The Abdus Salam
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Educational, Scientific and
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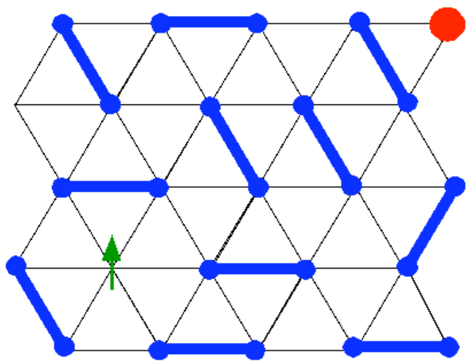
School and Workshop on Highly Frustrated Magnets and Strongly Correlated Systems: From Non-Perturbative Approaches to Experiments

30 July - 17 August, 2007

Quantum dimer models: liquidity, fractionalisation and topological order

Roderich Moessner
Oxford University, U.K.

Quantum dimer models: liquidity, fractionalisation and topological order



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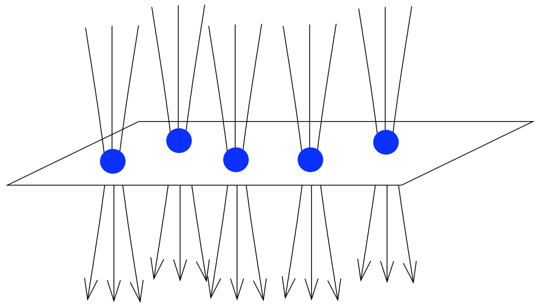
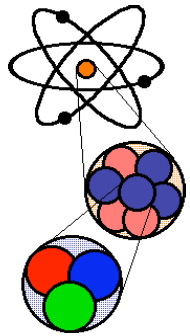


Many-body physics and collective behaviour

Complementary fundamental questions:

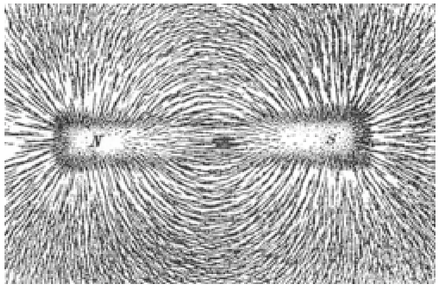


- What are *constituent* elements of matter, and their interactions?

  $\Rightarrow$ High-energy physics

- Given a set of degrees of freedom and their interactions: what is resulting *collective behaviour*?

  $\Rightarrow$ Many-body physics and complexity



Fluctuations and quantum dimer models

Fluctuations (thermal, quantum, . . .) destroy order.

$\Rightarrow$ what happens instead?

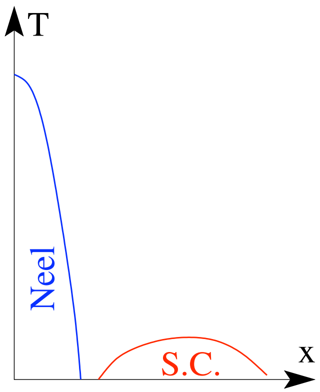
$\Rightarrow$ QDMs capture several aspects of new physics

Outline

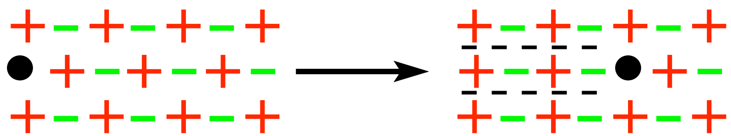
- historical perspective: high-temperature superconductors
  - spin liquids and fractionalisation
- quantum dimer models
  - phase diagram
  - liquidity and deconfinement
  - topological order
- Outlook

Background: short-range RVB physics

Basic problem of high-T_c: how do holes hop through an antiferromagnetic Mott insulator on square lattice?



