The Abdus Salam International Centre for Theoretical Physics

United Nations Educational, Scientific and Cultural Organization
IAEA International Atomic Energy Agency

1863-11

Advanced School and Conference on Statistics and Applied Probability in Life Sciences

24 September - 12 October, 2007

Generalized linear models III

Peter McCullagh

Department of Statistics
University of Chicago
Chicago IL 60637, USA

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Department of Statistics
University of Chicago

Trieste, October 2007

Outline

Binary response models

Model:

$Y_1, \ldots, Y_n$ independent Bernoulli variables
$E(Y_i) = \pi(x_i)$ is the mean
Need a model for the response function $\pi(x)$

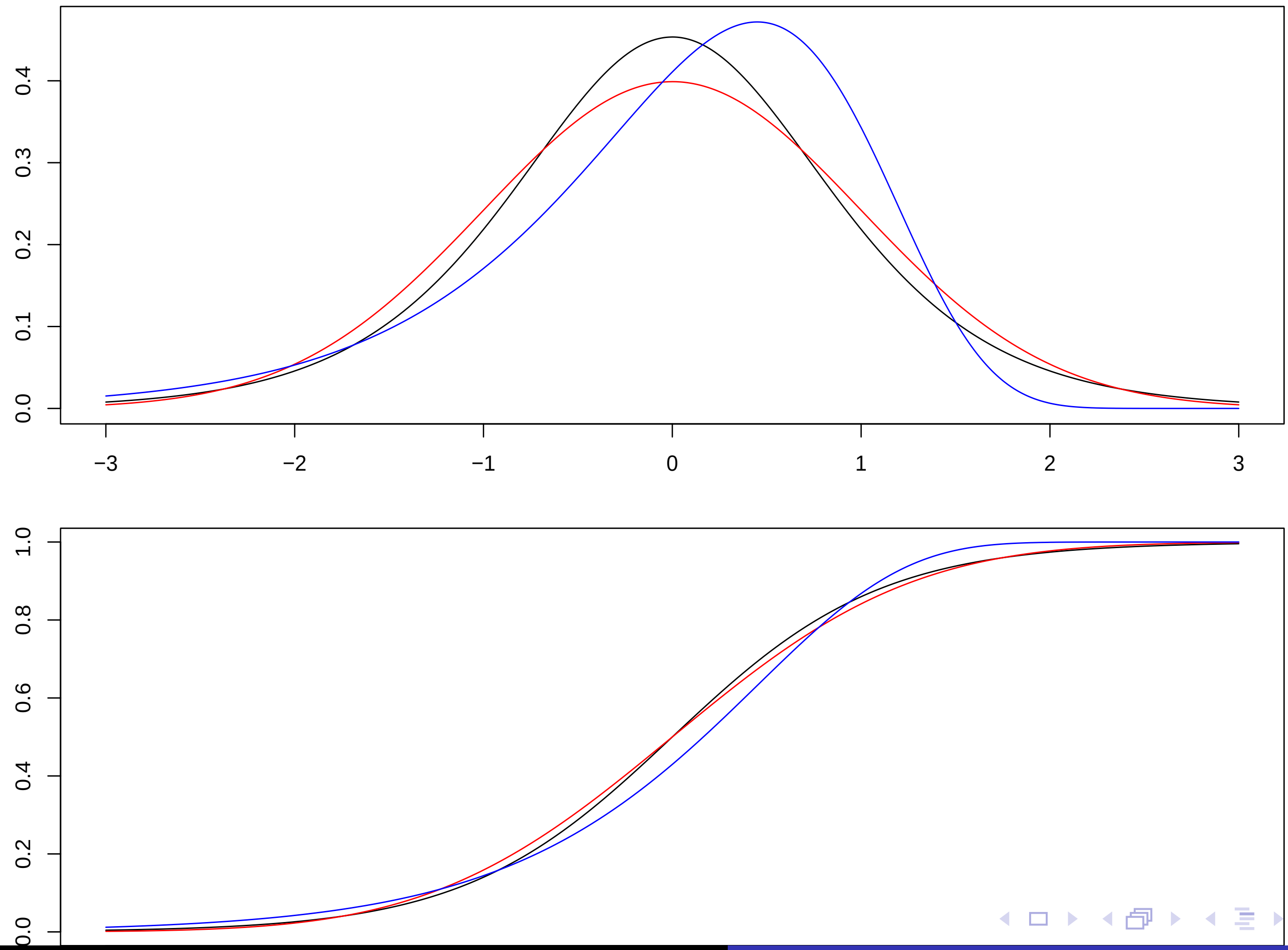
Constraint: $0 \le \pi(x) \le 1$

Standard response functions: $\pi(x) = F(\beta' x)$

F is a cumulative distribution function
$f = F'$ is a density function on $\mathcal{R}$

