Population and disease models

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4 Schistosomiasis.

4.1 Construction of the model.

We follow a simplified version of Macdonald's (1965) model. We consider a closed community of H human hosts, exposed to water area A containing N snails (dead snails are immediately replaced by new ones). Let $X(t)$ denote the number of parasites in the whole human host population, and let $Y(t)$ be the number of *infected* snails. Write $\Delta := N/A$ for the snail density, $\Sigma := H/A$ for the human density.

Transitions:

$$\begin{array}{lcll} X & \to & X+1, Y \text{ unchanged} & \text{at rate } \alpha H(Y/A); \\ X & \to & X-1, Y \text{ unchanged} & \text{at rate } \gamma X; \\ Y & \to & Y+1, X \text{ unchanged} & \text{at rate } \beta X(1-Y/N); \\ Y & \to & Y-1, X \text{ unchanged} & \text{at rate } \delta Y. \end{array}$$

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4.2 Deterministic treatment.

Define

$$\begin{aligned} x' &:= X/H = \text{average parasite load per human host;} \\ y &:= Y/N = \text{proportion of infected snails.} \end{aligned}$$

Then using an average rates approach gives the differential equations

$$\frac{dx'}{dt} = \alpha\Delta y - \gamma x'; \qquad \frac{dy}{dt} = \beta(\Sigma/\Delta)x'(1-y) - \delta y.$$

Set $R_0 := \alpha\beta\Sigma/\gamma\delta$.

Threshold theorem: If $R_0 \leq 1$, no endemic infection. If $R_0 > 1$, there is an endemic equilibrium

$$\bar{x}' = \frac{\alpha\Delta}{\gamma}(1 - 1/R_0); \qquad \bar{y} = (1 - 1/R_0). \tag{4.1}$$

4.3 Diffusion approximation

Practical interest centres on the behaviour near the deterministic equilibrium $(\bar{x}, \bar{y})$, where now, to match the earlier definition of MPP's, we replace $x' = X/H$ by $x = X/N = x'\Sigma/\Delta$. Starting with $\xi_0 = (\bar{x}, \bar{y})$, the CLT is valid over any finite time interval $[0, T]$. We then have

$$DF(\bar{x}, \bar{y}) = \begin{pmatrix} -\gamma & \alpha\Sigma \\ \beta(1-\bar{y}) & -\beta\bar{x} - \delta \end{pmatrix} = \begin{pmatrix} -\gamma & \alpha\Sigma \\ \beta/R_0 & -\delta R_0 \end{pmatrix} =: A,$$

the same for all t (equilibrium!), and

$$\sigma^2(t) = \sum_{j \in J} jj^T \lambda_j(\bar{x}, \bar{y}) = 2(1 - 1/R_0) \begin{pmatrix} \alpha\Sigma & 0 \\ 0 & \delta \end{pmatrix}.$$

The covariance matrix $\Sigma(t)$ satisfies the linear equation

$$\frac{d\Sigma}{dt} = A\Sigma + \Sigma A^T + \sigma^2. \tag{4.2}$$

Equilibrium distribution.

Not covered by the general CLT for MPP's, which needs $N \to \infty$ for *fixed* T.

The stochastic equilibrium is in any case $(0,0)$ with probability 1.

However, the $T \to \infty$ limit of the CLT can be shown (over asymptotically long time intervals) to be a good approximation to a *quasi–equilibrium* distribution. In particular, the equilibrium covariance matrix $\Sigma(\infty)$ can be found by solving (4.2) with the LHS set equal to zero.

4.4 Stochastic threshold.

If $Y \ll N$, the process has transition rates close to those of a Markov branching process in two dimensions (e.g. use a coupling). For this branching process:

1. Parasites (X-individuals) have lifetime offspring (infected snails) distribution with geometric distribution having mean β/γ;
2. Infected snails (Y-individuals) have lifetime offspring (parasites) distribution with geometric distribution having mean $\alpha\Sigma/\delta$.

The mean matrix for the imbedded two-dimensional Galton Watson process is

$$\begin{pmatrix} 0 & \beta/\gamma \\ \alpha\Sigma/\delta & 0 \end{pmatrix}.$$

If $R_0 > 1$, one can solve the extinction probability fixed point equation to find out the probability of early extinction, starting from (X_0, Y_0): this yields

$$\mathbb{P}_0[\text{rapid extinction}] \;=\; q_X^{X_0} q_Y^{Y_0},$$

with

$$q_X \;:=\; (1+\delta/\alpha\Sigma)/(1+\beta/\gamma); \quad q_Y \;:=\; (1+\gamma/\beta)/(1+\alpha\Sigma/\delta).$$

4.5 Parameter estimation.

1. Estimate R_0 using $\widehat{R}_0 = 1/(1-\bar{y})$;
2. Use the age-prevalence and age-egg output data to construct further parameter estimates.

Neither procedure suggests that this model is at all realistic.