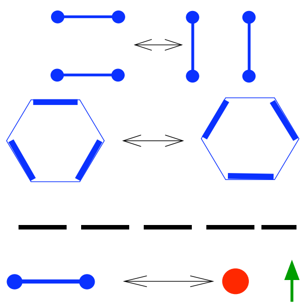
Hole motion is frustrated: hopping creates domain walls



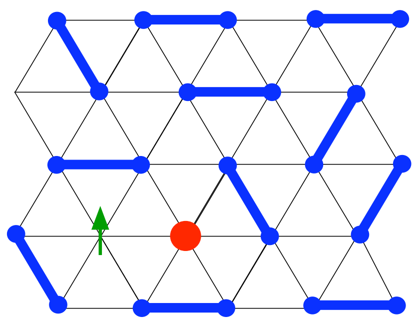
Possible resolution: magnet enters a different phase
resonating valence bond liquid phase
which breaks no symmetries. Neighbouring electrons form a singlet ("valence") bond, denoted by a dimer: $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle \sim$

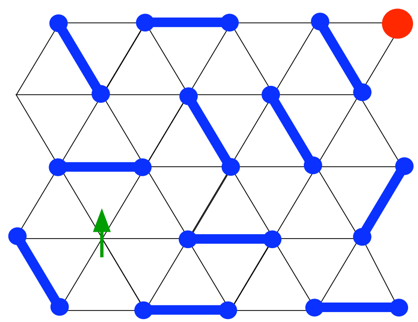
The basic RVB scenario - electron fractionalisation

Energetics	RVB	Neel
single pair	valence bond optimal	
higher coordination	energy from resonance	...each neighbour
hole doping	motion unimpeded	motion frustrated



- Basic resonance move is that of benzene
- Removing an electron $\longrightarrow$ holon + spinon



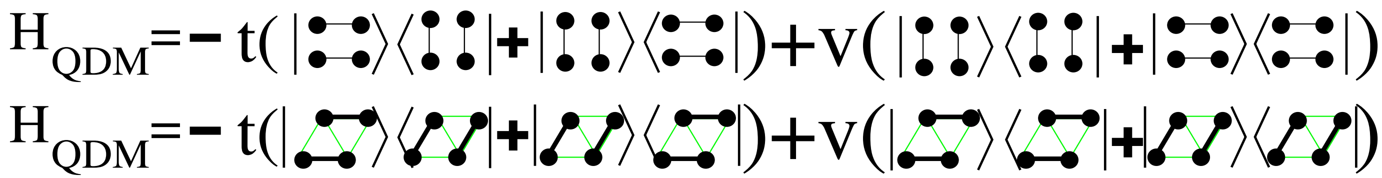


spinon and holon are deconfined

$\downarrow$

(bosonic) holons can condense

The Rokhsar-Kivelson quantum dimer model



- Hilbert space: exponentially numerous dimer coverings
- Resonance (t) and potential (v) term from uncontrolled approximation – one parameter: v/t
- RK point $v/t = 1$ is exactly soluble in $d = 2$ at $T = 0$:

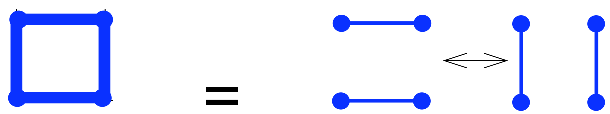
$$|0\rangle = \frac{1}{\sqrt{N_c}} \sum_c |c\rangle \rightarrow \langle \hat{P} \rangle = \frac{1}{N_c} \sum_{c,c'} \langle c|\hat{P}|c'\rangle = \frac{1}{N_c} \sum_c p_c$$

$\longrightarrow$ classical calculation for diagonal operators

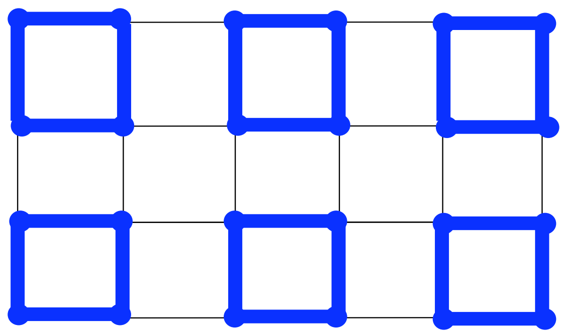
- $v/t > 1$ and limits of $v/t \rightarrow -\infty$ give solid (staggered and columnar, respectively) phases:

The enemy: order by disorder

- Consider $v = 0$: only term in H_{QDM} is kinetic term
- kinetic term gains energy from resonating plaquette:

