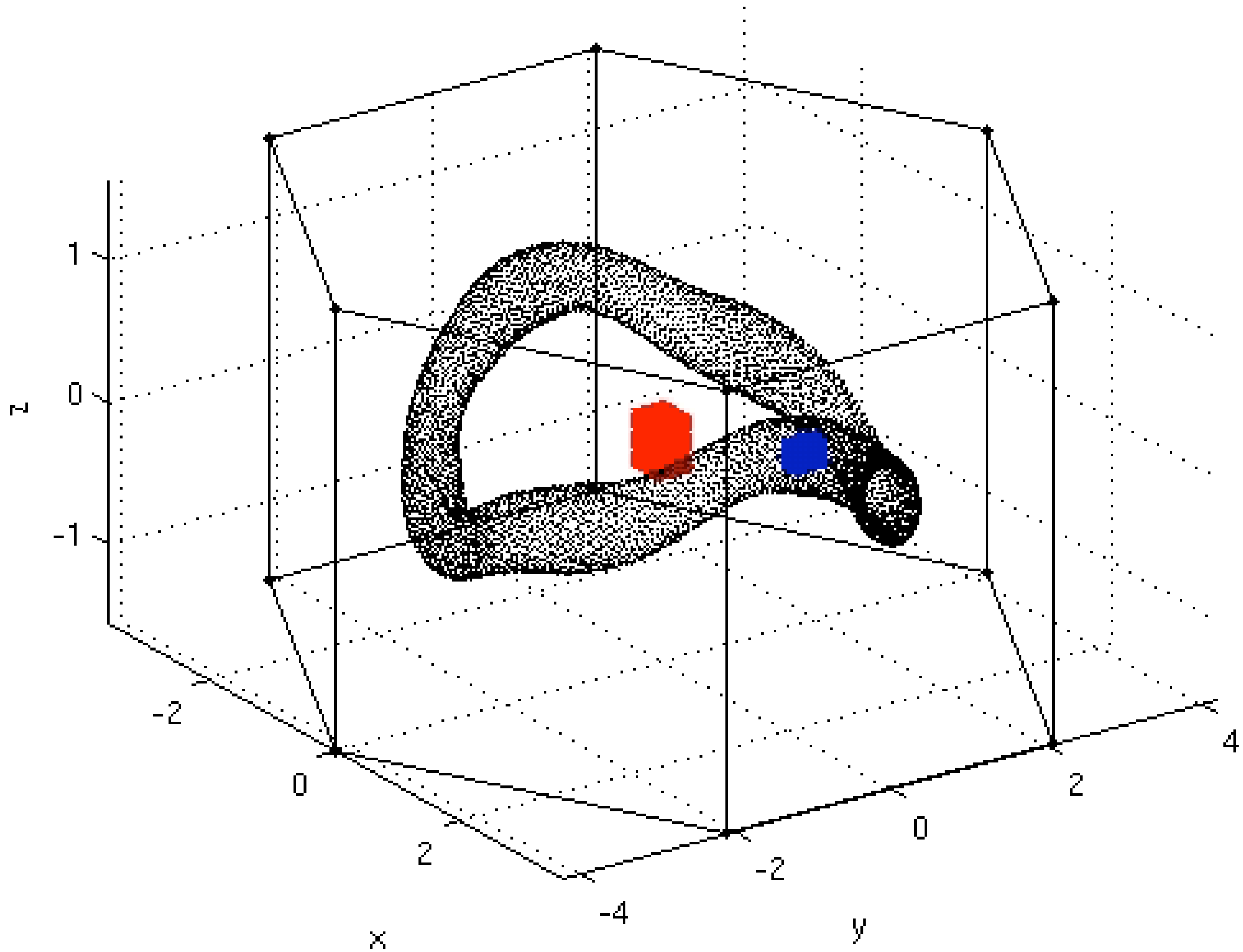
T = 1.172

T = 1.176

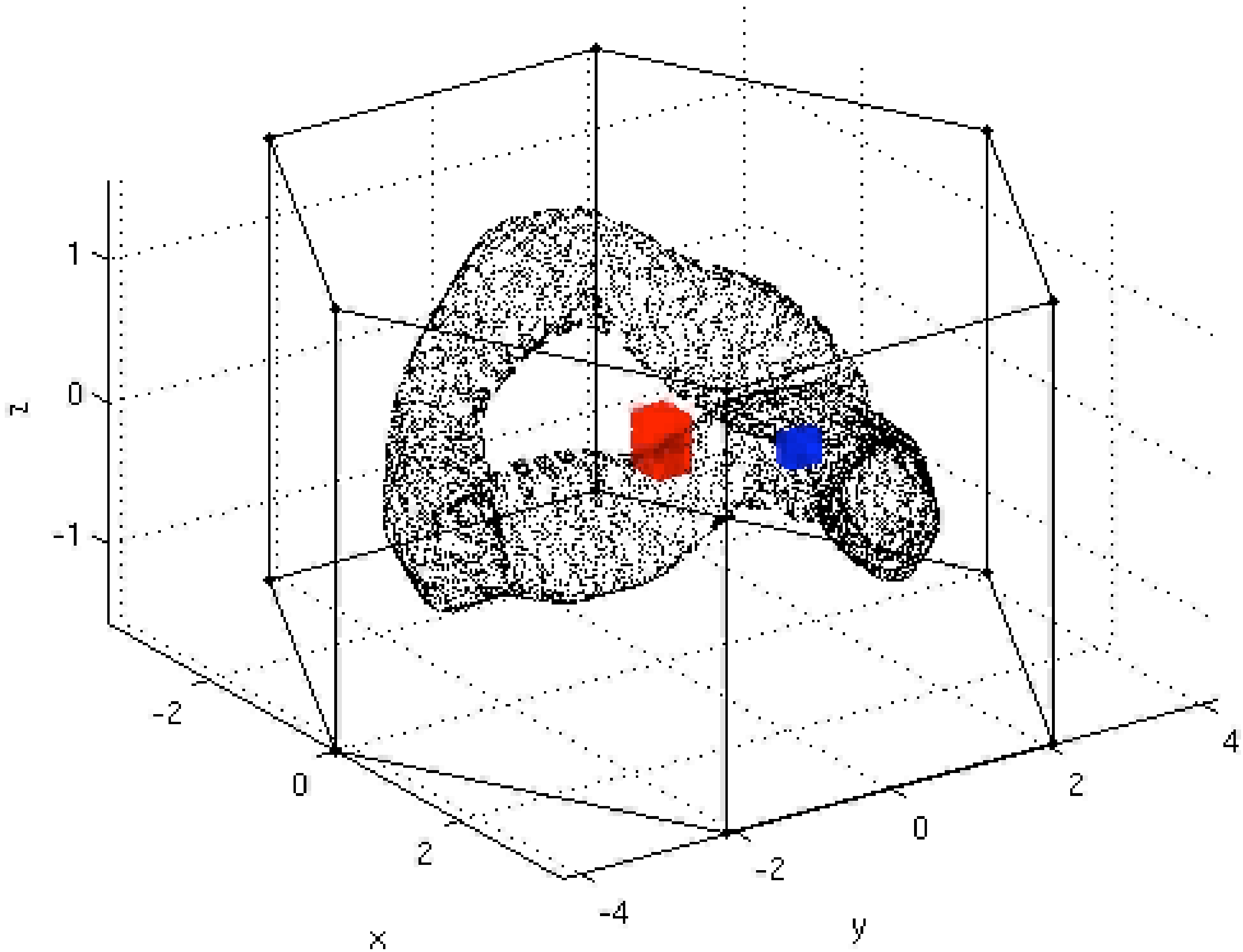
T = 1.18



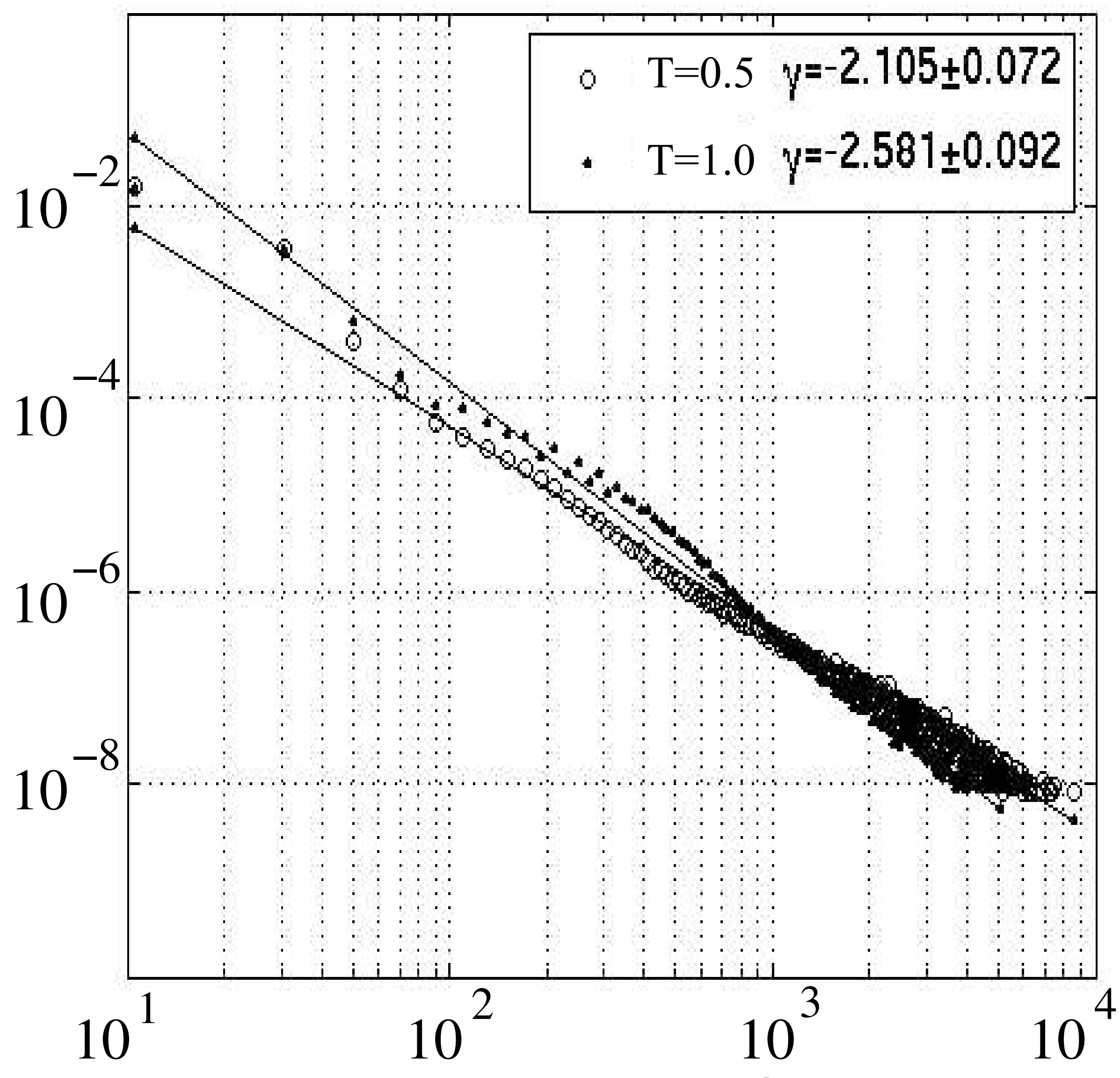
Confined & Regular



Confined Mixing

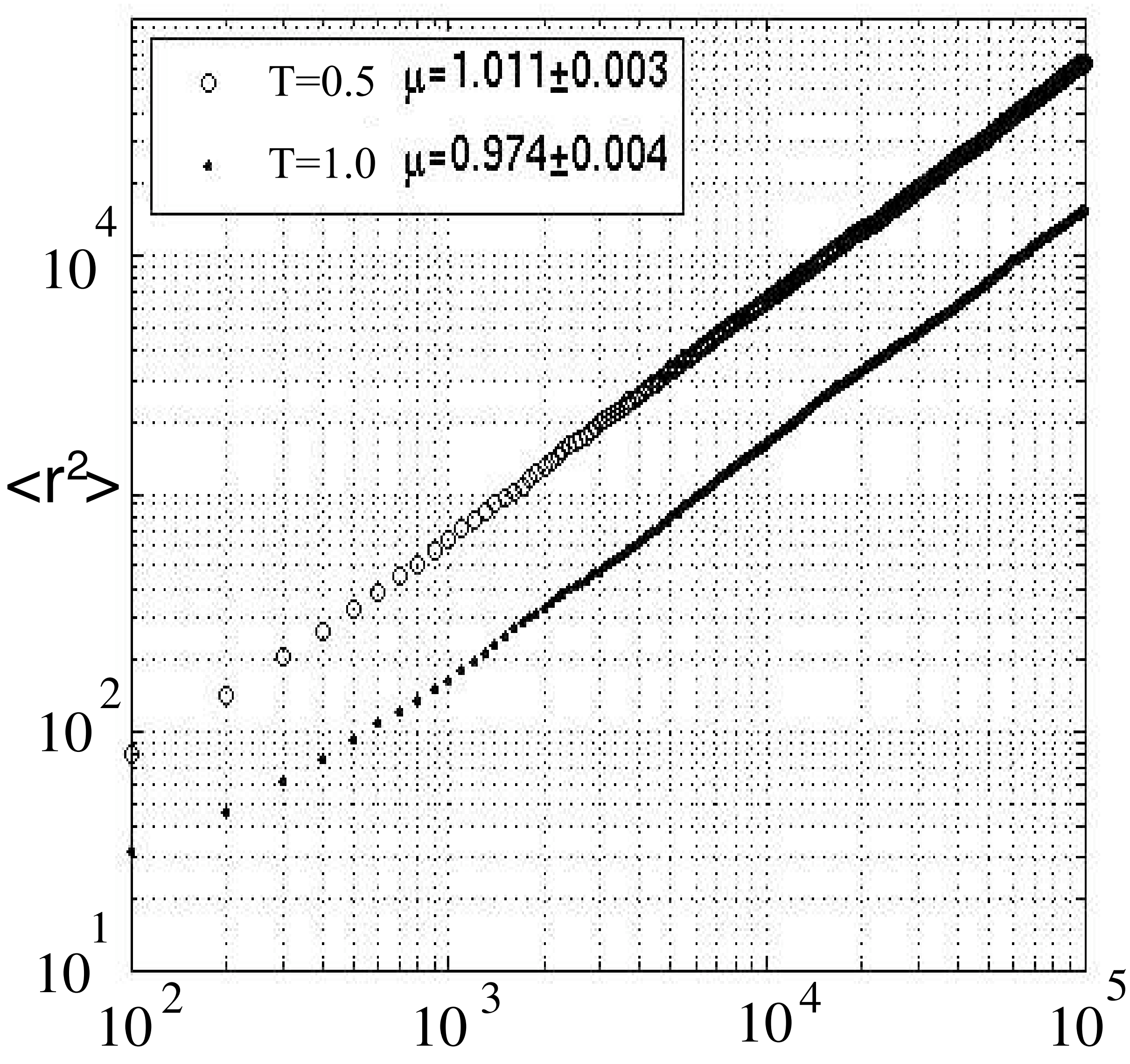


Exit Time Distribution



$$p \sim t^{\gamma}$$

Mean-Square Displacement



$$\langle r^2 \rangle \sim t^{\mu}$$

