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School and Workshop on Highly Frustrated Magnets and Strongly Correlated Systems: From Non-Perturbative Approaches to Experiments

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Linear Response Theory and Dynamical Correlations

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International Conference
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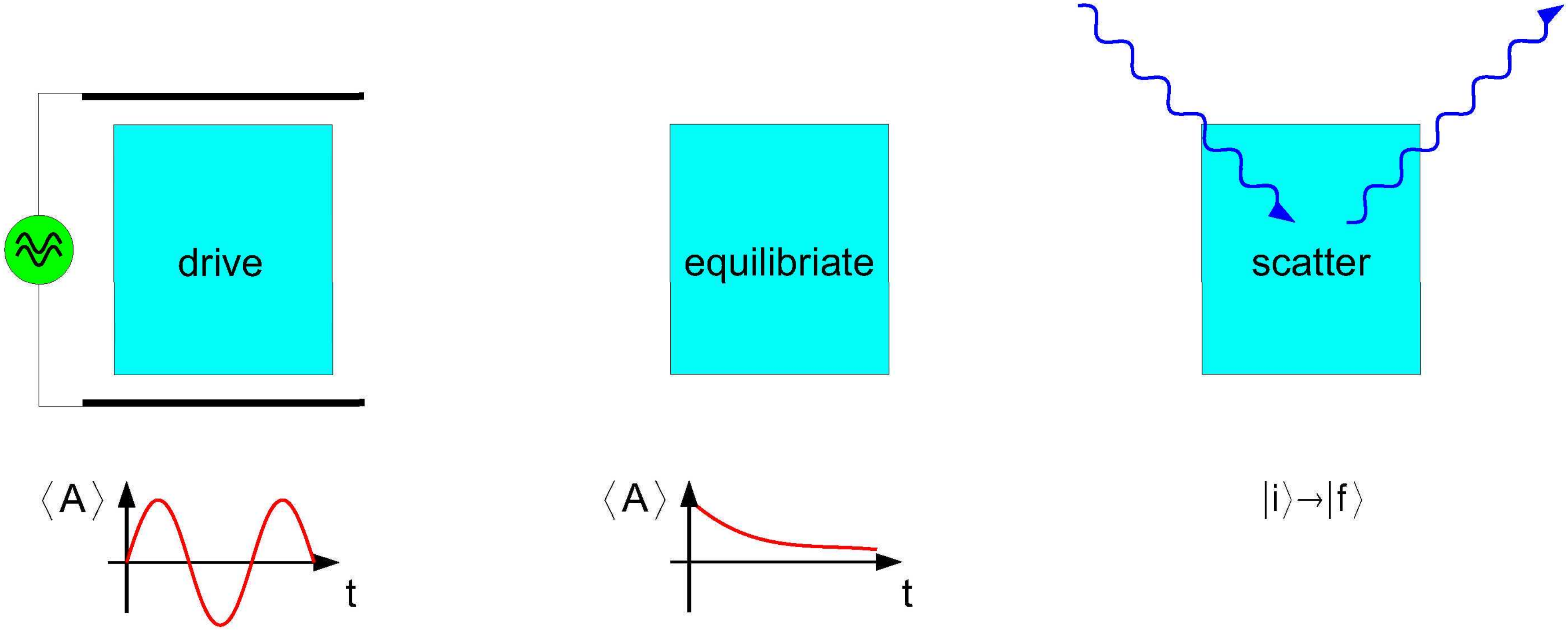
Three Types of Response (Functions)