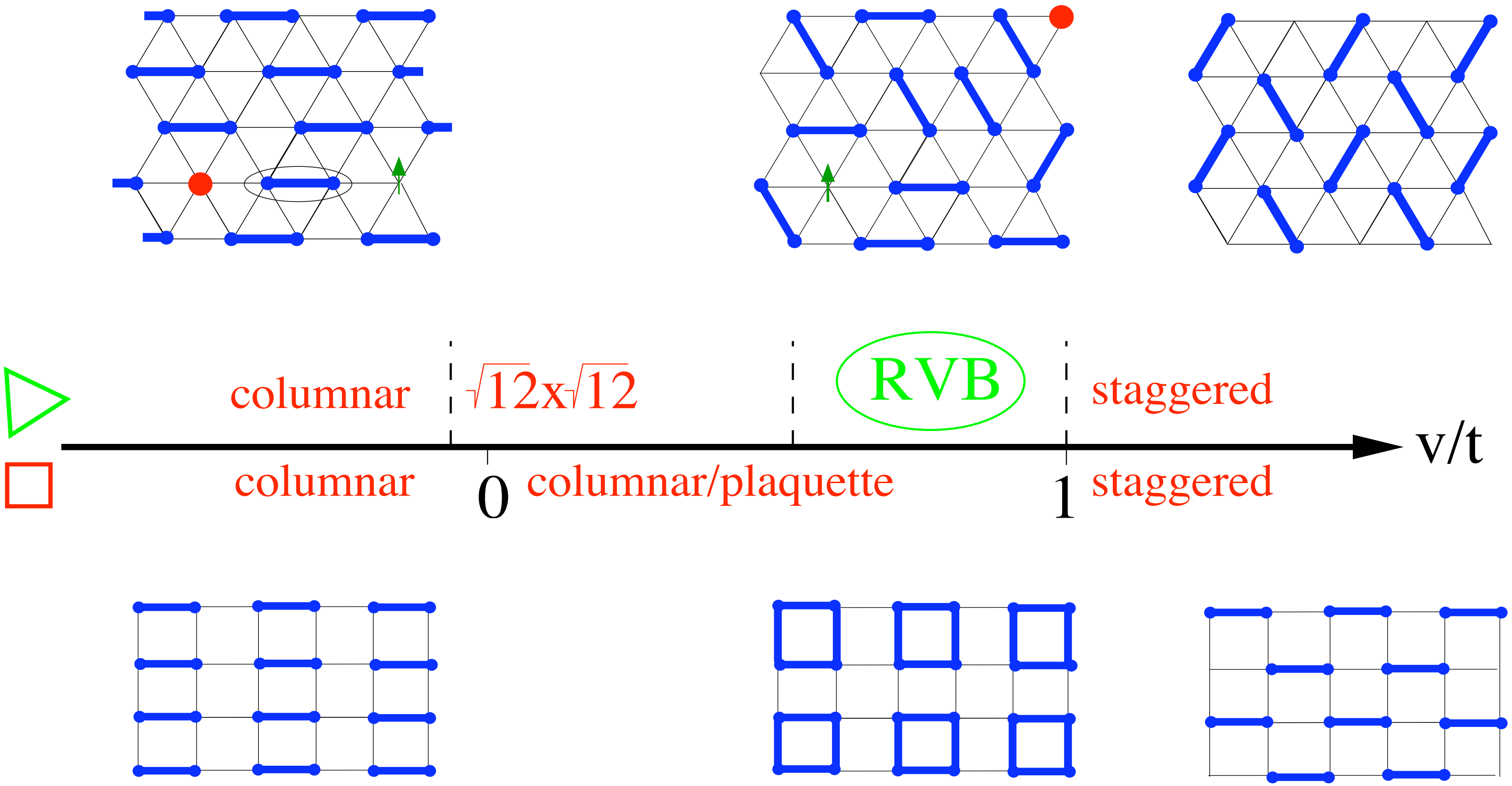


- Maximal energy gain $\longrightarrow$ dense packing
- Dense packing $\longrightarrow$ crystallinity
- Crystallinity $\longrightarrow$ symmetry breaking: ‘order by disorder’



- Plaquette solid: only variational guess!

Phase diagram for square and triangular lattices

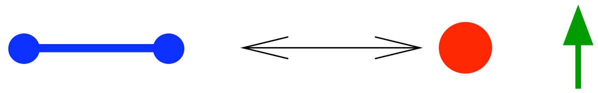


All phases on square lattice are confining RK; Sachdev; ...

