Summer School on Novel Quantum Phases and Non-Equilibrium Phenomena in Cold Atomic Gases

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Introduction to the theory of the BEC/BCS crossover

Wilhelm Zwerger
Technical University of Munich
problem: Fermions $\uparrow \downarrow$ with density $n = k_F^3 / 3\pi^2$ and attractive two particle interaction $V_{\uparrow \downarrow}(\vec{x}) = -\tilde{g} \cdot \delta(\vec{x})$

two-particle bound state: $g > g_c$

binding energy $\epsilon_b = \hbar^2 / ma_s^2$
Superfluidity of Fermions

1911 conventional SC’s Hg, Al, … $T_c \approx 1 - 23$ K

1960 pairing in nuclei, $\Delta \approx \text{MeV}$

1972 superfluid $^3$He, $T_c = 1$ mK (p-wave)

1975 neutron stars

1986 high-$T_c$ SC’s La$_2$Cu O$_4$, … $T_c = 35 - 138$ K

1991 Alkali-doped C$_{60}$ $T_c \approx 30$ K

1994 p-wave SC’s in Sr$_2$Ru O$_4$ $T_c = 1.5$ K

1998 color SC’s $\langle qq \rangle \neq 0$, $\Delta \approx \text{GeV}$
Feshbach-Resonances

closed channel bound state
couples resonantly

\[ a_s = a_{bg} \left( 1 - \frac{\Delta B}{B - B_0} \right) \]

scatt. ampl. \[ f(k) = \frac{-1}{1/a_s + r^* k^2 + i k} \rightarrow \frac{i}{k} \] at \( a_s = \infty \)

effective range \( r^* \ll \lambda_F, \) bound state for \( a_s > 0 \)
scattering length in $^6\text{Li}$

![Graph showing the scattering length in $^6\text{Li}$ as a function of magnetic field (G). The x-axis represents magnetic field in G, ranging from 600 to 1200 G. The y-axis represents the scattering length (1000 $a_0$), ranging from -10 to 10. The graph shows a sharp increase in scattering length around 800 G.]
BCS-LIMIT: weak coupling if \( k_F |a_s| < 0.5 \) (in practice)

Pairs form and condense at

\[
T_c \approx 0.3 \, T_F \exp \left( -\frac{\pi}{2k_F |a_s|} \right) \ll T_F \quad 2\Delta_0 = 3.52 \, k_B T_c
\]

Superfluidity is destroyed by pair-breaking

Pair size \( \xi_b = \xi_0 = \frac{\hbar v_F}{\pi \Delta_0} \) is large \( k_F \xi_b \gg 1 \)

\[
k_F \xi_0 \approx \begin{cases} 
10^3 & \text{supercond.} \\
10^2 & ^3\text{He} \\
10^1 & \text{high-}T_c 
\end{cases}
\]
BCS-transition in a harmonic trap

local density approx. \( \xi_b \ll R_{TF} \quad (\Delta_0 \gg \hbar \omega) \) valid

if \( N \gg N^* \approx \exp 3\pi/2k_F|a_s| = 10^5 \) at \( k_F|a_s| = 0.4 \)

\( N \ll N^* \): pairs formed within a single shell \( \Delta_0 \ll \hbar \omega \)

as in atomic nuclei (Bruun, Heiselberg, Mottelson 02)
BEC limit: strong coupling $|\epsilon_b| \gg \varepsilon_F \rightarrow k_F a_s \ll 1$

point bosons $\xi_b \approx a_s \rightarrow 0$ form far above $T_c$

pair size $k_F \xi_b \ll 1$

$\xi_b \neq$ coherence length $\xi_0$

condensation temperature $T_c = 0.218 T_F$ from ideal Bose gas with $n_B = n/2$ and $m_B = 2m$
**critical temperature**

Nozieres/SR '85

Drechsler/Zw. '92

attractive fermions turn into repulsive bosons

\[ a_{dd} = 0.60 \, a > 0 \quad \text{Petrov/Shlyapnikov/Salomon '03} \]