dynamical susceptibility

$\chi(z)$

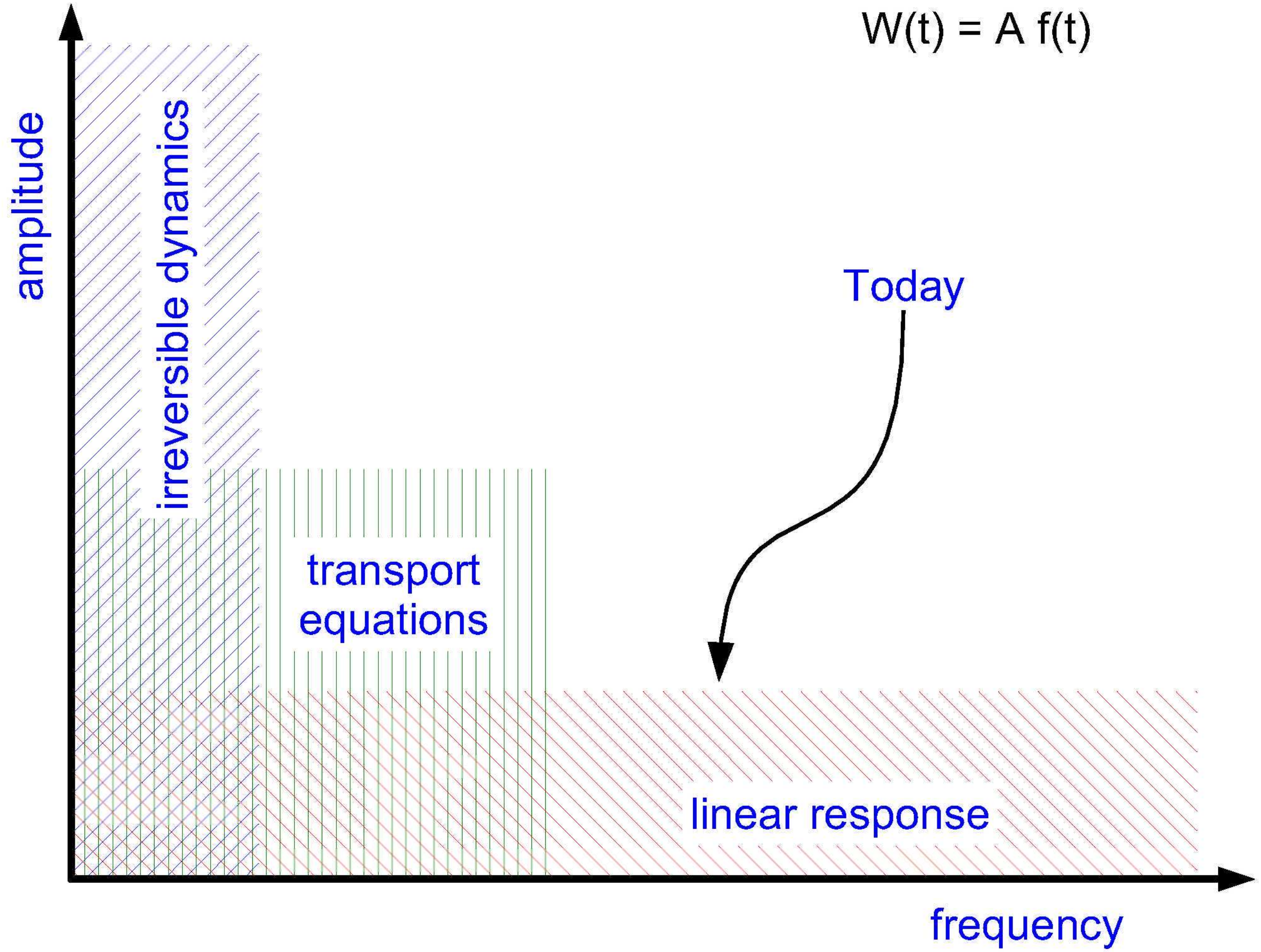
relaxation function

$\Phi(z)$

dynamical correlation function

$S(z)$

Regimes of Dynamic Processes



Syllabus

- von Neumann's Equation
- $\chi(z)$ Dynamical Susceptibility
- Spectral Functions
- Symmetries
- Mori Product
- Liouville Operator
- $\Phi(z)$ Relaxation Function
- Sum Rules and Moments
- $S(z)$ Dynamical Correlation Function
- Dissipation and Passivity
- Fluctuation-Dissipation Theorem
- Projection Technique

Literature:

D. Forster, "Hydrodynamic Fluctuations, Broken Symmetry and Correlation Functions", Addison-Wesley, 1990

E. Fick, G. Sauermann, "The quantum statistics of dynamic processes", Springer, 1990

Dynamics of the Statistical Operator: von Neumann's Equation*

- External perturbation

$$\tilde{H} = H + W(t)$$

- von Neumann's EQM for statistical operator

$$\dot{\rho}_S(t) = -i[\tilde{H}, \rho_S(t)]$$ * Schrödinger

$$A_H(t) = e^{i\tilde{H}t} A(t) e^{-i\tilde{H}t}$$

$$\dot{\rho}_H(t) = \rho_S(0)$$ * Heisenberg

$$A_D(t) = e^{iHt} A(t) e^{-iHt}$$

$$\dot{\rho}_D(t) = -i[W_D(t), \rho_D(t)]$$ * Dirac

- Switching „on“ adiabatically from -∞

$$W(t \to -\infty) = 0$$ „off“

$$\Rightarrow \quad [\rho_S(-\infty), H] = 0$$ equilib.

$$\Rightarrow \quad \rho_D(-\infty) = \rho_S(-\infty) \equiv \rho = \frac{1}{Z} e^{-\beta H}$$

- Linear regime $\|W\| \ll \|H\|$

$$\rho_D(t) = \rho - i \int_{-\infty}^{t} [W_D(t'), \rho] dt'$$

$$\cancel{+O(W^2)}$$

! $\hbar = k_B = 1$

Linear Response: Dynamical Susceptibility

- External perturbation

coordinates

forces

$$W(t) = -\sum_\nu A_\nu \, f_\nu(t)$$

coordinates,
momenta, axes,
various ops., ...

