[1] From spins to bosons: the Holstein-Primakoff transformation. Consider two quantum spins, $\mathbf{S}_1$ and $\mathbf{S}_2$, of magnitude S, interacting via the Hamiltonian

$$H = -J\mathbf{S}_1.\mathbf{S}_2 - H(S_1^z + S_2^z)$$

with $J > 0$.

(i) Use the standard theory for addition of angular momenta to find the exact energy levels.

(ii) Use the Holstein-Primakoff transformation and harmonic approximation to calculate the low-lying excitation energies, as follows. First, represent the spin operators $\mathbf{S}_i$ in terms of boson creation and annihilation operators $b_i^\dagger$ and b_i, using (Can you show that this works?)

$$S^z = S - b^\dagger b\,,$$

$$S^+ \equiv S^x + iS^y = (2S - b^\dagger b)^{1/2} b \approx \sqrt{2S} b\,,$$

and

$$S^- \equiv S^x - iS^y = b^\dagger(2S - b^\dagger b)^{1/2} \approx \sqrt{2S} b^\dagger\,.$$

In this way, you will get an approximate form for H which is quadratic in the boson operators. Diagonalise it using a unitary transformation.

(iii) Compare the exact and approximate calculations.

[2]. The kagome Heisenberg antiferromagnet. Consider the spin S Heisenberg antiferromagnet with nearest neighbour exchange J on the kagome lattice, Use spinwave theory in the harmonic approximation to treat excitations around a ground state in which all spins are co-planar.

Show that the spinwave frequencies are

$$\omega_n(\mathbf{q}) = JS\sqrt{2\lambda_n(\mathbf{q})(3 - \lambda_n(\mathbf{q}))}$$

where there are three branches, with

$$\begin{aligned}
\lambda_1(\mathbf{q}) &= 0 \\
\lambda_2(\mathbf{q}) &= \frac{3}{2}(1 - \sqrt{1 - A(\mathbf{q})}) \\
\lambda_3(\mathbf{q}) &= \frac{3}{2}(1 + \sqrt{1 - A(\mathbf{q})})
\end{aligned}$$

Here, $A(\mathbf{q}) = \frac{8}{9}\{1 - \cos(\pi q_1)\cos(\pi q_2)\cos(\pi[q_1 - q_2])\}$, and q_1, q_2 are the components of wavevector $\mathbf{q}$ along basis vectors of the reciprocal lattice.

[3]. The one-dimensional transverse field Ising model The intention in this question is to guide you through the exact solution of an interacting many-body problem in one space dimension. The solution uses two operator transformations – the Jordan-Wigner transformation and the Bogoluibov transformation – which are useful in many other contexts.

Consider a one-dimensional lattice with site-label m. Let σ^α, for $\alpha = x, y, z$ be the usual Pauli spin operators. The Hamiltonian for the one-dimensional transverse field Ising model is

$$H = -\Gamma \sum_m \sigma_m^z - J \sum_m \sigma_m^x \sigma_{m+1}^x .$$

(i) Discuss what the ground state would be as a function of J/Γ if σ^x and σ^z were components of a classical unit vector.

(ii) Let $a_m^\dagger$ and a_m be (spinless) fermion creation and annihilation operators. Show that these fermion operators can be written in terms of Pauli raising and lowering operators, $\sigma^\pm = (1/2)(\sigma_x \pm i\sigma_y)$, as

$$a_m = \exp(i\pi \sum_{j=1}^{m-1} \sigma_j^+ \sigma_j^-) \sigma_m^-$$

and

$$a_m^\dagger = \exp(-i\pi \sum_{j=1}^{m-1} \sigma_j^+ \sigma_j^-) \sigma_m^+ .$$

Show also that $a_m^\dagger a_m = (1 + \sigma_m^z)/2$

(iii) Write down expressions for $\sigma_m^\pm$ and σ_m^z in terms of the fermi operators. These constitute the Jordan-Wigner transformation.

(iv) Use these transformations to write H in terms of fermion operators.

(v) Use a Fourier transform and a Bogoluibov transformation to diagonalise the Hamiltonian. You should obtain

$$H = -\sum_k [(\Gamma + J\cos(k))(\alpha_k^\dagger \alpha_k + \alpha\dagger_{-k}\alpha_{-k} - 1) + iJ\sin(k)(\alpha_k^\dagger \alpha_{-k}^\dagger + \alpha_k \alpha_{-k})]$$

after the Fourier transformation alone, and

$$H = \sum_k \epsilon(k)(2c_k^\dagger c_k - 1)$$

after both transformations, with $\epsilon(k)^2 = \Gamma^2 + J^2 + 2\Gamma J \cos(k)$, where $\alpha^\dagger$, α, $c_k^\dagger$ and c_k are fermion creation and annihilation operators.

(vi) Hence show that the ground-state expectation value, $\langle \sigma_m^z \rangle$, is given by

$$\langle \sigma_m^z \rangle = \frac{1}{\pi} \int_0^\pi dk \frac{\Gamma + J\cos(k)}{\epsilon(k)} .$$