**Universality** for broad resonances \( k_F r^* \ll 1 \quad \text{Ho '04} \)

\[ \Delta_0 \approx 0.5 \, \epsilon_F, \quad \mu \approx 0.4 \, \epsilon_F \quad \text{at} \quad T = 0 \]

\[ T_c \approx 0.16 \, T_F, \quad \mu(T_c) \approx 0.42 \, \epsilon_F, \quad s(T_c) \approx 0.7 \, k_B \]
Outline:

I) BCS-BEC crossover in 3D
   1) Thermodynamics Haussmann, Rantner, Cerrito
   2) Universal properties at resonance

II) Exact solution of the 1D problem
   1) Bethe-Ansatz Fuchs, Recati
   2) Imbalanced case, FFLO

III) Imbalanced gases, RF-spectroscopy Punk
BCS groundstate (variational Ansatz for arbitrary $g$)

$$\psi_{\text{BCS},N} = \hat{\mathcal{A}} \{ \phi(12) \phi(34) \cdots \phi(N-1N) \}$$

ideal Bose gas of identical pairs

pair-state $\phi(1, 2) = \psi(\vec{x}) \cdot \chi(\vec{r}) \cdot \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$

internal wavefunction $\chi(\vec{r}) \sim \exp -r/\xi_b \rightarrow$ pair radius $\xi_b$

COM-coord. $\vec{x} \rightarrow$ inhomogeneous pair-amplitude $\psi(\vec{x})$

increase $g$: extended $\rightarrow$ local pairs $\rightarrow$ smooth crossover

(Keldysh 1965, Eagles 1969, Leggett 1980)
BCS Hamiltonian

\[ \hat{H}_{\text{BCS}} = -\frac{\tilde{g}}{V} b_0^\dagger b_0 \] (only \( q = 0 \) pairs)

exact ground state for fixed \( \mu \) is a coherent state

\[ |\psi\rangle_{\text{BCS}} = \prod_k (u_k|00\rangle_k + v_k|11\rangle_k) \sim \exp \sum_k \frac{u_k}{v_k} c_k^\dagger c_{-k}\downarrow \langle 0 | \]

at fixed \( N \) (even):

\[ |\psi_N\rangle_{\text{BCS}} \sim \left( \sum_k \frac{v_k}{u_k} c_k^\dagger c_{-k}\downarrow \right)^{N/2} |0\rangle \]

\[ \chi_k = \frac{v_k}{u_k} = \text{Fourier transf. of internal wavefunction} \]
gap parameter $\Delta_k$ follows from gap-equation

$$\frac{1}{\tilde{g}} = \frac{1}{V} \sum_k \frac{1}{\sqrt{(\epsilon_k - \mu)^2 + \Delta_k^2}}$$

chemical potential $\mu$ from number equation ($\mu_{BCS} = \epsilon_F$)

$$\langle N \rangle = 2 \sum_k v_k^2 = \sum_k \left( 1 - \frac{\epsilon_k - \mu}{\sqrt{(\epsilon_k - \mu)^2 + \Delta_k^2}} \right)$$

momentum distrib.

$$f_k = v_k^2 \to \frac{4\pi n \xi_b^3}{(1 + (k\xi_b)^2)^2}$$

since $\mu_{BEC} \to -\epsilon_b/2$
exact solution of $\hat{H}_{\text{BCS}}$ at $T = 0$ and fixed $N$

Richardson + Gaudin 1963, Ortiz/Dukelsky PRA 2005

exact dynamics Barankov/Levitov ...

are Cooper pairs Bosons?

antisymmetrization $\hat{A}$ reduces the condensate fraction $N_0/N \approx \Delta_0/\epsilon_F \ll 1$ in weak coupling, nevertheless

