The goal of this problem is to derive an effective model for a frustrated ladder in a magnetic field. The starting point is the Heisenberg model defined by the Hamiltonian:

$$H = \sum_n J_\perp \ \vec{S}_{1n} \cdot \vec{S}_{2n} - B \sum_n (S^z_{1n} + S^z_{2n}) + \sum_{\langle nm \rangle} \sum_{i,j=1,2} J_{ij} \ \vec{S}_{in} \cdot \vec{S}_{jm} \tag{1}$$

In the spin operators $\vec{S}_{in}$, the index i refers to the leg, while the index n refers to the rung. The summation over $\langle nm \rangle$ means over nearest neighboring rungs. We want to derive an effective Hamiltonian in the limit $J_\perp \gg J_{ij}$.

Part I - One-rung problem

In this part, we consider a single rung described by the Hamiltonian

$$H_n = J_\perp \ \vec{S}_{1n} \cdot \vec{S}_{2n} - B(S^z_{1n} + S^z_{2n}) \tag{2}$$

1) Using symmetry arguments, show that the states $|S\rangle_n = \frac{1}{\sqrt{2}}(| \uparrow_{1n} \downarrow_{2n}\rangle - | \downarrow_{1n} \uparrow_{2n}\rangle$, $|T_1\rangle_n = | \uparrow_{1n} \uparrow_{2n}\rangle$, $|T_0\rangle_n = \frac{1}{\sqrt{2}}(| \uparrow_{1n} \downarrow_{2n}\rangle + | \downarrow_{1n} \uparrow_{2n}\rangle$ and $|T_{-1}\rangle_n = | \downarrow_{1n} \downarrow_{2n}\rangle$ are eigenstates of H.

2) Determine their energies, and plot them as a function of B. Determine the critical value of the magnetic field B_c at which $|S\rangle_n$ and $|T_1\rangle_n$ are degenerate ground states.

Part II - Two-rung problem

We now consider two neighboring rungs n and m, and we partition the Hamiltonian as $H_{nm} = H_0 + V$ with:

$$H_0 = J_\perp(\vec{S}_{1n} \cdot \vec{S}_{2n} + \vec{S}_{1m} \cdot \vec{S}_{2m}) - B_c(S^z_{1n} + S^z_{2n} + S^z_{1m} + S^z_{2m}) \tag{3}$$

and

$$V = \sum_{i,j=1,2} J_{ij} \ \vec{S}_{in} \cdot \vec{S}_{jm} - (B - B_c)(S^z_{1n} + S^z_{2n} + S^z_{1m} + S^z_{2m}) \tag{4}$$

It will prove convenient to use two basis of the Hilbert space:

$$\{|\sigma_{1n} \ \sigma_{2n} \ \sigma_{1m} \ \sigma_{2m}\rangle, \ \sigma_{1n}, \sigma_{2n}, \sigma_{1m}, \sigma_{2m} = \uparrow \ \text{or} \ \downarrow\}, \tag{5}$$

and

$$\{|AB\rangle \equiv |A\rangle_n \otimes |B\rangle_m, \ A, B = S, T_1, T_0 \ \text{or} \ T_{-1}\}. \tag{6}$$

From the properties of individual rungs, it should be clear that $|SS\rangle$, $|ST_1\rangle$, $|T_1S\rangle$ and $|T_1T_1\rangle$ are degenerate ground states of H_0. We consider the limit $J_\perp \gg J_{ij}, B - B_c$, and we treat V with first-order degenerate perturbation theory.

1) Write the ground states in the basis $\{|\sigma_{1n}\ \sigma_{2n}\ \sigma_{1m}\ \sigma_{2m}\rangle\}$.

2) In this question, we consider the case $J_{11} \neq 0$, $J_{12} = J_{21} = J_{22} = 0$.

a) Calculate $\vec{S}_{1n} \cdot \vec{S}_{1m}|SS\rangle$, $\vec{S}_{1n} \cdot \vec{S}_{1m}|ST_1\rangle$, $\vec{S}_{1n} \cdot \vec{S}_{1m}|T_1S\rangle$ and $\vec{S}_{1n} \cdot \vec{S}_{1m}|T_1T_1\rangle$.

b) Using symmetry considerations, show that V only couples $|SS\rangle$ and $|T_1T_1\rangle$ with themselves.

c) Calculate the matrix elements $\langle SS|\vec{S}_{1n} \cdot \vec{S}_{1m}|SS\rangle$ and $\langle T_1T_1|\vec{S}_{1n} \cdot \vec{S}_{1m}|T_1T_1\rangle$.

d) Determine the remaining matrix elements $\langle ST_1|\vec{S}_{1n} \cdot \vec{S}_{1m}|ST_1\rangle$, $\langle T_1S|\vec{S}_{1n} \cdot \vec{S}_{1m}|T_1S\rangle$, $\langle ST_1|\vec{S}_{1n} \cdot \vec{S}_{1m}|T_1S\rangle$ and $\langle T_1S|\vec{S}_{1n} \cdot \vec{S}_{1m}|ST_1\rangle$.

e) In the subspace $\{|S\rangle, |T_1\rangle\}$ of a rung n, one defines the operator $\vec{\sigma}_n$ by:

$$\sigma_n^z|S\rangle = -\frac{1}{2}|S\rangle,\ \sigma_n^z|T_1\rangle = \frac{1}{2}|T_1\rangle,\ \sigma_n^+|S\rangle = |T_1\rangle,\ \sigma_n^+|T_1\rangle = 0,\ \sigma_n^-|S\rangle = 0,\ \sigma_n^-|T_1\rangle = |S\rangle. \quad (7)$$

Show that the matrix elements of V are the same as those of an effective Hamiltonian of the form:

$$H_{\text{eff}} = J^{xy}(\sigma_n^x\sigma_m^x + \sigma_n^y\sigma_m^y) + J^z\sigma_n^z\sigma_m^z + B_{\text{eff}}(\sigma_n^z + \sigma_m^z) + C \quad (8)$$

where J^{xy}, J^z, B_{eff} and C are functions of J_{11}, B and B_c to be determined.

3) One now considers the general case where all couplings J_{ij} can be non-zero.

a) Using symmetry considerations, express the matrix elements of $\vec{S}_{1n} \cdot \vec{S}_{2m}$, $\vec{S}_{2n} \cdot \vec{S}_{1m}$ and $\vec{S}_{2n} \cdot \vec{S}_{2m}$ in terms of those of $\vec{S}_{1n} \cdot \vec{S}_{1m}$.

b) Write down the effective Hamiltonian of two rungs.

Part III - Ladders

1) Show that, to first order, the effective Hamiltonian is the sum of two-rung Hamiltonians. Write down the full effective Hamiltonian to first order.

2) Determine the condition under which the effective model is a one-dimensional Heisenberg model in a field. Discuss the physics depending on where the system lies with respect to this condition.