Population and disease models

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2 Markov population processes.

2.1 Setting.

This chapter is substantially based on work of T. G. Kurtz – see the references.

Definition. A Markov population process (MPP) is a sequence of pure jump Markov processes $X_N(\cdot)$ on $\mathbb{Z}^d$ for some d, with transition rates given by

$$i \ \to \ i+j \qquad \text{at rate} \quad N\lambda_j(i/N), \qquad j \in J \subset \mathbb{Z}^d,$$

where the λ_j are *smooth* functions $\mathbb{R}^d \to \mathbb{R}_+$.

Here, we shall restrict ourselves to taking J to be a *finite* set, and shall assume that the λ_j's are all (fixed) functions of class C_2. One can relax these assumptions somewhat.

Exercise. Express the SIR-Markov epidemic as an MPP.

An MPP can be reformulated in terms of independent Poisson processes $\{P_j,\, j \in J\}$. Write $x_N(t) := N^{-1}X_N(t)$, and define the 'cumulative exposure to j-jumps' to be

$$\Lambda_{jN} \ := \ \int_0^t \lambda_j(x_N(u))\,du.$$

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Then the processes of j-jumps can be realized as $P_j(N\Lambda_{jN}(\cdot))$, $j \in J$. The MPP X_N can be realized as

$$X_N(t) \;=\; X_N(0) + \sum_{j\in J} jP_j(N\Lambda_{jN}(t)). \tag{2.1}$$

This construction is actually recursive in time — given the Poisson processes, the process X_N can be generated from them on a path by path basis, with no need to read the future...

2.2 The deterministic 'law of large numbers'.

Introduce the associated *deterministic equations*

$$\frac{d\xi}{dt} \;=\; F(\xi); \qquad \xi(0) = \xi_0, \tag{2.2}$$

where $F\colon \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$F(x) \;:=\; \sum_{j\in J} j\lambda_j(x).$$

Its integral form can be written as

$$\xi(t) \;=\; \xi(0) + \int_0^t F(\xi(u))\,du. \tag{2.3}$$

Note first that the deterministic integral equation (2.3) is very much like the Poisson construction (2.1) of X_N. Indeed, we have

$$\begin{aligned} x_N(t) &= x_N(0) + N^{-1}\sum_{j\in J} jP_j(N\Lambda_{jN}(t)) \\ &= x_N(0) + \sum_{j\in J} j\Lambda_{jN}(t) + w_N(t) \\ &= x_N(0) + \int_0^t F(x_N(u))\,du + w_N(t), \end{aligned} \tag{2.4}$$

where

$$w_N(t) \;:=\; N^{-1}\sum_{j\in J} j\widehat{P}_j(N\Lambda_{jN}(t)),$$

and $\widehat{P}_j(u) := P_j(u) - u$. This makes w_N a very nice process indeed: a vector valued martingale, with very nice sample paths. Subtracting (2.3) from (2.4), we get

$$x_N(t) - \xi(t) \;=\; x_N(0) - \xi(0) + \int_0^t \{F(x_N(u)) - F(\xi(u))\}\, du + w_N(t). \quad (2.5)$$

Suppose that $\|\lambda_j'\| := \sup_x |\lambda_j'(x)| < \infty$ for each j.

Then it follows that

$$|x_N(t) - \xi(t)| \;\le\; |x_N(0) - \xi(0)| + \int_0^t \|F'\|\, |x_N(u) - \xi(u)|\, du + |w_N(t)|.$$

Gronwall's inequality now implies that

$$\Delta_{NT}^{(1)} \;:=\; \sup_{0\le t\le T} |x_N(t) - \xi(t)| \;\le\; G_N e^{T\|F'\|},$$

where

$$G_N \;:=\; |x_N(0) - \xi(0)| + \sup_{0\le t\le T} |w_N(t)|.$$

Suppose also that $\|\lambda_j\| := \sup_x \lambda_j(x) < \infty$ for each j.