BCS-superfluidity is BEC of pairs!
Low energy description ground state is a neutral s-wave superfluid for arbitrary coupling excitations $\omega(q) = cq$ Bogoliubov/Anderson '58 universal thermodynamics at low temperatures

$$S(T)/V = \frac{2\pi^2}{45} \left(\frac{k_B T}{\hbar c}\right)^3$$

sound velocity $c$
Many-body theory  single channel model with pseudopotential $V_{\uparrow \downarrow}(x) = \bar{g} \delta(x)$  ren.  $g = 4\pi \hbar^2 a/m$

Luttinger/Ward '60  \[ \Omega = -T \ln Z = \Omega(\hat{G}) \]
\[ \Omega[\hat{G}] = \beta^{-1} \left( -\frac{1}{2} \text{Tr}\{-\ln \hat{G} + [\hat{G}^{-1} \hat{G} - 1]\} - \Phi[\hat{G}] \right) \]

DeDominicis/Martin '64  entropy $S[\hat{G}, \hat{\Gamma}]$ as a functional of one- and two-particle functions $\hat{G}(k, \omega)$ and $\hat{\Gamma}(K, \Omega)$
\[ \delta\Omega[\hat{G}, \hat{\Gamma}]/\delta\hat{G} = 0 \quad \text{and} \quad \delta\Omega[\hat{G}, \hat{\Gamma}]/\delta\hat{\Gamma} = 0 \]

variational principle for functions!
Ladder-approx. \[ \Phi[G] = \sum_{i=1}^{\infty} \left[ \sum_{l=1}^{i-1} + \sum_{l=1}^{i} \right] \]

enforce gapless nature by a modified coupling constant

critical temperature and entropy at \( T_c \)

at unitarity \[ \frac{T_c}{\epsilon_F} = 0.160 \quad \frac{S}{Nk_B} = 0.71 \]

Burovski et al '06 \[ \frac{T_c}{\epsilon_F} = 0.152(7) \quad \frac{S}{Nk_B} = 0.16 \]
**Universality** at $a = \infty$ \[ F/N = \varepsilon_F \cdot g(T/T_F) \] Ho '04

implies \[ p = 2U/3V \] as if interactions were $1/r^2$

chemical potential $\mu/\varepsilon_F = \xi(T/T_F) = \begin{cases} 0.36 \text{ at } T = 0 \\ 0.39 \text{ at } T = T_c \end{cases}$

determines cloud size in a trap \[ R_{TF} = R_{TF}^{(0)} \cdot \xi^{1/4} \]

and critical temp. \[ T_c/\varepsilon_F(N) = 0.16 \cdot \xi^{-1/2}(T_c) \approx 0.25 \]

**field theory** $\epsilon = 4 - d$ - expansion \[ \xi(T = 0) = \frac{1}{2} \varepsilon^{3/2} + \frac{\varepsilon^{5/2}}{16} \ln \epsilon + \ldots = 0.47 \text{ at } \epsilon = 1 \]

$1/N$ - expansion \[ \xi(T_c) = 0.59 \text{ at } N = 1 \] Nikolic/S. '06
cloud size (in situ)

<table>
<thead>
<tr>
<th>Experimental results</th>
<th>( \beta )</th>
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<tbody>
<tr>
<td>Bartenstein et al. [61]</td>
<td>(-0.68^{+0.13}_{-0.10})</td>
</tr>
<tr>
<td>Bourdel et al. [62]</td>
<td>(-0.64(15))</td>
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<tr>
<td>Duke [63]</td>
<td>(-0.49(4))</td>
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<tr>
<td>Partridge et al. [64]</td>
<td>(-0.54(5))</td>
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<thead>
<tr>
<th>Calculated values</th>
<th>( \beta )</th>
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<tr>
<td>Astrakharchik et al. [17]</td>
<td>(-0.58(1))</td>
</tr>
<tr>
<td>Carlson et al. [16]</td>
<td>(-0.56(1))</td>
</tr>
<tr>
<td>Hu, Liu, and Drummond [67]</td>
<td>(-0.599)</td>
</tr>
<tr>
<td>Perali et al. [65]</td>
<td>(-0.545)</td>
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<tr>
<td>Padé approximation [8,9]</td>
<td>(-0.67)</td>
</tr>
<tr>
<td>Present work</td>
<td>(-0.64)</td>
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vortices as a signature of superfluidity

