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Large N approaches and Schwinger Bosons.

Assa Auerbach
Physics Department, Technion,
Haifa 32000, Israel.
assa@physics.technion.ac.il

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Physics Department, Technion,
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assa@physics.technion.ac.il

1.1 SU(N) Heisenberg Models

The use of large N approximations to treat strongly interacting quantum systems been very extensive in the last decade. The approach originated in elementary particles theory, but has found many applications in condensed matter physics. Initially, the large N expansion was developed for the Kondo and Anderson models of magnetic impurities in metals. Soon thereafter it was extended to the Kondo and Anderson lattice models for mixed valence and heavy fermions phenomena in rare earth compounds [1, 2].

In these notes we shall formulate and apply the large N approach to the quantum Heisenberg model [3–6]. This method provides an additional avenue to the static and dynamical correlations of quantum magnets. The mean field theories derived below can describe both ordered and disordered phases, at zero and at finite temperatures, and they complement the semiclassical approaches.

Generally speaking, the parameter N labels an internal SU(N) symmetry at each lattice site (i.e., the number of "flavors" a Schwinger boson or a constrained fermion can have). In most cases, the large N approximation has been applied to treat spin Hamiltonians, where the symmetry is SU(2), and N is therefore not a truly large parameter. Nevertheless, the $1/N$ expansion provides an easy method for obtaining simple mean field theories. These have been found to be either surprisingly successful or completely wrong, depending on the system. For example: we shall see in Section 1.3 that the Schwinger boson mean field theory in one dimension works well for the ferromagnet and for the antiferromagnet of integer spin but fails for the half-odd integer spin antiferromagnet.

The large N approach handles strong local interactions in terms of constraints. It is not a perturbative expansion in the size of the interactions but rather a saddle point expansion which usually preserves the spin symmetry of the Hamiltonian. The hamiltonians are written as a sum of biquadratic forms $-\mathcal{O}^\dagger_{ij}\mathcal{O}_{ij}$ on each bond on the lattice. This sets up a natural mean field decoupling scheme using one complex Hubbard Stratonovich fields per bond.

At the mean field level, the constraints are enforced only on average. Their effects are systematically reintroduced by the higher-order corrections in $1/N$.

It turns out that different large N generalizations are suitable for different Heisenberg models, depending on the sign of couplings, spin size, and lattice. Below, we describe two large N generalizations of the Heisenberg antiferromagnet.

1.1.1 Bipartite Antiferromagnet

We consider the case of nearest neighbor antiferromagnetic interaction $J > 0$, on a bipartite lattice with sublattices A, B. A bond $\langle ij \rangle$ is defined such that $i \in A$ and $j \in B$. The antiferromagnetic bond operator is defined as

$$\mathcal{A}_{ij} \ = a_i b_j - b_i a_j. \tag{1.1}$$

The arrow $\rightarrow$ denotes the antisymmetry with respect to interchange of $i \rightarrow j$. We define a spin rotation by π about the y axis on sublattice B which sends

$$a_j \rightarrow -b_j \ , \quad b_j \rightarrow a_j \ . \tag{1.2}$$

This is a canonical transformation which preserves the constraint (**??**). The antiferromagnetic bond operator transforms into a symmetric operator:

$$\mathcal{A}_{ij} \ \rightarrow \mathcal{A}_{ij} \ = a_i a_j + b_i b_j. \tag{1.3}$$

The SU(2) Heisenberg model is written in the form

$$\begin{aligned} \mathcal{H} &= J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j \\ &= -\frac{J}{2} \sum_{\langle ij \rangle} \left(\mathcal{A}^\dagger_{ij} \mathcal{A}_{ij} - 2S^2 \right). \end{aligned} \tag{1.4}$$

H can be generalized to $N > 2$ models by adding Schwinger boson flavors. The constraint is generalized to (**??**), and the bond operator is generalized to

$$\mathcal{A}_{ij} \ \rightarrow \sum_{m=1}^{N} a_{im} a_{jm}. \tag{1.5}$$

The SU(N) antiferromagnetic bosons (AFM-B) Heisenberg model is

$$\begin{aligned} \mathcal{H}^{AFM-B}(N) &= -\frac{J}{N} \sum_{\langle ij \rangle} \left(\mathcal{A}^\dagger_{ij} \mathcal{A}_{ij} - NS^2 \right) \\ &= -\frac{J}{N} \sum_{\langle ij \rangle} \left(\sum_{mm'} S_i^{mm'} \tilde{S}_j^{m'm} - NS^2 \right), \end{aligned} \tag{1.6}$$

where

$$\tilde{S}_j^{mm'} = a^\dagger_{jm'} a_{jm} \tag{1.7}$$

are the generators of the *conjugate representation* on sublattice B. One should note that $\mathcal{H}^{AFM-B}$ of (1.6) is not invariant under uniform $SU(N)$ transformations U but only under staggered conjugate rotations U and $U^\dagger$ on sublattices A and B, respectively.

1.1.2 Non Bipartite (Frustrated) Antiferromagnets

Read and Sachdev [7] have extended the Schwinger bosons representation to an Sp(N) generalization. The resulting Hamiltonian be used to set up a mean field theory for the case of the antiferromagnet on any frustrated, non bipartite, lattice provided all the interactions are positive $J_{ij} > 0$. The "large N" version is given by replicating both a and b bosons to N flavors *each*. The physical model then corresponds to taking $N \to 1$ limit. This representation amounts to writing all interactions as negative biquadratic forms

$$\mathcal{H}^{Sp(N)} = -\frac{J}{N} \sum_{\langle ij \rangle} \left(\mathcal{A}^\dagger_{ij} \mathcal{A}_{ij} \right)$$
$$\mathcal{A}_{ij} = \sum_{m=1}^{N} \left(a_{im} b_{jm} - b_{im} a_{jm} \right) \tag{1.8}$$

The Sp(N) mean field theory requires a separate complex Hubbard Stratonovich field for every interaction range, plus a constraint field.