- Linear response of A^+_μ to force $f_\nu(t)$

$$\delta\langle A^+_\mu(t)\rangle = \mathrm{Tr}\left\{A^+_{\mu D}(t)(\rho_D(t) - \rho_D(-\infty))\right\}$$

$$\delta\langle A^+_\mu(t)\rangle = -i\int_{-\infty}^{t} \mathrm{Tr}\{A^+_{\mu D}(t)[W_D(t'),\rho]\}\,dt'$$

$$\delta\langle A^+_\mu(t)\rangle = i\sum_\nu \int_{-\infty}^{t} \langle[A^+_{\mu D}(t), A_{\nu D}(t')]\rangle_\rho f_\nu(t')\,dt'$$

'forget' D 'forget' ρ

- Dynamical susceptibility

$$\delta\langle A^+_\mu(t)\rangle = \sum_\nu \int_{-\infty}^{\infty} \chi_{\mu\nu}(t,t') f_\nu(t')\,dt'$$

$$\chi_{\mu\nu}(t,t') = \chi_{\mu\nu}(t-t',0) \equiv \chi_{\mu\nu}(t-t')$$

$$\boxed{\chi_{\mu\nu}(t,t') = i\Theta(t-t')\langle[A^+_\mu(t), A_\nu(t')]\rangle = i\Theta(t-t')\,\chi''_{\mu\nu}(t,t')}$$

Example :

$$\tilde{H} = \underbrace{\omega_0 b^+ b}_{H} + \underbrace{(b^+ + b) f(t)}_{W(t)} \quad ; \quad [b, b^+] = 1 \quad \text{Bosons} \; ; \quad \hbar = 1$$

$b^+ + b \equiv r$, $r^+ = r$

$$\Rightarrow \quad b(t) = e^{-i\omega_0 t} b$$

$$b^+(t) = e^{i\omega_0 t} b^+$$

$$\Rightarrow \quad \chi_{rr}(t) = i\Theta(t) \langle [e^{-i\omega_0 t} b + e^{i\omega_0 t} b^+ , b + b^+] \rangle$$

$$= \quad '' \quad (\langle [b, b^+] \rangle e^{-i\omega_0 t} + \langle [b^+, b] \rangle e^{i\omega_0 t})$$

$$\boxed{= i\Theta(t) \left(e^{-i\omega_0 t} - e^{i\omega_0 t} \right)}$$

No fct. of T ⚠

Spectral Functions

- Spectral function χ''

$$\boxed{\chi(t) := 2i\Theta(t)\chi''(t)}$$

- Laplace trafo

$$\int_{-\infty}^{\infty} \frac{\chi''(\omega)}{\omega - z}\frac{d\omega}{\pi} =$$

$\Im z > 0$

$$\int_{-\infty}^{\infty} dt\, \chi''(t) \int_{-\infty}^{\infty} \frac{d\omega}{\pi}\frac{e^{i\omega t}}{\omega - z} = \int_{0}^{\infty} dt\, 2i\chi''(t)e^{izt}$$

$$= \chi(z)$$

- Retarded / Advanced

$$\chi^{R/A}(u) = \chi(u \pm i0^{+}) =$$

$$P\int_{-\infty}^{\infty} \frac{\chi''(\omega)}{\omega - u}\frac{d\omega}{\pi} \pm i\chi''(u) = \chi'(u) \pm i\chi''(u)$$

- Fouriertrafo of linear response

$$\delta\langle A_{\mu}^{+}(\omega)\rangle = \sum_{\nu} \chi_{\mu\nu}^{R}(\omega) f_{\nu}(\omega)$$

- Adiabatic susceptibility

$$f_{\nu}(t) = \Theta(-t)e^{0^{+}t} f_{\nu}$$

$$\delta\langle A_{\mu}^{+}(t=0)\rangle = \sum_{\nu} \int_{0}^{\infty} \chi_{\mu\nu}(t')e^{-0^{+}t'}dt' f_{\nu}$$

$$= \sum_{\nu} \chi_{\mu\nu}^{R}(\omega = 0)\, f_{\nu}$$

adiabatic or
isolated χ
≠
isothermal χ

Example

$$\chi_{rr}(t) = i\,\Theta(t)\,\underbrace{\left(e^{-i\omega_0 t} - e^{+i\omega_0 t}\right)}$$

$$\leadsto \quad \chi''_{rr}(t) = \frac{1}{2}\cdot \quad \downarrow$$

$$\leadsto \quad \boxed{\chi''_{rr}(\omega)} = \frac{1}{2}\int dt\, e^{i\omega t}\left(\quad \downarrow \quad \right)$$