MIT
Cold Atoms in 1D  (Bloch, Weiss, Schmiedmayer, ..)

strong optical lattice
in two directions
typ. $\approx 10^3$ wires

pseudopotential $U(x) = g_1 \delta(x)$ with $g_1 = 2\hbar \omega_\perp \cdot a_s$

repulsive Bosons $\gamma = \frac{\epsilon_{int}}{\epsilon_{kin}} = \frac{g_1 n}{\hbar^2 m n^2} = \frac{2a_s}{n l_\perp^2}$

strong coupling at low 1D densities $n \ll a_s/l_\perp^2$

Tonks-limit $\gamma = \infty$: 1D Bosons with hard-core int.

$$\Psi_B(x_1 \ldots x_N) = \prod_{i<j} |\sin \left[ \frac{\pi(x_j - x_i)}{L} \right] | =$$

$$= |\Psi_F^{(0)}(x_1 \ldots x_N)| \leftrightarrow \text{free Fermions (Girardeau)}$$
Strongly interacting fermions in 1D

aritr. small attraction gives two-particle binding!

Gaudin/Yang '67: Bethe-Ansatz solution of

\[ H = -\frac{\hbar^2}{2m} \sum_\sigma \int \hat{\Psi}_\sigma^\dagger \partial_x^2 \hat{\Psi}_\sigma + \frac{g_1}{2} \sum_\sigma \int \hat{\Psi}_\sigma^\dagger \hat{\Psi}_\sigma^\dagger \hat{\Psi}_{-\sigma} \hat{\Psi}_{-\sigma} \]

dim.less interaction constant \( \gamma = mg_1/\hbar^2 n < 0 \)

\( H \) describes a Luther-Emery liquid: extended

pairs if \( |\gamma| \ll 1 \) transform to local pairs if \( |\gamma| \gg 1 \)

BEC-limit: pairs are hard-core bosons (TG-gas)
Olshanii '98: scattering theory in a single channel wire

eff. pot. between fermions with opposite spin \( g_1 \delta(x) \)

\[
g_1(a) = 2\hbar \omega_\perp \cdot \frac{a}{1 - A a / a_\perp}
\]

confinement induced resonance (CIR) at \( a = a_\perp / A \approx a_\perp \)

plus a bound state

for arbitrary sign of \( a \)

binding energy at CIR: \( |\epsilon_b| = 2\hbar \omega_\perp \gg \epsilon_F \rightarrow \)

BEC–side has unbreakable dimers with repulsive

interaction \( g_1 \rightarrow 2\hbar \omega_\perp \cdot 0.6 \, a \) (Mora et al '05)
\( \gamma < 0: \) increasing \(|\gamma|\) transforms Cooper-pairs to tightly bound molecules; size at CIR is \( \approx a_\perp \rightarrow \) hard core Bosons form Tonks gas

\( \gamma > 0: \) Boson size shrinks \( \rightarrow \) weakly interacting BEC

**Bethe-Ansatz** \((\text{Gaudin/Yang } \rightarrow \text{Lieb/Liniger})\)

\[
\psi_\sigma(x_1, \ldots x_N) = \sum_P A_\sigma\{P\} \exp i \sum_j k_{jp} x_j
\]