1.2 The Generating Functional

The generating Hamiltonian is defined by

$$\mathcal{H}[j] = \mathcal{H} - \sum_{imm'} j_{imm'}(\tau)\; a^\dagger_{im} a_{im'}, \tag{1.9}$$

where $\tau \in [0, \beta)$ is the imaginary time. The constraints (**??**) or (**??**) are enforced by the projector P_S, which commutes with $\mathcal{H}[j]$. The imaginary time generating functional (see (**??**)) is

$$Z[j] = \mathrm{Tr}\, P_S T_\tau \left[\exp\left(-\int_0^\beta d\tau\; \mathcal{H}[j] \right) \right]$$
$$= \lim_{\epsilon \to 0} \mathrm{Tr}\, T_\tau \prod_{\tau_n = \epsilon}^{\beta} \left[P_S(\tau) \exp\left(-\epsilon \mathcal{H}[j(\tau_n)] \right) \right]. \tag{1.10}$$

We use an integral representation of the constraint

$$P_S(\tau) \ = \int \mathcal{D}\lambda \ \exp\left[-i\epsilon \sum_{im} \lambda_i(\tau) \ (a^\dagger_{im} a_{im} - S)\right], \tag{1.11}$$

where the measure of the constraint field is

$$\int \mathcal{D}\lambda \ = \lim_{\epsilon\to 0} \prod_{i\tau} \epsilon \int_{-\pi/\epsilon}^{\pi/\epsilon} d\lambda_{i\tau} \ . \tag{1.12}$$

The exponential of (1.11) and $\mathcal{H}[j]$ can be combined in the exponent since they commute.

We construct a coherent states path integral for the generating functional which has unified notations for the Schwinger bosons and constrained fermion Hamiltonians:

$$\begin{aligned} Z[j] = & \int_{-\infty}^{\infty} \mathcal{D}\lambda \ \int \mathcal{D}^2\mathbf{z} \exp\Bigg\{ - \int_0^\beta d\tau \Big[\sum_{im} z^*_{im}\partial_\tau z_{im} + H[j] \\ & + i \sum_{im} \lambda_i(\tau)(z^*_{im} z_{im} - S)\Big] \Bigg\}, \end{aligned} \tag{1.13}$$

where $\mathbf{z}$ are complex variables. The Hamiltonian function is

$$H[j] \ = -\frac{J}{N} \sum_{\langle ij\rangle} \mathcal{Z}^*_{ij}\mathcal{Z}_{ij} \ - \sum_{imm'} j_{imm'}(\tau) z^*_{im} z_{im'}, \tag{1.14}$$

where

$$\mathcal{Z} \ = \sum_m z_{im} z_{jm} \tag{1.15}$$

where the AFM-B Hamiltonians were defined in (1.6).

1.3 Schwinger Bosons Mean Field Theory

Mean field theory is set up by decomposing the interaction terms in (1.15) on each bond with a dynamical Hubbard Stratonovich fields $Q_{ij}(\tau)$, which yields the functional integral representation of the generating functional (partition function):

$$\begin{aligned} Z[j] &= \int \mathcal{D}^2 Q \ \mathcal{D}\lambda \exp\left[-N\mathcal{S}[\lambda, Q, j]\right] \\ \mathcal{S} &= \ -\frac{\eta}{N} \mathrm{Tr}_{\tau i m} \left(\ln \hat{G}[j]\right) \ + \int_0^\beta d\tau \left(\sum_{\langle ij\rangle} \frac{|Q_{ij}|^2}{J} \ - iS \sum_i \lambda_i \right). \end{aligned} \tag{1.16}$$

where $\hat{G}$ is the Green function of the quadratic Fermion action. This expression is the starting point for a steepest descents expansion controlled by N as the large parameter. In the following, N is held as an independent parameter. We set $N \to 2$ when evaluating spin correlations for the physical Heisenberg model.

The SBMFT is given by replacing the auxiliary fields in the action of (1.16) by *static and uniform* saddle point parameters:

$$N\mathcal{S}\left[Q_{ij}(\tau), \lambda_i(\tau)\right] \to N\mathcal{S}_0(Q, -i\lambda) = \beta F^{MF}(Q, \lambda). \tag{1.17}$$

F^{MF} is the mean field free energy, which can be written as

$$F^{MF}(Q, \lambda) = -\beta^{-1} \ln \mathrm{Tr}_{im} \left[\exp\left(-\beta H^{MF}[Q, \lambda]\right)\right], \tag{1.18}$$

where H^{MF} is the mean field Hamiltonian of N decoupled boson flavors.

1.4 The Case of the Antiferromagnet

The mean field Hamiltonian is given by

$$\begin{aligned} H^{MF} = & \sum_{i,m} \lambda a^\dagger_{im} a_{im} + Q \sum_{\langle ij \rangle, m} \left(a^\dagger_{im} a^\dagger_{jm} + a_{im} a_{jm}\right) \\ & \qquad + N\mathcal{N} \frac{zQ^2}{2J} - N\mathcal{N} S\lambda \\ = & \sum_{\mathbf{k}m} \left[\lambda a^\dagger_{\mathbf{k}m} a_{\mathbf{k}m} + \frac{1}{2} zQ\gamma_{\mathbf{k}} \left(a^\dagger_{\mathbf{k}m} a^\dagger_{-\mathbf{k}m} + a_{\mathbf{k}m} a_{-\mathbf{k}m}\right)\right] \\ & \qquad + N\mathcal{N} \frac{zQ^2}{2J} - N\mathcal{N} S\lambda. \end{aligned} \tag{1.19}$$