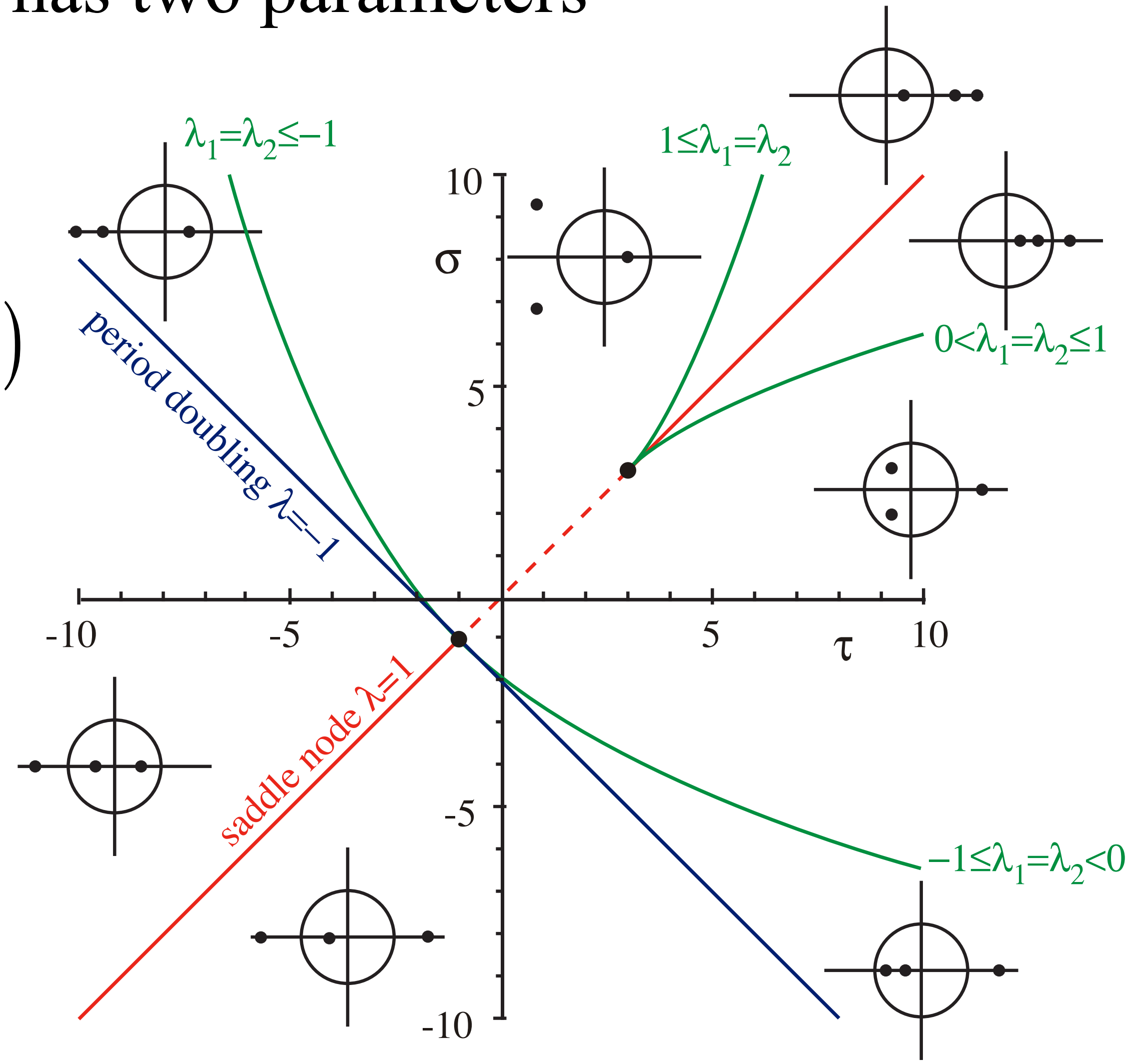
Volume Preserving Normal Forms

Bifurcations

Characteristic polynomial has two parameters

$$p(\lambda) = \lambda^3 - \tau\lambda^2 + \sigma\lambda - 1$$

$$\tau = Tr(Df),\ \ \sigma = \tfrac{1}{2}\left(\tau^2 - Tr(Df^2)\right)$$



$$\sigma = \tau :\ \ \lambda = 1$$

$$\sigma + \tau = -1 :\ \ \lambda = -1$$