\(N \rightarrow \infty: \) quasimomenta distribution function \( \rho(k) \)

\[
\pi \rho(k) = 1 + \int_{-K}^{K} dq \frac{\gamma \rho(k)}{n \gamma^2 + [(k - q)/n]^2}
\]

for both \( \gamma < 0 \) and \( \gamma > 0 \)
**excitations:**

a) **fermionic** for $\gamma < 0$

triplet excit. (2 spinons): $\omega_s(q) = \sqrt{(\Delta / 2\hbar)^2 + (v_s q)^2}$

gap $\Delta \to 2\Delta_{BCS} = \epsilon_F 16\pi^{-3/2}|\gamma|^{1/2}\exp(-\pi^2/2|\gamma|)$

pair size reaches $n^{-1}$ at $\gamma \approx -2$

spin gap at strong coupling $\Delta \to |\epsilon_b| = 2\hbar \omega_\perp$

b) **bosonic** collective excit. for arbitrary $\gamma$

Bogoliubov-Anderson mode: $\omega_c(q) = v_c \cdot q$
Experiments: Feshbach in 1D \((^{40}\text{K}, \text{Moritz '05})\)

\[\omega_z \approx 2\pi \cdot 250 \text{ Hz}, \text{ particle number } N \sim \frac{\epsilon_F}{\hbar \omega_z} \approx 100\]

\[\omega_\perp \approx 2\pi \cdot 69 \text{ kHz}, \text{ temperature } T \approx 0.2 T_F\]

RF-spectroscopy of two-particle binding
**Imbalanced 1D gases**  Hu/Liu/Drummond, Orso '07

Clogston-Chandrasekhar  \( h_c \equiv \Delta / 2 \)  where  \( n_\uparrow \neq n_\downarrow \)

saturation field  \( h_s \) beyond which  \( n_\downarrow \equiv 0 \)

\( T = 0 \) phase diagram

partially polarized phase

exhibits FFLO - order

\[
\langle \Psi_\uparrow^\dagger(x) \Psi_\downarrow^\dagger(x') \Psi_\downarrow(x) \Psi_\uparrow(x') \rangle \sim \exp\left[i Q(x - x') / |x - x'|^{1/K}\right]
\]
Novel phases in imbalanced Fermi gases

fermionic superfluids possess a pairing gap

measurable by RF-spectroscopy Chin et al ’04

a) what is actually measured near unitarity?

b) is the appearance of a 'gap' proof of superfluidity?
The graph shows the relationship between $(\Delta v/\text{kHz})^{1/2}$ and the magnetic field (G) for various points, indicated by symbols. The inset graph presents the ratio for comparison over the magnetic field range from 700 to 900 G.
RF-spectrum as a probe of superfluidity

\[ I(\omega) \sim \int dt \ e^{i\omega t} \langle [\hat{\Psi}_3^\dagger(t) \hat{\Psi}_1(t), \hat{\Psi}_1^\dagger(0) \hat{\Psi}_3(0)] \rangle \]

peak shift

\[ \hbar \omega = \frac{\langle H'_{12} \rangle}{N_2} (g_{13} - 1) \rightarrow \frac{\langle H'_{12} \rangle}{N_2} \frac{\pi}{2} \left( \frac{1}{a_{13}} - \frac{1}{a_{12}} \right) \]

energy

\[ E = 2 \sum_k \epsilon_k (n_k - C/k^4) \quad C = sk_F^4 \]

\[ \hbar \omega = s \cdot \frac{4\epsilon_F^2}{n_2} \left( \frac{1}{g_{12}} - \frac{1}{g_{13}} \right) \quad \text{Punk/Zw. } '07 \]

gives \[ \hbar \omega = \begin{cases} 2\epsilon_b (1 - a_{12}/a_{13}) & \text{BEC-limit} \\ -0.19 \hbar^2 k_F/ma_{13} & \text{at unitarity} \end{cases} \]
Imbalanced Fermi gases \textcolor{green}{Partridge, Zwierlein '06}

\[ N_\uparrow - N_\downarrow \neq 0 \quad \text{implies} \quad \mu_\uparrow - \mu_\downarrow \equiv 2\hbar \neq 0 \]
Clogston-Chandrasekhar field near unitarity