For $N = 2$, the SBMFT Hamiltonian resembles the Holstein–Primakoff spin wave Hamiltonian, except that here two Schwinger boson flavors replace the single Holstein–Primakoff boson. In close analogy to the spin wave problem, H^{MF} can be diagonalized by a canonical *Bogoliubov* transformation

$$\alpha_{\mathbf{k}m} = \cosh\theta_{\mathbf{k}} a_{\mathbf{k}m} - \sinh\theta_{\mathbf{k}} a^\dagger_{-\mathbf{k}m}, \tag{1.20}$$

or inversely,

$$a_{\mathbf{k}m} = \cosh\theta_{\mathbf{k}} \alpha_{\mathbf{k}m} + \sinh\theta_{\mathbf{k}} \alpha^\dagger_{-\mathbf{k}m}. \tag{1.21}$$

By inserting (1.21) in (1.19), one obtains a normal diagonal Hamiltonian in terms of the α bosons,

$$\begin{aligned} H^{MF} = & \frac{1}{2} \sum_{\mathbf{k}m} \Big[(\lambda \cosh 2\theta_{\mathbf{k}} + zQ\gamma_{\mathbf{k}} \sinh 2\theta_{\mathbf{k}})(\alpha^\dagger_{\mathbf{k}m} \alpha_{\mathbf{k}m} + \alpha_{\mathbf{k}m} \alpha^\dagger_{\mathbf{k}m}) \\ & + (\lambda \sinh 2\theta_{\mathbf{k}} + zQ\gamma_{\mathbf{k}} \cosh 2\theta_{\mathbf{k}})(\alpha^\dagger_{\mathbf{k}m} \alpha^\dagger_{-\mathbf{k}m} + \alpha_{\mathbf{k}m} \alpha_{-\mathbf{k}m}) \Big] \\ & + N\mathcal{N} \frac{zQ^2}{2J} - \mathcal{N} N \left(S + \frac{1}{2}\right) \lambda. \end{aligned} \tag{1.22}$$

Here, $\gamma_{\mathbf{k}} = \frac{2}{z}\sum_\eta \cos(\mathbf{k}\cdot\eta)$ is the lattice Fourier transform. Now, we choose $\theta_{\mathbf{k}}$ so that the anomalous terms $\alpha^\dagger\alpha^\dagger$ and $\alpha\alpha$ vanish. This amounts to the following condition on $\theta_{\mathbf{k}}$:

$$\tanh 2\theta_{\mathbf{k}} = -\frac{zQ\gamma_{\mathbf{k}}}{\lambda}. \tag{1.23}$$

Having solved for $\theta_{\mathbf{k}}$, we can substitute the hyperbolic functions in (1.22) by rational functions of the right-hand side of (1.23). This yields a normal and diagonal Hamiltonian:

$$\begin{aligned} H^{MF} &= \sum_{\mathbf{k}m}\omega_{\mathbf{k}}\left(\alpha^\dagger_{\mathbf{k}m}\alpha_{\mathbf{k}m}+\frac{1}{2}\right) + N\mathcal{N}\frac{zQ^2}{2J} - N\mathcal{N}\left(S+\frac{1}{2}\right)\lambda\ , \\ \omega_{\mathbf{k}} &= \sqrt{\lambda^2-(zQ\gamma_{\mathbf{k}})^2}. \end{aligned} \tag{1.24}$$

The mean field free energy is given by

$$F^{MF} = \beta^{-1}\sum_{\mathbf{k}m}\ln\left[2\sinh\left(\frac{\beta\omega_{\mathbf{k}}}{2}\right)\right] - \mathcal{N}N\left(S+\frac{1}{2}\right)\lambda + N\mathcal{N}\frac{zQ^2}{2J}. \tag{1.25}$$

The mean field equations are given by differentiating (1.25) with respect to λ and Q:

$$\frac{1}{\mathcal{N}}\sum_{\mathbf{k}}\frac{\lambda}{\sqrt{\lambda^2-(zQ\gamma_{\mathbf{k}})^2}}\left(n_{\mathbf{k}}+\frac{1}{2}\right) = S+\frac{1}{2}\ , \tag{1.26}$$

$$\frac{1}{\mathcal{N}}\sum_{\mathbf{k}}\frac{z^2{\gamma_{\mathbf{k}}}^2 Q}{\sqrt{\lambda^2-(zQ\gamma_{\mathbf{k}})^2}}\left(n_{\mathbf{k}}+\frac{1}{2}\right) = \frac{zQ}{J}. \tag{1.27}$$

The mean field ground state Ψ^{MF} is the vacuum of all α's,

$$\alpha_{\mathbf{k},m}\Psi^{MF}_0 = 0, \qquad \forall \mathbf{k}, m. \tag{1.28}$$

Using (1.20), one can write Ψ^{MF} explicitly in terms of the original Schwinger bosons as

$$\begin{aligned} \Psi^{MF} &= C\exp\left[\frac{1}{2}\sum_{ij}u_{ij}\left(\sum_m a^\dagger_{im}a^\dagger_{jm}\right)\right]|0\rangle\ , \\ u_{ij} &= \frac{1}{\mathcal{N}}\sum_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}_{ij}}\tanh\theta_{\mathbf{k}}. \end{aligned} \tag{1.29}$$

For $N=2$, using the *unrotated* operators $a^\dagger, b^\dagger$, Ψ^{MF} is the Schwinger bosons mean field state,

$$\Psi^{MF}_{N=2} = |\hat{u}\rangle = \exp\left[\sum_{i\in A, j\in B}u_{ij}\left(a^\dagger_i b^\dagger_j - b^\dagger_i a^\dagger_j\right)\right]|0\rangle. \tag{1.30}$$