$$\sigma = \tau = 2 :\ \ \lambda = (1,1,1)$$

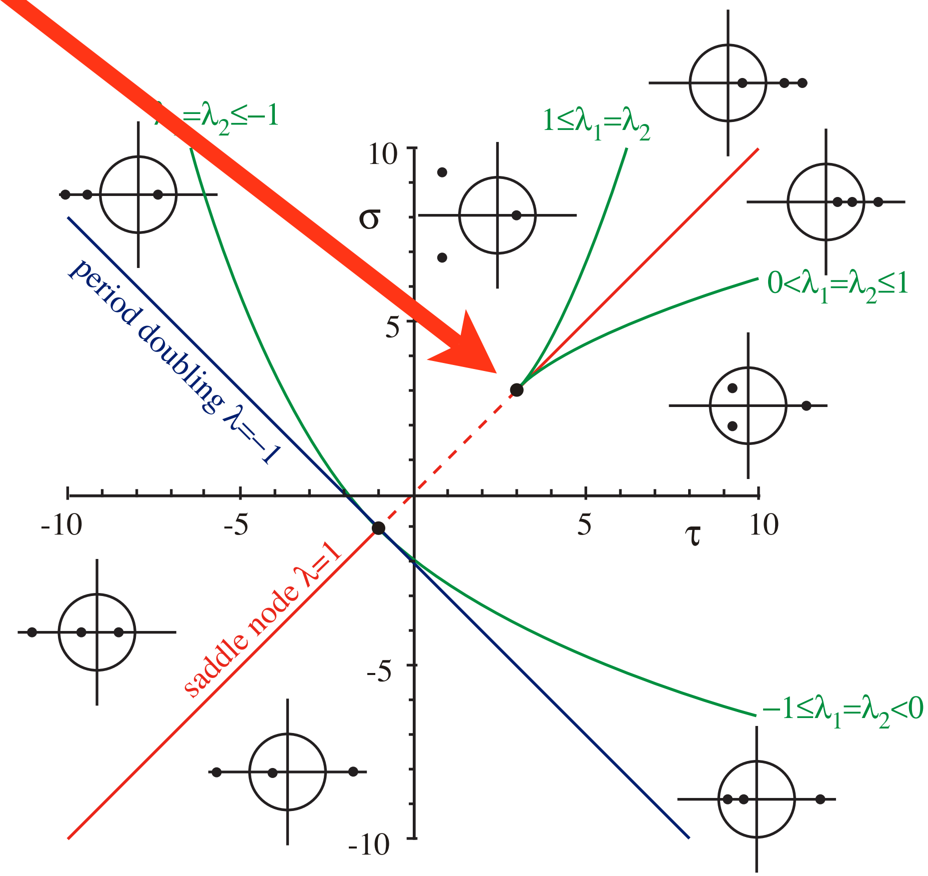
$$\sigma = \tau = -1 :\ \ \lambda = (-1,-1,1)$$

Bifurcations

(1,1,1) Normal Form

$$f(x,y,z)=\begin{pmatrix}1&1&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z+p(x,y)\end{pmatrix}$$

$$p=-\varepsilon+\mu y+ax^2+bxy+cy^2$$



Quadratic Volume Preserving Maps

Lomelí & Meiss, Nonlinearity 11 557(98).

Every quadratic volume preserving diffeomorphism with a quadratic inverse is conjugate to the a normal form

$$x' = \alpha + \tau x + z + Q(x, y)$$

$$y' = x$$

$$z' = y$$

Where $Q(x,y)$ is a quadratic form.

This map also arises in homoclinic bifurcations of 3D flows near a quadratic homoclinic tangency.