logit: $F(\eta) = e^\eta/(1 + e^\eta)$; $\eta = \log(\pi/(1-\pi))$
probit: $F(\eta) = \Phi(\eta)$; $\eta = \Phi^{-1}(\pi)$
c-log log: $F(\eta) = 1 - \exp(-e^\eta)$; $\eta = \log(-\log(1-\pi))$

Associated standardized densities



Logistic models

$$\mathrm{odds}(E) = \frac{\mathrm{pr}(E)}{\mathrm{pr}(\bar{E})}, \qquad \mathrm{pr}(E) = \frac{\mathrm{odds}(E)}{1 + \mathrm{odds}(E)}$$

$$\mathrm{logit}(E) = \log \mathrm{odds}(E)$$

$$\mathrm{logit}\,\pi(x) = \beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p$$

$$\mathrm{odds}(Y = 1; x) = \exp(\beta_0 + \beta_1 x_1 + \cdots + \beta_p x_p)$$

Parameter interpretation:

β_p is the change in the log odds... per unit change in x_p when $x_1, \ldots, x_{p-1}$ are held fixed

e^{β_p} is the multiplicative change in the odds per unit change in x_p...

The change in the probability is approximately

$$d\pi(x) = \pi(x)(1 - \pi(x))\beta_p dx_p < \beta_p dx_p/4$$

Example: Titanic survival rates

Data for 1316 passengers and 885 crew
Data listed by individual

Name	class	child/adult	female/male	survived
Andrews, C	1	0	1	1
Andrews, T	1	0	0	0
Rothschild, M	1	0	0	0
Young, M	1	0	1	1
Abelson, S	2	0	0	0
Abbott, E	3	0	0	0
⋮	⋮	⋮	⋮	⋮

Coding: class 0=crew; 1=first, 2, 3
adult=0, child=1; male=0, female=1; died=0, survived=1

Summary table

Summary data for 1316 passengers and 885 crew

	crew	class1	class2	class3	child	female	surv	pct
crew	885	0	0	0	0	23	212	24
class1	0	325	0	0	6	145	203	63
class2	0	0	285	0	24	106	118	41
class3	0	0	0	706	79	196	178	25
child	0	6	24	79	109	45	57	52
female	23	145	106	196	45	470	344	73
surv	212	203	118	178	57	344	711	32

Logistic model

Main effects only:

$$\begin{aligned} \mathrm{odds}(\text{survival}) &= \text{class effect} \times \text{sex effect} \times \text{age effect} \\ \log \mathrm{odds}(\text{survival}) &= \text{class effect} + \text{sex effect} + \text{age effect} \end{aligned}$$

Class treated as a 4-level factor
model formula used: `class+child+female`

Parameter estimates and S.E.s

	`class+child+female`		`class+child+female-1`	
(Intercept)	-1.23	0.08	–	–
crew	–	–	-1.23	0.080
class1	0.86	0.157	-0.38	0.136
class2	-0.16	0.174	-1.39	0.155
class3	-0.92	0.149	-2.15	0.127
child	1.06	0.244	1.06	0.244
female	2.42	0.140	2.42	0.140

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Fitted survival odds:

class1 adult female: $\exp(-0.38 + 2.42) = 7.69$, prob=0.89
class1 adult male: $\exp(-0.38) = 0.68$, prob=0.41
class2 adult male: $\exp(-1.39) = 0.25$, prob=0.20
class3 adult male: $\exp(-2.15) = 0.12$, prob=0.10
class3 male child: $\exp(-2.15 + 1.06) = 0.34$, prob=0.25

Does the additive model fit?

Children and females given preferential treatment

Does the degree of preference depend on class?