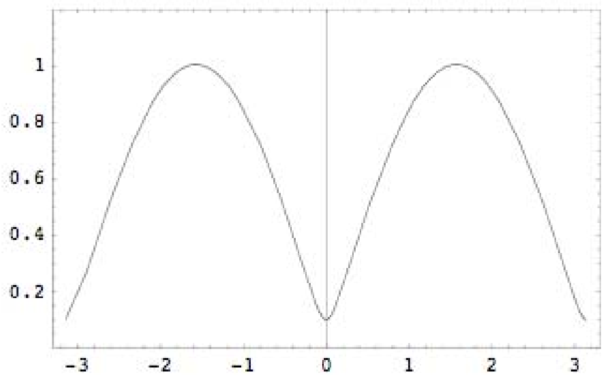

Fig. 1.1. Mean field dispersion ω_k, in the domain $-\pi < k < \pi$, for the one-dimensional antiferromagnet.

Ψ^{MF} contains many configurations with occupations different from $2S$ and is therefore not a pure spin state. As shown in Chapter **??**, under Gutzwiller projection it reduces to a valence bond state. Since

$$\tanh(\theta_{\mathbf{k}+\boldsymbol{\pi}}) = -\tanh(\theta_{\mathbf{k}}) , \tag{1.31}$$

where $\boldsymbol{\pi} = (\pi, \pi, \ldots)$, the bond parameters u_{ij} only connect sublattice A to B. Furthermore, one can verify that for the nearest neighbor model above, $u_{ij} \geq 0$, and therefore the valence bond states obey Marshall's sign.

Although Ψ^{MF} are manifestly rotationally invariant, they may or may not have long-range antiferromagnetic order. This depends on the long-distance decay of u_{ij}. As we shall see, the SBMFT ground state for the nearest neighbor model is disordered in one dimension and has long-range order in two dimensions.

For further calculations, it is convenient to introduce the parametrizations:

$$\begin{aligned} \omega_{\mathbf{k}} &\equiv c\sqrt{(\kappa/2)^2 + \frac{z}{2}(1 - \gamma_{\mathbf{k}}{}^2)}, \\ c &\equiv Q\sqrt{2z}, \\ \kappa &\equiv \frac{2}{c}\sqrt{\lambda^2 - (zQ)^2}, \\ t &= \frac{T}{zQ}. \end{aligned} \tag{1.32}$$

c, κ, t describe the spin wave velocity, the inverse correlation length, and the dimensionless temperature, respectively. In Fig. 1.2 the dispersion for the one-dimensional antiferromagnet is drawn. By (1.25), we see that near the zone center and zone corner the mean field dispersions are those of free massive relativistic bosons,

$$\omega_{\mathbf{k}}{}^{\gamma} \approx c\sqrt{(\kappa/2)^2 + |\mathbf{k}-\mathbf{k}_\gamma|^2}\ , \qquad \mathbf{k}_\gamma = 0, \boldsymbol{\pi}. \tag{1.33}$$

When the gap (or "mass" $c\kappa/2$) vanishes, $\omega_{\mathbf{k}}{}^{\alpha}$ are Goldstone modes which reduce to dispersions of antiferromagnetic spin waves.

The spin correlation function is given by inserting the α operators instead of a's in (**??**) using (1.21). This yields

$$\begin{aligned} S^{MF}(\mathbf{q}) &= \frac{1}{\mathcal{N}}\langle S^+_{\mathbf{q}} S^-_{-\mathbf{q}}\rangle_{MF} \\ &= \frac{1}{\mathcal{N}}\sum_{\mathbf{k}}\Bigg\{ \cosh\left[2\left(\theta_{\mathbf{k}}+\theta_{\mathbf{k}+\mathbf{q}+\boldsymbol{\pi}}\right)\right] \\ &\qquad \times \left(n_{\mathbf{k}}+\frac{1}{2}\right)\left(n_{\mathbf{k}+\mathbf{q}+\boldsymbol{\pi}}+\frac{1}{2}\right)-\frac{1}{4}\Bigg\}. \end{aligned} \tag{1.34}$$

Using (1.26), we confirm the large N limit of the sum rule (**??**),

$$\frac{1}{\mathcal{N}}\sum_{\mathbf{q}} S^{MF}(\mathbf{q}) = S(S+1). \tag{1.35}$$

For N=2, the mean field sum rule exceeds the exact result by a familiar factor of $\frac{3}{2}$.

The spatial dependence of the spin correlations at $\mathbf{x}_{ij} = \mathbf{x}_i - \mathbf{x}_j$ is given by

$$S^{MF}(\mathbf{x}_{ij}) \ = |f(\mathbf{x}_{ij})|^2 - |g(\mathbf{x}_{ij})|^2 - \frac{1}{4}\delta_{ij}, \tag{1.36}$$

where at low temperatures and long distances,

$$\begin{aligned} f(\mathbf{x}_{ij}) &= \mathcal{N}^{-1}\sum_{\mathbf{k}} \frac{\left(n_{\mathbf{k}}+\frac{1}{2}\right)e^{i\mathbf{k}\mathbf{x}_{ij}}}{\sqrt{1-\gamma_{\mathbf{k}}{}^2[\kappa^2/(2z)+1]^{-1}}} \\ &\approx 2zt\left(1+e^{i\boldsymbol{\pi}\mathbf{x}_{ij}}\right)\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{e^{i\mathbf{k}\mathbf{x}_{ij}}}{(\kappa/2)^2+|\mathbf{k}|^2} \\ &\propto \ \left(1+e^{i\boldsymbol{\pi}\mathbf{x}_{ij}}\right)(\mathbf{x}_{ij}|/\xi)^{-(d-1)/2}\exp\left(-|\mathbf{x}_{ij}|\kappa/2\right), \end{aligned} \tag{1.37}$$