(111) Normal Form

$$(x,y,z) \to (x+y, y+z+p, z+p)$$

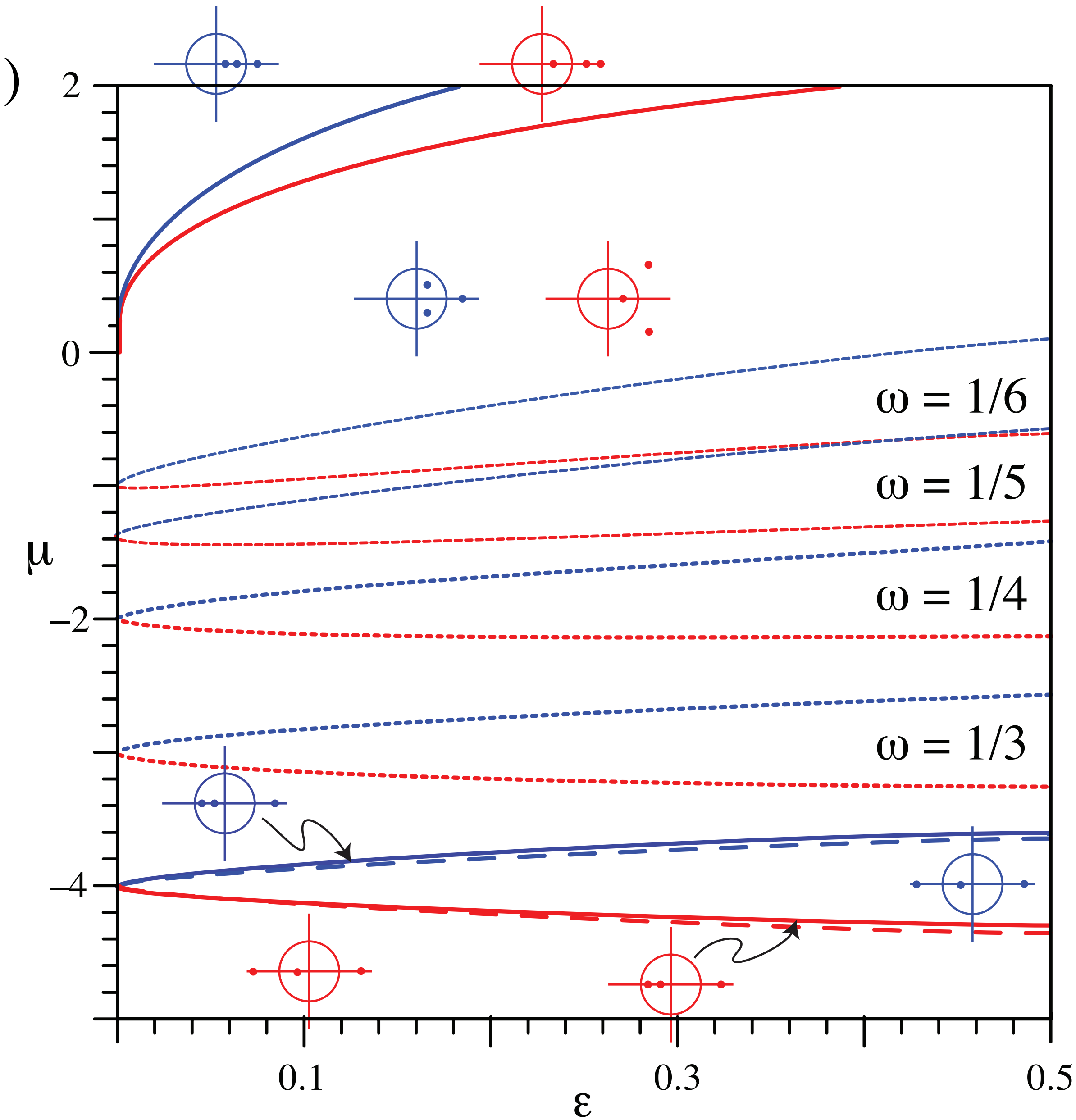
$$p = -\varepsilon + \mu y + x^2 + \tfrac{1}{2}xy + \tfrac{1}{2}y^2$$