$$= \boxed{\pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]}$$

Spectrum

$$\leadsto \quad \boxed{\chi^{R/A}_{rr}(\omega)} = \int d\omega' \frac{\left[\delta(\omega'-\omega_0) - \delta(\omega'+\omega_0)\right]}{\omega'-\omega \mp i0^+}$$

$$= \boxed{\frac{2\omega_0}{\omega_0^2 - (\omega \pm i0^+)^2}}$$

adiabatic χ_{iso}

$$\leadsto \quad \boxed{\chi_{iso} = \frac{2}{\omega_0}}$$

Symmetries

- shorthand $A^{+}_{\mu}=A_{\bar{\mu}}$

- complex conjugation:

$$\chi''_{\mu\nu}(\omega)^{*}=-\chi''_{\bar{\mu}\bar{\nu}}(-\omega)$$

- commutator:

$$\chi''_{\mu\nu}(\omega)=-\chi''_{\bar{\nu}\bar{\mu}}(-\omega)$$

if: $A_{\mu}=A_{\nu}=A^{+}=A$

$\Rightarrow \chi$ real & odd in ω

- parity under t-reversal

$$\pi A(t)\pi^{-1}=P^{t}\,A(-t);\;\; P^{t}=\pm 1$$

$$\chi''_{\mu\nu}(\omega)=-P^{t}_{\mu}P^{t}_{\nu}\chi''_{\bar{\mu}\bar{\nu}}(-\omega)$$

check complex conjugation

$$\chi''_{\mu\nu}(\omega)^{*}=$$

$$\left(\int_{-\infty}^{\infty}dt\,e^{i\omega t}\tfrac{1}{2}\langle[A^{+}_{\mu}(t),A_{\nu}]\rangle\right)^{*}=$$

$$\int_{-\infty}^{\infty}dt\,e^{-i\omega t}\tfrac{1}{2}\langle[A^{+}_{\nu},A_{\mu}(t)]\rangle=$$

$$-\int_{-\infty}^{\infty}dt\,e^{-i\omega t}\tfrac{1}{2}\langle[A^{++}_{\mu}(t),A^{+}_{\nu}]\rangle=$$

$$-\chi''_{\bar{\mu}\bar{\nu}}(-\omega)$$

Example

$$\chi''_{rr}(\omega) = \pi\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]$$

commutator:

$$-\chi''_{rr}(-\omega) = -\chi''_{\bar{r}\bar{r}}(-\omega) = \chi''_{rr}(\omega)$$

$$\bar{r} \mathrel{\hat{=}} r^{+} = r$$

complex conjugation

$$\chi''_{rr}(\omega)^{*} = \chi''_{rr}(\omega) = -\chi''_{\bar{r}\bar{r}}(-\omega)$$

t-reversal : $\pi r \pi^{-1} = +1\, r$

$$\chi''_{rr}(\omega) = -(+1)(+1)\,\chi''_{\bar{r}\bar{r}}(-\omega)$$

$$= -\chi''_{rr}(-\omega)$$

What about:

χ^{iso}_{px}

space

momentum

$$\chi''_{px}(\omega) = -(-1)(+1)\,\chi''_{\bar{p}\bar{x}}(-\omega)$$

$$= \chi''_{px}(-\omega)$$

↳ $\chi^{iso}_{px} = \int \frac{d\omega}{\pi}\, \frac{\chi''_{px}(\omega)}{\omega}$

$= 0$

↳ B.t.w.: what about

χ^{iso}_{PM} magnetization

diel. polarization

Mori Product

- Mori's scalar product

$$\boxed{\langle A_\mu | A_\nu \rangle = \int_0^\beta \langle \Delta A_\mu^+(\tau) \Delta A_\nu \rangle d\tau}$$

$$\Delta A = A - \langle A \rangle$$

$$A_\mu(\tau) = e^{\tau H} A_\mu e^{-\tau H}$$ imaginary time $\tau \in \Re$

1) bilinear ✓

2) Positive semidefinite

$$\langle A | A \rangle = \int_0^\beta \langle A^+(\tau) A \rangle d\tau = \frac{1}{Z} \sum_{l,m} |\langle l | A | m \rangle|^2 e^{-\beta E_m} \int_0^\beta e^{\tau (E_m - E_l)} d\tau \geqslant 0 \qquad =0 \Leftrightarrow A=0$$

$$\frac{e^{-\beta E_l} - e^{-\beta E_m}}{E_m - E_l} > 0$$

3) cc-conjugation

$$\langle A_\mu | A_\nu \rangle^* = \int_0^\beta \langle A_\nu^+ A_\mu(-\tau) \rangle d\tau = \langle A_\nu | A_\mu \rangle$$