Model formula: `child+class+female+female*class`

Parameter	estimates	S.E.
class0	-1.25	0.082
class1	-0.68	0.158
class2	-1.91	0.219
class3	-1.70	0.124
child	1.05	0.230
female	3.15	0.625
class1:female	1.09	0.820
class2:female	0.64	0.725
class3:female	-1.78	0.652

Fitted survival odds:

class1 adult female: $\exp(-0.68 + 3.15 + 1.09) = 35.16$ pr=0.97

class3 adult female: $\exp(-1.70 + 3.15 - 1.78) = 0.72$ pr=0.41

Formal test of significance

Likelihood ratio or deviance test: Deviances:

Model	Deviance	d.f.	Dev diff
`child+class+female`	2210	2195	
`child+class*female`	2143	2192	67 (3)
`class+child*female`	2192	2194	18 (1)
`class*child+female`	2174	2193	36 (2)
`all 2-factors`	2097	2187	113 (8)

All 2-factor interactions strongly significant.

Some findings:

Preferential treatment given to females;

Degree of preference higher in 1st class than 3rd

Effects of nuisance parameters

Binary matched pairs problem:

Subjects come in matched pairs (blocks of size 2)

Pair effects $\lambda_1, \ldots, \lambda_n$

Common treatment effect Δ

$$\mathrm{pr}(Y_{i1} = 1) = \frac{e^{\lambda_i}}{1 + e^{\lambda_i}}, \quad \mathrm{pr}(Y_{i2} = 1) = \frac{e^{\lambda_i + \Delta}}{1 + e^{\lambda_i + \Delta}}$$

Likelihood

$$\prod_i \frac{e^{\lambda_i y_{i1}}}{1 + e^{\lambda_i}} \frac{e^{\lambda_i y_{i2} + \Delta y_{i2}}}{1 + e^{\lambda_i + \Delta}} = \prod_i \frac{e^{\lambda_i y_{i\cdot}}}{(1 + e^{\lambda_i})(1 + e^{\lambda_i + \Delta})} \times e^{\Delta y_{\cdot 2}}$$

So $(\{y_{i\cdot}\}, y_{\cdot 2})$ is sufficient
and $(\{y_{i\cdot}\}$ is sufficient for fixed Δ.

Binary matched pairs (contd)

Maximum likelihood estimation: Sufficient statistic $(\{y_{i.}\}, y_{.2})$

$$
\begin{aligned}
E(Y_{i.};\lambda,\Delta) &= \frac{e^{\lambda_i}}{1+e^{\lambda_i}} + \frac{e^{\lambda_i+\Delta}}{1+e^{\lambda_i+\Delta}} \\
E(Y_{.2};\lambda,\Delta) &= \sum \frac{e^{\lambda_i+\Delta}}{1+e^{\lambda_i+\Delta}}
\end{aligned}
$$

Maximum likelihood equations for fixed Δ

$$
\hat\lambda_i(\Delta) = \begin{cases} -\infty & y_{i.} = 0 \\ -\Delta/2 & y_{i.} = 1 \\ \infty & y_{i.} = 2 \end{cases}
$$

Profile likelihood is a product over the mixed-response pairs

$$
\mathrm{pr}(0,1;\hat\lambda,\Delta) = \frac{e^{\Delta/2}}{\text{[illegible]}}
$$

Conditional likelihood

$$\begin{aligned}
\mathrm{pr}(Y_{i1}, Y_{i2} = (0,1)) &= \frac{e^{\lambda_i+\Delta}}{(1+e^{\lambda_i})(1+e^{\lambda_i+\Delta})} \\
\mathrm{pr}(Y_{i1}, Y_{i2} = (1,0)) &= \frac{e^{\lambda_i}}{(1+e^{\lambda_i})(1+e^{\lambda_i+\Delta})} \\
\mathrm{pr}(Y_{i1}, Y_{i2} = (0,1) \,|\, \mathrm{mixed}) &= \frac{e^{\lambda_i+\Delta}}{e^{\lambda_i}+e^{\lambda_i+\Delta}} \\
&= e^{\Delta}/(1+e^{\Delta})
\end{aligned}$$

Hence the mixed pairs are conditionally iid

The conditional distribution given $n_{10} + n_{01} = S$

$$n_{01} \sim \mathrm{Bin}(S, e^{\Delta}/(1+e^{\Delta}))$$

Conditional mle is $\hat{\Delta}_c = \log(n_{01}/n_{10}) = \hat{\Delta}/2$