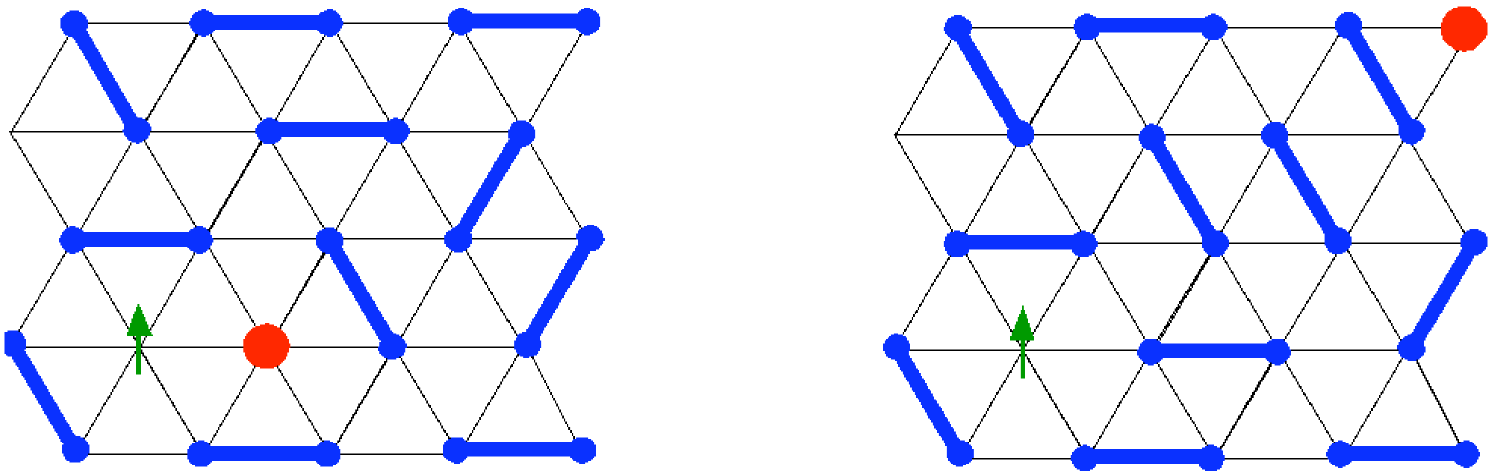
Triangular lattice has *bona fide* RVB phase

Liquidity and fractionalisation

- Removing an electron: holon (S=0) and spinon (q=0)

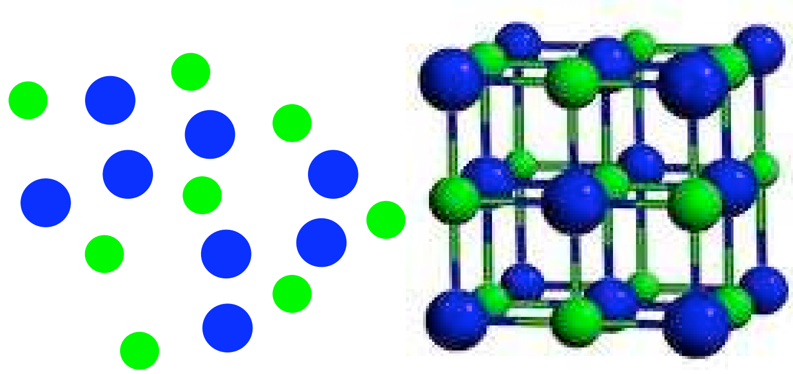


- Spinon and holons are deconfined: spin-charge separation

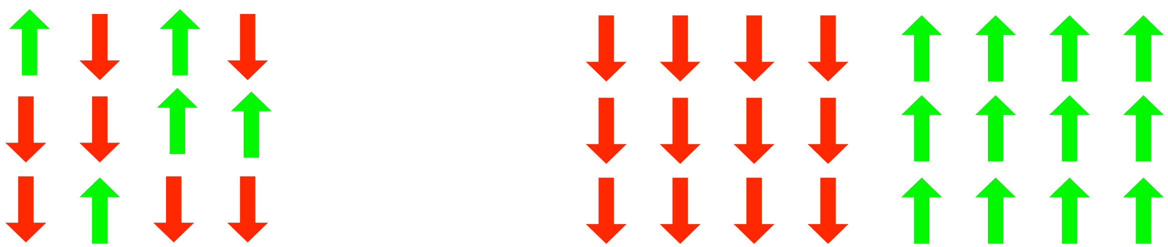


Anything beyond conventional order and disorder?