- isothermal susceptibility

$$\boxed{\chi_{\mu\nu}^T \equiv \frac{\partial \langle A_\mu^+ \rangle}{\partial f_\nu} = \langle A_\mu | A_\nu \rangle}$$

(see tutorial)

Example

$$b(\tau) = e^{-\omega_0 \tau} b$$

$$b^+(\tau) = e^{\omega_0 \tau} b^+$$

$$\rightsquigarrow \quad \langle + | + \rangle = \int_0^\beta \langle (e^{-\omega_0 \tau} b + e^{\omega_0 \tau} b^+)(b + b^+) \rangle d\tau$$

$$= \int_0^\beta \langle b b^+ \rangle e^{-\omega_0 \tau} d\tau + \int_0^\beta \langle b^+ b \rangle e^{\omega_0 \tau} d\tau$$

$$= (1 + n_\beta)\left(-\frac{1}{\omega_0}\right)\left(e^{-\beta\omega_0} - 1\right) + n_\beta \left(\frac{1}{\omega_0}\right)\left(e^{\beta\omega_0} - 1\right)$$

$$= \frac{1}{\omega_0} + \frac{1}{\omega_0} = \frac{2}{\omega_0}$$

$$\rightsquigarrow \quad \boxed{\chi_{\underline{I}so} = \chi^T}$$

Liouville Operator

- Linear operator on A_μ

$$L\,A_\mu(t) = [H, A_\mu(t)]$$

$$\dot{A}_\mu(t) = i L\, A_\mu(t)$$

$$A_\mu(t) = e^{iLt} A_\mu$$

- Liouville is hermitean

$$\langle A|LB\rangle = \langle B|L A\rangle^* = \langle L A|B\rangle$$

$$\int_0^\beta d\tau \langle A^+(\tau) LB\rangle =$$

$$\ldots\langle e^{\tau H} A^+ e^{-\tau H}[H,B]\rangle\ldots$$

$$\ldots\langle e^{\tau H}[H,A]^+ e^{-\tau H} B\rangle\ldots$$

- Kubo's identity

$$\langle A|LB\rangle = \langle[A^+, B]\rangle$$

$$\int_0^\beta d\tau \langle A^+(\tau) LB\rangle =$$

$$\ldots\langle e^{\tau L} A^+ LB\rangle = \ldots - \langle L e^{\tau L} A^+ B\rangle =$$

$$\ldots - \frac{d}{d\tau}\langle e^{\tau L} A^+ B\rangle = -\langle A^+(\beta) B\rangle + \langle A^+ B\rangle = \langle[A^+, B]\rangle$$

- Laplace trafo of dynamical susceptibility

$$\chi_{\mu\nu}(z) = i\int_0^\infty e^{izt}\langle[A_\mu^+(t), A_\nu]\rangle dt$$

$$= i\int_0^\infty \langle[A_\mu^+, e^{i(z-L)t} A_\nu]\rangle dt$$

$$\boxed{= -\langle[A_\mu^+, \frac{1}{z-L} A_\nu]\rangle}$$

Example

$$\langle x|x\rangle = \langle x| b + b^+\rangle$$

with: $-\frac{1}{\omega_0} L b = b$

$\frac{1}{\omega_0} L b^+ = b^+$

$$\rightsquigarrow \langle x|x\rangle = -\frac{1}{\omega_0}\langle x|Lb\rangle + \frac{1}{\omega_0}\langle x|Lb^+\rangle$$

$$= -\frac{1}{\omega_0}\langle[b+b^+, b]\rangle + \frac{1}{\omega_0}\langle[b+b^+, b^+]\rangle$$

$$= \frac{2}{\omega_0} \rightsquigarrow \boxed{\chi^T = \chi_{iso}}$$

$$\boxed{\chi_{xx}(z)} = -\left\langle\left[(b+b^+), \frac{1}{z-L}(b+b^+)\right]\right\rangle$$

$$= -\left\langle\left[b, \frac{1}{z-L}b^+\right]\right\rangle - \left\langle\left[b^+, \frac{1}{z-L}b\right]\right\rangle$$

$$= \frac{1}{z-\omega_0}\left(-\langle[b,b^+]\rangle\right) + \frac{1}{z+\omega_0}\left(-\langle[b^+,b]\rangle\right)$$

$$= \boxed{\frac{1}{\omega_0-z} - \frac{1}{-\omega_0-z}}$$

✓ o.k.

see spectral reps.