where we have used the long-distance asymptotic expansion at small κ and low temperatures. Similarly,

$$\begin{aligned} g(\mathbf{x}_{ij}) &= \mathcal{N}^{-1}\sum_{\mathbf{k}}\gamma_{\mathbf{k}} \frac{\left(n_{\mathbf{k}}+\frac{1}{2}\right)e^{i\mathbf{k}\mathbf{x}_{ij}}}{\sqrt{1-\gamma_{\mathbf{k}}{}^2[\kappa^2/(2z)+1]^{-1}}} \\ &\propto \ \left(1-e^{i\boldsymbol{\pi}\mathbf{x}_{ij}}\right)(|\mathbf{x}_{ij}|/\xi)^{-(d-1)/2}\exp\left(-|\mathbf{x}_{ij}|\kappa/2\right). \end{aligned} \tag{1.38}$$

Thus, for $\kappa > 0$,

$$S^{MF}(\mathbf{x}_{ij}) \ \propto \ e^{i\boldsymbol{\pi}\mathbf{x}_{ij}} \left(\frac{\xi}{|\mathbf{x}_{ij}|}\right)^{d-1} \exp\left(-|\mathbf{x}_{ij}|/\xi\right), \tag{1.39}$$

where the correlation length is $\xi = \kappa^{-1}$.

The uniform susceptibility is obtained directly from (1.34) using the identity

$$\chi_0^{MF} \ = \frac{1}{2T} S^{MF}(\mathbf{q}=0) \ = \frac{1}{T}\sum_{\mathbf{k}} n_{\mathbf{k}}(n_{\mathbf{k}}+1). \tag{1.40}$$

1.4.1 Long-Range Antiferromagnetic Order

In the absence of any magnetic fields, the ground state is a singlet. In two dimensions and higher, it is expected, based on extensive numerical studies, that nearest neighbor antiferromagnet has long-range order for all $S \geq \frac{1}{2}$. To investigate the possibility of spontaneously broken symmetry, we introduce an infinitesimal ordering field h which couples to the staggered magnetization. We restrict ourselves to $N=2$; thus

$$\begin{aligned} H^{MF} &\rightarrow H^{MF} - h\sum_i S_i^z \\ &= H^{MF} - h \sum_{i,s=-\frac{1}{2},\frac{1}{2}} s a^\dagger_{is} a_{is} \ , \end{aligned} \tag{1.41}$$

where we recall that the Schwinger bosons are defined using sublattice rotated spin directions in (1.2). Equation (1.41) can be diagonalized using spin dependent transformation angles $\theta_{\mathbf{k},s}$. Repeating the steps leading to (1.24), we obtain

$$\omega_{\mathbf{k},s} \ = \sqrt{(\lambda - sh)^2 - (zQ\gamma_{\mathbf{k}})^2}. \tag{1.42}$$

The spontaneous staggered magnetization is given by the limit

$$\begin{aligned} m_0 &= \lim_{h\rightarrow 0^+} m(h) \ , \\ m(h) &= \lim_{\mathcal{N}\rightarrow\infty} \frac{1}{\mathcal{N}} \sum_{i,s} s \ \langle a^\dagger_{is} a_{is} \rangle_h \\ &= \lim_{\mathcal{N}\rightarrow\infty} \frac{1}{\mathcal{N}} \sum_{\mathbf{k}s} \frac{\lambda - sh}{\sqrt{(\lambda - sh)^2 - (zQ\gamma_{\mathbf{k}})^2}} \left[n(\omega_{\mathbf{k},s}) + \frac{1}{2} \right]. \end{aligned} \tag{1.43}$$

The constraint equation (1.26) is

$$\frac{1}{2\mathcal{N}} \sum_{\mathbf{k},s} \frac{\lambda - sh}{\sqrt{(\lambda - sh)^2 - (zQ\gamma_{\mathbf{k}})^2}} \left[n(\omega_{\mathbf{k},s}) + \frac{1}{2} \right] = S + \frac{1}{2}. \tag{1.44}$$

Again, we parametrize the dispersions in terms of $c, \kappa, \tilde{h}$,

$$\omega_{\mathbf{k},s} \equiv c\sqrt{\kappa^2/4 - \left(s - \frac{1}{2}\right) z\tilde{h} + \frac{z}{2}(1-\gamma_{\mathbf{k}}{}^2) + \mathcal{O}(\tilde{h}^2)},$$
$$c = \sqrt{2z}Q,$$
$$\kappa = \frac{2}{c}\sqrt{(\lambda - \frac{1}{2}h)^2 - (zQ)^2},$$
$$\tilde{h} = h\frac{\lambda}{(zQ)^2}. \tag{1.45}$$

If

$$\lim_{\mathcal{N}\to\infty} \kappa > 0, \tag{1.46}$$

then both summands in (1.43), with $s = \pm\frac{1}{2}$, are continuous at $h = 0$, and therefore

$$\lim_{h\to 0^+} m(h,\kappa) \ = 0 \quad , \tag{1.47}$$

i.e., no spontaneous symmetry breaking. On the other hand, if

$$\kappa = \mathcal{O}(\mathcal{N})^{-1} \tag{1.48}$$

the $s = +\frac{1}{2}$ summand at $\mathbf{k} = 0, \boldsymbol{\pi}$ contributes a term of order $\mathcal{N}$. This yields a *macroscopic contribution* (i.e., order one) to the staggered magnetization (1.43). Since it also represents a macroscopic contribution to the Schwinger bosons density (1.44), we can say that there is Bose condensation [8] of the $s = +\frac{1}{2}$ bosons at $\mathbf{k} = 0, \boldsymbol{\pi}$ (see discussion after (**??**)). The order parameter for the condensate is thus

$$\langle a_{\mathbf{k}s}\rangle = \langle a^\dagger_{\mathbf{k}s}\rangle = \sqrt{\frac{\mathcal{N}m_0}{2}}\delta_{s,\frac{1}{2}}\left(\delta_{\mathbf{k},0} + \delta_{\mathbf{k},\boldsymbol{\pi}}\right). \tag{1.49}$$