Gas-crystal (e.g. rock salt):



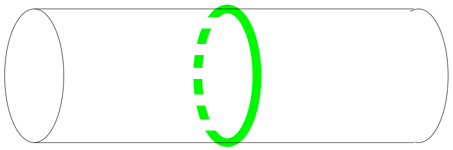
Paramagnet-ferromagnet



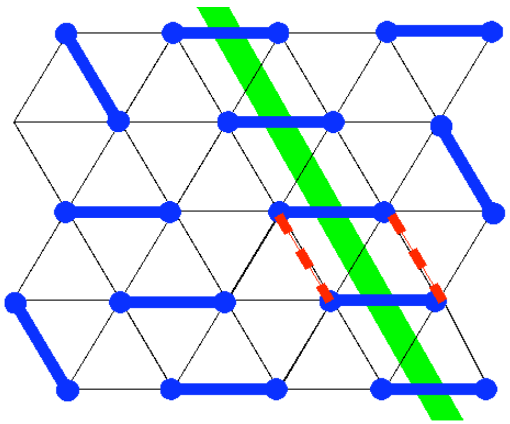
Anything else???

Liquidity and topological order

Topological order on surface of non-trivial topology (e.g. cylinder)



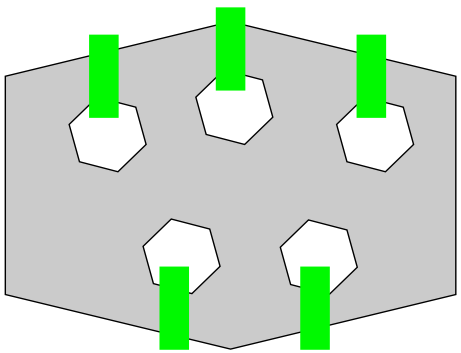
- Winding parity $\mathcal{P}$ with respect to cut is invariant under action of H_{RK}
  $\Rightarrow \mathcal{P}$ labels topological sectors
- Liquids locally indistinguishable $\Rightarrow$ ground states $|e\rangle, |o\rangle$ degenerate for $L \rightarrow \infty$:
  'topological degeneracy/order' Wen
- Unlike conventional order: degeneracy due to breaking of local symmetry

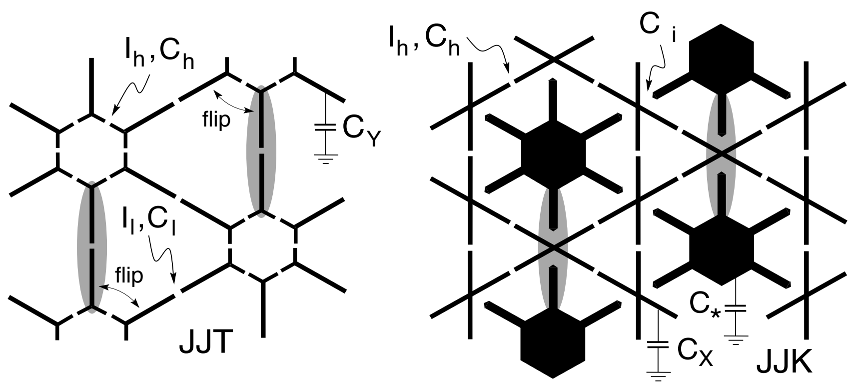


Topological quantum computing *Kitaev; Ioffe* et al.

Topological protection: Use $|\mathcal{P}\rangle = |e\rangle, |o\rangle$ as q-bit Kitaev

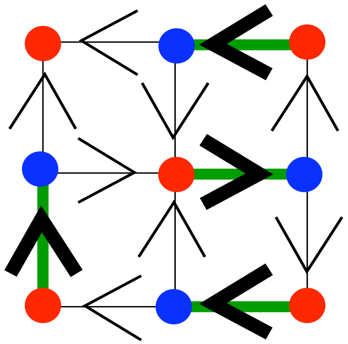
- Liquids locally indistinguishable: $E_N^e - E_N^o \propto \exp(-L)$
  $\Rightarrow$ local noise H_N <u>cannot</u> lead to dephasing
- Proposal is scalable: many cuts in single chip
- Implementation as Josephson-junction array Ioffe *et al.*
- Problem: logic gates; non-local operations, ...