We note that

$$\sup_{0\le t\le T} |w_N(t)| \;\le\; N^{-1} \sum_{j\in J} |j| \sup_{0\le t\le T} |\widehat{P}_j(Nt\|\lambda_j\|).$$

From properties of the Poisson process,

$$\mathbb{P}[\sup_{0\le u\le U} |\widehat{P}(u)| > c\sqrt{U}\log U] \;\le\; K/U^2,$$

for suitably chosen c, K. Hence, except on a set of probability at most $K\sum_{j\in J}(NT\|\lambda_j\|)^2 = O(N^{-2})$, we have

$$\begin{aligned} \sup_{0\le t\le T} |w_N(t)| \;&\le\; cN^{-1} \sum_{j\in J} |j| \sqrt{NT\|\lambda_j\|} \log\{NT\|\lambda_j\|\} \\ &=\; O(\sqrt{T/N}\log(NT)). \end{aligned}$$

It thus follows that, if $|x_N(0) - \xi(0)| = O(N^{-1/2}\log N)$ as $N \to \infty$, then

$$\Delta_{NT}^{(1)} \;=\; O(N^{-1/2}\log N) \quad \text{for each fixed } T.$$

This gives the deterministic equations as a large N limit, with an explicit approximation error rate.

2.3 Central limit theorem

Assume also that $\|\lambda_j''\| < \infty$ for all $j \in J$.

Returning to (2.5), we now have, for example,

$$|F(x_N(t)) - F(\xi(t)) - DF(\xi(t))(x_N(t) - \xi(t))| \;\le\; \|D^2F\|\{\Delta_{NT}^{(1)}\}^2 \;=\; O(N^{-1}(\log N)^2).$$

Write $DF(\xi(t)) = A(t)$, and let $\alpha(t)$ be such that

$$\frac{d}{dt}e^{\alpha(t)} \;=\; A(t)e^{\alpha(t)}, \qquad \alpha(0) = I.$$

Of course, if A were constant, we would have $\alpha(t) = e^{At}$. Then we can develop (2.5) to give

$$N^{1/2}\{x_N(t) - \xi(t)\} \;=\; N^{1/2}\{x_N(0) - \xi(0)\} + \int_0^t A(u)N^{1/2}\{x_N(u) - \xi(u)\}\,du + h_N(t),$$

with $h_N(t) := N^{1/2}\{w_N(t) + \varepsilon_N(t)\}$, and

$$\sup_{0 \le t \le T} |\varepsilon_N(t)| = O(N^{-1}(\log N)^2).$$

'Solving' this linear integral equation now gives

$$\begin{aligned} N^{1/2}\{x_N(t) - \xi(t)\} \;&=\; N^{1/2}w_N(t) + e^{\alpha(t)}N^{1/2}\{x_N(0) - \xi(0)\} \\ &\quad + e^{\alpha(t)}\int_0^t e^{-\alpha(u)}A(u)N^{1/2}w_N(u)\,du + O(N^{-1/2}(\log N)^2), \end{aligned}$$

uniformly in $0 \le t \le T$; this expresses $N^{1/2}\{x_N(t) - \xi(t)\}$ in terms of the much nicer process $N^{1/2}w_N$.

Define the even nicer process

$$\tilde{w}_N(t) \;:=\; N^{-1}\sum_{j \in J} j\widehat{P}_j(N\Lambda_j(t)),$$

where $\Lambda_j(t) := \int_0^t \lambda_j(\xi(u))\,du$ should be a close, but non-random, approximation to $\Lambda_{jN}(t)$. Indeed, we have

$$\begin{aligned} |\Lambda_{jN}(t) - \Lambda_j(t)| \;&=\; t\|\lambda_j'\| \sup_{0 \le u \le t} |x_N(u) - \xi(u)| \\ &\le\; T\|\lambda_j'\|\Delta_{NT}^{(1)} \end{aligned}$$