To evaluate $m_0 = \lim_{h\to 0} m(h)$, we subtract (1.43) from (1.44), eliminate the diverging $s = \frac{1}{2}$ summand, and obtain

$$m_0 = S + \frac{1}{2} - \lim_{h\to 0^+}\lim_{\mathcal{N}\to\infty}\mathcal{N}^{-1}\sum_{\mathbf{k}} \frac{2 + 2\tilde{h} + \kappa^2/4}{\sqrt{2\tilde{h} + \kappa^2/4 + 2(1-\gamma_{\mathbf{k}}{}^2)}}\left[n(\omega_{\mathbf{k}\frac{1}{2}}) + \frac{1}{2}\right]. \tag{1.50}$$

By keeping $h > 0$, we maintain a gap in the spectrum and in the denominator of the summand. Thus, we are allowed to replace (1.50) by an integral in the thermodynamic limit, and at $T = 0$ we can set $n(\omega_{\mathbf{k},+\frac{1}{2}}) = 0$ and $\kappa = 0$. This yields

$$m_0 = S + \frac{1}{2} - \frac{1}{2}\int \frac{d^d\mathbf{k}}{(2\pi)^d}\frac{1}{\sqrt{1-\gamma_{\mathbf{k}}{}^2}}. \tag{1.51}$$

The integral yields for cubic lattices in d dimensions the numerical results

$$m_0(d) = \begin{cases} 0 & d=1 \\ S-0.19660 & d=2 \\ S-0.078 & d=3 \end{cases} . \tag{1.52}$$

Notice that, in contrast to the ferromagnetic case, the ordered moment is always *less than* the classical value S. This is due to the quantum zero-point motion, which has its origin in the noncommutability of the Hamiltonian and the staggered magnetization. The SBMFT results for m_0 agree with low-order spin wave theory.

1.4.2 One Dimension

At zero temperature, we set $n_{\mathbf{k}}=0$ and expand (1.26) as

$$\begin{aligned} S+\frac{1}{2} &= \frac{1}{2}\int_{-\pi}^{\pi}\frac{dk}{2\pi}\frac{1}{\sqrt{1-\frac{1}{1+\kappa^2/8}\cos^2(k)}} \\ &= \frac{1}{\pi}K\left(\frac{1}{1+\kappa^2/8}\right) \\ &\sim \frac{1}{\pi}\ln\left(\frac{8\sqrt{2}}{\kappa}\right)+\mathcal{O}(\kappa), \end{aligned} \tag{1.53}$$

which results in

$$\kappa \approx \sqrt{32}\exp\left[-\pi\left(S+\frac{1}{2}\right)\right]. \tag{1.54}$$

In (1.52), we found that there cannot be long-range order in the SBMFT ground state of the one-dimensional antiferromagnet. Since κ decreases exponentially with S, we can neglect κ as we neglect higher-order corrections in S^{-1}. By subtracting (1.27) from (1.26) one obtains

$$\begin{aligned} c &= J\left(S+\frac{1}{2}-\frac{1}{2}\int_{-\pi}^{\pi}\frac{dk}{2\pi}\frac{\sin^2 k}{\sqrt{\kappa^2/8+\sin^2 k}}\right) \\ &\approx J\sqrt{2}\left[S+\frac{1}{2}-\frac{2}{\pi}+\mathcal{O}(\kappa,S^{-1})\right]. \end{aligned} \tag{1.55}$$

While c does not differ drastically from its classical value $c=JS$, the ground state correlations decay exponentially. The correlation length κ^{-1} as given by (1.54) agrees with the correlation length given by the continuum approach.

The mean field excitations are not physical excitations of the Heisenberg model (for example, they include constraint violating charge fluctuations). Nevertheless, the gap in their spectrum $c\kappa/2$ is consistent with the existence of Haldane's gap for one-dimensional integer spin chains. The physical magnon spectrum can be deduced from the peaks in Im, $S(\mathbf{q},\omega)$. Since spin one excitations involve at least two Schwinger bosons, the physical magnons have a gap of

$$\varDelta = c\kappa. \tag{1.56}$$

Thus, the SBMFT recovers Haldane's continuum results in the absence of a $\varTheta$ term [5].

One must beware that the SBMFT fails for half-odd integer spin chains. The effects of the $\varTheta$ term, which destroys Haldane's gap, are apparently absent in the mean field theory. The SBMFT has a nondegenerate ground state in violation of Lieb, Schultz, and Mattis' theorem for half-odd integer spins. Read and Sachdev overcame this problem by introducing the $\varTheta$ term into the large N theory [6]. They developed a continuum gauge theory for the Q,λ fluctuations (see bibliography).

1.4.3 Two Dimensions

In two dimensions, we expect no long-range order at finite temperatures due to Mermin and Wagner's theorem. Indeed, we shall find that, at low temperatures, the mean field equations yield a finite value for the inverse correlation length $\kappa(T,S)$.