A closer look at the square lattice

Orientation of dimers (from red to blue sublattice) is possible.



Analogy: dimer = flux $\vec{E}$

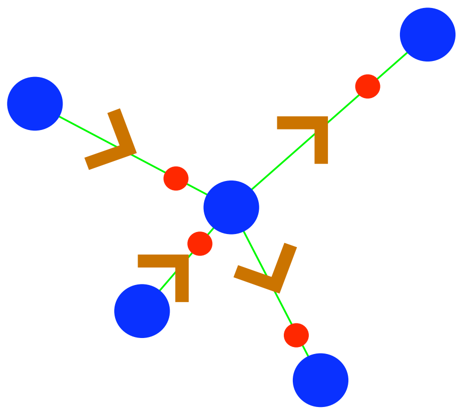
- Link with dimer $\rightarrow$ flux $\vec{E} = +3$
- Unoccupied link $\rightarrow$ flux $\vec{E} = -1$
- $\nabla \cdot \vec{E} = 0 \rightarrow \vec{E} = \nabla \times \vec{A} = \nabla \times h$

Vector potential $\vec{A}$ in $d = 2$ is simple scalar (height h) Youngblood et al.
Mapping to height takes care of hardcore constraint $\longrightarrow$ we can coarse-grain safely to get effective long-wavelength theory.

$$\mathcal{S}_q = \int (\partial_\tau \tilde{h})^2 - \rho_2 (\nabla \tilde{h})^2 - \rho_4 (\nabla^2 \tilde{h})^2 + \lambda \cos(2\pi \tilde{h})$$

<u>Crucial:</u> bipartitness$\Rightarrow$ height ($U(1)$ gauge) theory

Gauge theory for ice

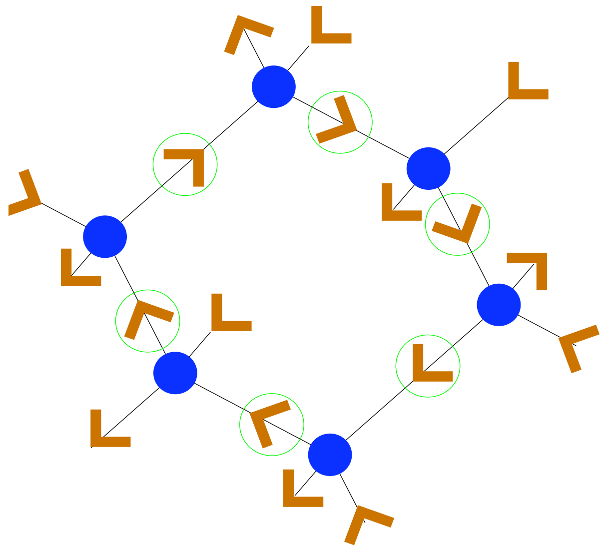


- Define 'flux' vector field on *links* of the ice lattice: $\mathbf{B}_i$
- Local constraint (ice rules) becomes conservation law (as in Kirchoff's laws)

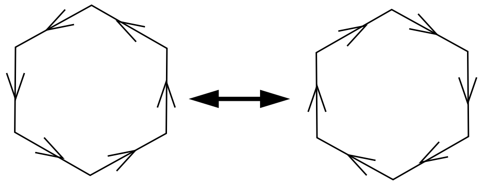
  $\Rightarrow$ gauge theory

$$\nabla \cdot \mathbf{B} = 0 \Longrightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

- Ice configurations differ by rearranging protons on a loop
- Amounts to reversing closed loop of flux $\mathbf{B}$
- Smallest loop: hexagon (six links)

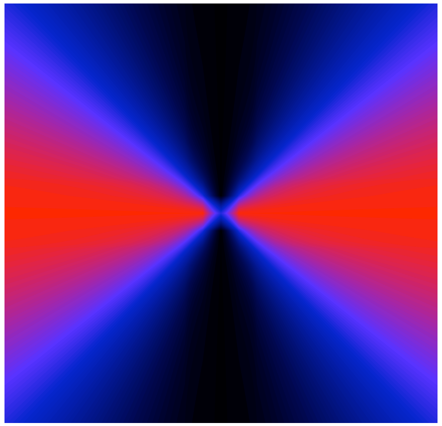


Long-wavelength analysis: coarse-graining



- Coarse-grain $\mathbf{B} \rightarrow \tilde{\mathbf{B}}$ with $\nabla \cdot \tilde{\mathbf{B}} = 0$
- 'Flippable' loops have zero average flux:
  low average flux $\Leftrightarrow$ many microstates
- Ansatz: upon coarse-graining, obtain energy functional of entropic origin:

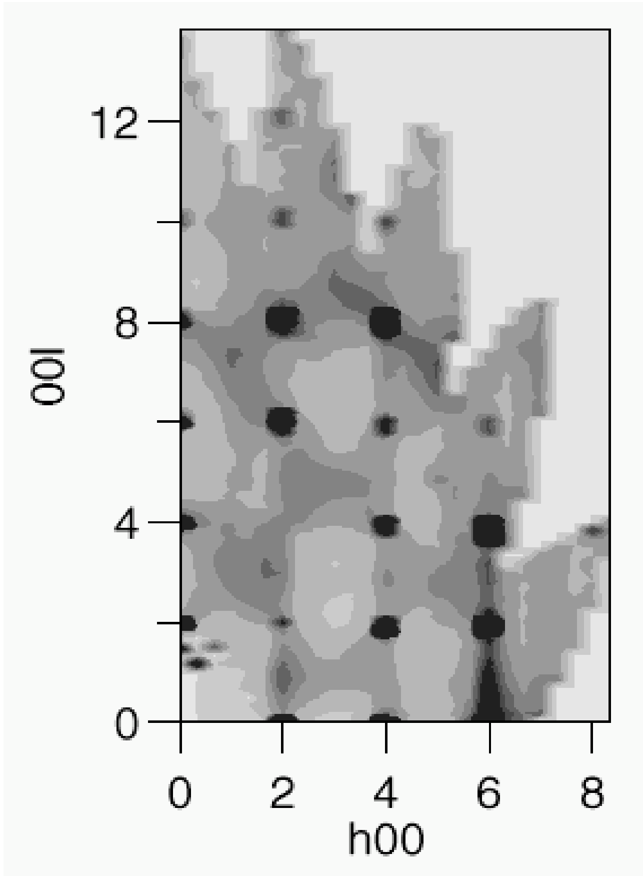
$$Z = \sum_{\mathbf{B}} \delta_{\nabla.\mathbf{B},0} \rightarrow \int \mathcal{D}\tilde{\mathbf{B}}\ \delta(\nabla \cdot \tilde{\mathbf{B}})\ \exp[-\frac{K}{2}\tilde{\mathbf{B}}^2]$$



- Artificial magnetostatics!
- Resulting correlators are transverse and algebraic (but not critical!): e.g.

$$\langle \tilde{B}_z(q)\tilde{B}_z(-q)\rangle \propto q_\perp^2/q^2 \leftrightarrow (3\cos^2\theta - 1)/r^3.$$

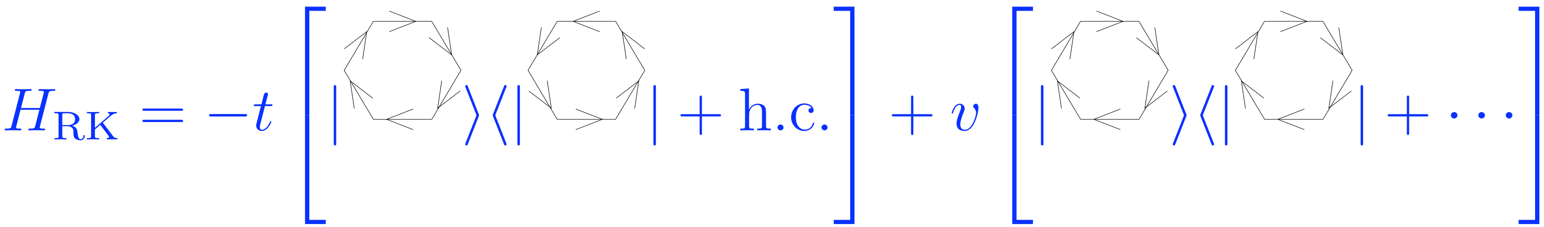
Bow-ties in the structure factor of ice



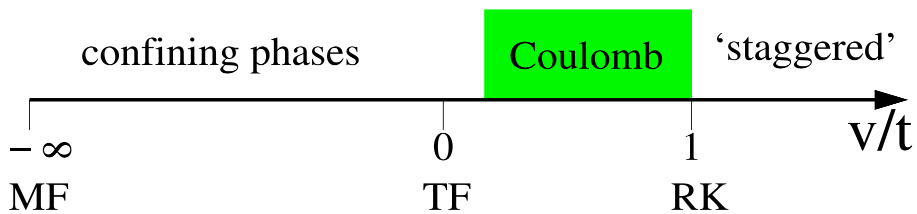
proton distribution in water ice, I_c Li *et al.*