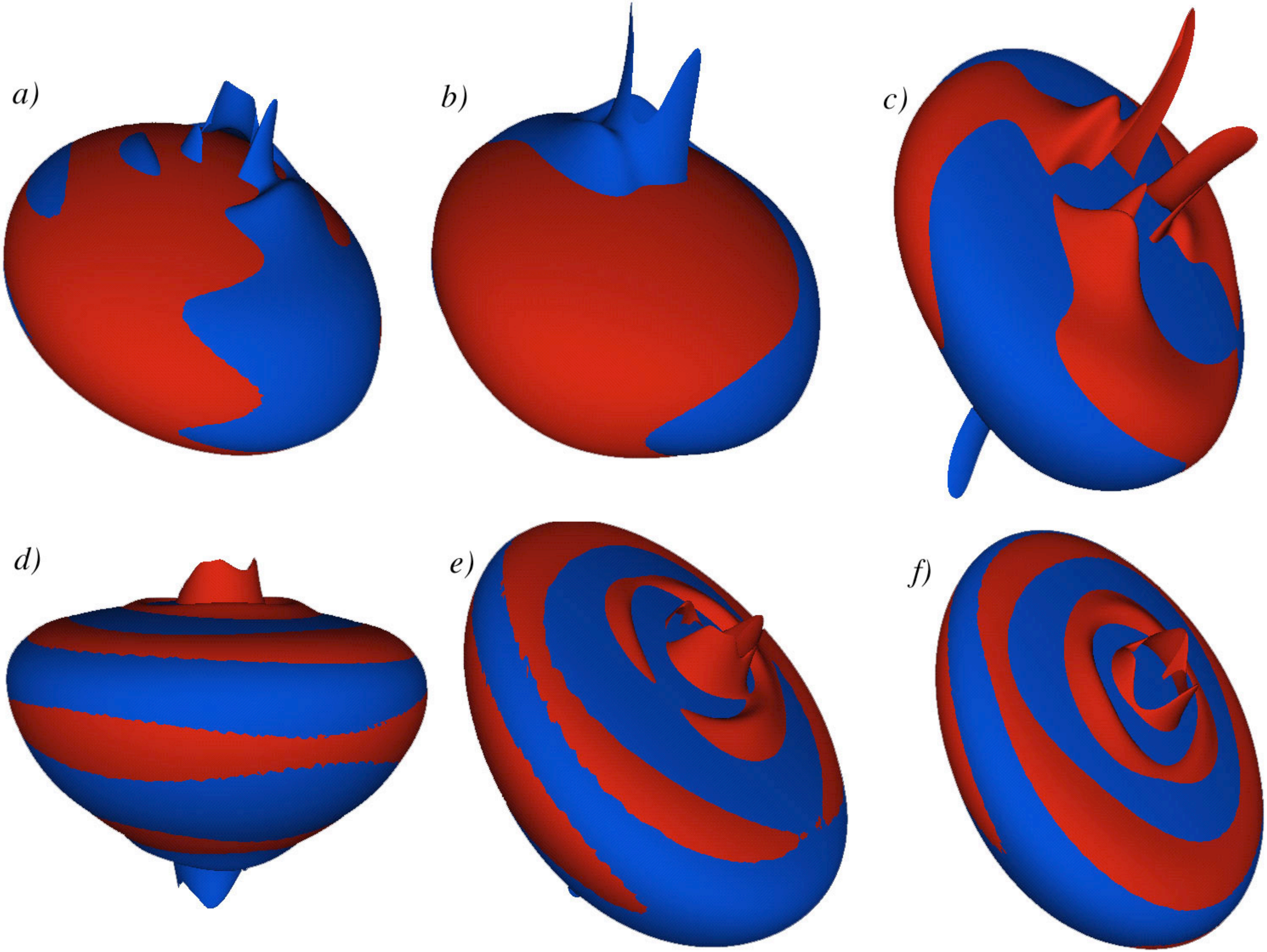
Two fixed points

$$(\pm\sqrt{\varepsilon},0,0)$$

Saddle-node-Hopf bifurcation along $\epsilon = 0$ and $-4<\mu<0$.

Period doubling at $\epsilon = 0$, $\mu = -4$.

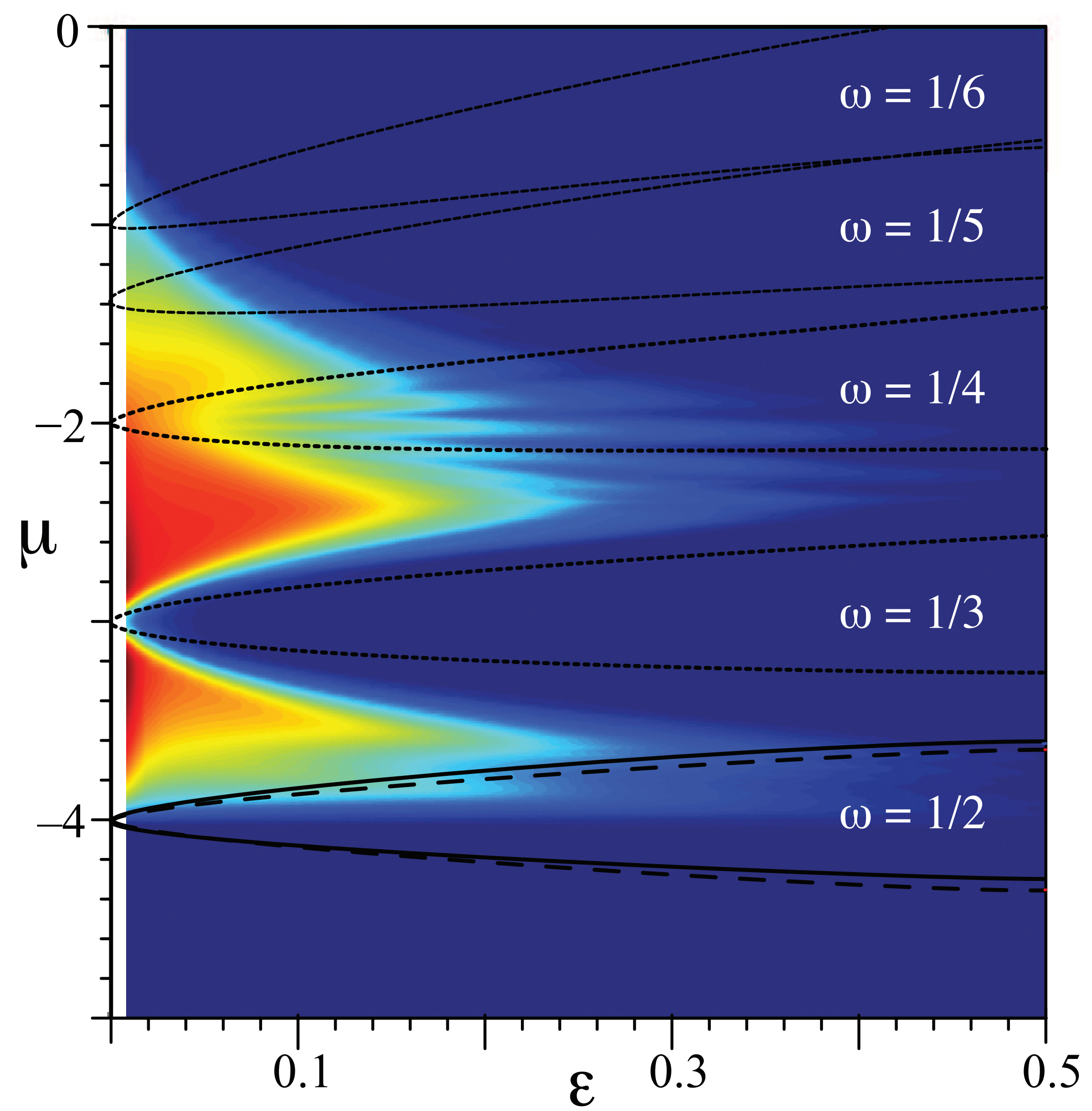




Lomelí, H. E. and J. D. Meiss (2000). “ Codimension One Melnikov Method” Chaos 10(1): 109-121.