The mean field equations can be solved to obtain $\kappa(T,S), c(T,S)$. It is convenient to use the fact that $\omega_{\mathbf{k}} = \omega(\gamma_{\mathbf{k}})$ and replace

$$\frac{1}{\mathcal{N}}\sum_{\mathbf{k}} F(\omega_{\mathbf{k}}) \rightarrow 2\int_0^1 d\gamma\; \rho(\gamma) F\left[\omega(\gamma)\right] , \tag{1.57}$$

where for the square lattice

$$\rho(\gamma) = \frac{2}{\pi^2} K(1-\gamma^2). \tag{1.58}$$

It turns out that at temperatures above $T_{max} > 0.91J$, the mean field equations have no nontrivial ($Q \neq 0$) solution [3] This reflects a failure of the SBMFT to describe the disordered phase at high temperatures, where nearest neighbor correlations are destroyed. At temperatures where the correlation length is large, i.e.,

$$\kappa << t << 1, \tag{1.59}$$

the constraint equation can be expanded following Takahashi:

$$\begin{aligned} S+\frac{1}{2} &= \frac{1}{2}\int_{-1}^1 d\gamma\,\rho(\gamma)\left(1+\kappa^2/8-\gamma^2\right)^{-\frac{1}{2}} \coth\left[(1+\kappa^2/8-\gamma^2)^{\frac{1}{2}}/2t\right] \\ &= \frac{t}{\pi}\left[\log\left(\frac{32}{\kappa^2}\right) - \log\left(\frac{2}{t}\right)\right] \\ &\qquad + \frac{1}{\pi^2}\int_{-1}^1 d\gamma (1-\gamma^2)^{-\frac{1}{2}} K(1-\gamma^2) + \mathcal{O}(t,\kappa). \end{aligned} \tag{1.60}$$

Similarly, by subtracting (1.27) from (1.26) and expanding the integrals to low order in κ, t, we obtain

$$\begin{aligned} & S+\frac{1}{2}-\frac{4Q}{J} \\ \geq\ & \frac{1}{2}\int_{-1}^{1} d\,\gamma\rho(\gamma)\left(1+\kappa^2/8-\gamma^2\right)^{\frac{1}{2}}\coth\left[(1+\kappa^2/8-\gamma^2)^{\frac{1}{2}}/2t\right] \\ \approx\ & \frac{1}{\pi^2}\int_{-1}^{1} d\gamma(1-\gamma^2)^{\frac{1}{2}}K(1-\gamma^2)+\mathcal{O}(t^3,\kappa)\ . \end{aligned} \tag{1.61}$$

By (1.51) and (1.52) the ordered moment is given by

$$m_0 = S+\frac{1}{2}-\frac{1}{\pi^2}\int_{-1}^{1} d\gamma(1-\gamma^2)^{-\frac{1}{2}}K(1-\gamma^2)\ = S-0.19660. \tag{1.62}$$

The spin wave velocity at zero temperature is

$$c=\sqrt{8}JSZ_c\ \ , \tag{1.63}$$

where the spin wave velocity renormalization factor is given using (1.61),

$$\begin{aligned} Z_c &= 1+S^{-1}\left[\frac{1}{2}-\frac{1}{\pi^2}\int_{-1}^{1} d\gamma(1-\gamma^2)^{\frac{1}{2}}K(1-\gamma^2)\right] \\ &= 1+0.078974/S\ . \end{aligned} \tag{1.64}$$

The asymptotic spin correlations are given by (1.39), where the correlation length is $\xi=\kappa^{-1}$. By inverting (1.60) and using (1.61), we obtain the temperature dependent correlation length

$$\xi=\frac{\sqrt{2}JSZ_c}{T}\exp\left[-\frac{2\pi Z_cJSm_0}{T}\right]\left[1+\mathcal{O}(t^2)\right]\ . \tag{1.65}$$

In comparing (1.65) to the continuum approximation value, we can determine the renormalized stiffness constant $\rho_s(S)$ for the square lattice model as follows:

$$\rho_s\ =\lim_{T\to 0}\frac{T}{2\pi}\log(\xi)=JSm_0Z_c, \tag{1.66}$$

where m_0 and Z_c are given by (1.62) and (1.64), respectively. It properly recovers the classical value $\rho_s\to JS^2$ in the limit of large S.

1.5 Staggered Magnetization in the Layered Antiferromagnet

For layered a square lattice antiferromagnets with in-plane exchange $J=1$ and very weak interlayer coupling $\alpha << 1$. We expect long range magnetic order at a finite Néel temperature T_N. Ordering $M = \langle (-1)^i S_i^z \rangle \neq 0$ occurs when the in plane correlation length ξ, which diverges exponentially at low T, will produce an effective coupling between layers of order $\alpha\xi^2(T_N) \sim 1$. This means in effect that the coarse grained spins start to interact as if in an isotropic three dimensional cubic lattice which orders at that $T \leq 1$. The interlayer mean field theory, introduced by Scalapino, Imry and Pincus (SIP) [9] in the 70's, can be derived quantitatively using the SBMFT. Here we follow Keimer *et.al.* [10], and Ofer *et. al* [11], to compute the temperature dependent staggered magnetization, in the range $T \in [0, T_N]$.

The Hamiltonian is given by

$$\mathcal{H} = \left(\sum_{l,i,\eta} \mathbf{S}_i^l \cdot \mathbf{S}_{i+\eta_\parallel}^l + \alpha \sum_{l,i,\eta_\perp} \mathbf{S}_i^l \cdot \mathbf{S}_i^{l+\eta_\perp} \right) \tag{1.67}$$

The interplane coupling is decomposed using Hartree-Fock staggered magnetization field:

$$\alpha S_{li}^z S_{l'i}^z \simeq (-1)^i h \,(S_{li}^z - S_{l'i}^z) - \frac{h^2}{\alpha}, \tag{1.68}$$

Self consistency is achieved therefore when

$$h = 2\alpha M(T, h) \tag{1.69}$$

where $M(T, h)$ is the staggered magnetization response to an ordering staggered field h of a single layer.