'Quantum ice': artificial electrodynamics

- Hilbert space: (classical) ice configurations
- Add coherent quantum dynamics for hexagonal loop:



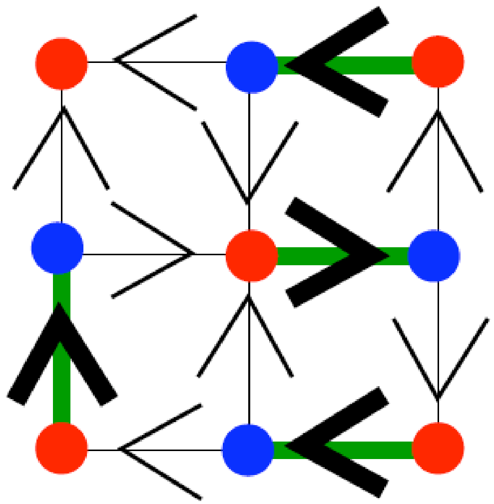
- Effective long-wavelength theory $\mathcal{H}_q = \int \tilde{\mathbf{E}}^2 + c^2 \tilde{\mathbf{B}}^2$ Maxwell
- This describes the Coulomb phase of a $U(1)$ gauge theory:
  - gapless photons, speed of light $c^2 \propto t - v$
  - deconfinement
  - microscopic model!



- Artificial electrodynamics with ice as 'ether'

Tutorial I: Winding number for bipartite lattices

- Orient each bond to point from sublattice A to B
- Assign $z-1$ units of flux to each dimer, -1 unit to each empty link
- Show that the total flux crossing a cut remains invariant under local dynamics



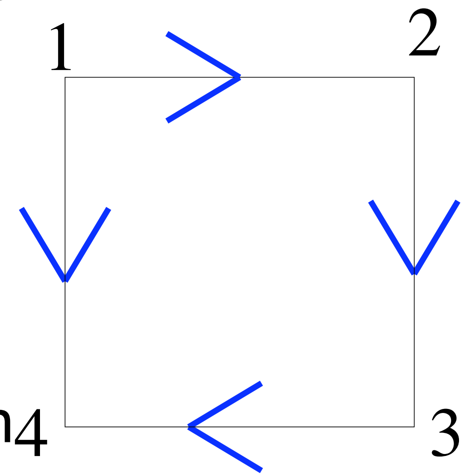
Tutorial II: Enumeration of dimer coverings

Define Grassman variables η_i on sites of **square plaquette**

- $\eta_i \eta_j = -\eta_j \eta_i \; ; \; \int d\eta \;\; \eta^\alpha = \delta_{\alpha,1}$

- Show that the number of dimer coverings equals:

$$N_d = \int \prod_{k=1}^{4} d\eta_k \;\exp[\frac{1}{2}\eta_i A_{ij}\eta_j] = \sqrt{|\det A|}$$



where $A_{ij} = \pm 1$ according to the direction of the arrow between i and j.

Square lattice: define antisymmetric $A_{jk} = 1 \; (i)$ if j is the left (bottom) neighbour of k.

- How can the above formula be used to calculate N_d?

- This can be done generally for planar lattices (Kasteleyn's thm)

Tutorial III: Overlap matrix

A ‘transition graph’ is obtained by superposing two dimer configurations, $|i\rangle$ and $|j\rangle$, and erasing all doubly occupied links.

- If a dimer stands for an SU(2) singlet, show that the overlap is given by

$$S_{ij} \equiv \langle i|j\rangle = \prod_{\alpha} 2x^{L_\alpha}$$

where the product runs over all loops in the graph, and L_α is the length of a loop.

- Show that the $\{|M\rangle\}$ with $M = 1 \dots N_d$

$$|M\rangle = \sum_i (S^{-1/2})_{Mi}|i\rangle$$

form an orthonormal set (if S is invertible).