Bounded Orbits

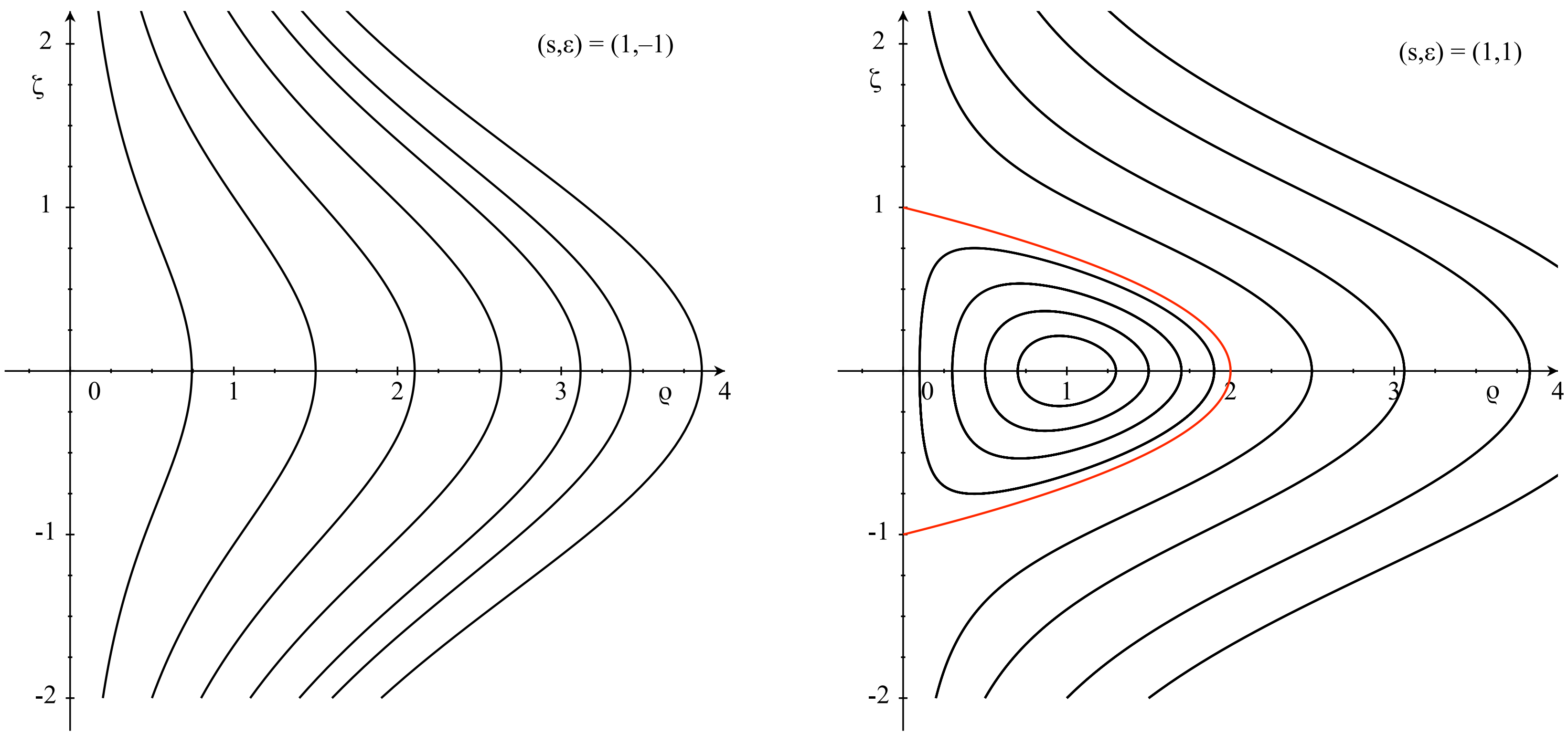
Fraction of orbits in a cube of size $\sqrt{\epsilon}$ that remain bounded for 100 iterations.