To determine $M(T, h)$, we write The mean field dispersion for spin half, in a tetragonal lattice is given by is given by

$$\omega_{k\pm} = C\sqrt{(1 + \Delta + \frac{1}{2}(h \pm h))^2 - \gamma_\parallel^2} \tag{1.70}$$

where $\gamma_\parallel = (\cos(k_x) + \cos(k_y))/2$. Since $\alpha << 1$, the effects of the three dimensional coupling on the mean field equation for Q is negligible and C acquires its 2D value $C = 2.32J$. The staggered field h splits the two SB dispersions of a and b bosons such that only the b bosons can condense when the $\omega_{k,-} = 0$. Let us define the integrals

$$I^\pm = \sum_{\mathbf{k},\pm} \left(n(\omega_{\mathbf{k},\pm}) + \frac{1}{2} \right) \frac{C(1 + \Delta + (h \pm h)/2)}{\omega_{\mathbf{k},\pm}} \tag{1.71}$$

Then $\Delta(h, T)$ is solved by the constraint equation

$$2S + 1 = I^+(\Delta, h, T) + I^-(\Delta, h, T) \tag{1.72}$$

Δ is the intrinsic gap parameter which in the absence of the ordering field reduces to $\Delta_0 = \xi_{2D}^{-2}/16$, where ξ_{2D} is the 2D correlation length given in (1.65). In the presence of the ordering field, i.e. $h > 0$, $\Delta(h, T)$ can be solved

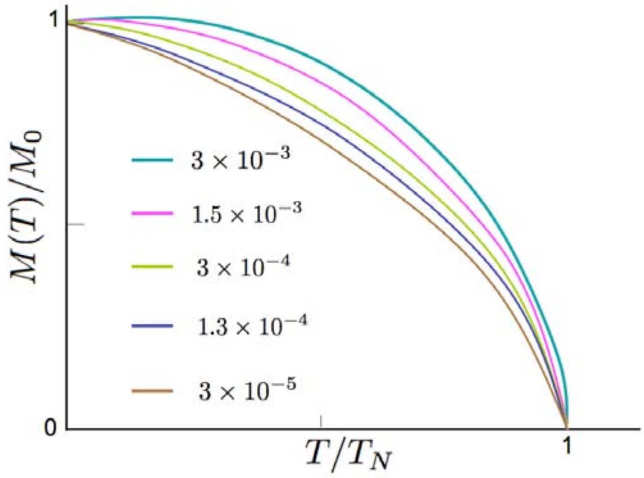

Fig. 1.2. Numerical solution, from Ofer *et.al.* [11], of the scaled staggered magnetization $M(T)$ of the layered antiferomagnet for various values of α.

numerically using Eq. (1.72). Using this solution we can then solve for the self consistent field $h(T)$,

$$\frac{h}{2\alpha} = S + \frac{1}{2} - I^{+}(\Delta(h,T),h,T) \tag{1.73}$$

Extracting T_N is relatively easy, since as $T \to T_N$, $h \to 0$, and $I^{\pm}(h)$ can be expanded to linear order in h. In fact at T_N, the self consistency equation reads

$$\begin{aligned} h(T_N) &= 2\alpha\chi^s(T_N)h(T_N) \\ 2\alpha\chi^s_{2D}(T_N) &= 1 \end{aligned} \tag{1.74}$$

where the 2D staggered susceptibility goes as $\xi^2_{2D}(T_N)$, which is agreement with SIP theory. Our calculation yields

$$T_N = \frac{2M_0\pi}{\log\left(4\alpha/M_0\pi^2\log(4\alpha/\pi)\right)} \tag{1.75}$$

The numerically determined temperature dependence $M(T) = h(T)/(2\alpha)$ are shown in Fig.1.2).

1.6 Exercises

1. Using the explicit solutions for Q and λ at zero tempearture, prove that the ground state energy for the SBMFT is given by

$$E^{MF} = \lim_{T\to 0} F^{MF} = -N\mathcal{N}\frac{zQ^2}{2J}. \tag{1.76}$$

2. Prove that a staggered magnetic field term (1.41) for the antiferromagnetic SBMFT results in the dispersions $\omega_{\mathbf{k}s}$ of (1.42). *Hint: Find the zeros of the determinant of the quadratic matrix*

$$L = \sum_s (z^\dagger_{\mathbf{k}s}, \bar{z}_{-\mathbf{k}s}) \begin{pmatrix} \omega - \lambda + \frac{1}{2}h_s & z\gamma_{\mathbf{k}}Q \\ z\gamma_{\mathbf{k}}Q & -\omega - \lambda + \frac{1}{2}h_s \end{pmatrix} \begin{pmatrix} z_{\mathbf{k}s} \\ \bar{z}^\dagger_{-\mathbf{k}s} \end{pmatrix}, \tag{1.77}$$

where z and $\bar{z}$ are coherent state variables of sublattices A and B, respectively.

3. For the antiferromagnetic model, add a *uniform* magnetic field to H^{MF} of (1.19) as follows:

$$H^{MF} \to H^{MF} - h \sum_{is=\pm\frac{1}{2}} s e^{i\boldsymbol{\pi}\mathbf{x}_i} a^\dagger_{is} a_{is}. \tag{1.78}$$

Allow the constraint field to have uniform and staggered components:

$$\lambda_i = \lambda + \exp(i\boldsymbol{\pi}\mathbf{x}_i)\lambda_s. \tag{1.79}$$

Show that the mean field ground state energy is minimized for $\lambda_s = -\frac{1}{2}h$. Discuss how λ_s can be interpreted as a uniform precession at angular frequency $\omega = -\frac{1}{2}h$ of all spins in the xy plane.

4. Using the results of the previous exercise, derive the uniform susceptibility χ_0^{MF} of (1.40) as a second derivative of the free energy with respect to h_0. *Note that it is necessary to keep the temperature finite before taking the thermodynamic limit. Why?*

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