Example $\chi_{ISO} \neq \chi^T$

$$\tilde{H} = \underbrace{BS^z}_{H} + \underbrace{b(t)S^z}_{W(t)} \quad , \quad S=\frac{1}{2}$$

$$LS^z = 0 \qquad \boxed{S^z \equiv \text{conserved}}$$

$$\boxed{\langle S^z | S^z \rangle} = \int_0^\beta \langle S^z(\tau) S^z \rangle - \beta \langle S^z \rangle^2$$

$$= \beta \left(\langle S^{z2} \rangle - \langle S^z \rangle^2 \right)$$

$$= \boxed{\frac{\beta}{4} \frac{1}{ch^2\left(\frac{\beta B}{2}\right)}}$$

$$\boxed{\chi(z)} = \langle [S^z, \frac{1}{z-L} S^z] \rangle$$

$$= \frac{1}{z} \langle [S^z, S^z] \rangle$$

$$= \boxed{0}$$

$$\boxed{\chi_{ISO} \neq \chi^T}$$

Relaxation Function

- t=0: equlibrium with forces ≠0

$$\rho_D(t=0)=\rho_S(t=0)=\frac{1}{Z}e^{-\beta(H-\sum_\nu A_\nu f_\nu)}$$

t>0: switch off forces

$$\frac{d}{dt}\langle A^+_\mu(t)\rangle= i\mathrm{Tr}\{[H,A^+_\mu(t)]\rho_D(0)\}=$$

$$i\mathrm{Tr}\{[\textstyle\sum_\nu A_\nu f_\nu, A^+_\mu(t)]\rho_D(0)\}=$$

$$-i\sum_\nu \mathrm{Tr}\{[A^+_\mu(t),A_\nu]\rho_D(0)\}\ f_\nu$$

$$\langle A^+(t)\rangle=-i\sum_\nu \Phi_{\mu\nu}(t)f_\nu$$

$$\boxed{\Phi_{\mu\nu}(t)=i\Theta(t)(\chi^T_{\mu\nu}-\int_0^t \chi_{\mu\nu}(t')dt')}$$

- Laplace trafo

$$\int_0^\infty \Phi_{\mu\nu}(t)e^{izt}=$$

$$\boxed{\Phi_{\mu\nu}(z)=\frac{1}{z}(\chi_{\mu\nu}(z)-\chi^T_{\mu\nu})}$$

$$\frac{1}{z}\chi_{\mu\nu}(z)=-\frac{1}{z}\langle[A^+_\mu,\frac{1}{z-L}A_\nu]\rangle$$

$$=\frac{1}{z}\langle A_\mu|(-L)\frac{1}{z-L}A_\nu\rangle$$

$$=\frac{1}{z}\langle A_\mu|(z-L-z)\frac{1}{z-L}A_\nu\rangle$$

$$=\frac{1}{z}\langle A_\mu|A_\nu\rangle-\langle A_\mu|\frac{1}{z-L}A_\nu\rangle$$

$$\boxed{\Phi_{\mu\nu}(z)=-\langle A_\mu|\frac{1}{z-L}A_\nu\rangle}$$

...by the way

$\Phi''_{\mu\nu}(t)=\frac{1}{2}\langle A_\mu|e^{-iLt}A_\nu\rangle$ spectral function of relaxation function

$\Phi''_{\mu\nu}(\omega)=\pi\langle A_\mu|\delta(\omega-L)A_\nu\rangle$ Fouriertrafo

$\Im\Phi^R_{\mu\nu}(\omega) = \Im\Phi_{\mu\nu}(\omega+i0^+) = -\Im\langle A_\mu|\frac{1}{\omega+i0^+-L}A_\nu\rangle = \pi\langle A_\mu|\delta(\omega-L)A_\nu\rangle$ O.K. ✓

$$\boxed{\chi''_{\mu\nu}(\omega)=\omega\Phi''_{\mu\nu}(\omega)}$$

beware if $LA_\nu=0 \Rightarrow$

$\frac{\chi''_{\mu\nu}(\omega)}{\omega} \neq \Phi''_{\mu\nu}(\omega)$

Sum Rules and Moments

- n-th moment of the spectrum

$$\boxed{\int \frac{d\omega}{\pi}\, \omega^n \chi''_{\mu\nu}(\omega) =}$$

$$\int \frac{d\omega}{\pi}\, \omega^{n+1} \Phi''_{\mu\nu}(\omega) =$$

$$\int d\omega\, \omega^{n+1} \langle A_\mu | \delta(\omega - L) A_\nu \rangle =$$

$$\langle A_\mu | L^{n+1} A_\nu \rangle =$$

$$\boxed{\langle [A_\mu^+, \underbrace{[H,[H,\ldots,}_{\text{n commutators with H}} A_\nu]\ldots]] \rangle}$$