Zwierlein et al '06 disappearance of vortex lattice beyond a critical imbalance \( \delta_c \approx 0.77 \)

Lobo et al '06 \( h_c \approx 0.96 \mu \rightarrow h_c \approx 0.4 \epsilon_F \)

saturation field \( h_s \) to a fully polarized Fermi gas

Chevy '06: \( h_s \geq 0.6 \mu_\uparrow \) implies \( h_s \geq 1.27 \epsilon_F \)
Radio-frequency offset [kHz]

RF in imbalanced gases

Schunck et al '07
Conclusions

1) The BCS-BEC crossover in the balanced case is qualitatively well understood, precise results for universal ratio’s and dynamical prop. are still open.

2) Imbalanced gases exhibit novel phases, like a 'magnetized superfluid' or (possibly) FFLO order. FFLO appears accessible in 1D imbalanced gases.
Collective mode damping and viscosity

shear viscosity $\eta$ from sound damping (hydrodynamic)

$$\omega(q) = cq - i \frac{2\eta}{3\rho} \cdot q^2 + \ldots$$

$$\alpha_\eta = \frac{\eta}{\hbar n} \sim m(v)\ell / \hbar \to k_F \ell \geq (6\pi)^{-1} \quad \text{Shuryak '04}$$

is the unitary Fermi gas a perfect liquid? \text{Son '07}

quantum hydrodynamics

$$\hat{H}_0 = \frac{\epsilon}{2} \int \left[ \frac{\rho}{c} (\nabla \varphi)^2 + \frac{\rho}{\rho} \Pi^2 \right]$$

phase $\hat{\varphi}(x)$ and density fluctuations $\hat{\Pi}(x)$

$$\hat{H}' = \frac{1}{6} \int \left[ \nabla \varphi \nabla \varphi + (\nabla \varphi)^2 \Pi + \Pi (\nabla \varphi)^2 + \frac{d}{d\rho} \left( \frac{c^2}{\rho} \right) \Pi^3 \right]$$

self-energy

$$\Sigma(q, \omega = cq) = -i \frac{\epsilon}{2\rho} \cdot q^2 \quad \text{at} \quad T = 0$$
\[ \zeta = \alpha_\zeta \hbar n \quad \text{with} \quad \alpha_\zeta = \frac{1}{4} \sqrt{\frac{\pi}{4} - \frac{1}{2}} \frac{c}{v_F} \left\{ 3 + \frac{v_F^2}{c^2} \frac{d}{dv_F} \left( \frac{c^2}{v_F} \right) \right\} \]

at unitarity (CIR)

\[ \alpha_\zeta = \frac{1}{2} \sqrt{\pi/2} - 1 = 0.38 \]

\[ T \neq 0 \quad \text{Im} \quad \Sigma(q, \omega = cq) \sim \sqrt{T} \cdot q^{3/2} \quad \text{Andreev '80} \]

sound damping is not hydrodynamic at finite \( T \)

dynamical scaling \( \Gamma_q = \frac{\hbar q^2}{2m} \Phi(q\xi_T) \quad \text{with} \)

\[ \xi_T = \frac{\hbar c}{T} \quad \text{and} \quad \Phi(x) = \begin{cases} \alpha_\zeta & \text{for } q\xi_T \gg 1 \\ 3.7 \alpha_\zeta / \sqrt{x} & \text{for } q\xi_T \ll 1 \end{cases} \]
partially polarized, normal phase \( h_c < h < h_s \)

poles of pair scattering amplitude \( \Gamma_{\uparrow,\downarrow}(\mathbf{q} = 0, \omega) \)

excitations at \( 2h - \Omega_+ \) and \( -2h + \Omega_- \)

binding energy of \( \uparrow, \downarrow \)-pairs approaches \( 0.6 \mu_\uparrow \) for \( h \gg h_c \) (Chevy '06)