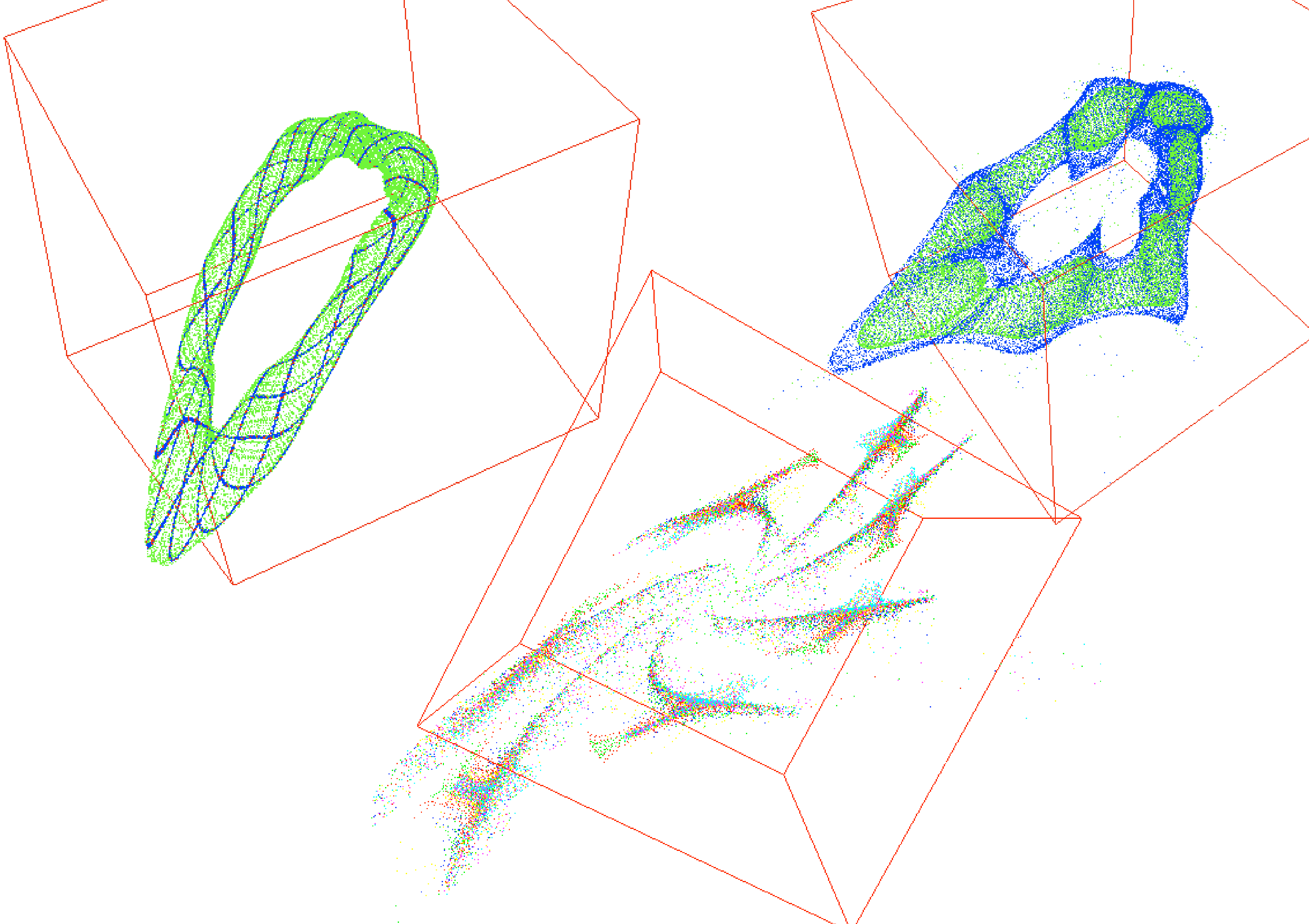


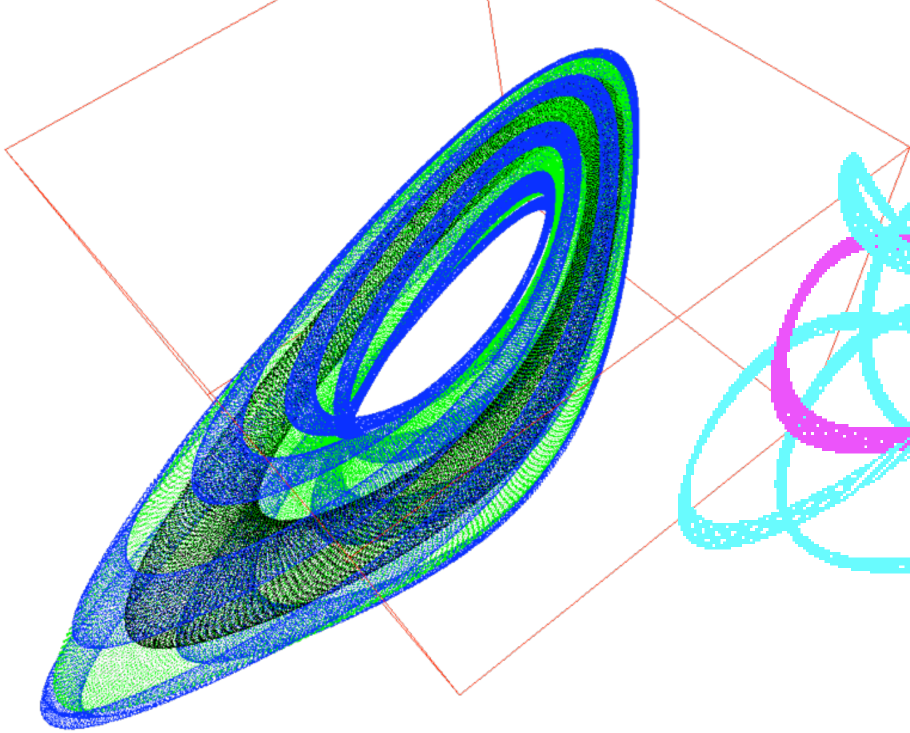
Saddle-node-Hopf

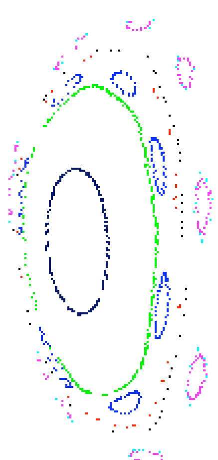
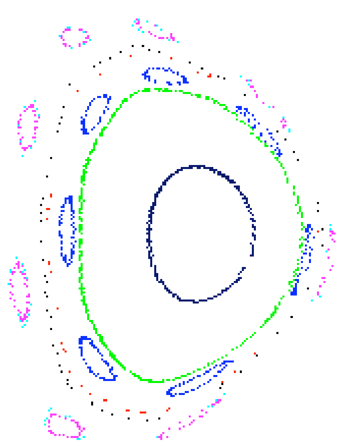
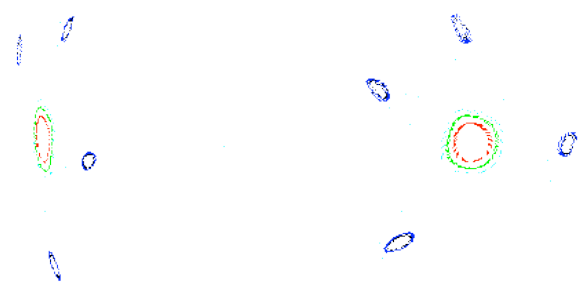
Normal form near $\left(e^{2\pi i\omega}, e^{-2\pi i\omega}, 1\right)$, ω irrational

$$\begin{aligned} r' &= r\left(1-2\gamma z\right) \\ \theta' &= \theta+\omega+\tau z \qquad\qquad +O(3) \\ z' &= -\delta+z+\beta r+\gamma z^2 \end{aligned}$$







Many more Questions

- Can experimentalists see the invariant tori in convection experiments?
- How to better quantify and control transport for 3D Systems
- How are invariant tori created/destroyed?
- Are there remnants of invariant tori in the chaotic seas? Cantori?
- What parameters optimize the mixing?
- Can one develop robust algorithms for finding tori and their invariant manifolds?
- Generalizations of KAM theory exist, but critical tori not studied.