- Typical example: „f sum-rule“

polarization: $d = e\,x$

$$\int \frac{d\omega}{\pi}\, \omega\, \chi''_{xx}(\omega) = \langle [x,[H,x]] \rangle$$

electrons with renorm. mass m^*: $H = \frac{p^2}{2m^*}$

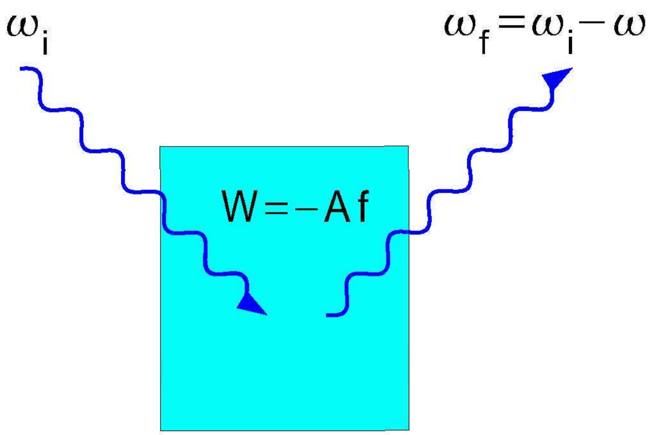
$$\int \frac{d\omega}{\pi}\, \omega\, \chi''_{dd}(\omega) = \frac{e^2 \hbar^2}{m^*}$$

spectrum of dielectric susceptibility

similar stuff for optical conductivity

Dynamical Correlation Fuctions

- Scattering: neutrons, light, ...



$$|l\rangle \rightarrow |m\rangle$$

- Intensity from „golden rule“

$$I(\omega) = \frac{2\pi}{Z} \sum_{l,m} e^{-\beta E_l} |\langle m|A|l\rangle|^2 \delta(\omega - E_m + E_l)$$

$$\delta(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} dt$$

$$\ldots \frac{1}{Z} \sum_{lm} \int_{-\infty}^{\infty} e^{i\omega t} e^{-\beta E_l} \langle l|e^{iE_l t} A e^{-iE_m t}|m\rangle \langle m|A|l\rangle dt =$$

$$\ldots \frac{1}{Z} \sum_{lm} \int_{-\infty}^{\infty} e^{i\omega t} e^{-\beta E_l} \langle l|e^{iHt} A e^{-iHt}|m\rangle \langle m|A|l\rangle dt =$$

$$\ldots \int_{-\infty}^{\infty} e^{i\omega t} \langle A(t) A\rangle dt$$

$$\boxed{I(\omega) \propto \int_{-\infty}^{\infty} e^{i\omega t} \langle A(t) A\rangle dt \equiv S_{AA}(t) \qquad \text{dynamical correlation function}}$$

Dissipation and Passivity

$$\tilde{H}_S(t) = H - \sum_\mu A_\mu f_\mu(t)$$

hermitean   real

- Rate of energy change

$$\frac{dW}{dt} = \mathrm{Tr}\{\rho_S(t)\dot{\tilde{H}}_S(t)\} + \mathrm{Tr}\{\dot{\rho}_S(t)\tilde{H}_S(t)\}$$

$$= -\sum_\mu \langle A_\mu \rangle(t)\dot{f}_\mu(t)$$

- Total energy absorbed

$$W(\infty) - W(-\infty) =$$

$$\sum_\mu \int_{-\infty}^{\infty} f_\mu(t) \frac{d\langle A_\mu \rangle(t)}{dt} dt =$$

$$\sum_{\mu\nu} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f_\mu(-\omega)(-i\omega)\chi_{\mu\nu}(\omega) f_\nu(\omega) =$$

$$\boxed{\sum_{\mu\nu} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} f_\mu(-\omega)\, \omega\chi''_{\mu\nu}(\omega)\; f_\nu(\omega)}$$

- Dissipative systems

$$\boxed{\omega\chi''_{\mu\nu}(\omega) = \omega^2 \Phi''_{\mu\nu}(\omega) \geqslant 0}$$

Fluctuation-Dissipation Theorem

- note

$$\langle B A \rangle = \frac{1}{Z} \mathrm{Tr}\{e^{-\beta H} B A\}$$

$$= \langle A(\beta) B \rangle = \langle e^{\beta L} A B \rangle$$

where: $A(x) = e^{xH} A e^{-xH}$

- with 'real' time dependence

$$\langle B A(it) \rangle = \langle e^{(it+\beta)L} A B \rangle$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \langle B A(it) \rangle dt = e^{-\beta\omega} \langle \delta(\omega + L) A B \rangle$$

or

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \langle [A(it), B]_{\pm} \rangle dt$$

$$= (1 \pm e^{-\beta\omega}) \langle \delta(\omega + L) A B \rangle$$

$$= (1 \pm e^{-\beta\omega}) \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \langle A(it) B \rangle$$