for all $0 \le t \le T$. Then $w_N(t)$ and $\tilde{w}_N(t)$ differ only because of the shifts in the arguments of the Poisson processes, and, for a centred Poisson process $\widehat{P}$, we have

$$\sup_{0\le u\le U}\sup_{0\le v\le\delta}|\widehat{P}(u+v)-\widehat{P}(u)| \;\le\; c\sqrt{\delta}\log(U/\delta),$$

except with probability $(\delta/U)^2$. Hence, recalling the definitions of w_N and $\tilde{w}_N$, we obtain

$$\begin{aligned}\Delta^{(2)}_{NT} \;&:=\; \sup_{0\le t\le T}|w_N(t)-\tilde{w}_N(t)| \;\le\; cN^{-1}\sum_{j\in J}|j|\left\{\|\lambda'_j\|TN\Delta^{(1)}_{NT}\right\}^{1/2}\log N\\ &=\; O(N^{-3/4}(\log N)^{3/2}),\end{aligned}$$

except with probability of order $O(N^{-1})$.

Now we can approximate the nicer process $\tilde{w}_N$ using the Komlos–Major–Tusnady theorem: for suitably chosen c, one can construct a standard Brownian motion B_j such that

$$\sup_{0\le u\le U}|\widehat{P}_j(Nu)-B_j(Nu)| \;\le\; c\log(NU)$$

except on a set of probably at most N^{-1}. This enables one to establish that, except with probability of at most order $O(N^{-1})$,

$$\sup_{0\le t\le T}|N^{1/2}\tilde{w}_N(t)-W(t)| \;=\; O(N^{-1/2}\log N),$$

where

$$W(t) \;:=\; \int_0^t \sigma(u)\,dB(u),$$

B is a d-dimensional standard Brownian motion, and

$$\sigma^2(u) \;:=\; \sum_{j\in J} jj^T\lambda_j(\xi(u)).$$

Putting this all back into the expression for $N^{1/2}\{x_N(t)-\xi(t)\}$ in terms of the process $N^{1/2}w_N$, we obtain the following

Central limit theorem:

$$\begin{aligned}N^{1/2}\{x_N(t)-\xi(t)\} \;&=\; W(t)+e^{\alpha(t)}N^{1/2}\{x_N(0)-\xi(0)\}\\ &\qquad + e^{\alpha(t)}\int_0^t e^{-\alpha(u)}A(u)W(u)\,du + O(N^{-1/4}(\log N)^{3/2})\\ &=\; N^{1/2}\{x_N(0)-\xi(0)\} + \int_0^t e^{\alpha(t)-\alpha(v)}\sigma(v)\,dB(v) + O(N^{-1/4}(\log N)^{3/2}),\end{aligned}$$

this last by partial integration.

Note that this diffusion approximation has covariance matrix given by

$$\Sigma(t) \;:=\; \int_0^t e^{\alpha(t)-\alpha(v)}\sigma^2(v)e^{[\alpha(t)-\alpha(v)]^T}\,dv,$$

yielding the differential equation

$$\frac{d\Sigma}{dt} \;=\; A\Sigma + \Sigma A^T + \sigma^2$$

as an alternative means of calculation. The approximation also has infinitesimal drift $A(t)w$ at position w at time t – a time-transformed multidimensional Ornstein–Uhlenbeck process.

Notes. In terms of the SIR-epidemic process, there are two problems. First, the final size is not (directly) covered, because there is always the restriction to fixed, finite T: $T \to \infty$ is not easily deduced, in general. Secondly, the value $\xi(0)$ for the epidemic starting with e.g. only one initial infective is the point $(1, 0)$, and the differential equation then remains in this state for all time, whatever the value of R_0. This gives no counterpart to the deterministic thrshold theorem. Thus there are problems with applying the above result in epidemic theory that still need to be overcome, although the general approximation is extremely useful.