- Relation between *spectrum of dynamical suscetibility* and *correlation function* ≡ relation between *dissipation* and *fluctuation*

$$\boxed{\chi''^{\pm}_{AB}(\omega) = \frac{1}{2}(1 \pm e^{-\beta\omega}) S_{AB}(\omega)}$$

- detailed balance

$$\chi''_{AB}(\omega) = -\chi''_{\bar{B}\bar{A}}(-\omega)$$

$$\boxed{e^{-\beta\omega} S_{AB}(\omega) = S_{\bar{B}\bar{A}}(-\omega)}$$

Stokes Anti-Stokes

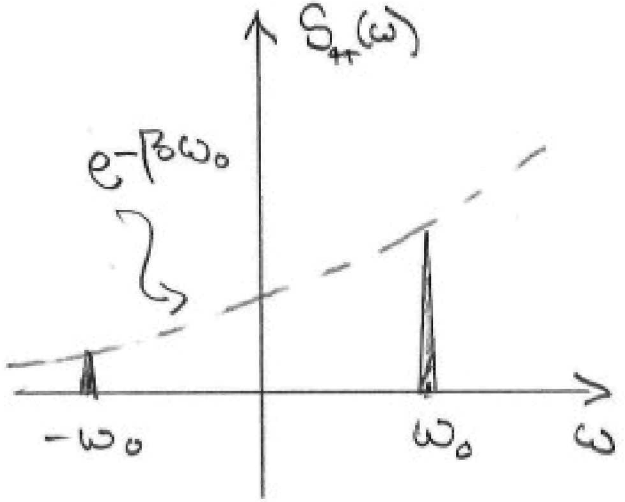
↓

Example

$$S_{rr}(t) = \langle (e^{-i\omega_0 t} b + e^{i\omega_0 t} b^+)(b + b^+) \rangle$$

$$= e^{-i\omega_0 t} \langle b b^+ \rangle + e^{i\omega_0 t} \langle b^+ b \rangle$$

$$= \underbrace{(1+n_{\omega_0})\, e^{-i\omega_0 t} + n_{\omega_0} e^{i\omega_0 t}}_{\text{is fct. of } T}$$



↘ $S_{rr}(\omega) = 2\pi\left[(1+n)\,\delta(\omega-\omega_0) + n\,\delta(\omega+\omega_0)\right]$

$$\Rightarrow \boxed{\frac{1}{2} S_{rr}(\omega)\cdot\left(1-e^{-\beta\omega}\right)} = \pi\left[\left(1-e^{-\beta\omega_0}\right)\cdot\frac{1}{1-e^{-\beta\omega_0}}\,\delta(\omega-\omega_0)\right.$$

$$\left. + \left(1-e^{\beta\omega_0}\right)\frac{1}{e^{\beta\omega_0}-1}\,\delta(\omega+\omega_0)\right]$$

$$= \boxed{\chi''_{rr}(\omega)}$$

Projection Technique

- Recursion / 'Dyson' equation

$$\boxed{\langle A_\mu | \frac{1}{z-L} A_\nu \rangle = [z\mathbf{1} - \Omega - \Sigma(z)]^{-1}\, \chi}$$

- Isothermal susceptibility

$$[\chi]_{\mu\nu} = \langle A_\mu | A_\nu \rangle$$

- Frequency term

$$[\Omega\chi]_{\mu\nu} = \langle A_\mu | L A_\nu \rangle = \langle [A_\mu^+, A_\nu] \rangle$$

- Memory function

$$[\Sigma(z)\chi]_{\mu\nu} = \langle L A_\mu | Q \frac{1}{z-QLQ} QLA_\nu \rangle$$

- Projectors

$$P = \sum_{\mu\nu} |A_\mu\rangle \langle A_\mu | A_\nu \rangle^{-1} \langle A_\nu | \,; \qquad Q = 1 - P$$

Quicky:

1) $\frac{1}{z-L} = \frac{1}{z-LQ}(z-L+LP)\frac{1}{z-L} =$

$$\frac{1}{z-LQ} + \frac{1}{z-LQ}\,LP\,\frac{1}{z-L}$$

2) $-(1 - \langle A | \frac{1}{z-LQ} LA \rangle \frac{1}{\langle A|A \rangle})\Phi(z) =$

$$\langle A | \frac{1}{z-LQ} A \rangle$$

3) $-[1 - \frac{1}{z}(\langle A|LA\rangle + \langle A|LQ\frac{1}{z-LQ}LA\rangle) \times$

$$\frac{1}{\langle A|A\rangle}]\Phi(z) = \frac{1}{z}\langle A|A\rangle$$

Applications of the Projection Technique ...

... another lecture ...

... and one of the tutorials ...

Finito !